THE PACKING AND COVERING FUNCTIONS OF SOME SELF-SIMILAR FRACTALS

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Abstract

For a self-similar set K satisfying a certain separation condition, the number $N(\varepsilon)$ of points in a maximal ε-separated subset and the number $M(\varepsilon)$ of ε-balls needed to cover satisfy $N(\varepsilon) \sim$ const $\cdot\varepsilon^{-D}$ and $M(\varepsilon) \sim$ const $\cdot\varepsilon^{-D}$ as $\varepsilon \to 0$ through a certain multiplicative group. Here D is the Hausdorff dimension of K. Furthermore, the empirical distribution of points in a maximal ε-separated set converges weakly to normalized D-dimensional Hausdorff measure on K.

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1. Introduction

Self-Similar sets in $\mathbb{R}^d$ occur as the limit sets (equivalently, the minimal closed invariant sets) of certain semigroups of contractive Euclidean similarity transformations ([4], [7]). The purpose of this note is to describe the asymptotic behavior as $\varepsilon \to 0$ of the number $N(\varepsilon)$ of points in a maximal ε-separated subset and the number $M(\varepsilon)$ of ε-balls needed to cover a self-similar set, and to investigate the relationships between maximal packings, minimal coverings, and Hausdorff measure. The functions $N(\varepsilon)$ and $M(\varepsilon)$ are used to define the *packing* and *covering dimensions* (often called the *capacity* and *metric entropy*): see below.

A *similarity transformation* $S : \mathbb{R}^d \to \mathbb{R}^d$ has the form $S = rJ$, where $J : \mathbb{R}^d \to \mathbb{R}^d$ is an isometry and $r > 0$ is a scalar; if $0 < r < 1$ then S is called *contractive*. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$ be a finite set of contractive similarity transformations. Then for any sequence $i_1, i_2, \cdots$ of indices and any $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} S_{i_1} S_{i_2} \ldots S_{i_n} x \triangleq k_{i_1, i_2 \ldots}$$

exists, and the limit is independent of x ([4], sec. 3; two different sequences $i_1, i_2, \ldots$ and $i_1', i_2', \ldots$ may yield the same limit). Let

$$K = \{k_{i_1 i_2 \ldots}\}$$

be the set of all possible limit points: this set will be the principal object of study in this paper.

Most of the fractals in [7], sec. 6–8, 14 arise in this manner. Some examples:

(1) Let $S_1 x = rx$ and $S_2 x = rx + 1 - r$, where $0 < r \le \frac{1}{2}$. If $r = \frac{1}{3}$ then K is the Cantor set; if $r = \frac{1}{2}$ then K is the unit interval ([7], plate 81).

(2) Let $S_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$S_1(x_1, x_2) = (x_1/2, x_2/2)$$

$$S_2(x_1, x_2) = (1/2 + x_1/2, x_2/2)$$

$$S_3(x_1, x_2) = (1/4 + x_1/2, \sqrt{3}/4 + x_2/2);$$

then K is the "Sierpinski gasket" ([7], p. 142).

(3) Let $a_1 = (0,0)$, $a_2 = (1/3,0)$, $a_3 = (1/2, \sqrt{3}/6)$, $a_4 = (2/3,0)$, and $a_5 = (1,0)$. Let S_i $(i = 1,2,3,4)$ be the unique similarity transformation of $\mathbb{R}^2$ mapping $\overline{a_1 a_5}$ onto $\overline{a_i a_{i+1}}$ and having positive determinant. Then K is the "Koch snowflake" ([7], pp. 42–43).

The set K is always compact ([4], sec. 3), as are the images

$$K_{i_1 i_2 \ldots i_n} \triangleq S_{i_1} S_{i_2} \ldots S_{i_n} K.$$

In the examples above the sets $K_1, K_2, \ldots, K_N$ are either pairwise disjoint or have "small" overlaps. In the former case the set K is totally disconnected and each point $x \in K$ has a unique representation $x = k_{i_1 i_2 \ldots}$; in the latter case, some points have multiple representations and K may be arcwise connected. It is always the case that $K = \bigcup_{i=1}^{N} K_i$.

Say that $\mathcal{S}$ satisfies the *open set condition* [4] if there exists a nonempty open subset U of $\mathbb{R}^d$ such that $S_i U \subset U$ for each i and $S_i U \cap S_j U = \emptyset$ if $i \neq j$. If U can be chosen so that $U \cap K \neq \emptyset$, say that $\mathcal{S}$ satisfies the *strong open set condition.* Notice that this holds in the examples above.

Write $S_i = r_i J_i$, where $0 < r_i < 1$ and J_i is an isometry. The *similarity dimension* of $\mathcal{S}$ ([4],[6]) is the unique $D > 0$ such that

$$\sum_{i=1}^{N} r_i^D = 1.$$

Let $H^D(\cdot)$ be the D-dimensional Hausdorff measure on $\mathbb{R}^d$ ([4]).

Theorem 0 ([4]): *If $\mathcal{S}$ satisfies the open set condition then $0 < H^D(K) < \infty$ and $H^D(K_i \cap K_j) = 0$ for $i \neq j$.*

Thus, D is the Hausdorff dimension of K. Since $H^D(K_i \cap K_j) = 0$ it follows that

$$H^D(K_{i_1 i_2 \ldots i_n}) = (r_{i_1} r_{i_2} \cdots r_{i_n})^D H^D(K).$$

Therefore, if one chooses indices $i, i_2, \ldots$ at random from the set $\{1, 2, \ldots, N\}$ according to the multinomial distribution $\{r_1^D, r_2^D, \ldots, r_N^D\}$, then the random point $k_{i_1 i_2 \ldots}$ will be "uniformly distributed" on K relative to D-dimensional Hausdorff measure.

Call a finite subset F of K *ε-separated* if dist $(x, x') \geq \varepsilon$ for all $x, x' \in F$ such that $x \neq x'$. Let $N(\varepsilon)$ be the maximum cardinality of an ε-separated subset of K; this will be called the *packing function.* Call a finite subset F of K an *ε-covering* if for every $y \in K$ there exists $x \in F$ such that dist $(x, y) < \varepsilon$. Let $M(\varepsilon)$ be the minimum cardinality of an ε-covering subset of K; this will be called the *covering function.* The *packing* and *covering dimensions* D_P and D_C are defined by

$$D_P = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon^{-1}},$$

$$D_C = \lim_{\varepsilon \to 0} \frac{\log M(\varepsilon)}{\log \varepsilon^{-1}},$$

provided these limits exist. (The covering dimension was introduced in [5], the packing dimension in [8]. They are usually called the *metric entropy* and *capacity.*) A simple argument shows that $N(3\varepsilon) \leq M(\varepsilon) \leq N(\varepsilon)$, so $D_p = D_C$ whenever either limit exists.

Theorem 1: *Assume that the strong open set condition holds.*

(a) *If the additive group generated by* $\log r_1, \log r_2, \ldots, \log r_N$ *is dense in* $\mathbb{R}$, *then there exist constants* $C, C' > 0$ *such that as* $\varepsilon \to 0$

$$N(\varepsilon) \sim C\varepsilon^{-D} \quad \textit{and} \tag{1.1}$$

$$M(\varepsilon) \sim C'\varepsilon^{-D}. \tag{1.2}$$

(b) *If the additive group generated by* $\log r_1, \log r_2, \ldots, \log r_n$ *is* $h\mathbb{Z} (h > 0)$ *then for each* $\beta \in [0, h)$ *there exist constants* $C_\beta, C'_\beta > 0$, *uniformly bounded, such that as* $n \to \infty$

$$N(e^{-nh+\beta}) \sim C_\beta \exp\{D(-nh + \beta)\} \quad \textit{and} \tag{1.3}$$

$$M(e^{-nh+\beta}) \sim C'_\beta \exp\{D(-nh + \beta)\}. \tag{1.4}$$

Observe that case (6) obtains for the Cantor set, the Koch snowflake, and the Sierpinski gasket.

Corollary: *If the strong open set condition holds then*

$$D = D_P = D_C.$$

This answers a query in [1]. (After writing this note I learned that this relation is part of the folklore: see, for example, [9].)

Let F_ε be an ε-separated subset of K having maximum cardinality, and let G_ε be an ε-covering subset of K having minimum cardinality. Define Borel probability measures $\mu_\varepsilon(\nu_\varepsilon)$ on K by putting mass $1/N(\varepsilon)$ $\big(1/M(\varepsilon)\big)$ at each point of $F_\varepsilon(G_\varepsilon)$.

Theorem 2: *If the strong open set condition holds then as* $\varepsilon \to 0$

$$\mu_\varepsilon \xrightarrow{\mathcal{D}} \frac{H^D}{H^D(K)} \quad \textit{and} \tag{1.5}$$

$$\nu_\varepsilon \xrightarrow{\mathcal{D}} \frac{H^D}{H^D(K)}. \tag{1.6}$$

Theorems 1 and 2 help clarify the relations between packings, coverings, and Hausdorff masures. Maximal ε-separated sets and minimal ε-separated sets are usually very difficult to find. In the totally disconnected case (i.e., $K_i \cap K_j = \emptyset$ for $i \neq j$) one may give an algorithm for obtaining an ε-separated set whose cardinality is within $0(1)$ of $N(\varepsilon)$. In general one may produce an ε-separated set whose cardinality is within $0(\varepsilon^{-D+\delta})$ of $N(\varepsilon)$ for some $\delta > 0$. The proofs below should suggest how this may be done.

In proving Theorems 1–2, I shall consider only the packing function $N(\varepsilon)$. The same arguments apply to the covering function $M(\varepsilon)$.

2. Totally Disconnected K

This case is particularly simple. Assume that $K_1, K_2, \ldots, K_N$ are pairwise disjoint; since each K_i is compact there exists $\delta > 0$ such that if $x \in K_i$ and $x' \in K_j$, $i \neq j$, then $\operatorname{dist}(x, x') > \delta$. Now if $\varepsilon < \delta$ then one may obtain an ε-separated subset of maximum cardinality by finding maximal ε-separated subsets of $K_1, K_2, \ldots, K_N$ and taking their union. Since $K_i = S_i K$ is similar to K, a maximal ε- separated subset of K_i is similar to a maximal εr_i^{-1}-separated subset of K, and therefore its cardinality is $N(\varepsilon r_i^{-1})$. Hence, if $\varepsilon < \delta$ then $N(\varepsilon) = \sum_{i=1}^N N(\varepsilon r_i^{-1})$. It follows that

$$N(\varepsilon) = \sum_{i=1}^{N} N(\varepsilon r_i^{-1}) + L(\varepsilon) \tag{2.1}$$

for all $\varepsilon > 0$. Since $N(\varepsilon)$ is a nonincreasing integer-valued function that is zero for all sufficiently large ε, $L(\varepsilon)$ is a piecewise continuous function with only finitely many discontinuities that vanishes for $0 < \varepsilon < \delta$.

Equation (2.1) may be rewritten as a renewal equation ([3], ch. 11) in the following manner. Define

$$Z(a) = e^{-aD} N(e^{-a})$$

for $a > 0$; since $\sum_{i=1}^N r_i^D = 1$, it follows from (2.1) that

$$Z(a) = z(a) + \int_{(0,a]} Z(a-x)F(dx), \quad a > 0,$$

where $F(dx)$ is the probability measure that puts mass r_i^D at $-\log r_i, i = 1, 2, \ldots, N$. Because F has finite support and L is piecewise continuous with only finitely many discontinuities, z is also piecewise continuous with only finitely many discontinuities. Moreover, z has compact support in $[0, \infty)$ since L vanishes in $(0, \delta)$. Therefore, z is directly Riemann integrable ([3], ch. 11).

There are now two cases, the nonlattice case and the lattice case, corresponding to (a) and (b) of Theorem 1. In the nonlattice case the renewal theorem ([3], ch. 11) implies that

$$\lim_{a\to\infty} Z(a) = \int_0^\infty z(x)dx \Big/ \sum_{i=1}^N r_i^D \log r_i^{-1}.$$

This is equivalent to (1.1). In the lattice case the renewal theorem ([2], ch. 13) implies that for $0 \leq \beta < h$

$$\lim_{n\to\infty} Z(nh+\beta) = \sum_{n=1}^{\infty} z(nh+\beta) \Big/ \sum_{i=1}^N r_i^D \log r_i^{-1}$$

This is equivalent to (1.3). Note that the constants C_β must be uniformly bounded because $N(\varepsilon)$ is nonincreasing.

3. The General Case

If $K_1, K_2, \ldots, K_N$ are not pairwise disjoint then the argument of the preceding section fails because the union of ε-separated subsets of $K_1, \ldots K_N$ will not generally be ε-separated. Nevertheless, since $K = \bigcup_{i=1}^N K_i$,

$$N(\varepsilon) \leq \sum_{i=1}^{N} N(\varepsilon r_i^{-1}).$$

Define

$$L(\varepsilon) = \sum_{i=1}^{N} N(\varepsilon r_i^{-1}) - N(\varepsilon).$$

Proposition 1: *Assume that the strong open set condition holds. Then there exist constants* $\gamma > 0, \delta > 0$ *such that*

$$L(\varepsilon) \leq \gamma \varepsilon^{\delta - D}.$$

The proof is deferred to sec. 5.

Define, as in sec. 2, $Z(a) = e^{-aD} N(e^{-a})$, and write

$$Z(a) = z(a) + \int_{(0,a]} Z(a-x) F(dx)$$

where $F(dx)$ puts mass r_i^D at $\log r_i^{-1}$, $i = 1, 2, \ldots, N$. Observe that for all sufficiently large a, $z(a) = -e^{-aD} L(e^{-a})$. Moreover, since $N(\varepsilon)$ is a nonincreasing, nonnegative integer valued function and $F(dx)$ has finite support, $z(a)$ is a piecewise continuous function with only finitely many discontinuities in any finite interval. Proposition 1 implies that

$$|z(a)| \leq \gamma e^{-a\delta}$$

for all sufficiently large a. It follows that $z(a)$ is directly Riemann integrable. Therefore, in the nonlattice case

$$\lim_{a \to \infty} Z(a) = \int_0^\infty z(x) dx \Big/ \sum_{i=1}^{N} r_i^D \log r_i^{-1},$$

and in the lattice case

$$\lim_{n\to\infty} Z(nh+\beta) = \sum_{n=1}^{\infty} z(nh+\beta)/\sum_{i=1}^{N} r_i^D \log r_i^{-1}$$

for every $\beta \in [0,h)$. This proves (1.1) and (1.3). As before, the constants C_β are uniformly bounded because $N(\varepsilon)$ is nonincreasing.

4. Maximal Packings and Hausdorff Measure

Recall that μ_ε is the probability measure that puts mass $1/N(\varepsilon)$ at each point of a maximal ε-separated set.

Proposition 2: *Assume that the strong open set condition holds. For each pair of distinct sequences $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$,*

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(K_{i_1 i_2 \ldots i_n} \cap K_{j_1 j_2 \ldots j_n}) = 0.$$

The proof will be given in sec. 5.

Since the support of μ_ε is an ε-separated subset of K, and since K_i is similar to K, it follows that

$$\mu_\varepsilon(K_i) \le \frac{N(\varepsilon r_i^{-1})}{N(\varepsilon)}, \quad i = 1, 2, \ldots, N. \tag{4.1}$$

For small ε, $\sum_{i=1}^{N} \left(N(\varepsilon r_i^{-1})/N(\varepsilon)\right) \sim 1$ by Theorem 1, and $\mu_\varepsilon(K_i \cap K_j) = o(1)$ for $i \ne j$, by Proposition 2. Since $\mu_\varepsilon(K) = 1$ and $K = \cup K_i$, (4.1) implies that

$$\begin{aligned} \mu_\varepsilon(K_i) &\sim N(\varepsilon r_i^{-1})/N(\varepsilon) \\ &\sim r_i^D = H^D(K_i)/H^D(K). \end{aligned}$$

Now the sets $K_{i_1 i_2 \ldots i_n}$ are all similar to K, so by an easy induction argument

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(K_{i_1 i_2 \ldots i_n}) = H^D(K_{i_1 i_2 \ldots i_n})/H^D(K)$$

for each sequence $i_1, i_2, \ldots, i_n$. Since $K = \cup K_{i_1 i_2 \ldots i_n}$ and diam $K_{i_1 i_2 \ldots i_n} \le (\max_{1\le i\le N} r_i)^n \to 0$, it follows easily that for any continuous function $f : K \to \mathbb{R}$

$$\lim_{\varepsilon\to 0} \int_K f d\mu_\varepsilon = \int_K f(x) H^D(dx)/H^D(K).$$

This proves (1.5).

5. The Key Estimate

Assume that the strong open set condition holds. Let $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$. Define $Q_{ij}(\varepsilon)$ to be the maximum cardinality of an ε-separated subset F of K_i such that for each $x \in F$, $\mathrm{dist}(x, K_j) \leq \varepsilon$.

Proposition 3: *There exists $\delta > 0$ such that as $\varepsilon \to 0$,*

$$Q_{ij}(\varepsilon) = 0(\varepsilon^{\delta - D}). \tag{5.1}$$

Proposition 3 implies Proposition 1. To see this observe that one gets an ε-separated subset of K by taking maximal ε-separated subsets of K_i, $i = 1, 2, \ldots, N$, deleting all points from K_i within ε of $\cup_{j: j \neq i} K_j$, then taking the union. Thus,

$$N(\varepsilon) \geq \sum_{i=1}^{N} N(\varepsilon r_i^{-1}) - \sum_{i \neq j} \sum Q_{ij}(\varepsilon),$$

and Proposition 1 follows.

Proposition 3 also implies Proposition 2. First notice that to prove Proposition 2 it suffices, since $K_{i_1} \supset K_{i_1 i_2} \supset \ldots$, to establish that if $i \neq j$ then

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(K_{i_1 i_2 \ldots i_n i} \cap K_{i_1 i_2 \ldots i_n j}) = 0.$$

Recall that $\mu_\varepsilon(G)$ is $N(\varepsilon)^{-1} \times$ the cardinality of $F_\varepsilon \cap G$, where F_ε is a maximal ε-separated subset of K. Since

$$K_{i_1 i_2 \ldots i_n i} \cap K_{i_1 i_2 \ldots i_n j} \quad = S_{i_1} S_{i_2} \ldots S_{i_n}(K_i \cap K_j)$$

and since $S_{i_1} \ldots S_{i_n}$ is a similarity transformation that contracts distances by a factor of $r_{i_1} r_{1_2} \ldots r_{i_n} = \rho$,

$$\mu_\varepsilon(K_{i_1 i_2 \ldots i_n i} \cap K_{i_1 i_2 \ldots i_n j}) \leq \{Q_{ij}(\varepsilon \rho^{-1}) + Q_{ji}(\varepsilon \rho^{-1})\}/N(\varepsilon).$$

Proposition 3 and Theorem 1 imply that this converges to 0 as $\varepsilon \to 0$.

6. Proof of the Key Estimate

Recall that the open set condition holds if there is an open set $U \subset \mathbb{R}^d$ such that $S_i U \subset U$ for each i and $S_i U \cap S_j U = 0$ for $i \neq j$. Let $U_{i_1 i_2 \ldots i_n} = S_{i_1} S_{i_2} \ldots S_{i_n} U$. If the open set condition holds then

(a) $U \supset U_{i_1} \supset U_{i_1 i_2} \supset \cdots$;

(b) $K_{i_1 i_2 \ldots i_n} \subset \overline{U}_{i_1 i_2 \ldots i_n}$;

(c) $K_{j_1 j_2 \ldots j_n} \cap U_{i_1 i_2 \ldots i_n} = \emptyset$ unless $(i_1, \ldots, i_n) = (j_1, \ldots, j_n)$

([4], sec. 5.2 (3)).

If the open set U can be chosen so that $U \cap K \neq \emptyset$ then the strong open set condition holds. Assume that this is the case. Then there exists a point $k_{j_1 j_2 \ldots} \in U$. Now the diameters of the sets $K_{j_1 j_\varepsilon \ldots j_n}$ converge to zero as $n \to \infty$, and $k_{j_1 j_2 \ldots}$ is an element of each; consequently, there exists a finite sequence $j_1, j_2, \ldots, j_p$ such that

$$K_{j_1 j_2 \ldots j_p} \subset U.$$

Since $K_{j_1 j_2 \ldots j_p}$ is compact there exists $\alpha > 0$ such that

$$\operatorname{dist}(x, U^c) > \alpha \qquad \forall x \in K_{j_1 j_2 \ldots j_p}.$$

It follows upon applying the similarity transformation $S_{i_1} S_{i_2} \ldots S_{i_n}$ that for any sequence $i_1, i_2, \ldots, i_n$

$$K_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_p} \subset U_{i_1 i_2 \ldots i_n}$$

and that for each $x \in K_{i_1 i_2 \ldots i_n j_1 \ldots j_p}$

$$\operatorname{dist}(x, U^c_{i_1 i_2 \ldots i_n}) > \alpha r_{i_1} r_{i_2} \ldots r_{i_n}. \tag{6.1}$$

Let $j \in \{1,2,\ldots,N\}$ and let $i_1, i_2, \ldots, i_n$ be a finite sequence such that $i_1 \neq j$ and $\alpha r_{i_1} r_{i_2} \ldots r_{i_n} > \varepsilon$. If $x \in K_{i_1 i_2 \ldots i_n}$ and $\text{dist}(x, K_j) \leq \varepsilon$ then the sequence $j_1, j_2, \ldots, j_p$ cannot occur in $i_1, i_2, \ldots, i_n$, because of (6.1) and the fact that $U_{i_1 i_2 \ldots i_n} \cap K_j = \emptyset$.

Now let F be an ε-separated subset of K_i such that for each $x \in F$, $\text{dist}(x, K_j) \leq \varepsilon$ (where $i \neq j$). Each $x \in F$ lies in a set $K_{i_1 i_2 \ldots i_m}$ such that $i_1 = i$ and

$$r_{i_1} r_{i_2} \cdots r_{i_m} \text{ diam } K < \varepsilon \leq r_{i_1} r_{i_2} \cdots r_{i_{m-1}} \text{ diam } K; \tag{6.2}$$

since $\text{diam } K_{i_1 i_2 \ldots i_m} = r_{i_1} \ldots r_{i_m} \text{ diam } K < \varepsilon$ and F is ε-separated, each $x \in F$ has its own unique sequence $i_1, i_2, \ldots, i_m$ satisfying (6.2). Let $r_* = \max(r_1, r_2, \ldots, r_N) < 1$ and let $q \geq 1$ be an integer such that $r_*^{q-1} \text{ diam } K < \alpha$; then (6.2) implies that $\alpha r_{i_1} r_{i_2} \cdots r_{i_{m-q}} > \varepsilon$. Consequently, if $x \in F \cap K_{i_1 i_2 \ldots i_m}$ and (6.2) holds then by the preceding paragraph the sequence $j_1, j_2, \ldots, j_p$ does not occur in $i_1, i_2, \ldots, i_{m-q}$. Therefore, the cardinality of F, and hence $Q_{ij}(\varepsilon)$, is bounded above by the number $A(\varepsilon)$ of distinct sequences $i_1, i_2, \ldots, i_m$ satisfying (6.2) such that the sequence $j_1, j_2, \ldots, j_p$ does not occur in $i_1, i_2, \ldots, i_{m-q}$. It remains to show that

$$A(\varepsilon) = 0(\varepsilon^{\delta - D}) \tag{6.3}$$

as $\varepsilon \to 0$ for some $\delta > 0$.

Define $B(\varepsilon)$ to be the number of distinct sequences $i_1 i_2, \ldots, i_n$ such that the sequence $j_1, j_2, \ldots, j_p$ does not occur in $i_1, i_2, \ldots, i_n$ and $r_{i_1} r_{i_2} \cdots r_{i_n} > \varepsilon$. Then

$$A(\varepsilon) \leq N^q B(\varepsilon / \text{diam } K);$$

consequently, to prove (6.3) it suffices to show that for some $D^* < D$

$$B(\varepsilon) = 0(\varepsilon^{-D^*}). \tag{6.4}$$

The function $B(\varepsilon)$ is a nonincreasing, nonnegative integer-valued function of $\varepsilon > 0$. Each sequence $i_1, i_2, \ldots, i_n$ counted in $B(\varepsilon)$ begins with some $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_p)$, provided

$\varepsilon < (\min_{1\le i\le N} r_i)^p$, so

$$B(\varepsilon) \le \sum_{(i_1,\ldots,i_p)\ne(j_1,\ldots,j_p)} B(\varepsilon/r_{i_1}r_{i_2}\ldots r_{i_p}) \tag{6.5}$$

for all $\varepsilon < (\min_{1\le i\le N} r_i)^p$. Let D^* be the unique real number such that

$$\sum_{(i_1,\ldots,i_p)\ne(j_1,\ldots,j_p)} (r_{i_1}r_{i_2}\ldots r_{i_p})^{D*} = 1. \tag{6.6}$$

Notice that $D^* < D$ because

$$\sum_{(i_1,\ldots,i_p)} (r_{i_1}r_{i_2}\ldots r_{i_p})^{D} = \left(\sum_i r_i^D\right)^p = 1.$$

Define $Z(x) = e^{-xD^*}B(e^{-x})$; then by (6.5)

$$Z(x) \le \sum_{(i_1,\ldots,i_p)\ne(j_1,\ldots,j_p)} Z\big(x+\log(r_{i_1}r_{i_2}\ldots r_{i_p})\big)(r_{i_1}\ldots r_{i_p})^{D*} \tag{6.7}$$

for all sufficiently large $x \in \mathbb{R}$. Moreover, for each $a \in \mathbb{R}$, $Z(x)$ is bounded on $(-\infty, a]$, because $B(\varepsilon) = 0$ for large ε. It now follows from (6.6) and (6.7) that for all sufficiently large $a \in \mathbb{R}$,

$$\begin{aligned}&\sup\{Z(x) : x \le a + \min_{(i_1,\ldots,i_p)} \log(r_{i_1}r_{i_2}\ldots r_{i_p})^{-1}\}\\ &\le \sup\{Z(x) : x \le a\}.\end{aligned}$$

Therefore, $Z(x)$ is bounded on $\mathbb{R}$. This proves (6.4).

7. Concluding Remarks

(1) The methods used here may also be used to determine the asymptotic behavior of various other functions. For example, let $x \in \mathbb{R}^d\backslash K$ be a point in the complement of K whose orbit $\mathcal{O}(x) = \{S_{i_1}S_{i_2}\ldots S_{i_n}x\}$ is disjoint from K; define $Q(\varepsilon) = \#\{y \in \mathcal{O}(x)$: distance $(y,K) \ge \varepsilon\}$. Then $Q(\varepsilon)$ satisfies an asymptotic relation analogous to (1.1)-(1.4).

(2) The methods of this paper rely heavily on the *strict* self-similarity of K. For fractals with some *approximate* self-similarity, such as limit sets of Kleinian groups, the analogous problems are considerably harder, but similar results obtain (cf. [6]).

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