Travelling Waves in Inhomogeneous Branching Brownian Motions II

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ABSTRACT

We study an inhomogeneous branching Brownian motion in which individual particles execute standard Brownian movements and reproduce at rates depending on their locations. The rate of reproduction for a particle located at x is $\beta(x) = b + \beta_0(x)$, where $\beta_0(x)$ is a nonnegative, continuous, integrable function. Let $M(t)$ be the position of the rightmost particle at time t; then as $t \to \infty$, $M(t) -$ med $\big(M(t)\big)$ converges in law to a location mixture of extreme value distributions. We determine med $\big(M(t)\big)$ to within a constant $+\, o(1)$. The rate at which med $\big(M(t)\big) \to \infty$ depends on the largest eigenvalue λ of a differential operator involving $\beta(x)$; the cases $\lambda < 2, \lambda = 2$, and $\lambda > 2$ are qualitatively different.

KEY WORDS AND PHRASES: Inhomogeneous branching Brownian motion, travelling wave, extreme value distribution, Feynman-Kac formula.

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RUNNING HEAD: Travelling Waves II

1. Statement of Principal Results

An inhomogeneous branching Brownian motion (IBBM) is a branching process in which individual particles execute independent Brownian motions and undergo binary fission at a rate $\beta(x)$ depending on the spatial position x. At time $t = 0$ there is a single particle located at $x = 0$. In [6] (cf. also [4]) we showed that if β is continuous and $\int_{-\infty}^{\infty} \beta(x)dx < \infty$ then the distribution of $M(t)$, the position of the rightmost particle at time t, approaches a travelling wave as $t \to \infty$. In this paper we consider the case $\beta(x) = b + \beta_0(x)$, where $b > 0$, $\beta_0(x) \geq 0$, β_0 is continuous, and $\int_{-\infty}^{\infty} \beta_0(x)dx < \infty$.

The asymptotic behavior of the distribution of $M(t)$ depends on $\beta(x)$ primarily through the largest eigenvalue λ and corresponding L^2-eigenfunction $\varphi(x)$ (normalized so that $\varphi(0) = 1$) of the differential operator $g(x) \to \frac{1}{2}g''(x) + b^{-1}\beta(x/\sqrt{b})g(x)$. Since $b^{-1}\beta(x/\sqrt{b}) \geq 1$ and $\int_{-\infty}^{\infty} \beta_0(x)dx < \infty$, $\lambda \geq 1$; also $\varphi(x) > 0$ everywhere (cf. [3], Ch. 9). Set $\mu = \lambda\{2(\lambda - 1)\}^{-\frac{1}{2}}$.

THEOREM: *If $\lambda > 2$ there exists a cumulative distribution function $F(x)$ such that as $t \to \infty$*

$$P\{M(t) \leq \sqrt{b}\mu t + x\} \to F(x) \tag{1.1}$$

for all $x \in \mathbb{R}$. If $\lambda = 2$ there exists a c.d.f. $F(x)$ such that as $t \to \infty$

$$P\{M(t) \leq \sqrt{b}\left(\sqrt{2}t - (\frac{1}{2\sqrt{2}}) \log t\right) + x\} \to F(x) \tag{1.2}$$

for all $x \in \mathbb{R}$. If $\lambda < 2$ there exists a c.d.f. $F(x)$ such that

$$P\{M(t) \leq \sqrt{b}\left(\sqrt{2}t - (3/2\sqrt{2}) \log t\right) + x\} \to F(x) \tag{1.3}$$

for all $x \in \mathbb{R}$.

This result should be compared with the corresponding result for homogeneous ($\beta \equiv 1$) branching Brownian motion (BBM). For this process it is known (cf. [8], also [1]) that the distribution of the position of the rightmost particle at time t approaches a travelling wave, and the median m_t satisfies $m_t = \sqrt{2}t - (3/2\sqrt{2}) \log t + \text{constant} + o(1)$ as $t \to \infty$. Comparing (1.1)–(1.3) when $b = 1$, one sees that if $\lambda > 2$ the right edge travels much faster than for BBM, if $\lambda = 2$ the right edge travels a little faster, and if $\lambda < 2$ it travels at essentially the same rate. Thus there is a "threshold" effect: when the enhancement $\beta_0(x)$ of the base reproductive rate 1 becomes sufficiently "large" that the eigenvalue λ crosses from $\lambda < 2$ to $\lambda > 2$, suddenly the production of extra particles allows the process to outrun the homogeneous BBM.

We shall refer to the different cases $\lambda > 2$, $\lambda = 2$, and $\lambda < 2$ as the supercritical, critical, and subcritical cases, respectively. We shall assume throughout the rest of the paper that $b = 1$; the general case may be recovered by rescaling time and space. Furthermore, we shall only consider the special case where $\beta_0(x)$ has compact support: the general case may be obtained by a modification of the argument similar to that in sections 6–7 of [6]. For our analysis of the critical case we shall borrow some delicate estimates from Bramson [2]. For the subcritical and supercritical cases, however, no such heavy machinery will be needed.

2. The Basic Argument for the Critical and Supercritical Cases

Consider an IBBM with branching rate function $\beta(x) = 1 + \beta_0(x)$, where β_0 is continuous, has compact support, $\beta_0 \geq 0$, and $\beta_0(x) > 0$ somewhere. Using an auxiliary randomization, classify the particles of the IBBM as "blue" or "red" as follows: if a particle is born at postion x, label it blue with probability $\beta_0(x)/\beta(x)$ and red with probability $1/\beta(x)$. The original particle, which is born at $x = 0$, $t = 0$, is labelled blue. Observe that all blue particles other than the original are born in the (compact) support of β_0.

Clearly, each particle (blue or red) produces red descendants at constant rate 1. Thus, for each blue particle, the movements and reproductive histories of itself and its direct red descendants(i.e., those without an intermediate blue parent) constitute a *homogeneous* BBM (constant birth rate 1). Thus, the IBBM is the superposition of a sequence of homogeneous BBMs. If $M_i(t)$ is the position of the rightmost particle among the ith blue particle and its direct red descendants, then

$$M(t) = \max\left(M_1(t), M_2(t), \ldots, M_{N_B(t)}(t)\right),$$

where $N_B(t)$ is the number of blue particles born by time t.

The process of blue particle births looks approximately like an inhomogeneous, doubly stochastic Poisson process for large t. This follows from a theorem of Watanabe [9], which implies that for each bounded interval J

$$\lim_{t\to\infty} N(t;J)/e^{\lambda t} = Z\int_J \varphi(x)dx \quad a.s. \tag{2.1}$$

Here $N(t;J)$ is the number of IBBM particles in J at time t, $\lambda > 1$ (since $\beta_0(x) > 0$ somewhere) is the leading eigenvalue of the differential operator $g \to \frac{1}{2}g'' + \beta g$, $\varphi(x)$ the corresponding L^2-eigenfunction such that $\varphi(0) = 1$, and

$$Z = \lim_{t\to\infty} e^{-\lambda t}\int_{\mathbb{R}} \varphi(x)N(t;dx) > 0 \quad \text{a.s.}$$

Because of (2.1) and the continuity of β_0 and φ, blue particles are born in dx at rate (approximately)

$$Ze^{\lambda t}\beta_0(x)\varphi(x)dx$$

for large t. Therefore, conditionally on the value of Z, the process of blue particle births is approximately an inhomogeneous Poisson process. It also follows that the number $N_B(t)$ of blue particles born by time t grows like $Z\lambda^{-1}e^{\lambda t}\int \beta_0(x)\varphi(x)dx$.

Consider now a process in which blue particles are born in spacetime at rate

$$ze^{\lambda t}\beta_0(x)\varphi(x)dxdt \qquad (t \geq 0),$$

where $z > 0$ is a constant, and each blue particle gives rise to a homogeneous BBM of red particles started at the birthplace (in spacetime) of the blue particle. Call this process a "Poisson wave" of BBMs. Observe that the process of blue particle births is an inhomogeneous Poisson process. The evolutions of the various BBMs in the wave are independent of each other and of the blue particle birth process. Let $M_i^*(t)$ be the position at time t of the rightmost descendant of the i^{th} blue particle, $N_B^*(t)$ the number of blue particles born by time t, and $M^*(t) = \max\left(M_1^*(t), \ldots, M_{N_B^*(t)}^*(t)\right)$. For each $t > 0$ the positions $M_i^*(t), M_2^*(t), \ldots, M_{N_B^*(t)}^*(t)$ constitute a Poisson point process on $\mathbb{R}$ with intensity measure

$$z\{\int_0^t \int_{\mathbb{R}} e^{\lambda s}\beta_0(x)\varphi(x)v(t-s, y-x)dxds\}dy$$

where

$$v(t,x) = -\frac{\partial}{\partial x}u(t,x)$$

and $1 - u(t,x)$ is the cumulative distribution function of the position of the rightmost particle in a homogeneous BBM at time t. Thus

$$P\{M^*(t) \leq y\} = \exp\{-z\int_0^t \int_{\mathbb{R}} e^{\lambda s}\beta_0(x)\varphi(x)u(t-s, y-x)dxds\}. \tag{2.2}$$

In Proposition 2 (sec. 3) we will prove that if $\lambda > 2$ (the supercritical case) and $\mu = \lambda\{2(\lambda-1)\}^{-\frac{1}{2}}$ then

$$\lim_{t\to\infty}\int_0^t\int_{\mathbb{R}} e^{\lambda s}\beta_0(x)\varphi(x)u(t-s,\mu t+y-x)\,dx\,ds = K_1 e^{-\sqrt{2(\lambda-1)}y}$$

for a certain constant $0 < K_1 < \infty$. Consequently,

$$\lim_{t\to\infty} P\{M^*(t) \le \mu t + y\} = \exp\{-zK_1 e^{-\sqrt{2(\lambda-1)}y}\}. \tag{2.3}$$

In Proposition 3 (sec. 3) we will prove that if $\lambda = 2$ (the critical case) then

$$\lim_{t\to\infty}\int_0^t\int_{\mathbb{R}} e^{2s}\beta_0(x)\varphi(x)u(t-s,x_t+y-x)\,dx\,ds = K_0 e^{-\sqrt{2}y} \tag{2.4}$$

where $x_t - m_t \to \infty$ as $t \to \infty$, m_t being the median of the distribution of the rightmost particle in a homogeneous BBM. Hence

$$\lim_{t\to\infty} P\{M^*(t) \le x_t + y\} = \exp\{-zK_0 e^{-\sqrt{2}y}\}. \tag{2.5}$$

Observe that in both the supercritical and the critical case the "center" of the wave (μt in the supercritical case, x_t in the critical case) diverges to ∞ faster than m_t. This implies in both cases, for any $t_* < \infty$,

$$\lim_{t\to\infty} P\{\text{leading particle at time } t \text{ is a direct descendant of a blue particle born before time } t_*\} = 0, \tag{2.6}$$

because with probability 1 only finitely many blue particles are born before t_*. But this is true also of the IBBM, and we have already remarked that for large time the production of blue particles in the IBBM, conditional on Z, is nearly the same as in the Poisson wave of BBMs. Consequently (2.6) must hold for the IBBM as well. Now (2.6) implies that for large t the distribution of $M(t)$ is practically unaffected by the blue particles born before t_*; therefore, conditional on $Z = z$ the distribution of $M(t)$ is essentially the same as that of $M^*(t)$. It follows from (2.3) and (2.5) that

$$\lim_{t\to\infty} P\{M(t) \le \mu t + y\} = E\exp\{-ZK_1 e^{-\sqrt{2(\lambda-1)}y}\} \quad (\lambda > 2) \tag{2.7}$$

and

$$\lim_{t\to\infty} P\{M(t) \le x_t + y\} = E\exp\{-ZK_0 e^{-\sqrt{2}y}\} \quad (\lambda = 2). \tag{2.8}$$

A rigorous proof of (2.7) and (2.8) may be fashioned from this argument by coupling the IBBM with Poisson waves of BBMs as in [6], sec. 5. (In this coupling, blue particles from the IBBM are paired with blue particles from the Poisson wave; once two blue particles are paired, they give rise to the same BBM of red descendants.) We shall not give the details of the construction.

In the subcritical case (2.6) does not hold, so the distribution of $M(t)$ cannot be approximated by mixing distributions of $M^*(t)$. For this case a different approach is needed: cf. sec. 4.

3. Asymptotics for the Critical and Supercritical Cases

The cumulative distribution function $1-u(t,x)$ of the position of the rightmost particle in a *homogeneous* ($\beta \equiv 1$) BBM solves the KPP/Fisher equation

$$u_t = \frac{1}{2}u_{xx} + u(1-u)$$

with the initial condition $u(0,x) = 1\{x < 0\}$ (cf. [8], [1]). It is known that $u(t,x)$

approaches a travelling wave with velocity $\sqrt{2}$: in particular

$$u(t, m_t + x) \to w(x) \quad \text{as} \quad t \to \infty, \tag{3.1}$$

where $u(t, m_t) = \frac{1}{2}$, $m_t = \sqrt{2}t - (3/2\sqrt{2}) \log t +$ constant $+ o(1)$ as $t \to \infty$, and $w(x) \sim Cxe^{-\sqrt{2}x}$ as $x \to \infty$ [2]. Moreover, the Feynman-Kac formula implies that

$$u(t,x) = e^t E^x \exp\{-\int_0^t u(t-s, X(s))ds\} 1\{X(t) < 0\} \tag{3.2}$$

where $X(s)$ is a Brownian motion started at x under E^x [8],[2]. Conditioning on $X(t)$ and reversing time, we may rewrite (3.2) as

$$e^{-t} u(t,x) = \int_{-\infty}^{0} p(t,x,y) g(t,y,x) dy \tag{3.3}$$

where

$$\begin{aligned} p(t,x,y) &= (2\pi t)^{-\frac{1}{2}} e^{-(x-y)^2/2t}, \\ g(t,y,x) &= E^{t,y,x} \exp\{-\int_0^t u(s, X(s))ds\} \end{aligned} \tag{3.4}$$

and under $E^{t,y,x}$ $X(s), 0 \le s \le t$ is a Brownian motion conditioned by $X(0) = y$, $X(t) = x$.

PROPOSITION 1: For each $\mu \ge \sqrt{2}$ and $t > 0$

$$u(t, \mu t) \le \mu^{-1} (2\pi t)^{-\frac{1}{2}} \exp\{-t(\mu^2/2 - 1)\}. \tag{3.5}$$

Furthermore, there exist constants C_μ depending continuously on $\mu \in (\sqrt{2}, \infty)$ such that

$$u(t, \mu t) \sim C_\mu t^{-\frac{1}{2}} \exp\{-t(\mu^2/2 - 1)\} \tag{3.6}$$

as $t \to \infty$, uniformly for μ in any compact subset of $(\sqrt{2}, \infty)$.

This is a large deviations theorem for the rightmost particle in a homogeneous BBM. Similar results may be obtained for solutions of $u_t = \frac{1}{2}u_{xx} + f(u)$ for certain f by similar methods.

PROOF: Since $u \geq 0$, (3.2) implies that

$$u(t, t\mu) \leq e^t P^{\mu t}\{X(t) < 0\},$$

from which (3.5) follows easily.

Let $X(s)$, $0 \leq s \leq t$, be a Brownian motion conditioned by $X(0) = y$, $X(t) = \mu t$. If $\mu > \sqrt{2}$ and s_0 is large then with probability near 1 the path $X(s)$, $s_0 \leq s \leq t$ lies entirely above the straight line from $(0,0)$ to $(t, (\sqrt{2} + \varepsilon)t)$, $\varepsilon < \mu - \sqrt{2}$. Relation (3.5) therefore implies that for sufficiently large s_0

$$1 \leq \frac{E^{t,y,\mu t} \exp\{-\int_0^{s_0} u(s, X(s))ds\}}{E^{t,y,\mu t} \exp\{-\int_0^{t} u(s, X(s))ds\}} \leq 1 + \eta$$

where η is small and $t \geq s_0$.

As $t \to \infty$ the distribution of $X(s)$, $0 \leq s \leq s_0$ conditional on $X(0) = y$, $X(t) = \mu t$ approaches that of a Brownian motion started at y with drift μ. Hence

$$\lim_{t \to \infty} E^{t,y,\mu t} \exp\{-\int_0^{s_0} u(s, X(s))ds\}$$

$$= E^y_\mu \exp\{-\int_0^{s_0} u(s, X(s))ds\},$$

where under E^y_μ, $X(s)$ is a Brownian motion started at y with drift μ. Once again, if $\mu > \sqrt{2}$ and s_0 is large then with P^y_μ-probability near 1 the path $X(s)$, $s_0 \leq s < \infty$ lies entirely above the line $x = (\sqrt{2} + \varepsilon)s$. In view of (3.5),

$$1 \leq \frac{E^y_\mu \exp\{-\int_0^{s_0} u(s, X(s))ds\}}{E^y_\mu \exp\{-\int_0^{\infty} u(s, X(s))ds\}} \leq 1 + \eta.$$

Letting $s_0 \to \infty$ (and therefore $\eta \to 0$) we obtain

$$\lim_{t\to\infty} E^{t,y,\mu t} \exp\{-\int_0^t u\big(s, X(s)\big)ds\}$$

$$= E_\mu^y \exp\{-\int_0^\infty u\big(s, X(s)\big)ds\} > 0.$$

A routine argument based on likelihood ratios shows that this holds uniformly for y in any compact subset of $(-\infty, \infty)$ and μ in any compact subset of $(\sqrt{2}, \infty)$.

We now apply (3.3), expanding the exponent in $p(t, y, \mu t)$, to obtain

$$\sqrt{2\pi t}\; e^{-t+\mu^2 t/2} u(t, \mu t)$$

$$= \int_{-\infty}^0 e^{\mu y - y^2/2t} E^{t,y,\mu t} \exp\{-\int_0^t u\big(s, X(s)\big)ds\}dy$$

$$\to \int_{-\infty}^0 e^{\mu y} E_\mu^y \exp\{-\int_0^\infty u\big(s, X(s)\big)ds\}dy.$$

This implies (3.6). □

Recall that for a Poisson wave of BBMs the distribution of the position of the rightmost particle at time t is given by (2.2). The following result shows that this distribution converges to a travelling wave in the supercritical case.

PROPOSITION 2: Assume $\lambda > 2$; let $\mu = \lambda\{2(\lambda - 1\}^{-\frac{1}{2}}$. As $t \to \infty$,

$$\int_0^t \int_{\mathbb{R}} e^{\lambda s} \beta_0(x)\varphi(x) u(t - s, \mu t + y - x)dxds$$
$$\to K_1 e^{-\sqrt{2(\lambda-1)}y}, \tag{3.7}$$

where

$$K_1 = C_{2\mu(\lambda-1)/\lambda}\big(\pi/(\lambda-1)\big)^{\frac{1}{2}} \int_{\mathbb{R}} \beta_0(x)\varphi(x)e^{\sqrt{2(\lambda-1)}x}dx, \tag{3.8}$$

uniformly for y in any compact subset of $\mathbb{R}$.

PROOF: Consider the function

$$h(\overline{s}) = (\lambda-1)\overline{s} - \mu^2/2(1-\overline{s}), \quad 0 \le \overline{s} \le 1.$$

The maximum of h on [0,1] is attained at $\overline{s}_* = (\lambda-2)/2(\lambda-1) > 0$, and

$$h(\overline{s}_*) = -1, h'(\overline{s}_*) = 0, h''(\overline{s}_*) = -4(\lambda-1)^2/\lambda.$$

The relation (3.6) implies that for $0 \le s \le (1-\varepsilon)t$, any $\varepsilon > 0$, as $t \to \infty$

$$e^{\lambda s}u(t-s,\mu t) \sim C_{\mu(1-s/t)^{-1}}(t-s)^{-\frac{1}{2}}\exp\{\lambda s + (t-s) - \mu^2t^2/2(t-s)\}$$

$$= C_{\mu(1-\overline{s})^{-1}}(1-\overline{s})^{-\frac{1}{2}}t^{-\frac{1}{2}}\exp\{t\big(1+h(\overline{s})\big)\},$$

where $\overline{s} = s/t$. Similarly, (3.5) implies that for $(1-\varepsilon)t \le s \le t$

$$e^{\lambda s}u(t-s,\mu t) \le \mu^{-1}(1-\overline{s})^{-\frac{1}{2}}t^{-\frac{1}{2}}\exp\{t\big(1+h(\overline{s})\big)\}.$$

Therefore,

$$\int_0^t e^{\lambda s}u(t-s,\mu t)ds \sim t^{\frac{1}{2}} \int_{\overline{s}_*-\delta}^{\overline{s}_*+\delta} C_{\mu(1-\overline{s})^{-1}}(1-\overline{s})^{-\frac{1}{2}}e^{t(1+h(\overline{s}))}d\overline{s}$$

$$\sim C_{\mu(1-\overline{s}_*)^{-1}}(1-\overline{s}_*)^{-\frac{1}{2}}\big(-2\pi/h''(\overline{s}_*)\big)^{\frac{1}{2}}$$

$$= C_{2\mu(\lambda-1)/\lambda}\big(\pi/(\lambda-1)\big)^{\frac{1}{2}}$$

by Laplace's method of asymptotic expansion. A similar but somewhat messier analysis shows that

$$\int_0^t e^{\lambda s} u(t-s, \mu t + x) ds \sim C_{2\mu(\lambda-1)/\lambda} \left(\pi/(\lambda-1)\right)^{\frac{1}{2}} e^{-\sqrt{2(\lambda-1)}x}$$

uniformly for x in any compact set. Since $\beta_0(x)$ has compact support, (3.7) follows directly. □

The result (3.5) implies that there is a solution x_t of the equation

$$\int_0^t e^{2s} u(t-s, x_t) ds = \frac{1}{2}. \tag{3.9}$$

LEMMA 1: $\lim_{t\to\infty} u(t, x_t) = 0$.

PROOF: Suppose $u(t, x_t) \geq \varepsilon$ for arbitrarily large t. Since $w(x) \sim Cxe^{-\sqrt{2}x}$ as $x \to \infty$, (3.1) would imply that $u(t-s, x_t) \geq \varepsilon e^{-2s}/2$ for $C_0 \leq s \leq C_1(t)$, with $C_1(t) \to \infty$ as $t \to \infty$. But this contradicts (3.9). □

Lemma 1 implies that $x_t - m_t \to \infty$ as $t \to \infty$. An easy calculation based on (3.5) shows that $\limsup_{t\to\infty} x_t/t \leq \sqrt{2}$; since $m_t/t \to \sqrt{2}$, it follows that $x_t/t \to \sqrt{2}$.

PROPOSITION 3: As $t \to \infty$

$$x_t - m_t = \frac{1}{\sqrt{2}} \log t + \textit{ constant } + 0(1) \quad \textit{and} \tag{3.10}$$

$$\int_0^t e^{2s} u(t-s, x_t + x) ds \to \frac{1}{2} e^{-\sqrt{2}x} \tag{3.11}$$

uniformly for x *in any compact subset of* $\mathbb{R}$.

Since β_0 has compact support, (2.4) follows from (3.11), and we get that the distribution of the position of the rightmost particle in a Poisson wave of BBMs approaches a travelling wave as $t \to \infty$ in the critical case $\lambda = 2$.

PROOF: First we show that the primary contribution to the integral in (3.11) comes from $0 \le s \le K\sqrt{t}$, where K is large. For $z \ge \sqrt{2}t$, (3.5) implies that $u(t,z) \le (4\pi t)^{-\frac{1}{2}} \exp\{t - z^2/2t\}$. Let $y_t = m_t + (1/\sqrt{2}) \log t$; recall that $m_t = \sqrt{2}t - (3/2\sqrt{2}) \log t$. Hence, for large t and all x in a bounded set

$$\int_{K\sqrt{t}}^{t} e^{2s} u(t-s, y_t + x) ds$$

$$\le C \int_{K\sqrt{t}}^{\infty} t^{-\frac{1}{2}} e^{-s^2/t} ds = C \int_{K}^{\infty} e^{-s^2} ds$$

for some constant $C < \infty$. By choosing K sufficiently large we can make this arbitrarily small.

Next we appeal to results of Bramson [2] to analyze $e^{2s}u(t-s, y_t+x)$ for $0 \le s \le K\sqrt{t}$. It follows from (8.64) and (8.65) of [2] that for $r >> 0$, $r/t < \delta$, and $m_t + 8r \le z \le \sqrt{2}t + K\sqrt{t}$,

$$\begin{aligned} & B(\delta)^{-1}\gamma(r)^{-1}C(r)(z - m_t) \exp\{-\sqrt{2}(z - m_t) - (z - \sqrt{2}t)^2/2t\} \\ & \le u(t,z) \\ & \le B(\delta)\gamma(r)C(r)(z - m_t) \exp\{-\sqrt{2}(z - m_t) - (z - \sqrt{2}t)^2/2t\} \end{aligned}$$

where $C(r) = \sqrt{2/\pi} \int_0^\infty y e^{\sqrt{2}y} u(r, y + \sqrt{2}r) dy < \infty$ and $B(\delta) \to 1$, $\gamma(r) \to 1$ as $\delta \to 0$,

$r \to \infty$. Consequently, as $t \to \infty$

$$\int_0^{K\sqrt{t}} e^{2s} u(t-s, y_t + x) ds$$

$$\sim C_\infty e^{-\sqrt{2}(y_t - m_t + x)} \int_0^{K\sqrt{t}} (y_t - m_t + x + \sqrt{2}s) e^{\frac{-(y_t + x - \sqrt{2}(t-s))^2}{2(t-s)}} ds$$

$$\sim C_\infty e^{-\sqrt{2}x} t^{-1} (t \int_0^K \sqrt{2}s' e^{-(s')^2} ds'),$$

where $C_\infty = \lim_{r \to \infty} C(r) \in (0, \infty)$. The relations (3.10) and (3.11) now follow directly, with the fact that the constant coefficient in (3.11) is 1/2 following from the definition of x_t in (3.9). □

4. The Subcritical Case

Consider now the subcritical case $\lambda < 2$. As in the critical and supercritical cases we assume that the branching rate function $\beta(x) = 1 + \beta_0(x)$, where $\beta_0(x) \geq 0$, $\beta_0(x) > 0$ somewhere, and β_0 has support in a bounded interval J^*. Since $\beta \geq 1$ the rate of production of new particles in the IBBM is $\geq$ that for a homogeneous ($\beta \equiv 1$) BBM. Consequently, if $M(t)$ is the position of the rightmost particle in the IBBM

$$P\{M(t) \geq x\} \geq u(t, x) \quad \forall t \geq 0, \ x \in \mathbb{R}, \tag{4.1}$$

and therefore the distribution of $M(t)$ "travels" at least as fast as the travelling wave for the BBM.

Recall the classification of IBBM particles as "red" or "blue" (sec. 2). For a given blue particle call a red descendant a *direct* descendant if there are no intermediate blue descendants. Recall that the direct descendants of a blue particle constitute a homogeneous BBM.

PROPOSITION 4: For any $\varepsilon > 0$ there exists $t_\varepsilon < \infty$ such that for all $t \geq 0$

$$P\{\text{ rightmost particle at time } t \text{ is descended from a blue particle born after time } t_\varepsilon\} \leq \varepsilon.$$

PROOF: Watanabe's theorem (cf. (2.1)) implies that $N(t, J^*) \sim C'Ze^{\lambda t}$. Blue particles are only born in J^*, and the intensity of the blue particle birth process is $\int \beta_0(x)N(t, dx)$. Consequently, there is a constant $C'' < \infty$ and a time $s < \infty$ such that

$$P\{\#\text{ blue births during } [s+j, s+j+1] \leq C''e^{\lambda(s+j)} \forall j = 0, 1, 2, \ldots\} \geq 1 - \varepsilon/3. \quad (4.2)$$

Consider now a blue particle born at time $r \geq 0$. The position at birth is in J^*, hence to the left of $\xi \triangleq \sup J^*$. The direct descendants make up a homogeneous BBM; the probability that the position of the rightmost direct descendant is $\geq x$ at time $t \geq r$ is $\leq u(t-r, x-\xi)$. It therefore follows from (4.2) that if $M_\tau(t)$ is the position of the rightmost particle at time t descended from a blue particle born after time $\tau \geq s$ then

$$P\{M_\tau(t) \geq x\} \leq \varepsilon/3 + \int_\tau^t C''e^{\lambda r}u(t-r, x-\xi)dr. \quad (4.3)$$

Recall that $u(t, m_t + x) \to w(x)$ as $t \to \infty$, $w(0) = \frac{1}{2}$, $w(x) \sim Cxe^{-\sqrt{2}x}$ as $x \to \infty$, and $m_t = \sqrt{2}t - (3/2\sqrt{2})\log t +$ constant $+ o(1)$ as $t \to \infty$. In fact, somewhat more is true:

$$u(t, m_t + x) \uparrow w(x) \quad \text{for } x > 0, \quad (4.4)$$

$$u(t, m_t + x) \downarrow w(x) \quad \text{for } x < 0 \quad (4.5)$$

(cf. [2], p. 32, Cor. 1). Choose x so that $w(x) \geq 1 - \varepsilon/3$; then (4.5) implies that $u(t, m_t + x) \geq 1 - \varepsilon/3 \quad \forall t > 0$ and (4.1) implies that

$$P\{M(t) > m_t + x\} \geq 1 - \varepsilon/3.$$

Now consider $\int_T^t C'' e^{\lambda r} u(t-r, m_t + x - \xi) dr$. If t is sufficiently large then for $0 \leq r \leq \sqrt{t} \log t$, $m_t = m_{t-r} + \sqrt{2} r + o(1)$, so by choosing t_ε large we can make (by (4.4))

$$\begin{aligned} &\int_{t_\varepsilon}^{\sqrt{t}\log t} C'' e^{\lambda r} u(t-r, m_t + x - \xi) dr \\ &\leq \int_{t_\varepsilon}^{\sqrt{t}\log t} C''' e^{\lambda r} (x - \xi + \sqrt{2} r) e^{-\sqrt{2} r} dr \\ &\leq \varepsilon/6, \end{aligned}$$

since $\lambda < 2$. On the other hand, the inequality (3.5) implies that as $t \to \infty$

$$\int_{\sqrt{t}\log t}^{t} e^{\lambda r} u(t-r, m_t + x - \xi) dr \to 0.$$

Hence, for t_ε sufficiently large

$$P\{M_{t_\varepsilon}(t) > m_t + x\} \leq 2\varepsilon/3$$

and

$$P\{M(t) > m_t + x\} \geq 1 - \varepsilon/3$$

for all $t \geq t_\varepsilon$. This proves the Proposition. □

Consider a branching diffusion process that evolves as follows. Until time t_ε the movements and fissions occur in exactly the same manner as in the IBBM; after time t_ε individual particles move and reproduce as in homogeneous BBM. Thus in the new process individual particles execute independent Brownian motions; before time t_ε the instantaneous rate of reproduction of a particle located at x is $\beta(x)$, whereas after time t_ε the reproduction rate of any particle is 1. Let $\tilde{X}_1(t), \tilde{X}_2(t), \ldots, \tilde{X}_{\tilde{N}(t)}(t)$ be the positions of the particles in this new process at time t, and let $\tilde{M}(t) = \max_{1 \le i \le \tilde{N}(t)} \tilde{X}_i(t)$.

The conditional distribution of $\tilde{M}(t)$, $t > t_\varepsilon$, given the positions $\tilde{X}_1(t_\varepsilon), \ldots, \tilde{X}_{\tilde{N}(t_\varepsilon)}(t_\varepsilon)$ is as follows:

$$\begin{aligned} P\{\tilde{M}(t) \le x | \tilde{X}_1(t_\varepsilon), \ldots \tilde{X}_{\tilde{N}(t_\varepsilon)}(t_\varepsilon)\} \\ = \Pi_{i=1}^{\tilde{N}(t)} \Big(1 - u\big(t - t_\varepsilon, x - \tilde{X}_i(t_\varepsilon)\big)\Big) \end{aligned}$$

Now as $t \to \infty$, $u(t - t_\varepsilon,\ m_t + x) \to w(x + \sqrt{2}t_\varepsilon)$; consequently, as $t \to \infty$

$$P\{\tilde{M}(t) \le m_t + x\} \to E\Pi_{i=1}^{\tilde{N}(t_\varepsilon)} \Big(1 - w\big(x - \tilde{X}_i(t_\varepsilon) + \sqrt{2}t_\varepsilon\big)\Big). \tag{4.6}$$

Thus the distribution of $\tilde{M}(t)$ approaches a travelling wave, and the "location" m_t of the wave is the same as for homogeneous BBM.

Now consider the IBBM. By Proposition 4 the distribution of $M(t)$ is very little different from that of $\tilde{M}(t)$, in particular,

$$\begin{aligned} P\{\tilde{M}(t) \ge x\} \le P\{M(t) \ge x\} \\ \le P\{\tilde{M}(t) \ge x\} + \varepsilon \quad \forall x. \end{aligned} \tag{4.7}$$

Moreover, the distribution of $\big(X_1(t_\varepsilon), \ldots, X_{N(t_\varepsilon)}(t_\varepsilon)\big)$ is the same as that of $\big(\tilde{X}_1(t_\varepsilon), \ldots, \tilde{X}_{\tilde{N}(t_\varepsilon)}(t_\varepsilon)\big)$, because the two processes evolve the same way up to time t_ε. Now (4.6) and (4.7) imply that for each $x \in \mathbb{R}$

$$v(x) = \lim_{\varepsilon \to 0} E\Pi_{i=1}^{N(t_\varepsilon)} \Big(1 - w\big(x - X_i(t_\varepsilon) + \sqrt{2}t_\varepsilon\big)\Big) > 0 \tag{4.8}$$

exists, and

$$\lim_{t \to \infty} P\{M(t) \le m_t + x\} = v(x).$$

REMARK: The convergence in (4.8) follows on other grounds. For homogeneous BBM the process $\Pi_{i=1}^{N(t)}\left(1 - w\big(x - X_i(t) + \sqrt{2}t\big)\right)$ is a bounded, positive martingale (cf. [5]). Because of the occasional production of "blue" particles and the fact that $0 < 1 - w < 1$, for the IBBM, $\Pi_{i=1}^{N(t)}\left(1 - w\big(x - X_i(t) + \sqrt{2}t\big)\right)$ is a supermartingale. Hence (4.8). Observe that in the subcritical case this supermartingale has a strictly positive limit, whereas in the critical and supercritical cases it converges to zero.

5. Concluding Remarks

(1) The methods of this paper can be adapted to show that $M(t)/t \to \mu$ a. s. in the supercritical case and $M(t)/t \to \sqrt{2}$ a. s. in the critical and subcritical cases.

(2) We have assumed throughout that the IBBM starts with a single particle located at $x = 0$ at time $t = 0$. Replacing $\beta(x)$ by $\beta(x - x_0)$, we see that our results are also valid for an IBBM started with a single particle at x_0 at time $t = 0$. However, the shape of the limiting wave-form changes with x_0. In a subsequent paper [7] we shall investigate the dependence on initial conditions in a more general framework.

(3) It is natural to inquire about the possibility of characterizing those continuous functions $\beta(x)$ such that an IBBM run with branching rate $\beta(x)$ will exhibit the travelling wave effect. We believe that if $\lim_{x\to\infty} \beta(x)$ and $\lim_{x\to-\infty} \beta(x)$ exist and are finite then the travelling wave phenomenon should occur. It seems unlikely that this is a necessary condition, though. We conjecture that a sufficient condition for the existence of a travelling wave is that for every $\varepsilon > 0$, $\lim_{x\to\infty} \int_x^{x+\varepsilon} \beta(y)dy$ exists and is finite, $\limsup_{x\to\infty} x^{-1} \int_{-x}^{0} \beta(y)dy < \infty$. We have been able to prove that if $\beta(x) \to \infty$ as $x \to \infty$ then there can be no travelling wave.

(4) Burgess Davis asked us about the "genealogy" of the rightmost particle at time t. Our methods lead to the following picture. In the supercritical case, the rightmost particle at time t (t large) is descended from a particle located at distance $0(1)$ away from the origin at time $\{(\lambda-2)/2(\lambda-1)\}t \pm 0(\sqrt{t}\log t)$, but has no ancestors that were $0(1)$ away from the origin at any time after $\{(\lambda-2)/2(\lambda-1)\}t + t^{\frac{1}{2}+\varepsilon}$. In the critical and subcritical cases, the rightmost particle at time t has no ancestors that were within $0(1)$ of the origin after time $t^{\frac{1}{2}+\varepsilon}$.

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