ISOTONIC SELECTION WITH RESPECT TO A CONTROL: A BAYESIAN APPROACH

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ABSTRACT

Let $\pi_1, \pi_2, \ldots, \pi_k$ be k experimental populations and π_0 is a standard or a control population. The π_i are characterized by the associated distribution functions F_{θ_i}, $i = 0, 1, \ldots, k$. The θ_i of the experimental populations are unknown; however, it is known that $\theta_1 \le \theta_2 \le \cdots \le \theta_k$. The value of θ_0 may or may not be known. The goal is to select all the experimental populations that are superior to π_0 (i.e. all π_i for whcih $\theta_i \ge \theta_0$). Under a fairly general loss function a Bayes rule with isotonic property (i.e. if π_i is selected, then any π_j for $j > i$ is also selected) is derived in a form convenient for applications. Two special cases of the loss function are discussed. One of these loss functions is used to discuss applications to discrete (with Poisson example) as well as continuous exponential class of distributions.

Key Words and Phrases: subset selection, comparison with a control, isotonic property, Bayes rule, discrete and continuous exponential families.

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1. INTRODUCTION

Let $\pi_1, \pi_2, \ldots, \pi_k$ denote k experimental populations where π_i has the associated distribution F_{θ_i}, $i = 1, 2, \ldots, k$. The parameters θ_i are unknown; however, it is known that $\theta_1 \le \theta_2 \le \cdots \le \theta_k$. This is typical, for example, in experiments involving different dose levels of a drug where the treatment effects will have a known ordering. These k experimental populations are compared with the population π_0 having the associated distribution function F_{θ_0}. Two cases arise: (1) the case of a <u>known standard</u> θ_0, and (2) the case of an <u>unknown control</u> θ_0. Any experimental population π_i is defined to be <u>superior</u> to π_0 if $\theta_i \ge \theta_0$ and <u>inferior</u> to π_0 otherwise. Our goal is to select all populations that are superior to π_0. Since the ordering among the experimental populations is known, it is reasonable to require a selection procedure to have the property: If π_i is selected, then any π_j with $j > i$ is also selected. This is the so-called <u>isotonic property</u>. If we were to propose rules rather heuristically, one would propose rules based on isotonic estimators of $\theta_1, \ldots, \theta_k$. Such procedures have been investigated in the recent years by Gupta and Yang (1984) in the case of normal means (common variance σ^2 may be known or unknown), by Gupta and Huang (1984) in the case of binomial populations with success probabilities θ_i, and by Gupta and Leu (1986) in the case of two-parameter exponential populations with location parameters (guarantee times) θ_i and a common (known or unknown) scale parameter. Huang (1984) has considered the problem in a nonparametric setup. All these papers consider both cases of a known standard and an unknown control.

In the present paper, our interest is to derive a Bayes rule having the isotonic property for selecting the populations superior to π_0. Under a fairly general loss function the Bayes rule is derived in Section 2 in a form convenient for applications. Two special cases of the loss function are discussed in Section 3. This is followed by discussions of the results under one of the special loss functions to the cases of discrete as well as continuous exponential classes of distributions.

2. BAYES PROCEDURE

Let $\Omega_1 = \{\underset{\sim}{\theta} | \theta_0 \le \theta_1\}$, $\Omega_i = \{\underset{\sim}{\theta} | \theta_{i-1} < \theta_0 \le \theta_i\}$, $i = 2,3,\ldots,k$, $\Omega_{k+1} = \{\underset{\sim}{\theta} | \theta_k < \theta_0\}$, and $\Omega = \bigcup_{i=1}^{k+1} \Omega_i$. Since $\Pr\{\theta_1 \le \theta_2 \le \cdots \le \theta_k\} = 1$, we restrict our attention to the "isotonic" action space $\mathcal{A} = \{1,2,\ldots,k+1\}$. For $1 \le i \le k$, action i corresponds to selecting populations π_j, $i \le j \le k$, as superior to π_0, and excluding the remaining ones as inferior to π_0. Action k+1 corresponds to declaring all the k populations as inferior.

For $\underset{\sim}{\theta} \in \Omega_m$, $1 \le m \le k+1$, and any action a, $1 \le a \le k+1$, we define the loss function $L(\underset{\sim}{\theta}, a)$ by

$$L(\underset{\sim}{\theta}, a) = \begin{cases} \sum_{i=a}^{m-1} L_{m,i}^{(1)}(\theta_0 - \theta_i) & \text{if } a < m, \\ \sum_{j=m}^{a-1} L_{m,j}^{(2)}(\theta_j - \theta_0) & \text{if } a > m, \\ 0 & \text{if } a = m, \end{cases} \tag{1}$$

where

$$L_{m,i}^{(1)}(y) \begin{cases} > 0 \text{ and nonincreasing in } i \text{ for } i < m, \\ = 0 \text{ for } i \geq m, \end{cases}$$

$$L_{m,j}^{(2)}(y) \begin{cases} = 0 \text{ for } j < m, \\ > 0 \text{ and nondecreasing in } j \text{ for } j \geq m \end{cases}$$

and $L_{m,\ell}^{(i)}(y)$, $i = 1,2$, are nondecreasing in y.

We will consider here only the case of known θ_0. The unknown θ_0 case is analogous and involves only straightforward modifications. Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_k)$ be the sample observation belonging to the sample space $\mathfrak{X}$. A decision rule $\underset{\sim}{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1})$ is a measurable mapping from $\mathfrak{X}$ to $[0,1]^{k+1}$ such that $\sum_{i=1}^{k+1} \delta_i(\underset{\sim}{x}) = 1$ for each $\underset{\sim}{x} \in \mathfrak{X}$. The value of $\delta_i(\underset{\sim}{x})$ is the probability of taking action i given $\underset{\sim}{X} = \underset{\sim}{x}$.

Let $f(\underset{\sim}{x}|\underset{\sim}{\theta})$ be the joint probability density function (p.d.f.) of $\underset{\sim}{X}$ given $\underset{\sim}{\theta}$, and let $g(\underset{\sim}{\theta})$ denote the prior distribution of $\underset{\sim}{\theta}$ over Ω. Let

$$f(\underset{\sim}{x}) = \int_{\Omega} f(\underset{\sim}{x}|\underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{\theta} \tag{2}$$

and

$$g(\underset{\sim}{\theta}|\underset{\sim}{x}) = f(\underset{\sim}{x}|\underset{\sim}{\theta}) g(\underset{\sim}{\theta}) / f(\underset{\sim}{x}). \tag{3}$$

Then $f(\underset{\sim}{x})$ is the marginal joint p.d.f. of $\underset{\sim}{X}$ and $g(\underset{\sim}{\theta}|\underset{\sim}{x})$ is the posterior p.d.f. of $\underset{\sim}{\theta}$.

For any decision rule $\underset{\sim}{\delta}$, the associated Bayes risk $r(\underset{\sim}{\delta}, g)$ is given by

$$r(\underset{\sim}{\delta}, g) = \sum_{m=1}^{k+1} \int_{\Omega_m} \int_{\mathfrak{X}} \sum_{a=1}^{k+1} \delta_a(\underset{\sim}{x}) L(\underset{\sim}{\theta}, a) f(\underset{\sim}{x} | \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{x} d\underset{\sim}{\theta}. \quad (4)$$

By the usual interchanging of summations and integrals (justified under suitable regularity conditions), letting

$$R(a | \underset{\sim}{x}) = \sum_{m=1}^{k+1} \int_{\Omega_m} L(\underset{\sim}{\theta}, a) g(\underset{\sim}{\theta} | \underset{\sim}{x}) d\underset{\sim}{\theta}, \quad a = 1, 2, \ldots, k+1, \quad (5)$$

and using (3), we can rewrite (4) as

$$r(\underset{\sim}{\delta}, g) = \sum_{a=1}^{k+1} \int_{\mathfrak{X}} \delta_a(\underset{\sim}{x}) R(a | \underset{\sim}{x}) f(\underset{\sim}{x}) d\underset{\sim}{x}. \quad (6)$$

It now follows that $\underset{\sim}{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1})$ is a Bayes rule if $\sum_{a \in A(\underset{\sim}{x})} \delta_a(\underset{\sim}{x}) = 1$, where

$$A(\underset{\sim}{x}) = \{a | R(a | \underset{\sim}{x}) = \min_{1 \le a' \le k+1} R(a' | \underset{\sim}{x})\}. \quad (7)$$

In order to obtain more insight for implementation of this Bayes rule, let

$$D(a | \underset{\sim}{x}) = R(a+1 | \underset{\sim}{x}) - R(a | \underset{\sim}{x}), \quad a = 1, 2, \ldots, k. \quad (8)$$

Defining (any sum) $\sum_{m=r}^{s} \equiv 0$ for $s < r$, we write (5) as

$$R(a | \underset{\sim}{x}) = \sum_{m=1}^{a-1} \int_{\Omega_m} L(\underset{\sim}{\theta}, a) g(\underset{\sim}{\theta} | \underset{\sim}{x}) d\underset{\sim}{\theta} + \sum_{m=a+1}^{k+1} \int_{\Omega_m} L(\underset{\sim}{\theta}, a) g(\underset{\sim}{\theta} | \underset{\sim}{x}) d\underset{\sim}{\theta}. \quad (9)$$

Using (1) and (9) in (8), it is easy to see that

$$D(a | \underset{\sim}{x}) = \sum_{m=1}^{a} \int_{\Omega_m} L^{(2)}_{m,a}(\theta_a - \theta_0) g(\underset{\sim}{\theta} | \underset{\sim}{x}) d\underset{\sim}{\theta}$$
$$- \sum_{m=a+1}^{k+1} \int_{\Omega_m} L^{(1)}_{m,a}(\theta_0 - \theta_a) g(\underset{\sim}{\theta} | \underset{\sim}{x}) d\underset{\sim}{\theta}. \quad (10)$$

Lemma 2.1. For any given $\underset{\sim}{x} \in \mathfrak{X}$, $D(a|\underset{\sim}{x})$ is nondecreasing in a.

Proof. For $a = 1,2,\ldots,k$, by using (10), we obtain

$$\begin{aligned}
&D(a+1|\underset{\sim}{x}) - D(a|\underset{\sim}{x}) \\
&= \sum_{m=1}^{a} \int_{\Omega_m} [L^{(2)}_{m,a+1}(\theta_{a+1}-\theta_0) - L^{(2)}_{m,a}(\theta_a-\theta_0)] g(\underset{\sim}{\theta}|\underset{\sim}{x}) d\underset{\sim}{\theta} \\
&\quad + \int_{\Omega_{a+1}} L^{(2)}_{a+1,a+1}(\theta_{a+1}-\theta_0) g(\underset{\sim}{\theta}|\underset{\sim}{x}) d\underset{\sim}{\theta} \\
&\quad + \sum_{m=a+2}^{k+1} \int_{\Omega_m} [L^{(1)}_{m,a}(\theta_0-\theta_a) - L^{(1)}_{m,a+1}(\theta_0-\theta_{a+1})] g(\underset{\sim}{\theta}|\underset{\sim}{x}) d\underset{\sim}{\theta} \\
&\quad + \int_{\Omega_{a+1}} L^{(1)}_{a+1,a}(\theta_0-\theta_a) g(\underset{\sim}{\theta}|\underset{\sim}{x}) d\underset{\sim}{\theta}. \qquad (11)
\end{aligned}$$

For $m \le a$,

$$\begin{aligned}
&L^{(2)}_{m,a+1}(\theta_{a+1}-\theta_0) - L^{(2)}_{m,a}(\theta_a-\theta_0) \\
&= [L^{(2)}_{m,a+1}(\theta_{a+1}-\theta_0) - L^{(2)}_{m,a}(\theta_{a+1}-\theta_0)] \\
&\quad + [L^{(2)}_{m,a}(\theta_{a+1}-\theta_0) - L^{(2)}_{m,a}(\theta_a-\theta_0)] \\
&\ge 0,
\end{aligned}$$

the differences inside the brackets being nonnegative by virtue of the fact that $\theta_{a+1} - \theta_0 \ge \theta_a - \theta_0 \ge 0$ for $m \le a$, and of the properties of the loss component.

By a similar reasoning, we can see that

$$L^{(1)}_{m,a}(\theta_0-\theta_a) - L^{(1)}_{m,a+1}(\theta_0-\theta_{a+1}) \ge 0 \text{ for } m \ge a+2.$$

Consequently, $D(a+1|\underset{\sim}{x}) - D(a|\underset{\sim}{x}) \ge 0$ for $a = 1,2,\ldots,k$. This completes the proof of the lemma. ∎

We can use Lemma 1 and obtain a more convenient form for our Bayes rule. Let $B(\underset{\sim}{x}) = \{a|D(a|\underset{\sim}{x}) = 0\}$. If $B(\underset{\sim}{x}) \ne \emptyset$, let $a^* = \max_{a \in B(\underset{\sim}{x})} a$ and $A^*(\underset{\sim}{x}) = B(\underset{\sim}{x}) \cup \{a^*+1\}$. Then the Bayes rule $\underset{\sim}{\delta}$ can be expressed as follows:

If $B(\underset{\sim}{x})$ is not empty, randomize your decision over the set $A^{*}(\underset{\sim}{x})$. If $B(\underset{\sim}{x})$ is vacuous, then choose action b where b is the smallest a for which $D(a|\underset{\sim}{x}) > 0$. If such an integer b does not exist, then choose action k+1.

Remark 1. Because of the monotonicity of $D(a|\underset{\sim}{x})$ in a, the set $B(\underset{\sim}{x})$ is either vacuous or it consists of consecutive members of the set $\{1,2,\ldots,k\}$. We can define a nonrandomized rule, by taking the action corresponding to the smallest member of nonempty $B(\underset{\sim}{x})$.

3. TWO SPECIAL LOSS FUNCTIONS

We consider two special cases of the loss function $L(\underset{\sim}{\theta},a)$ given by (1):

$$L_1(\underset{\sim}{\theta},a) = \begin{cases} \sum_{i=a}^{m-1} (\theta_0-\theta_i) & \text{if } a < m, \\ \sum_{j=m}^{a-1} (\theta_j-\theta_0) & \text{if } a > m, \\ 0 & \text{if } a = m, \end{cases} \tag{12}$$

and

$$L_2(\underset{\sim}{\theta},a) = \begin{cases} (m-a)c_1 & \text{if } a < m, \\ (a-m)c_2 & \text{if } a > m, \\ 0 & \text{if } a = m, \end{cases} \tag{13}$$

where c_1 and c_2 are known positive constants.

For the loss function $L_1(\underset{\sim}{\theta}, a)$, rewriting (10) we get

$$D(a|\underset{\sim}{x}) = \sum_{m=1}^{a} \int_{\Omega_m} (\theta_a-\theta_0)g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta} - \sum_{m=a+1}^{k+1} \int_{\Omega_m} (\theta_0-\theta_a)g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta}$$

$$= \sum_{m=1}^{k+1} \int_{\Omega_m} (\theta_a-\theta_0)g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta}$$

$$= \int_{\Omega} (\theta_a-\theta_0)g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta}$$

$$= E[\theta_a|\underset{\sim}{x}] - \theta_0 .$$

So $E[\theta_{a+1}|\underset{\sim}{x}] - E[\theta_a|\underset{\sim}{x}] = D(a+1|\underset{\sim}{x}) - D(a|\underset{\sim}{x}) \geq 0$. The set $B(\underset{\sim}{x})$ associated with the Bayes rule is: $B(\underset{\sim}{x}) = \{a|E[\theta_a|\underset{\sim}{x}] = \theta_0\}$. A nonrandomized version of the rule is: Select π_i as superior to π_0 if and only if $E[\theta_i|\underset{\sim}{x}] \geq \theta_0$. Thus, to obtain the Bayes rule, we need only to evaluate the posterior means $E[\theta_i|\underset{\sim}{x}]$, $i = 1,2,\ldots,k$.

As for the loss function $L_2(\underset{\sim}{\theta}, a)$, we get

$$D(a|\underset{\sim}{x}) = \sum_{m=1}^{a} \int_{\Omega_m} c_2 g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta} - \sum_{m=a+1}^{k+1} \int_{\Omega_m} c_1 g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta}$$

$$= \sum_{m=1}^{k+1} \int_{\Omega_m} c_2 g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta} - \sum_{m=a+1}^{k+1} \int_{\Omega_m} (c_1+c_2)g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta}$$

$$= c_2 - (c_1+c_2)\int_{\Omega'} g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta},$$

where $\Omega' = \bigcup_{m=a+1}^{k+1} \Omega_m$. Thus

$$D(a|\underset{\sim}{x}) \geq 0 \iff \int_{\Omega'} g(\underset{\sim}{\theta}|\underset{\sim}{x})d\underset{\sim}{\theta} \leq \frac{c_1}{c_1 + c_2} .$$

4. TWO CLASSES OF DENSITY FUNCTIONS

We consider two exponential classes of density functions $f(\underset{\sim}{x}|\underset{\sim}{\theta})$, one discrete and the other continuous. Let $f_i(x|\theta_i)$ denote the conditional p.d.f. of X_i given θ_i, and let $f_i(x)$ denote the marginal p.d.f. of X_i, i.e. $f_i(x) = \int f_i(x|\theta_i)g_i(\theta_i)d\theta_i$, where $g_i(\theta_i)$ denotes the prior marginal p.d.f. of θ_i. We assume that, conditional on $\underset{\sim}{\theta}$, the X_i are independently distributed. Thus $f(\underset{\sim}{x}|\underset{\sim}{\theta}) = \prod_{i=1}^{k} f_i(x_i|\theta_i)$.

First, we consider the discrete class of densities of the form

$$f_i(x|\theta_i) = \begin{cases} \theta_i^x h(x)\beta(\theta_i), & x = 0,1,\ldots,\ 0 \le \theta_i < d, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

where $h(x) > 0$ for all x, and d may be finite or infinite.

Under the loss function $L_1(\underset{\sim}{\theta},a)$ in (12), a straightforward computation, for $X_a = x_a$, gives

$$E[\theta_a|x_a] = \frac{h(x_a)f_a(x_a+1)}{h(x_a+1)f_a(x_a)}, \tag{15}$$

which is the posterior mean of θ_a given $X_a = x_a$. Considering the posterior mean of θ_a given $\underset{\sim}{X} = \underset{\sim}{x}$, we get

$$\begin{aligned} E[\theta_a|\underset{\sim}{X} = \underset{\sim}{x}] &= \frac{\int_\Omega \theta_a \prod_{i=1}^{k} [\theta_i^{x_i} h(x_i)\beta(\theta_i)]g(\underset{\sim}{\theta})d\underset{\sim}{\theta})}{f(\underset{\sim}{x})} \\ &= \frac{h(x_a)f(\underset{\sim}{x}(a))}{h(x_a+1)f(\underset{\sim}{x})}, \end{aligned} \tag{16}$$

where $\underset{\sim}{x}(a) = (x_1,\ldots,x_{a-1},x_a+1,x_{a+1},\ldots,x_k)$. Now, from (15) and (16), we obtain

$$E[\theta_a|\underset{\sim}{X} = \underset{\sim}{x}] = \frac{f_a(x_a)f(\underset{\sim}{x}(a))}{f_a(x_a+1)f(\underset{\sim}{x})}E[\theta_a|x_a]. \tag{17}$$

Thus, the "isotonic" posterior mean $E[\theta_a|\underset{\sim}{X} = \underset{\sim}{x}]$ can be viewed as a weighted result of the posterior mean $E[\theta_a|x_a]$ with weight $\frac{f_a(x_a)f(\underset{\sim}{x}(a))}{f_a(x_a+1)f(\underset{\sim}{x})}$. Hence, for the Bayes rule, it suffices to compute $E[\theta_a|x_a]$, $f_a(x_a)$, and $f(\underset{\sim}{x})$.

As an illustration of the above, we consider the following example of Poisson populations.

Example. For Poisson populations,

$$f_a(x|\theta_a) = e^{-\theta_a}\frac{\theta_a^x}{x!}, \quad x = 0,1,\ldots, \quad a = 1,2,\ldots,k.$$

It is assumed that $\theta_1, \theta_2, \ldots, \theta_k$ have a joint distribution identical to that of the order statistics from k independent gamma random variables, each having scale parameter α and shape parameter m. In other words,

$$g(\underset{\sim}{\theta}) = \begin{cases} k!\prod_{i=1}^{k}\frac{\alpha^m}{\Gamma(m)}\theta_i^{m-1}e^{-\alpha\theta_i}, & \theta_1 \le \theta_2 \le \cdots \le \theta_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$g_a(\theta_a) = \frac{k!}{(a-1)!(k-a)!}H^{a-1}(\theta_a)[1-H(\theta_a)]^{k-1}h(\theta_a),$$

$$a = 1,2,\ldots,k,$$

where

$$h(\theta) = \frac{\alpha^m}{\Gamma(m)}\theta^{m-1}e^{-\alpha\theta}, \quad \theta > 0,$$

and

$$H(\theta) = \int_0^\theta h(y)dy.$$

Thus

$$f_a(x_a) = \int_0^\infty f_a(x_a|\theta_a)g_a(\theta_a)d\theta_a,$$

$$E[\theta_a|x_a] = \frac{\int_0^\infty \theta f_a(x_a|\theta)g_a(\theta)d\theta}{f_a(x_a)},$$

and

$$f(\underset{\sim}{x}) = \int_\Omega k! \prod_{i=1}^k \left[\frac{\alpha^m}{\Gamma(m)x_i!}\theta_i^{x_i+m-1} e^{-\theta_i(1+\alpha)}\right]d\underset{\sim}{\theta}.$$

In general, it is hard to evaluate the function $f(\underset{\sim}{x})$. However, if m is a positive integer, one could compute it by obtaining a recursive formula.

Next, we consider the class of densities

$$f_i(x|\theta_i) = \begin{cases} e^{-\theta_i x}\beta(\theta_i)h(x), & x > a,\ \theta_i > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

where a may be finite or infinite, and $h(x) > 0$ for $x > a$. Let $g_i(\theta_i)$ denote the marginal p.d.f. of θ_i. Then

$$f_i(x) = \int e^{-x\theta_i}h(x)\beta(\theta_i)g_i(\theta_i)d\theta_i. \tag{19}$$

Dividing both sides of (19) by h(x) and then differentiating with respect to x, we get

$$\int \theta_i e^{-x\theta_i}\beta(\theta_i)g_i(\theta_i)d\theta_i = -\frac{d}{dx}\frac{f_i(x)}{h(x)}. \tag{20}$$

Now, using (19) and (20), we can write

$$E[\theta_i | x_i] = \frac{\int \theta_i e^{-x_i\theta_i}\beta(\theta_i)g_i(\theta_i)d\theta_i}{\int e^{-x_i\theta_i}\beta(\theta_i)g_i(\theta_i)d\theta_i}$$

$$= \frac{h'(x_i)}{h(x_i)} - \frac{f_i'(x_i)}{f_i(x_i)}, \qquad (21)$$

where $h'(x) = (d/dx)h(x)$. Further,

$$E[\theta_a | \underset{\sim}{x}] = \frac{1}{f(\underset{\sim}{x})} \int_\Omega \theta_a f(\underset{\sim}{x} | \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{\theta}, \qquad (22)$$

where

$$f(\underset{\sim}{x}) = \int_\Omega f(\underset{\sim}{x} | \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{\theta}. \qquad (23)$$

As in the case of $E[\theta_i | x_i]$, we have

$$E[\theta_a | \underset{\sim}{x}] = \frac{\frac{1}{h(x_a)} \int_\Omega \theta_a f(\underset{\sim}{x} | \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{\theta}}{f(\underset{\sim}{x})/h(x_a)}$$

$$= -\frac{d}{dx_a}\left[\frac{f(\underset{\sim}{x})}{h(x_a)}\right] \Big/ \frac{f(\underset{\sim}{x})}{h(x_a)} . \qquad (24)$$

Carrying out the differentiation in (24) and using (21), we obtain

$$E[\theta_a | \underset{\sim}{x}] = \frac{f'(x_a)}{f(x_a)} - \frac{f_{(a)}(\underset{\sim}{x})}{f(\underset{\sim}{x})} + E[\theta_a | x_a] \qquad (25)$$

where $f_{(a)}(\underset{\sim}{x}) = (d/dx_a)f(\underset{\sim}{x})$. Thus, by evaluating the quantities on the right-hand side of (25), we can implement the Bayes rule using the loss function $L_1(\underset{\sim}{\theta}, a)$ in (12).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\pi_1, \pi_2, \ldots, \pi_k$ be k experimental populations and π_0 is a standard or a control population. The π_i are characterized by the associated distribution functions F_{θ_i}, $i = 0,1,\ldots,k$. The θ_i of the experimental populations are unknown; however, it is known that $\theta_1 \le \theta_2 \le \ldots \le \theta_k$. The value of θ_0 may or may not be known. The goal is to select all the experimental populations that are superior to π_0 (i.e. all π_i for which $\theta_i \ge \theta_0$). Under a fairly general loss function a Bayes rule with isotonic property (i.e. if π_i is selected, then any π_j for $j > i$ is also selected) is derived in a form convenient for applications. Two special cases of the loss function are

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) discussed. One of these loss functions is used to discuss applications to discrete (with Poisson example) as well as continuous exponential class of distributions.