A One-Dimensional Infiltration Game

by

S. P. Lalley
Purdue University

Technical Report 87-17

Department of Statistics
Purdue University

October 1987

A One-Dimensional Infiltration Game

by

S. P. Lalley
Purdue University

ABSTRACT

Minimax strategies are obtained for an infiltration game in which one player must move through a one dimensional interval defended by the other player.

1. Rules of the Game

This zero-sum game has two players, the *infiltrator* and the *defender*; it is played on the discrete set of sites $x = 0, 1, 2, \ldots, M+1$, in discrete time $t = 0, 1, 2, \ldots, N+1$. Movement by either player is required to be *continuous*, i.e., increments in position (per unit time) are restricted to $0, +1$, or -1. The defender's position is restricted to the sites $x = 1, 2, \ldots, M$.

The objective of the infiltrator, yielding a payoff of 1 if achieved, is to move from his base at $x = 0$ to a target at $x = M+1$ without being detected by the defender: this must be accomplished within $N+1$ units of time ($N \geq M$). Neither player receives any information concerning his opponent's moves until the time when (if) detection occurs. The defender cannot detect the infiltrator unless they simultaneously occupy the same site; at each time the two players occupy the same site, detection occurs with probability $(1-\lambda)$, where $0 < \lambda < 1$.

This game was proposed by S. Gal [1] (section 4.6) in a somewhat more general form. It may be considered a crude model of a naval operation in which a submarine or smuggler attempts to move through a narrow channel to the open sea.

2. Minimax Strategies

It is obvious that if the defender does not move during the course of the game then regardless of the infiltrator's strategy the probability of successful infiltration is no greater than λ (since the infiltrator's position must coincide with that of the defender at least once). Thus λ is a bound on the minimax value of the game. As we shall see, the minimax value is actually appreciably smaller than λ except when $N >> M$, so the passive defense is decidedly suboptimal.

The "Admiral Farragut" strategy for the infiltrator is defined as follows. A starting time T is chosen at random (with uniform distribution) from $\{0, 1, 2, \ldots, N - M\}$. The infiltrator waits at his base until time T, then proceeds "full speed ahead" toward the target. Thus the trajectory of the infiltrator is

$$X(t) = \begin{cases} 0 & \text{if} \quad t \leq T \\ t - T & \text{if} \quad T < t \leq T + M + 1, \\ M + 1 & \text{if} \quad T + M + 1 < t \leq N + 1. \end{cases} \tag{1}$$

The "orderly fallback" strategy for the defender is defined as follows. Let k, r be the unique nonnegative integers such that

$$M - 1 = k(N - M + 1) + r, \qquad 0 \leq r < N - M + 1. \tag{2}$$

Let $\xi_0, \xi_1, \xi_2, \ldots, \xi_{N-M}$ be the random variables obtained by sampling without replacement from an urn with r balls marked $k + 1$ and $N - M + 1 - r$ balls marked k (thus $\xi_0, \xi_1, \ldots, \xi_{N-M}$ are exchangeable, with $P\{\xi_i = k + 1\} = 1 - P\{\xi_i = k\} = r/(N - M + 1)$ and $\xi_0 + \xi_1 + \ldots + \xi_{N-M} = M - 1$). The orderly fallback strategy calls for the defender to occupy site $x = 1$ at times $t = 0$ and $t = 1$, retreat at full speed for ξ_0 time units, wait 1 time unit, then retreat at full speed for ξ_1 time units, wait 1 time unit, etc. The trajectory of the defender is $Y(0) = 1$, then

$$Y(t) = \begin{cases} t & \text{if} \quad 1 \leq t \leq 1 + \xi_0, \\ t - (m - 1) & \text{if} \quad m + \sum_{i=0}^{m-2} \xi_i \leq t \leq m + \sum_{i=0}^{m-1} \xi_i, \\ M & \text{if} \quad t \geq N; \end{cases} \tag{3}$$

see Figure 1.

(Figure 1 here)

THEOREM: *The Admiral Farragut strategy and the orderly fallback strategy are minimax strategies for the infiltrator and the defender, respectively. If these strategies are used, the probability of* *successful infiltration is*

$$\lambda^{k+1}(\lambda r/(N-M+1)+(1-r/(N-M+1))).$$

Notice that both strategies are mixed (i.e., $X(t)$ and $Y(t)$ are *random* paths) except when $r=0$, in which case the orderly fallback strategy is pure.

Observe that the quantity $\big(\lambda r/(N-M+1)+(1-r/(N-M+1)\big)$ is a convex combination of the real numbers λ and 1, hence lies between them. Therefore, the value of the game (given in the theorem) is between λ^{k+2} and λ^{k+1}. Observe also that the integer k defined by (2) must satisfy

$$\frac{M-1}{N-M+1}\geq k>\frac{M-1}{N-M+1}-1.$$

Consequently, if $N>>M$ then $k=0$, and the value of the game is

$$\lambda\Big(\lambda(M-1)/(N-M+1)+\big(1-(M-1)/(N-M+1)\big)\Big).$$

3. Proof of the Theorem

Define ν to be the number of times the positions of the infiltrator and the defender coincide. Since the infiltrator survives each such encounter with probability λ,

$$P\{\text{successful infiltration}\}=E\lambda^{\nu}. \tag{4}$$

Recall that $x(t)$ is continuous if $x(t+1)-x(t)=0,1,$ or -1 for every $t=0,1,\ldots$.

LEMMA 1: *Let $x(t)$ be any continuous trajectory on the set $\{0,1,\ldots,M+1\}$ of sites such that $x(0)=0$ and $x(N+1)=M+1$. If the defender uses the orderly fallback strategy and the infiltrator follows the trajectory $x(t)$, then*

$$P\{\nu\geq k+1\}=1, \tag{5}$$

$$P\{\nu \geq k+2\} \geq r/(N-M+1). \tag{6}$$

PROOF: The path $x(t)$ must cross the path $Y(t)$ somewhere, from below to above, since $x(0) < Y(0)$ and $x(N+1) > Y(N+1)$ and since $x(\cdot), Y(\cdot)$ are continuous, with $Y(\cdot)$ nondecreasing. At any time s such that $x(s) = Y(s)$ but $x(s-1) < Y(s-1)$ the defender is just beginning one of his fallback periods ξ_i. Since $\xi_i \geq k$, the trajectory $x(t)$, in passing through $Y(t)$, must coincide with $Y(t)$ for at least $k+1$ time units. This proves (5).

Let $\tau = \inf\{t \geq 1 : x(t) = Y(t)\}$. Let σ be the unique nonnegative integer such that $\tau = \sigma + 1 + \sum_{i=0}^{\sigma-1} \xi_i$: thus at time τ the defender is just about to begin his σ^{th} fallback. Conditional on the values of $\tau, \sigma, \xi_0, \xi_1, \ldots, \xi_{\sigma-1}$, the values of $\xi_\sigma, \xi_{\sigma+1}, \ldots, \xi_{N-M}$ are obtained by random sampling (without replacement) of the *remaining* $N - M - \sigma + 1$ balls in the urn. Now in sampling without replacement the distribution of the first ball chosen is the same as that of the last ball chosen; therefore,

$$P\{\xi_\sigma = k+1 | \tau, \sigma, \xi_0, \xi_1, \ldots, \xi_{\sigma-1}\} = P\{\xi_{N-M} = k+1 | \tau, \sigma, \xi_0, \xi_1, \ldots, \xi_{\sigma-1}\}.$$

Taking expectations in this relation, we obtain the *unconditional* formula

$$P\{\xi_\sigma = k+1\} = P\{\xi_{N-M} = k+1\}.$$

But

$$P\{\xi_{N-M} = k+1\} = P\{\xi_0 = k+1\} = r/(N-M+1),$$

so

$$P\{\xi_\sigma = k+1\} = r/(N-M+1). \tag{7}$$

Now either $x(t) \geq Y(t)$ for all $t \geq \tau$ or $x(t') < Y(t')$ for some $t' > \tau$. In the former case $x(t) = Y(t)$ for $\tau \leq t \leq \tau + \xi_\sigma$, hence $\nu \geq \xi_\sigma + 1$. In the latter case $x(t)$ must coincide with $Y(t)$ for at least $k+1$ time units *after* t', by the argument of the first paragraph, hence $\nu \geq k+2$. Therefore (6) follows from (7). ///

LEMMA 2: *The minimum of* $\sum_{j=0}^{N-M} \lambda^{\nu_j}$ *over the set of* $(N-M+1)$ *- tuples of nonnegative integers* $(\nu_0, \nu_1, \ldots, \nu_{N-M})$ *satisfying* $\sum_{j=0}^{N-M} \nu_j \leq N$ *is*

$$\lambda^{k+1}(\lambda r + N - M + 1 - r).$$

The minimum is achieved by exactly those $(N-M+1)$ *- tuples which satisfy*

$$\sum_{j=0}^{N-M} 1\{\nu_j = k+2\} = r, \tag{8}$$

$$\sum_{j=0}^{N-M} 1\{\nu_j = k+1\} = N - M + 1 - r. \tag{9}$$

PROOF: Suppose $a < a_* \leq b_* < b$ are integers such that $a + b = a_* + b_*$. Then $b - b_* = a_* - a$, so $\lambda^{a_*} - \lambda^a = \lambda^a(\lambda^{a_*-a} - 1) < \lambda^{b_*}(\lambda^{b-b_*} - 1) = \lambda^b - \lambda^{b_*}$, and hence $\lambda^{a_*} + \lambda^{b_*} < \lambda^a + \lambda^b$.

Now the set of $(N-M+1)$ - tuples of nonnegative integers $(\nu_0, \nu_1, \ldots, \nu_{N-M})$ satisfying $\nu_0 + \nu_1 + \ldots + \nu_{N-M} \leq N$ is a finite set, so the minimum of $\Sigma_{j=0}^{N-M} \lambda^{\nu_j}$ over this set is in fact attained. Since $0 < \lambda < 1$, the minimum can *only* be attained at an n-tuple $(\nu_0, \nu_1, \ldots, \nu_{N-M})$ satisfying $\nu_0 + \nu_1 + \ldots + \nu_{N-M} = N$. (If $\nu_0 + \nu_1 + \ldots + \nu_{N-M} < N$, then there exist nonnegative integers $\nu_0^*, \nu_1^*, \ldots, \nu_{N-M}^*$ such that $\nu_j \leq \nu_j^* \;\; \forall\, j = 0, 1, \ldots, N-M$; $\nu_j < \nu_j^*$ for *some* j; and $\nu_0^* + \nu_1^* + \ldots + \nu_{N-M}^* \leq N$. But then $\Sigma_{j=0}^{N-M} \lambda^{\nu_j^*} < \Sigma_{j=0}^{N-M} \lambda^{\nu_j}$.) Moreover, if the minimum is attained at $(\nu_0, \nu_1, \ldots, \nu_{N-M})$, then for all $i, j = 0, 1, \ldots, N-M$, the difference $\nu_i - \nu_j$ must be 0, 1, or -1. (If $\nu_i \leq \nu_j + 2$ for some i, j, let $\nu_i^* = \nu_i + 1, \nu_j^* = \nu_j - 1$, and $\nu_\ell^* = \nu_\ell$ for $\ell \neq i$ or j. Then by the result of the preceding paragraph, $\Sigma_{\ell=0}^{N-M} \lambda^{\nu_\ell^*} < \Sigma_{\ell=0}^{N-M} \lambda^{\nu_\ell}$.) Now if $\nu_i - \nu_j = 0, 1$, or -1 for all $0 \leq i, j \leq N-M$ then there exists an integer n such that $\nu_i = n+1$ or $n+2$ for every $i = 0, 1, \ldots, N-M$. Let $\alpha = \sum_{i=0}^{N-M} 1\{\nu_i = n+2\}$, so $N - M + 1 - \alpha = \sum_{i=0}^{N-M} 1\{\nu_i = n+1\}$.

Since $\nu_0 + \nu_1 + \cdots + \nu_{N-M} = N$, we have

$$\alpha(n+2) + (N - M + 1 - \alpha)(n+1) = N \Longleftrightarrow$$

$$\alpha + (N - M + 1)n = M - 1 \Longleftrightarrow$$

$$n = k \quad \text{and} \quad \alpha = r, (\text{ or, if } r = 0, \text{ perhaps } n = k - 1 \text{ and } \alpha = N - M + 1),$$

by (2). Therefore, the minimum can only be achieved at an $(N-M+1)$–tuple $(\nu_0, \nu_1, \ldots, \nu_{N-M})$ satisfying (8) and (9).

Finally, it is obvious that if $(\nu_0, \nu_1, \ldots, \nu_{N-M})$ satisfies (8) and (9), then

$$\sum_{i=0}^{N-M} \lambda^{\nu_i} = \lambda^{k+1}(\lambda r + N - M + 1 - r). \quad ///$$

PROOF OF THE THEOREM: It follows immediately from (4) - (6) that if the defender uses the orderly fallback strategy then the probability of successful infiltration is no greater than $\lambda^{k+1}(\lambda r + N - M + 1 - r)/(N - M + 1)$ no matter what pure strategy (and hence, no matter what mixed strategy) the infiltrator uses.

Suppose the infiltrator uses the Admiral Farragut strategy and the defender uses the orderly fallback strategy. Then the trajectory of the infiltrator will meet the trajectory of the defender at the beginning of the defender's T^{th} fallback period and coincide with it throughout the T^{th} fallback period. Because the Admiral Farragut strategy calls for the infiltrator to move at maximum speed, the positions of the infiltrator and the defender will *not* coincide except during the T^{th} fallback period. Therefore $\nu = 1 + \xi_T$; by (4) the probability of successful infiltration is $\lambda^{k+1}(\lambda r + N - M + 1 - r)/(N - M + 1)$, since $P\{\xi_T = k\} = 1 - P\{\xi_T = k+1\} = 1 - r/(N - M + 1)$. Hence, if the defender uses the orderly fallback strategy then the infiltrator has no better strategy than the Admiral Farragut strategy.

Let $y(t)$ be any continuous trajectory on $\{1, 2, \ldots, M\}$. Define

$$\nu_j = \sum_{t=1}^{N} 1\{y(t) = t - j\}, \qquad j = 0, 1, \ldots, N - M;$$

clearly $\Sigma_{j=0}^{N-M}\nu_j \leq N$. Suppose the defender follows the trajectory $y(t)$ and the infiltrator uses the Admiral Farragut strategy. Then by (4) the probability of successful infiltration is

$$\sum_{j=0}^{N-M} \lambda^{\nu_j}/(N-M+1)$$

(ν_T is the number of times $y(t)$ coincides with $X(t)$). But Lemma 2 implies that $\Sigma_{j=0}^{N-M}\lambda^{\nu_j}$ attains its minimum value exactly when (8) and (9) hold. Since (8) and (9) hold for every path $y(t)$ in the support of the orderly fallback strategy, this proves that when the infiltrator uses the Admiral Farragut strategy (where $\nu_j = \xi_j + 1$), the defender has no better strategy than the orderly fallback strategy. ///

4. Concluding Remark

The infiltrator does not benefit from knowledge of the current or past position(s) of the defender. More precisely, if the game is played according to the same rules *except* that the infiltrator's move at time $t+1$ is allowed to depend on $Y(0), Y(1), \ldots, Y(t)$ (these being the positions of the defender at times $0, 1, \ldots, t$) then the Admiral Farragut and orderly fallback strategies are still minimax strategies for the infiltrator and the defender, respectively. The proof is almost exactly the same as that given in section 3 for the original game. The only substantial difference is that Lemma 1 must be replaced by an analogous statement for paths $X(t, Y(0), Y(1), \ldots, Y(t-1))$ (however, the same basic argument works).

Acknowledgements. The author benefitted from discussions with D.V. Chudnovsky, G.V. Chudnovsky, and H. Robbins. The author also wishes to thank a referee for catching two mistakes and making helpful suggestions about the exposition.

References

[1] S. Gal, *Search Games*, Academic Press, New York, 1980.

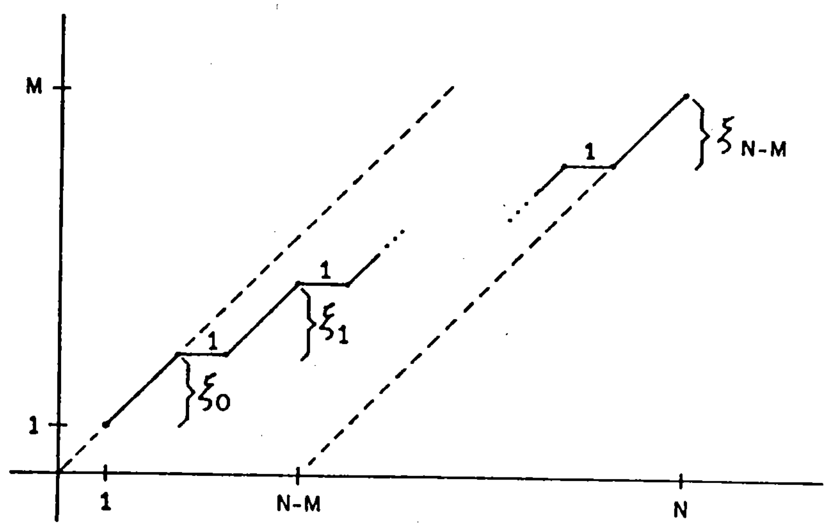

FIGURE 1