Conditioned Brownian Motion in Planar Domains

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Technical Report #87-29

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January 1988

AMS 1980 Subject Classification Primary 60J65, 60J45, Secondary 30C85
[1]Supported by an NSF Grant

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Summary

Let Ω be a simply connected planar domain which is not the entire plane, and let Z_t, $t \geq 0$, be two dimensional Brownian motion started at a point of Ω and either conditioned to exit Ω at a given boundary point or conditioned to hit a given point of Ω before exiting Ω. Let τ be the lifetime of Z. We study $\tau, \int_0^\tau f(Z_t)dt$, and $\int_0^\tau (f(Z_t), g(Z_t)) \cdot dZ_t$. It is shown that Z behaves almost independently in two Whitney squares of Ω which are far apart in the sense of P. Jones, and that Var $\tau \leq c\delta(\Omega)$ area Ω, where $\delta(\Omega)$ is the supremum of the areas of the discs contained in Ω.

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1. Introduction. This paper studies standard complex Brownian motion started at a point x in a connected Greenian domain of the complex plane $\mathbb{C}$ and either conditioned to exit Γ at a given point y of its Martin boundary or conditioned to hit a point y in Γ before leaving Γ. We use $Z(x,y,\Gamma) = \{Z_t(x,y,\Gamma), 0 \le t < \infty\}$, to designate either of these processes. Formally, $Z(x,y,\Gamma)$ is the h process with h respectively $K_\Gamma(\cdot,y)$ and $G_\Gamma(\cdot,y)$, where K_Γ is the Martin kernel function of Γ and G_Γ is the Green function of Γ. These are the two basic h processes in Γ; all h processes in Γ are mixtures of them. An extensive discussion of h processes and their uses in potential theory may be found in Doob [6]. See Durrett [7] for an elementary account as well as a description of some of the ways h processes arise in connection with complex variables and partial differential equations. There is more detail on h processes in the next section of this paper.

If A is a Borel subset of $\mathbb{C}$, the area, closure, diameter, complement, and boundary of A are respectively denoted $\sigma(A), \overline{A}$, $\operatorname{diam}(A)$, A^c, and ∂A, and the (minimum) Euclidean distance between A and another Borel set B is written $d(A,B)$. The lifetime of $Z(x,y,\Gamma)$ is denoted $\tau(x,y,\Gamma)$. In [4] Cranston and McConnell answer a question of Chung by proving there is an absolute constant K such that

$$E\tau(x,y,\Gamma) \le K\sigma(\Gamma). \tag{1.1}$$

Here we study the processes $Z(x,y,\Gamma)$ under the restriction that Γ is simply connected. Several of our results are related to (1.1). Throughout this paper Ω stands for a simply connected domain which is not the entire plane, that is, which has a Green function, and we suppress x, y, and Ω in the notation by putting $Z = Z(x,y,\Omega)$ and $\tau = \tau(x,y,\Omega)$. We use c, C, c_p, etc. for positive absolute constants, not necessarily the same at each occurrence.

If Q is a square contained in Ω we call it a Whitney square (for Ω) if $\operatorname{diam}(Q) \le d(Q,\Omega^c) \le 4 \operatorname{diam}(Q)$. See [13] for a proof that Ω is a union of Whitney squares $Q_i, 1 \le i < \infty$, which have disjoint interiors. Such a collection of squares is called a Whitney decomposition of Ω. If Q and R are both Whitney squares we define, after P. Jones [11], $\rho(Q,R) = 0$ if $Q = R$, and if $Q \ne R, \rho(Q,R)$ is the smallest integer n such that there exist Whitney squares $S_1, S_2, \ldots, S_n$ satisfying $S_1 \cap Q \ne \phi, S_n \cap R \ne \phi$, and $S_i \cap S_{i+1} \ne \phi, 1 \le$

$i < n$. If Q is a Whitney square we denote $\int_0^\tau I(Z_t \epsilon Q)dt$, the total time Z spends in Q, by T_Q, and let P_Q stand for the probability Z ever hits Q. If $Q_i, 1 \le i < \infty$, is a Whitney decomposition of $\Omega, \tau = \Sigma T_{Q_i}$. Now the main results of the paper are stated.

Theorem 1.1. Let Q be a Whitney square. Then

(1.2) $$ET_Q \le CP_Q\sigma(Q).$$

If furthermore either y is a boundary point or one of x, y is a distance at least $\text{diam}(Q)/2$ from Q, then

(1.3) $$ET_Q \ge cP_Q\sigma(Q).$$

Most of this paper is devoted to proving the following theorem.

Theorem 1.2. If Q and R are Whitney squares then

(1.4) $$|\text{Cov }(T_Q, T_R)| \le Ce^{-c\rho(Q,R)}\sigma(Q)\sigma(R)(P_Q + P_R).$$

The following theorem is derived from Theorem 1.2, using the inequality Var $(\Sigma X_i) \le \Sigma$ Var $X_i + \sum_{i \ne j} |Cov(X_i, X_j)|$. Let $\delta(\Omega)$ be the supremum of the areas of the discs contained in Ω. This quantity has appeared in several contexts in the analysis of simply connected domains. See, for example, [9] and [12]. We prove

Theorem 1.3. If $\delta(\Omega) < \infty$ then either $P(\tau = \infty) = 1$ or $E\tau < \infty$. If $E\tau < \infty$ then

(1.5) $$\text{Var } \tau \le C\delta(\Omega)E\tau.$$

We also show there is a constant c such that if $\delta(\Omega) < \infty$, there exist z and w in Ω satisfying

(1.6) $$\text{Var } \tau(w, z, \Omega) \ge c\delta(\Omega)E\tau(w, z, \Omega).$$

Of course (1.5), together with (1.1), implies Var $\tau \le C\delta(\Omega)\sigma(\Omega)$. It is known that $P(\tau = \infty)$ is either 1 or 0, whether or not $\delta(\Omega) < \infty$ (see [6]).

We remark that Var τ has an analytic formulation. Assume that $y \in \Omega$ for simplicity. Let $f_t(x,\cdot)$ be the transition density of Brownian motion killed when it hits $\partial\Omega$, that is, the fundamental solution of the heat equation with boundary values zero. Put $h(t) = f_t(x,y)$ and $r(t) = h(t)/\int_0^\infty h(t)dt = h(t)/G_\Omega(x,y)$. Then $r(t)$ is the density of τ, so Var $\tau = \int_0^\infty (t-\mu)^2 r(t)dt$, where $\mu = \int_0^\infty r(t)dt$. Especially, if Var τ is small, the graph of h looks more or less like a spike at μ.

The proofs of Theorems 1.1 and 1.2 can be easily altered to treat other functionals of the paths of Z. We now give several examples of this. If f is a real valued measurable function on Ω and Q is a square contained in Ω put $f^*(Q) = sup_{z \epsilon Q}|f(z)|$,

$$T_Q^f = \int_0^\tau f(Z_t)I(Z_t \epsilon Q)dt, \text{ and}$$

$$W_Q^f = \int_0^\tau f(Z_t)I(Z_t \epsilon Q)dX_t,$$

where X_t is the real part of Z_t. The following inequalities, which hold for Q and R Whitney squares, are proved in Section 5, or more precisely, it is pointed out how the arguments of Sections 2, 3, and 4 can be altered to prove them. Note that (1.7) below follows immediately from (1.2). Also, we use $f^*(Q)$ here for brevity. In Section 5 we point out that $f^*(Q)$ can be replaced in the following four inequalities by quantities which are sometimes much smaller.

(1.7) $$E|T_Q^f| \leq cP_Q f^*(Q)\sigma(Q).$$

(1.8) $$|\text{Cov }(T_Q^f, T_R^f)| \leq Ce^{-c\rho(Q,R)} f^*(Q)f^*(R)\sigma(Q)\sigma(R)(P_Q + P_R).$$

(1.9) $$E|W_Q^f| \leq cP_Q f^*(Q) \text{ diam}(Q).$$

(1.10) $$|\text{Cov }(W_Q^f, W_R^f)| \leq Ce^{-c\rho(Q,R)} f^*(Q)f^*(R) \text{ diam}(Q) \text{ diam}(R)(P_Q + P_R).$$

Note that (1.1) implies that, for any z in Γ,

$$P(\tau(z,y,\Gamma) \geq 2K\sigma(\Gamma)) \leq E\tau(z,y,\Gamma)/[2K\sigma(\Gamma)] \leq 1/2,$$

which, upon using the strong Markov property for conditioned Brownian motion (see [6] and the next section) gives

$$P(\tau(x,y,\Gamma) \geq (j+1)2K\sigma(\Gamma) | \tau(x,y,\Gamma) \geq j2K\sigma(\Gamma)) \leq 1/2, j = 0,1,2,\ldots$$

implying

(1.11) $$P(\tau(x,y,\Gamma) \geq m2K\sigma(\Gamma)) \leq 2^{-m}, m = 0,1,2,\ldots$$

Putting $K_p = \sum_{m=1}^{\infty} 2^{-m-1}(2Km)^p$, (1.11) gives

(1.12) $$E\tau(x,y,\Gamma)^p \leq K_p\sigma(\Gamma)^p, p > 0.$$

This proof of (1.12) is known. Similar extensions of the weakened versions of (1.2), (1.7), and (1.9), in which P_Q is replaced by 1, follow in the same way. More precise information concerning the tail of the distribution of $\tau(x,y,\Gamma)$ may be found in [5].

A different proof of (1.1) appears in [2], and (1.1) has been generalized and extended in a number of ways. See [3] for references and a discussion. With the exception of (1.3), which will be seen to be trivial, and of course (1.1), if we alter the statements of the results stated earlier by dropping the condition that Ω be simply connected, none remains valid. Also, no reasonable analog of any of these results except Theorem 1.1 holds for arbitrary simply connected domains in $\mathbb{R}^n$.

2. Notation and preliminaries. With one exception, indicated below, the notation introduced in the first section is retained. If μ and ν are measures on the same σ-algebra $\mathcal{F}, \mu \leq \nu$ means $\mu(A) \leq \nu(A), A \in \mathcal{F}$. If X is a complex valued random variable on the probability space $(M, \mathcal{A}, \eta)$ and $H \in \mathcal{A}$ then $\eta(X, H)$ stands for the Borel measure α given by

$$\alpha(B) = \eta(\{X \in B\} \cap H), B \text{ a Borel set,}$$

and if $H = M$ the "H" is omitted, so that $\eta(X)$ is the distribution of X. Furthermore if f is a nonnegative Borel measurable function on $\mathbb{C}$ and ν is a Borel measure on $\mathbb{C}$ then $f\nu$ stands for the Borel measure β given by

$$\beta(B) = \int_B f d\nu.$$

As we mentioned, we use $c, C, c_1, c_2, \dots$ for absolute positive constants, and with enough extra work we could replace every one of these constants by an explicit number. The constants c, C may be different in each occurrence. Sometimes, it is helpful to keep track of constants locally, and when this is the case we use c_1, c_2, etc. If a positive constant has properties in addition to being positive, for example being an integer or being less than one, it is designated by some letter other than c.

The letters x and y are not used indiscriminately but solely to designate starting and ending points of conditioned motions. Without loss of generality we assume from now on $y \in \Omega$, and so the theorems of the first section will be proved only under this assumption. The proofs are readily altered to handle boundary points y, or, the results for a boundary point may be proved from the analogous results for each of a sequence of interior points approaching the boundary point. We shorten G_Ω to G.

The symbol Δ designates a cemetary or trap state a distance 1 from all points of $\mathbb{C}$. Following Doob, $Z_t, t \geq 0$, denotes a generic stochastic process with right continuous paths having limits from the left (r.c.l.l.), or, more precisely, our underlying sample space is always the set W of r.c.l.l. functions on $[0, \infty)$ which take values in $\mathbb{C} \cup \{\Delta\}$, and $Z_t(\omega) = \omega(t)$, if $\omega \in W$. When $Z_t \neq \Delta$ we use X_t for ReZ_t and Y_t for ImZ_t. Different processes arise from different probability measures on W. A subscript but no superscript as in P_z, E_z, Var_z, etc., where $z \in \mathbb{C}$, or P_μ, E_μ, etc. with μ a probability measure on $\mathbb{C}$, designates the measure on W which makes $Z_t, t \geq 0$, standard complex Brownian motion, started, respectively, at z or with initial distribution μ. A subscript, which still designates an initial position, together with a superscript which is a superharmonic function of Ω, as in P_y^g, denotes the h process in Ω with function g. The subject of the rest of the paper is thus $P_x^{G(\cdot,y)}$. This combination of sub- and superscripts occurs so often we delete them both, so that P stands for $P_x^{G(\cdot,y)}$, E stands for $E_x^{G(\cdot,y)}$, etc. In any case, the superscript $G(\cdot, y)$ is always abbreviated to y, that is $P_z^{G(\cdot,y)} = P_z^y$. This slightly changes the notation of the first section, in that for example we no longer talk about $Z(z, w, \Omega)$ or, as will be seen, about $\tau(z, w, \Omega)$, although abbreviated notations such as $P(Z_t \in A)$ and $E\tau$ mean the same in both schemes.

The σ-fields $\mathcal{F}(u)$ of subsets of W are defined by $\mathcal{F}(u) = \sigma(Z_s, s \leq u)$, and stopping times will be $\mathcal{F}(t), t \geq 0$, stopping times, that is, defined in terms of the paths of the process, so that they make sense simultaneously for all processes considered. We define $\tau = inf\{t : lim_{s \to t} Z_s = y\}$ (always inf $\phi = \infty$), and observe that, with P probability 1, τ is the lifetime of Z (see [6]). If $R \subset \mathbb{C}$ we put $\tau_R = inf\{t : Z_t \in R^c\}$. The shift operator from W to W is defined by $\theta_t(w)(s) = w(s+t), w \in W$. We always omit the qualifier a.s.. The strong Markov property is abbreviated to sMp, and $sMp(\nu)$ means the sMp is to be applied at the stopping time ν. If f is a nonnegative function on an interval I satisfying $0 < \int_I f(t)dt < \infty$, its normalization, that is, the function on I given by $f(u)/\int_I f(t)dt$, will be denoted $N(f)$. The normalization of a finite positive measure μ is defined analogously and denoted $N(\mu)$. We will make frequent use of the conformal invariance of Brownian paths, and the conformal invariance of harmonic measure. See [7] for more detail on these topics. Next we discuss P, that is, we discuss the started at x and conditioned to hit y before leaving Ω process. Under P, Z is a strong Markov process with stationary transition densities given by

$$p_t(z,w) = f_t(z,w)G(w,y)/G(z,y), \tag{2.1}$$

where f_t is the density of standard Brownian motion killed upon leaving Ω, that is, if A is a Borel subset of $\Omega, P_z(Z_t \in A, t < \tau_\Omega) = \int_A f_t(z,w)d\sigma(w)$. For each $t > 0, P(t > \tau) = 1 - \int_\Omega p_t(x,w)dw > 0$, and $Z_t = \Delta$ if $t \geq \tau$. If η is a stopping time and $\Lambda \in \mathcal{F}(\eta)$ then

$$P(\Lambda \cap \{\eta < \tau\}) = \int_{\Lambda \cap \{\eta < \tau_\Omega\}} G(Z_\eta, y)dP_x/G(x,y). \tag{2.2}$$

This is equation (2.1) on page 672 of [6].

Now Theorem 1.1 will be proved. We have, using (2.1),

$$\begin{aligned} E_z^y T_Q &= \int_Q \int_0^\infty p_t(z,w)dtd\sigma(w) \\ &= \int_Q G(w,y) \int_0^\infty f_t(z,w)dtd\sigma(w)/G(z,y) \\ &= \int_Q G(w,y)G(z,w)d\sigma(w)/G(z,y), \end{aligned} \tag{2.3}$$

where p_t and f_t are as in (2.1).

First we prove (1.2). It suffices to show that

$$E_z^y T_Q \leq C\sigma(Q), z \in Q. \tag{2.4}$$

For let $\gamma = inf\{t : Z_t \in Q\}$. If (2.4) holds then the $sMp(\gamma)$ gives

$$ET_Q = EE_{Z_\gamma}^y T_Q I(\gamma < \tau) \leq EC\sigma(Q)I(\gamma < \tau) = C\sigma(Q)P_Q.$$

To prove (2.4), suppose first, and throughout this paragraph, that $d(y, Q) \geq$ diam$(Q)/4$. Harnack's inequality then implies

$$c < G(w, y)/G(z, y) < C, w, z \in Q,$$

so that, using (2.3), we have

(2.5)

$$c\int_Q G(z, w)d\sigma(w) \leq E_z^y T_Q \leq C\int_Q G(z, w)d\sigma(w), \text{ if } z \in Q \text{ and } d(y, Q) \geq \text{ diam}(Q)/4.$$

Now

$$\int_Q G(a, w)d\sigma(w) = E_a T_Q, a \in \Omega. \tag{2.6}$$

Furthermore, there is an absolute constant $\varepsilon < 1$ such that

$$P_a(T_Q \geq \sigma(Q)) < \varepsilon, a \in \Omega. \tag{2.7}$$

An argument involving conditioning on the first hitting time of Q and using the sMp shows that it suffices to prove (2.7) for $a \in Q$, and we see that $1 - \varepsilon$ may be taken as the probability that standard Brownian motion makes a loop enclosing the disc of radius 5 diam(Q) around its starting point by time $\sigma(Q)$, for if this starting point is in Q, then parts of this loop must lie outside Ω, since Q is Whitney. Now (2.7) and an argument similar to the one which gave (1.11) yields

$$P_a(T_Q \geq m\sigma(Q)) < \varepsilon^m, a \in \Omega, m = 1, 2, \ldots,$$

so that

$$E_a T_Q \le C\sigma(Q), \tag{2.8}$$

and (1.2) in the case $d(y,Q) \ge \text{ diam}(Q)/4$ follows from (2.8), (2.6), and (2.5).

To complete the proof of (2.4) we now handle the case $d(y,Q) \le \text{ diam}(Q)/4$, which is assumed throughout this paragraph. Since Q is Whitney, $d(y,\Omega^c) \ge (3/4)\text{ diam}(Q)$ so if $z \in Q$, Harnack's inequality gives

$$G(w,y) \le cG(z,y) \text{ if } |w-y| = \min\,(|z-y|/2,\ \text{diam}(Q)/4),$$

implying, by the maximum principle, that

$$G(w,y) \le cG(z,y) \text{ if } |w-y| \ge \min\,(|z-y|/2,\ \text{diam}(Q)/4).$$

Similarly,

$$G(w,z) \le cG(z,y) \text{ if } |w-z| \ge \min\,(|z-y|/2,\ \text{diam}(Q)/4).$$

Thus, since either $|w-y|$ or $|w-z|$ is at least as large as $|z-y|/2$, we have

$$G(w,y)G(z,w)/G(z,y) \le c[G(w,y)+G(z,w)],$$

and (2.6), (2.8), and the symmetry of G imply

$$E_z^y T_Q = \int_\Omega G(w,y)G(z,w)dw/G(z,y) \le C\sigma(Q).$$

This finishes the proof of (1.2).

Now we turn to the proof of (1.3). The time reversal of Brownian motion conditioned to go from x to y in Ω is Brownian motion conditioned to go from y to x in Ω (for the precise statement see page 682 of [6]), so we may and do assume with no loss of generality that $d(y,Q) \ge \frac{1}{2}\text{ diam}(Q)$.

It suffices to prove that

$$E_z^y T_Q \ge c\sigma(Q), \text{ if } z \in Q \text{ and } d(y,Q) \ge \frac{1}{2}\text{ diam}(Q), \tag{2.9}$$

for if z is not in Q we can condition on the first hitting time of Q and use the sMp. Using (2.5), we see that (2.9) is equivalent to

$$\int_Q G(z,w)d\sigma(w) \geq c\sigma(Q), \text{ if } z \in Q \text{ and } d(y,Q) \geq \frac{1}{2} \operatorname{diam}(Q). \tag{2.10}$$

Let Θ be the disc of radius diam(Q) about z. Then $G_\Theta(z,w) \leq G(z,w)$, and direct computation gives $\int_Q G_\Theta(z,w)d\sigma(w) \geq c\sigma(Q)$, proving (2.10) and thus (1.3).

3. Some harmonic analysis. In this section we use an old technique of studying harmonic measure in subdomains of an infinite strip and then translating these results to Ω by conformal mapping. If z is a point in a domain Γ, harmonic measure with respect to Γ and the point z, that is $P_z(Z_{\tau_\Gamma})$, the distribution of the exit position from Γ of a standard Brownian motion started at z, will be denoted μ_z^Γ. We put $S = \{z : -\frac{1}{2} < Imz < \frac{1}{2}\}$, $L_a = S \cap \{Rez = a\}$, a real, $R_a = S \cap \{-a < Rez < a\}$, $a > 0$, and $H_a = S \cap \{Rez < a\}, a > 0$. For $-\frac{1}{2} < s < \frac{1}{2}$ and $t > 0$, the function on $(-\frac{1}{2}, \frac{1}{2})$ which is the continuous version of the density, with respect to linear Lebesgue measure, of $P_{is}(Y_{\tau_{R_t}}, \{X_{\tau_{R_t}} = t\})$, is denoted $f_{s,t}$, and we put $h_{s,t} = N(f_{s,t})$ (recall that N stands for normalization), so that $h_{s,t}$ is the density of the imaginary part of the point where a standard Brownian motion started at is exits R_t, given that the exit position is on the right side. For $q > 0, \mu_{is}^{H_q}(L_q)$ is the probability that Brownian motion started at is exits H_q on its right side. We have

Lemma 3.1. Let n be a positive integer. If $q > n$ and $-\frac{1}{2} < s < \frac{1}{2}$ then $\mu_{is}^{H_q}(L_q) \leq 2^{-n}$.

Proof. Clearly, by symmetry,

$$P_{u+iv}(X_t - u = 1 \text{ for some } t \text{ such that } Y_s - v < 1 \text{ if } s \leq t) = \frac{1}{2}.$$

Thus $\mu_{is+k}^{H_{k+1}}(L_{k+1}) \leq \frac{1}{2}, 0 \leq k < \infty, -\frac{1}{2} < s < \frac{1}{2}$, since the range of imaginary values in S is 1. Together with the sMp this gives

$$P_{is}(Z_{\tau_{H_{k+1}}} \in L_{k+1} | Z_{\tau_{H_k}} \in L_k) \leq \frac{1}{2}, k \geq 1, \text{ and } P_{is}(Z_{\tau_{H_1}} \in L_1) \leq \frac{1}{2}. \tag{3.1}$$

and since $\{Z_{\tau_{H_a}} \in L_a\} \subset \{Z_{\tau_{H_b}} \in L_b\}$ if $a > b > 0$, (3.1) and induction give $\mu_{is}^{H_n}(L_n) \leq 2^{-n}, n \geq 1$, which implies $\mu_{is}^{H_q}(L_q) \leq 2^{-n}$, if $q \geq n$. □

With a little more work, the 2^{-n} in the statement of Lemma 3.1 could have been replaced by α^n, where $\alpha = \mu_0^{H_1}(L_1) < \frac{1}{2}$.

Lemma 3.2. There is an absolute constant $\eta < 1$ such that, if n is a positive integer, $q \geq n$, and $-\frac{1}{2} < s < \frac{1}{2}$, then

$$(3.2) \qquad (1-\eta^n)h_{0,q} < h_{s,q} < (1+\eta^n)h_{0,q}.$$

Proof. Let D be the unit disc. Let $\Psi_q = \Psi$ be a function analytic in R_q which maps $\overline{R}_q$ univalently and continuously onto $\overline{D}$, which maps 0 to 0, and which maps $\partial R_q \cap \{Re z \geq 0\} = N_q$ onto $\partial D \cap \{Im z \geq 0\} = M$. Then $\Psi(L_0)$ is the interval (-1,1) of the real axis, since the points of L_0 are the only points in R_q such that $\mu_z^{R_q}(N_q) = 1/2$, and similarly the points of (-1,1) are the only points in D such that $\mu_z^D(M) = 1/2$, and harmonic measure is preserved under conformal mapping.

Now $\Psi(L_q) = A_q$ is an arc of the unit circle which is of the form

$$A_q = \{e^{i\theta} : \pi/2 - a_q < \theta < \pi/2 + a_q\}.$$

To show this it suffices to show that

$$\Psi(\partial R_q \cap \{Im z = 1/2, Re z > 0\}) \text{ and } \Psi(\partial R_q \cap \{Im z = -1/2, Re z > 0\})$$

are arcs of the same length; this follows from the fact that by symmetry $\mu_0^{R_q}(\partial R_q \cap \{Im z = 1/2, Re z > 0\})$ and $\mu_0^{R_q}(\partial R_q \cap \{Im z = -1/2, Re z > 0\})$ are the same, that $\Psi(0) = 0$, and that μ_0^D is normalized length on ∂D.

Now $\mu_0^D(A_q) = \mu_0^{R_q}(H_q)$. Thus, by Lemma 3.1, we have $0 < a_q < \pi 2^{-n}$. Lemma 3.2 is a statement about harmonic measures and ratios of harmonic densities, both preserved under conformal mapping. Also, μ_z^D is of course the measure on ∂D which has density with respect to $d\theta/2\pi$ given by the Poisson kernal $P(z, e^{i\theta}) = (1-|z|^2)/|z - e^{i\theta}|^2$. Thus Lemma 3.2 is equivalent to the statement that there is a constant $\eta < 1$, not depending on q, such that if $e^{i\theta} \in A_q$ and z is real,

$$(3.3) \qquad (1-\eta^n)(2a_q)^{-1} < P(z, e^{i\theta}) \Big/ \int_{\pi/2-a_q}^{\pi/2+a_q} P(z, e^{i\theta})d\theta < (1+\eta^n)(2a_q)^{-1}.$$

This readily verified from the equation for $P(z, e^{i\theta})$, and the fact that $0 < a_q < \pi 2^{-n}$. □

From now on, η will be a number less than one, which satisfies (3.2). If $\Psi_s(t), s \in I, t \in J$ is a nonnegative measurable (as a function of (s,t)) function on $I \times J$, where I and J are intervals, and γ is a measure on the Borel sets of I, then $\int_I \Psi_s(t) d\gamma(s) = g(t), t \in J$, is called a mixture of the functions Ψ_s.

Lemma 3.3. Let $\Psi_s, s \in I$, and g be as above and suppose all these functions have positive and finite integrals. If α and β are functions such that $\alpha \leq N(\Psi_s) \leq \beta, s \in I$, then $\alpha \leq N(g) \leq \beta$.

The proof of this lemma is easy and omitted.

Now define $\phi_{s,t}$ and $\lambda_{s,t}$ exactly as $f_{s,t}$ and $h_{s,t}$ were defined except that R_t is replaced by H_t. That is, $\phi_{s,t}$ is the density with respect to Lebesgue measure of $P_{is}(Y_{\tau_{H_t}}, \{Z_{\tau_{H_t}} \in L_t\})$ and $\lambda_{s,t} = N(\phi_{s,t})$.

Lemma 3.4. Let η be as in Lemma 3.2. If n is a positive integer and $q \geq n$ then,

(3.4) $$(1-\eta^n)h_{0,q} < \lambda_{s,q} < (1+\eta^n)h_{0,q}, -1/2 < s < 1/2, \text{ and}$$

(3.5) $$(1-\eta^k)h_{0,k} < \lambda_{s,q} < (1+\eta^k)h_{0,k}, 1 \leq k \leq n.$$

Proof. Recall we denote ReZ_t by X_t, and put

$$T_0 = 0,$$
$$T_1 = \inf\{t > 0 : |X_t| = q\},$$
$$T_2 = \inf\{t > T_1 : X_t = 0\},$$
$$T_{2i+1} = T_1(\theta_{T_{2i}}), i \geq 1, \text{ and}$$
$$T_{2i+2} = T_2(\theta_{T_{2i}}), i \geq 1.$$

Let $\gamma_i = P_{is}(Z_{T_{2i}}, \{T_{2i} < \tau_S\})$. Since $\phi_{s,q}$ is the density of

$$\sum_{i=0}^{\infty} P_{is}\left(Y_{min(T_{2i+1},\tau_S)}, \{T_{2i} < \tau_S, Z_{min(T_{2i+1},\tau_S)} \in L_q\}\right)$$

and since $min(T_{2i+1}, \tau_S)$ on $\{T_{2i} < \tau_S\}$ is the first time after T_{2i} that Z exits R_q, we have, by the sMp, that $\phi_{s,q}$ is a mixture of the functions $f_{r,\cdot}, -\frac{1}{2} < r < \frac{1}{2}$, the mixing measure being $\sum_{i=0}^{\infty} \gamma_i$. Now (3.4) follows from Lemmas 3.2 and 3.3

We prove (3.5) for $k = 1$ only, the other cases being similar. Let $\xi = \inf\{t : Z_t \in L_{q-1}\}$. If $z = (q-1) + ia \in L_{q-1}, P_z(Y_{\tau_{H_q}}, \{Z_{\tau_{H_q}} \in L_q\})$ has density $f_{a,1}$ on (-1/2, 1/2). The sMp (ξ) implies $\phi_{s,q}$ is a mixture of the functions $f_{a,1}, -1/2 < a < 1/2$, the mixing measure being $P_{is}(Y_\xi, \{\xi < \tau_S\})$. Now (3.5) follows from (3.4) in the case $q = 1$, and Lemma 3.3. □

The next two lemmas concern images of Whitney squares under conformal mappings of Ω onto S. Rodrigo Bañuelos has pointed out to us that both follow from the results of [8].

Lemma 3.5. Let ϕ be a univalent conformal mapping from Ω onto S. Let Q be a Whitney square and put $a = \max\{Rez : z \in \phi(Q)\}$ and $b = \min\{Rez : z \in \phi(Q)\}$. Then $a - b < c$.

Proof. Let A be the circle of radius diam$(Q)/2$ around the center of Q and let B be the circle of radius diam(Q) around the center of Q. Let $\varepsilon > 0$ be the probability that the path of a standard Brownian motion started inside A makes a closed loop, which contains the interior of A and which lies between A and B, before it hits B. This probability does not depend on the starting point inside A, nor does it depend on Q. We have $P_z(Z_t, 0 \le t < \tau_\Omega$, makes a closed loop enclosing $Q) \ge \varepsilon, z \in Q$. This implies that

(3.6) $$P_w(Z_t, 0 \le t < \tau_S, \text{ makes a closed loop around } \phi(Q)) \ge \varepsilon, w \in \phi(Q).$$

Lemma 3.5 now follows from the observation that, by Lemma 3.1, for any w in S, $P_w(\max_{0 \le t < \tau_s}(X_t - Rew) \ge n) \le 2^{-n}$, since if $w \in \phi(Q)$ and $Rew = b$, (3.6) and this last inequality implies that $a - b \le \alpha$, where α is the smallest integer such that $2^{-\alpha} < \varepsilon$. □

Lemma 3.6. There is a positive number c_1 and a positive integer k, both absolute constants, such that if f is a univalent conformal map of S onto Ω, if $Q_i, 1 \le i < \infty$, is a Whitney decomposition of Ω, and if I is an interval of the real axis of length c_1, then $f(I) \cap Q_i \neq \phi$ for at most k of the Q_i.

Proof: Let h be a fixed one to one conformal map of the unit disc D onto S such that $h(0) = 0$, and put $\gamma = |h'(0)| > 0$. Now $f(h(z))$ maps D univalently onto Ω so Koebe's Theorem (see [9]) shows that

$$|f'(0)|\gamma = |f(h(0))'| \leq 4d(f(h(0)), \Omega^c) = 4d(f(0), \Omega^c).$$

Similarly, replacing h in the above by $h + a$, we get

$$|f'(a)| \leq 4\gamma^{-1}d(f(a), \Omega^c), \text{ a real.} \tag{3.7}$$

Now define g on the real line by $g(r) = d(f(r), \Omega^c)$. Then $g'(r) \leq |f'(r)| \leq 4\gamma^{-1}g(r)$, by (3.7), implying $g(r) \leq g(a)e^{4(r-a)/\gamma}, r > a$, so that

$$g(r) \leq eg(a) \text{ if } 0 \leq r - a \leq \gamma/4, \text{ a real.} \tag{3.8}$$

Now, for a fixed, define $\Psi_a(r) = \Psi(r) = |f(r) - f(a)|$. Then

$$\Psi'(r) \leq |f'(r)| \leq 4g(r)/\gamma \leq 4eg(a)/\gamma \text{ if } 0 \leq r - a \leq \gamma/4.$$

Thus

$$\Psi(r) = \int_a^r \Psi'(s)ds \leq g(a)/2 \text{ if } 0 \leq r - a \leq \gamma/8e. \tag{3.9}$$

Now if $w \in \Omega$, the number of squares Q_i which can touch the disc of radius $d(w, \Omega^c)/2$ about w is at most $\pi(6.5)^2 \cdot 200 \leq 27,000$. For the diameter of any such square is at most $4 \cdot (3/2)d(w, \Omega^c)$ and at least $d(w, \Omega^c)/10$, so all the squares are contained in the disc of radius (6.5) $d(w, \Omega^c)$ around w, and have area at least $d(w, \Omega^c)^2/200$, and are disjoint. Thus (3.8) and (3.9) imply we may take the c_1 of Lemma 3.6 as $\gamma/8e$ and the k as 27,000. □.

4. Covariance. In this section Theorem 1.2 is proved. The following lemma essentially handles the case where $\rho(Q, R)$ is small.

Proposition 4.1. If Q and R are Whitney squares then

$$|\text{Cov }(T_Q, T_R)| \leq c\sigma(Q)\sigma(R)(P_Q + P_R).$$

Proof. As was mentioned in the first section, the proof of (1.12) from (1.1) can be easily adapted to use (1.2) to prove that if G is a Whitney square

$$E_z^y T_G^p \leq c_p \sigma(G)^p, 0 < p < \infty, z \in \Omega. \tag{4.1}$$

Conditioning on $\{\gamma < \tau\}$, where $\gamma = \inf\{t : Z_t \in G\}$, the $sMp(\gamma)$ and (4.1) in the case $p = 2$ gives

$$E(T_G^2 | \gamma < \tau) \leq c_2 \sigma(G)^2,$$

so that $ET_G^2 \leq c_2\sigma(G)^2 P(\gamma < \tau) = c_2\sigma(G)^2 P_G$.

$$\begin{aligned} \text{Thus } |\text{Cov }(T_Q, T_R)| &\leq (ET_Q^2)^{1/2}(ET_R^2)^{1/2} \\ &\leq c_2^2 \sigma(Q)\sigma(R)(P_Q P_R)^{1/2} \\ &\leq c_2^2 \sigma(Q)\sigma(R)(P_Q + P_R). \quad \square \end{aligned}$$

Now let U, V be Whitney squares and suppose $\Psi_{U,V} = \Psi$ is a conformal mapping of Ω onto S such that both $\Psi(U)$ and $\Psi(V)$ contain points of the real axis. Suppose that the two intervals $\{Rez : z \in \Psi(U)\}$ and $\{Rez : z \in \Psi(V)\}$ are a distance 15m or less apart, where m is a positive integer. Then there is an interval of the real axis, of length at most $15m + 2c$, which contains points of both $\Psi(U)$ and $\Psi(V)$, where c is the same constant c appearing in the statement of Lemma 3.5. Thus, by Lemma 3.6, $\rho(U, V) \leq [(15m + 2c)c_1^{-1} + 1]k \leq Cm$, so we have the following lemma.

Lemma 4.2. There is a constant c_2 such that if Q and R are Whitney squares, if m is a positive integer, and if $\rho(Q, R) \geq c_2 m$, then if ϕ maps Ω univalently onto S and $\phi(Q), \phi(R)$ both contain points of the real axis there is a real number α such that one of $\phi(Q), \phi(R)$ lies to the left of $L_{\alpha-1}$ and the other lies to the right of $L_{\alpha+3m+1}$, and that furthermore neither of $\phi(x), \phi(y)$ belongs to that part of S lying between $L_{\alpha-1}$ and $L_{\alpha+3m+1}$.

The following proposition, together with Proposition 4.1, proves Theorem 1.2.

Proposition 4.3. There are absolute constants c and C such that if c_2 is the c_2 guaranteed by Lemma 4.2, and Q and R are Whitney squares satisfying $\rho(Q,R) \geq c_2 m$ for a positive integer m, then

$$\text{Cov}\,(T_Q, T_R) \leq Ce^{-cm}\sigma(Q)\sigma(R)(P_Q + P_R). \tag{4.2}$$

Proof. Several lemmas will be needed in the course of this proof. We let ϕ and α be a function and number (both of which may be considered fixed) with the properties described in the statement of Lemma 4.2. We also assume without loss of generality, by symmetry, that both $\phi(x)$ and $\phi(Q)$ lie to the left of $L_{\alpha-1}$. Define the curve $K_t \subset \Omega$ by $K_t = \phi^{-1}(L_t)$. We say a point $z \in \Omega$ or set $A \subset \Omega$ is to the left or right of K_t if $\phi(z)$ or $\phi(A)$ is to the left or right of L_t in S.

Define the stopping times γ and ξ by $\gamma = \inf\{t : Z_t \in K_{\alpha+m}\}$ and $\xi = \inf\{t : Z_t \in K_{\alpha+2m}\}$, put $\gamma_0 = 0, \gamma_{2i+1} = \gamma(\theta_{\gamma_{2i}}), i \geq 0$, and $\gamma_{2i} = \xi(\theta_{2i-1}), i \geq 1$, and let $u_i = \min\,(\gamma_i, \tau), i \geq 1$. We also put

$$\Delta_0 = \int_{u_0}^{u_1} I(Z_t \in Q)dt,$$

$$\Delta_i = \int_{u_{2i-1}}^{u_{2i}} I(Z_t \in Q)dt, i \geq 1,$$

and

$$\Psi_i = \int_{u_{2i}}^{u_{2i+1}} I(Z_t \in R)dt, i \geq 1,$$

so that (recall that x is to the left of $K_{\alpha-1}$) we have

$$T_Q = \sum_{i=0}^{\infty} \Delta_i \text{ and } T_R = \sum_{i=1}^{\infty} \Psi_i.$$

In the following two lemmas the constant η is the same one that appears in the statements of Lemmas 3.2 and 3.4, and we put $\eta^* = (1+\eta)/(1-\eta)$.

Lemma 4.4. Let j be an integer exceeding 2 and let $a_1, a_2, \ldots, a_j$ be real numbers such that $a_1 = a_j, a_{i+1} \neq a_i, 1 \leq i < j$, and $\sum_{i=2}^{j} |a_i - a_{i-1}| = \beta$ is an integer. Suppose also that $|Re\phi(z) - a_1| \geq 1$. Put $T_0 = 0, T_i = \inf\{t \geq T_{i-1} : Z_t \in K_{a_i}\}, 1 \leq i \leq j$. Then

$$P_z^y(Z_{T_j}, \{T_j < \tau\}) \leq \eta^* 2^{-\beta} P_z^y(Z_{T_1}, \{T_1 < \tau\}). \tag{4.3}$$

Proof. First note that (4.3) is equivalent to

$$P_z(Z_{T_j}, \{T_j < \tau_\Omega\}) \le \eta^* 2^{-\beta} P_z(Z_{T_1}, \{T_1 < \tau_\Omega\}), \tag{4.4}$$

for multiplying (4.4) by $G(\cdot, y)/G(x,y)$ gives, via (2.2), (4.3). The conformal invariance of Brownian paths implies that if $S_0 = 0, S_i = \inf\{t \ge S_{i-1} : Z_t \in L_{a_i}\}, 1 \le i \le j$, then (4.4) is equivalent to

$$P_{\phi(z)}(Z_{S_j}, \{S_j < \tau_S\}) \le \eta^* 2^{-\beta} P_{\phi(z)}(Z_{S_1}, \{S_1 < \tau_S\}). \tag{4.5}$$

Assume that z lies to the left of a_1 without loss of generality. Let $b_1 = |Re\phi(z) - a_1|, b_i = |a_i - a_{i-1}|, 2 \le i \le j$. Put $v = Im\phi(z)$, and let $U_0 = 0, U_i = \inf\{t \ge U_{i-1} : Z_t = \sum_{n=1}^{i} b_n\}$. Translations of Brownian paths are Brownian paths, so

$$P_{\phi(z)}(Y_{S_1}, \{S_1 < \tau_S\}) = P_{iv}(Y_{U_1}, \{U_1 < \tau_S\}). \tag{4.6}$$

Recall Y is the imaginary part of Z. If $a_2 > a_1$, then translation invariance again gives

$$P_{\phi(z)}(Y_{S_2}, \{S_2 < \tau_S\}) = P_{iv}(Y_{U_2}, \{U_2 < \tau_S\}), \tag{4.7}$$

and if $a_2 < a_1$ we can use the fact that Brownian motion started on L_{a_1} remains Brownian motion under reflection around L_{a_1}, so that by the sMp (S_1), Brownian motion reflected after S_1 is still Brownian motion, to conclude that (4.7) holds. Proceed inductively to conclude

$$P_{\phi(z)}(Y_{S_j}, \{S_j < \tau_S\}) = P_{iv}(Y_{U_j}, \{U_j < \tau_S\}). \tag{4.8}$$

Now on $\{Z_0 = iv\}$, min $(U_j, \tau_S) = \tau_{H_{\beta+b_1}}$, noting $\beta + b_1 = \sum_{i=1}^{j} b_i$, where H_t is as defined in Section 3. Thus the densities of $P_{\phi(z)}(Y_{S_i}, \{S_i < \tau\}), i = 1, j$, are ϕ_{v,b_1} and $\phi_{v,b_1+\beta}$, respectively. Of course, since both $P_{\phi(z)}(Z_{S_i}, \{S_i < \tau_S\}), i = 1, j$, are concentrated on $\{Rez = a_1\}$ to compare them it suffices to compare the densities of $P_{\phi(z)}$ $(Y_{S_i}, \{S_i < \tau_S\})$. Now $\int_{-1/2}^{1/2} \phi_{v,b_1}(u)du = P_{iv}(U_1 < \tau_S)$, and $\int_{-1/2}^{1/2} \phi_{v,b_1+\beta}(u)du = P_{iv}(U_j < \tau_S)$. Recalling $\lambda_{v,t} = \phi_{v,t}/\int_{-1/2}^{1/2} \phi_{v,t}(u)du$, (3.5) with $k = 1$ gives

$$\lambda_{v,b_1+\beta} \le (1+\eta)h_{0,1} \le [(1+\eta)/(1-\eta)]\lambda_{v,b_1} = \eta^* \lambda_{v,b_1},$$

so that $\phi_{v,b_1+\beta} \leq \eta^* P_{iv}(U_j < \tau_S)/P_{iv}(U_1 < \tau_S)\phi_{v,b_1}$. Now the $sMp(U_1)$ and Lemma 3.1 give

$$P_{iv}(U_j < \tau_S) < P_{iv}(U_j < \tau_S | U_1 < \tau_S) P(U_1 < \tau_S) < 2^{-\beta} P(U_1 < \tau_S),$$

so $\phi_{v,b_1+\beta} \leq \eta^* 2^{-\beta} \phi_{v,b_1}$. Thus, (4.6), and (4.8) give (4.5), and the lemma is proved. □.

Now for r real and $q > 0$ let γ_r^q be the probability measure concentrated on L_r given by

$$\gamma_r^q(\{r + iy : a < y < b\}) = \int_a^b h_{0,q}(s)ds, -1/2 < a < b < 1/2,$$

so that $\gamma_r^q = N(P_{r \pm q}\ (Z_\alpha, \{\alpha < \tau_S \text{ and } Z_\alpha \in \mathrm{L}_r\}))$, where $\alpha = inf\{t : Z_t \in L_r \text{ or } L_{r \pm 2q}\}$. Define the measure ε_r^q on K_r by $\varepsilon_q^r(A) = \gamma_r^q(\phi(A))$. Conformal invariancce of harmonic measure implies $\varepsilon_r^q = N(P_{\phi^{-1}(r \pm q)}(Z_\beta, \{\beta < \tau_\Omega \text{ and } Z_\beta \in K_r\}$, where $\beta = inf\{t > 0 : Z_t \in K_r \text{ or } K_{r \pm 2q}\}$. Recall η remains the η guaranteed in Lemmas 3.2 and 3.4, and $\eta^* = (1 + \eta)/(1 - \eta)$.

Lemma 4.5. *Let r be a real number and let $z \in \Omega$, such that $|Re\phi(z) - r| \geq n$ for a positive integer n. Let $V = \inf\{t \geq 0 : Z_t \in K_r\}$. Put $g(w) = G(w,y)/G(z,y), w \in \Omega$. Then*

$$[(1 - \eta^n)/(1 + \eta^n)] N(g\varepsilon_r^n) < N(P_z^y(Z_V, \{V < \tau\})) < [(1 + \eta^n)/(1 - \eta^n)] N(g\varepsilon_r^n), \tag{4.10}$$

and

$$(1/\eta^*) N(g\varepsilon_r^1) < N(P_z^y(Z_V, \{V < \tau\}) < \eta^* N(g\varepsilon_r^1). \tag{4.11}$$

Proof. By (3.4), conformal invariance of harmonic measure, and translation invariance of Brownian paths, we have

$$(1 - \eta^n) N(\varepsilon_r^n) \leq N(P_z(Z_V, \{V < \tau_\Omega\})) \leq (1 + \eta^n) N(\varepsilon_r^n). \tag{4.12}$$

Since $gP_z(Z_V, \{V < \tau_\Omega\}) = P_z^y(Z_V, \{V < \tau\})$, by (2.2), (4.10) follows easily, from (4.12), using no properties of g except its nonnegativity. Inequality (4.11) follows similarly from (3.5). □

Lemma 4.6. Let s be a real number. Suppose x and y both lie to the left of K_{s-1} and a Whitney cube V lies to the right of K_{s+m}. Then $P_V \leq \eta^* 2^{-2m}$. The same result holds if x and y are to the right of K_{s+m+1} and V is to the left of K_s.

Proof. Under the supposition of the first sentence of Lemma 4.6, let $\xi_1 = \inf \{t : Z_t \in K_s\}, \xi_2 = \inf \{t > \xi_1 : Z_t \in K_{s+m}\}$, and $\xi_3 = \inf \{t > \xi_2 : Z_t \in K_s\}$. Now by Lemma 4.4,

$$P(\xi_3 < \tau) \leq \eta^* 2^{-2m} P(\xi_1 < \tau) \leq \eta^* 2^{-2m}.$$

By the placement of x and y,

$$P_V \leq P(\xi_2 < \tau) = P(\xi_3 < \tau) \leq \eta^* 2^{-2m}.$$

The proof of the second sentence in the lemma is similar. □

We will finish the proof of Proposition (4.3) by proving

$$|\text{Cov}\ (\Delta_0 + \Delta_1, T_R)| < Ce^{-cm}\sigma(Q)\sigma(R)(P_Q + P_R) \tag{4.13}$$

and

$$|\text{Cov}\ (\sum_{i=2}^{\infty} \Delta_i, T_R)| < Ce^{-cm}\sigma(Q)\sigma(R)(P_Q + P_R). \tag{4.14}$$

We begin with the proof of (4.13). We have two cases to consider: y (as well as x) to the left of $K_{\alpha-1}$ and y to the right of $K_{\alpha+3m+1}$. First we assume y is to the left of $K_{\alpha-1}$. Lemma 4.6 gives $P_R \leq \eta^* 2^{-2m}$, and (1.2) gives $ET_R \leq c2^{-2m}\sigma(R)$ and $ET_Q \leq cP_Q\sigma(Q)$, implying

$$E(\Delta_0 + \Delta_1)ET_R \leq ET_Q ET_R \leq C2^{-2m}\sigma(Q)\sigma(R)P_Q. \tag{4.15}$$

Furthermore, again by Lemma 4.6, $P_z^y(Z_t \in R \text{ for some } t < \tau) \leq \eta^* 2^{-2m}$ if $z \in K_{\alpha+2m}$, so $E_z^y T_R \leq c2^{-2m}\sigma(R)$ if $z \in K_{\alpha+2m}$, by (1.2), and thus by the sMp, the fact that $\Delta_0+\Delta_1$ is $\mathcal{F}(u_2)$ measurable, and that $Z_{u_2} \in K_{\alpha+2m}$ on $\{u_2 < \tau\}$, we have

$$
\begin{aligned}
E(\Delta_0+\Delta_1)T_R &= E(\Delta_0+\Delta_1)E(T_R|\Delta_0+\Delta_1) \\
&= E(\Delta_0+\Delta_1)E[E(T_R|\mathcal{F}(u_2))I(u_2<\tau)|\Delta_0+\Delta_1) \\
&\leq E(\Delta_0+\Delta_1)E(c2^{-2m}\sigma(R)|\Delta_0+\Delta_1) \\
&= c2^{-2m}\sigma(R)E(\Delta_0+\Delta_1) \\
&\leq c2^{-2m}\sigma(R)ET_Q \\
&\leq c2^{-2m}\sigma(R)\sigma(Q)P_R. \qquad (4.16)
\end{aligned}
$$

Together with (4.15), this proves (4.13) in case y is to the left of $K_{\alpha-1}$.

Next we prove (4.13) in case y lies to the right of $K_{\alpha+3m+1}$. Here $P(\gamma_2 < \tau) = 1$. Let $\Psi = \inf\{t : Z_t \in K_{\alpha+3m}\}$. Now also $P(\Psi < \tau) = 1$ and $P_z^y(\Psi < \tau) = 1$ if $z \in K_{\alpha+2m}$. Using (4.10), and noting that normalization of the middle term is not necessary in this case since it is already a probability measure, we get

$$
(4.17) \qquad [(1-\eta^n)/(1+\eta^n)]N(g\varepsilon_r^n) \leq P_z^y(Z_\Psi) \leq [(1+\eta^n)/(1-\eta^n)]N(g\varepsilon_r^n),
$$

if $z \in K_{\alpha+2m}$ or $z = x$ (of course $P_x^y = P$). Now two measures close to the same measure are close to each other, and from (4.17) we get

$$
[(1-\eta^n)/(1+\eta^n)]^2 P(Z_\Psi) \leq P_z^y(Z_\Psi) \leq [(1+\eta^n)/(1-\eta^n)]^2 P(Z_\Psi),
$$

implying, since $ET_R = EE(T_R|Z_\Psi) = \int E_z^y T_R dP(Z_\Psi)(z)$, with an analogous inequality for $E_z^y T_R$,

$$
(4.18) \qquad [(1-\eta^n)/(1+\eta^n)]^2 ET_R \leq E_z^y T_R \leq [(1+\eta^n))/(1-\eta^n)]^2 ET_R, z \in K_{\alpha+2m}.
$$

Thus

$$
(4.19) \qquad [(1-\eta^n)/(1+\eta^n)]^2 ET_R \leq E(T_R|\mathcal{F}(\gamma_2)) \leq [(1+\eta^n))/(1-\eta^n)]^2 ET_R,
$$

and, since $\Delta_0 + \Delta_1$ is $\mathcal{F}(\gamma_2)$ measurable, (4.19) still holds with $E(T_R|\mathcal{F}(\gamma_2))$ replaced by $E(T_R|\Delta_0+\Delta_1)$. Thus, since $E(\Delta_0+\Delta_1)T_R = E(\Delta_0+\Delta_1)E(T_R|\Delta_0+\Delta_1)$, we have, by (1.2),

$$\begin{aligned}|\text{Cov }(\Delta_0+\Delta_1, T_R)| &= |E(\Delta_0+\Delta_1)T_R - E(\Delta_0+\Delta_1)ET_R| \\ &\le ([(1+\eta^n)/(1-\eta^n)]^2 - 1)E(\Delta_0+\Delta_1)ET_R \\ &\le ([(1+\eta^n)/(1-\eta^n)]^2 - 1)ET_Q ET_R \\ &\le c([(1+\eta^n)/(1-\eta^n)]^2 - 1)P_Q\sigma(Q)P_R\sigma(R),\end{aligned}$$

from which (4.13) in this case, the last we had to do to prove (4.13), follows.

Finally we prove (4.14). Lemma 4.4 implies

$$P(u_{2i+1} < \tau) \le \eta^* 2^{-2mi} P(u_1 < \tau) \le \eta^* 2^{-2mi}, i \ge 1, \text{ and} \tag{4.20}$$

using the sMp (u_{2i+1}), and (1.2), we have

$$\begin{aligned}E\sum_{k=i+1}^{\infty} \Delta_k &= EE_{Z_{u_{2i+1}}} T_Q I(u_{2i+1} < \tau) \\ &\le c\sigma(Q)P(u_{2i+1} < \tau) \\ &\le c2^{-2mi}\sigma(Q), i \ge 1.\end{aligned}$$

Again using (1.2), we have

$$E(\sum_{i=2}^{\infty} \Delta_i)ET_R \le c2^{-2m}\sigma(Q)\sigma(R)P_R. \tag{4.21}$$

The proof of (4.14) will be completed by showing

$$E\left(\sum_{i=2}^{\infty} \Delta_i\right) T_R \le Ce^{-cm}\sigma(Q)\sigma(R)(P_Q+P_R). \tag{4.22}$$

Now $\sum\limits_{i=2}^{\infty} \Delta_i T_R = \sum\limits_{i=2}^{\infty} \Delta_i \left(\sum\limits_{j=i}^{\infty} \Psi_j\right) + \sum\limits_{i=1}^{\infty} \Psi_i \left(\sum\limits_{j=i+1}^{\infty} \Delta_j\right)$. We have

$$E\left(\sum_{j=i}^{\infty} \Psi_j | \mathcal{F}_{u_{2i}}\right) = E^y_{Z_{u_{2i}}} T_R \le cP_R\sigma(R) \text{ on } \{u_{2i} < \tau\}, \text{ so}$$

$$E\left(\sum_{j=i}^{\infty}\Psi_j|\Delta_i\right)\le cP_R\sigma(R) \text{ on } \{\Delta_i>0\}, \text{ and}$$

$$E\Delta_i\sum_{j=i}^{\infty}\Psi_j\le cP_R\sigma(R)E\Delta_i\le cP_R\sigma(R)2^{-2m(i-1)}\sigma(Q).$$

Thus

$$E\sum_{i=2}^{\infty}\Delta_i\left(\sum_{j=i}^{\infty}\Psi_j\right)\le c2^{-2m}\sigma(Q)\sigma(R)P_R. \tag{4.23}$$

Similarly (this involves also proving an analog of (4.20)) we have

$$E\sum_{i=2}^{\infty}\Psi_i\left(\sum_{j=i+1}^{\infty}\Delta_j\right)\le c2^{-2m}\sigma(Q)\sigma(R)P_Q. \tag{4.24}$$

To complete the proof of (4.22) we will show

$$E\left(\Psi_1\sum_{j=2}^{\infty}\Delta_j\right)\le Ce^{-cm}\sigma(Q)\sigma(R)(P_Q+P_R). \tag{4.25}$$

We divide the proof of (4.25) into the same two cases we considered earlier. If y is to the right of $K_{\alpha+3m+1}$, then

$$E\left(\sum_{j=2}^{\infty}\Delta_j|\mathcal{F}(u_3)\right)\le E^{y}_{Z_{u_3}}T_Q\le c2^{-2m}\sigma(Q) \text{ on } \{u_3<\tau\},$$

using Lemma 4.6 and the fact that Z_{u_3} on $\{u_3<\tau\}$ and, under our current assumption, y, are both to the right of $K_{\alpha+m-1}$ while Q is to the left of $K_{\alpha-1}$. This, together with the $\mathcal{F}(u_3)$ measurability of Ψ_1, gives both $E(\sum_{j=2}^{\infty}\Delta_j|\Psi_1)\le c2^{-2m}\sigma(Q)$, and

$$\begin{aligned}E(\Psi_1\sum_{j=2}^{\infty}\Delta_j)&\le E\Psi_1c2^{-2m}\sigma(Q)\\&\le ET_Rc2^{-2m}\sigma(Q)\\&\le CP_R\sigma(R)2^{-2m}\sigma(Q).\end{aligned}$$

Now we prove (4.25) under the assumption that y is to the left of $K_{\alpha-1}$. Let $\tilde{u} = \min(\inf\{t : Z_t \in K_\alpha\}, \tau)$ and $\hat{u} = \min(\inf\{t > u_3 : Z_t \in K_\alpha\}, \tau)$. Now (4.11) gives that $P(Z_{\tilde{u}}, \{\tilde{u} < \tau\})/P(\tilde{u} < \tau)$ lies between $1/\eta^*$ and η^* times $N(g\xi^1_\tau)$, and also that $P(Z_{\hat{u}} | \mathcal{F}(u_3))$, by which we mean a version of the regular conditional distribution (see [2]), almost surely lies between these same bounds on $\{u_3 < \tau\}$. Due to the placement of y, $P(\hat{u} < \tau) = P(u_3 < \tau)$ so the regular conditional distributions above are already probability measures and are not changed by normalization. Since Ψ_1 is $\mathcal{F}(u_3)$ measurable, we have that $P(Z_{\hat{u}} | \Psi_1)$ lies between these same bounds on $\Psi_1 > 0$. Two probability measures which lie between $1/\eta^*$ and η^* of the same measure are close to each other: We have

(4.26) $$(1/\eta^*)^2 P(Z_{\hat{u}} | \Psi_1) \le P(Z_{\tilde{u}}, \{\tilde{u} < \tau\})/P(\tilde{u} < \tau) \le (\eta^*)^2 P(Z_{\hat{u}} | \Psi_1) \text{ on } \{\Psi_1 > 0\}.$$

$$\begin{aligned}\text{Now } ET_Q &\ge E\int_{\tilde{u}}^{\infty} I(Z_t \in Q)dt \\ &= EE^y_{Z_{\tilde{u}}} T_Q I(\tilde{u} < \tau) \\ &= \int E^y_z T_Q dP(Z_{\tilde{u}}, \{\tilde{u} < \tau\})(z).\end{aligned}$$

Using (4.26), we have, on $\{\Psi_1 > 0\}$

(4.27) $$\begin{aligned} E(\sum_{i=2}^{\infty} \Delta_i | \Psi_1) &= E(\int_{\hat{u}}^{\infty} I(Z_t \in Q)dt | \Psi_1) \\ &= E(E^y_{Z_{\hat{u}}} T_Q | \Psi_1) \\ &\le E\int E^y_z T_Q dP(Z_{\hat{u}} | \Psi_1)(z) \\ &\le (\eta^*)^2 ET_Q / P(\tilde{u} < \tau).\end{aligned}$$

By Lemma 4.6, if $z \in K_\alpha$, and we still assume y is to the left of $K_{\alpha-1}, E^y_z T_R \le \eta^* 2^{-2m} \sigma(R)$, yielding

$$\begin{aligned} E\Psi_1 &\le EE^y_{Z_{\tilde{u}}} T_R I(\tilde{u} < \tau) \\ &\le \eta^* 2^{-2m} \sigma(R) P(\tilde{u} < \tau).\end{aligned}$$

Using (4.27), this last inequality, and (1.2) we get

$$\begin{aligned} E(\Psi_1 \sum_{i=2}^{\infty} \Delta_i) &= E\Psi_1 E(\sum_{i=2}^{\infty} \Delta_i | \Psi_1) \\ &\leq E\Psi_1 (\eta^*)^2 ET_Q / P(\tilde{u} < \tau) \\ &\leq (\eta^*)^3 2^{-2m} \sigma(R) ET_Q \\ &\leq c2^{-2m} \sigma(R) \sigma(Q) P_Q, \end{aligned}$$

and the proof of (4.25), and thus (4.22), is now finished. Together with (4.21), (4.22) gives (4.14), and this completes the proof of Theorem 1.2.

5. Conclusion.

We first prove the following lemma.

Lemma 5.1 If R is a Whitney square then

$$ET_R^2 \leq cET_R\sigma(R).$$

Proof. Theorem 1.1 gives $E_z^y T_R \leq c_1\sigma(R), z \in \Omega$, and essentially the same argument that proved (1.11) yields

$$P\left(T_R \geq 2(k+1)c_1\sigma(R) | T_R \geq 2kc_1\sigma(R)\right) \leq \frac{1}{2}, k \geq 0.$$

This shows that, if $A = \{T_R \geq 2c_1\sigma(R)\}$, then $P\left(T_R \geq 2nc_1\sigma(R), A\right) \leq 2^{-n}P(A)$ so that

$$ET_R^2 I(A) \leq c\sigma(R)^2 P(A) \leq cET_R I(A)\sigma(R).$$

Clearly $ET_R^2 I(A^c) \leq cET_R\sigma(R)$ since $T_R \leq \sigma(R)$ on A^c. □

Proof of (1.5). First we prove that if Ω is any domain and if $Q_i, i \geq 1$, is a Whitney decomposition of Ω then

$$\text{(5.1)} \qquad \text{Var}\left(\sum_{i=1}^{n} T_{Q_i}\right) \leq c\delta(\Omega) E\left(\sum_{i=1}^{n} T_{Q_i}\right).$$

Note that there is an absolute constant N such that the number of $Q_i, 1 \leq i < \infty$, which satisfy $\frac{1}{2}\text{diam}(Q_i) \geq d(y, Q_i)$ is at most N. Suppose without loss that the Q_i of

$Q_1, \ldots, Q_k$ which satisfy $\frac{1}{2}\text{diam}(Q_i) \geq d(y, Q_i)$ are the last ones, that is, there is an integer $k_0, k_0 \leq k, k - k_0 \leq N$, such that

$$\frac{1}{2}\text{diam}(Q_i) \leq d(y, Q_i), 1 \leq i \leq k_0. \tag{5.2}$$

Suppose further, without loss of generality, that

$$P_{Q_i} \geq P_{Q_{i+1}}, 1 \leq i < k_0. \tag{5.3}$$

Now it is easy to see that the total area of all Whitney squares V of Ω such that $\rho(S, V) \leq j$ is at most $cj^2\delta(\Omega)$, since $\rho(S,V)\delta(\Omega)^{1/2} \geq cd(S,V)$. Therefore if m is an integer, $1 \leq m < k_0$, (1.4), and, in the last step (using (5.2)), (1.3), imply

$$\begin{aligned}
\sum_{j=m+1}^{k_0} |\text{ Cov } (T_{Q_m}, T_{Q_j})| &\leq \sum_{i=m+1}^{\infty} Ce^{-c\rho(Q_m, Q_i)} P_{Q_m}\sigma(Q_m)\sigma(Q_i) \\
&\leq P_{Q_m}\sigma(Q_m) \sum_{j=1}^{\infty} Cj^2 e^{-cj}\delta(\Omega) \\
&\leq CP_{Q_m}\sigma(Q_m)\delta(\Omega) \\
&\leq C'ET_{Q_m}\delta(\Omega).
\end{aligned}$$

Thus $\displaystyle\sum_{\substack{i<j \\ 1\leq i,j\leq k_0}} |\text{Cov } (T_{Q_i}, T_{Q_j})| \leq c \sum_{m=1}^{k_0} ET_{Q_m}\delta(\Omega).$

Furthermore, using (1.4) in the case where Q and R are the same square, and (1.3),

$$\text{Var } T_{Q_i} \leq cP_{Q_i}\sigma(Q_i)^2 \leq c\delta(\Omega)ET_{Q_i}, 1 \leq i \leq k_0,$$

so $\text{Var}(\sum_{i=1}^{k_0} T_{Q_i}) \leq c\delta(\Omega) \sum_{i=1}^{k_0} ET_{Q_i}$. Together with Lemma 5.1 and the fact that $\text{Var}\left(\sum_{i=1}^{n} W_i\right) \leq n \sum_{i=1}^{n} \text{Var } W_i$, we get

$$\text{Var} \sum_{i=1}^{k} T_{Q_i} \leq 2 \text{ Var} \sum_{i=1}^{k_0} T_{Q_i} + 2 \text{ Var} \sum_{i=k_0+1}^{k} T_{Q_i} \leq c \sum_{i=1}^{k_0} ET_{Q_i}\delta(\Omega) + NC \sum_{i=k_0+1}^{k} ET_{Q_i}\delta(\Omega)$$

proving (5.1). □

Inequality (1.5) follows immediately from (1.7). To show, as claimed in Theorem 1.3, that if $\delta(\Omega) < \infty$ then $E\tau < \infty$ implies $P(\tau = \infty) = 1$, we note that, by Chebyshev's inequality, if $A_i, i \geq 1$, are random variables such that $A_i \leq A_{i+1}$ and Var $A_i \leq KEA_i$ for some constant K not depending on i, then $lim_{n\to\infty} EA_n = \infty$ implies $lim_{n\to\infty} A_n = \infty$, and observe that, by (5.1), $\sum_{j=1}^{i} T_{Q_j}, i \geq 1$, satisfies all these hypothesis if $E\tau = \sum_{j=1}^{\infty} ET_{Q_i} = \infty$.

Let R be a square and let Δ be the disc of radius diam(R) around the center of R. Then $E_z \int_0^{\tau_\Delta} I(Z_t \in R)dt \leq E_z\tau_\Delta \leq c\sigma(R)$. This, together with the fact that $\int_0^t I(Z_s \in R)ds$ is an additive functional of the path of Z which changes only when the path is in R, were key to the proofs of (1.2) and (1.4). Now, using the notation of (1.9), (1.10), and (1.11) we have

$$|E_z \int_0^{\tau_\Delta} f(Z_t)I(Z_t \in R)dt| \leq cf^*(R)\sigma(R).$$

Furthermore,

$$\begin{aligned} E_z|\int_0^{\tau_\Delta} f(Z_t)I(Z_t \in R)dX_t| &\leq \left[E_z\left(\int_0^{\tau_\Delta} f(Z_t)I(Z_t \in R)dX_t\right)^2\right]^{1/2} \\ &= \left[E_z\int_0^{\tau_\Delta} (f(Z_t)I(Z_t \in R))^2 dt\right]^{1/2} \\ &\leq \left[E_z\int_0^{\tau_\Delta} f^*(R)^2 I(Z_t \in R)dt\right]^{1/2} \\ &\leq \left[cf^*(R)^2\sigma(R)\right]^{1/2} \\ &\leq cf^*(R) \text{ diam}(R). \end{aligned}$$

Using these inequalities, (1.8), (1.9), and (1.10) may be proved by straightforward application of the proofs of Theorems 1.2, 1.1, and 1.2 respectively. We may replace $f^*(Q)$ in (1.7) and (1.8) by

$$f^{\#}(Q) = \sup{}_{z\in Q} E_z \int_0^{\tau_\Delta} |f(Z_t)|I(Z_t \in Q)dt/\sigma(Q),$$

where Δ is the disc of radius diam(Q) around the center of Q, and we may replace $f^*(Q)$ in (1.9) and (1.10) by

$$f^{+}(Q) = \left[sup_{z\in Q} E_z\left(\int_0^{\tau_\Delta} f(Z_t)dX_t\right)^2\right]^{1/2} /\text{diam}(Q).$$

Finally, we turn to the example connected with (1.6). We will be brief. First note that if Δ is a disc with center z, then $\mathrm{Var}_z\tau_\Delta = \alpha\sigma(\Delta)^2$, where α is a positive constant not depending on Δ. Furthermore, by symmetry, $\mathrm{Var}_z\tau_\Delta$ conditioned on $Z_{\tau_\Delta} = w$ still equals $\alpha\sigma(\Delta)^2$, for any point $w \in \partial\Delta$.

Suppose $\delta(\Omega) < \infty$, let J be a disc of area $\delta(\Omega)/2$ contained in Ω, and let r stand for the radius of J. Let w be the center of J, and z be any point satisfying $|z - w| = r/2$. Let M be the disc of radius $r/2$ around w. Let $\gamma = \inf\{t : lim_{s\to t} Z_s = z\}$. Then under $P_w^{G(\cdot,z)}, \gamma$ is the lifetime of our process. (In the notation of the introduction, we used $\tau(z,w,\Omega)$ for this lifetime.) Now $\gamma = \tau_M + (\gamma - \tau_M)$. Furthermore, under $P_w^{G(\cdot,z)}$, given $Z_{\tau_M}, (\gamma - \tau_M)$ is independent of τ_M, so that the conditional variance under $P_w^{G(\cdot,z)}$ of γ given Z_{τ_M} is at least $\alpha\sigma(M)^2$, so that

$$\mathrm{Var}_w^{G(\cdot,z)}\gamma \geq \alpha\sigma(M)^2 \geq c\delta(\Omega)^2. \tag{5.3}$$

To finish the proof of (1.6) we will show

$$E_w^{G(\cdot,z)}\gamma \leq c\delta(\Omega). \tag{5.4}$$

Let A be the disc of radius $(3/4)r$ around z. Harnack's inequality implies $G(a,z)/G(w,z) \leq C_1, a \in A$. A winding argument similar to that used to prove (1.7) shows there is a constant $c_1 > 1$ such that $P_a(|Z_t - Z_0| > c_1 r$ for some $t < \tau_\Omega) < 1/2, a \in \Omega$, which, upon using the kind of iteration argument used to establish Lemma 3.1, gives $P_a(|Z_t - Z_a| > kc_1 r$ for some $t < \tau_\Omega) < 2^{-k}, k = 0, 1, 2, \ldots$. In particular $P_a(Z_t \in \partial A$ for some $t < \tau_\Omega) < 2^{-(k-1)}$ if $|a - z| > kc_1 r$. Now if $\lambda = \inf\{t : Z_t \in \partial A\}$,

$$G(a,z) = E_a G(Z_\gamma, z) I(\lambda < \tau_\Omega) \text{ if } a \text{ is outside } A.$$

Thus $G(a,z) \leq C_1 2^{-(k-1)} G(w,z)$ if $|a - z| > kc_1 r$, which implies, using (2.1),

$$P_w^z(|Z_t - z| \geq kc_1 r \text{ for some } t < \gamma) < c2^{-(k-1)}.$$

Now let $H_0 = \{|Z_t - z| < c_1 r, t < \gamma\}$, and if $n \geq 1$ put

$$H_n = \{|Z_t - z| \geq nc_1 r \text{ for some } t < \gamma \text{ but } |Z_t - z| < (n+1)c_1 r \text{ for some } t < \gamma\}.$$

Then $P_w^z(H_n) < C_1 2^{-(n-1)}$, and it is easy to show that $E_w^z(\gamma|H_n) < c\pi[(n+1)c_1 r]^2$, using the main result of [4] (essentially (1.1)]. For all the action takes place in a subdomain of the disc or radius $(n+1)c_1 r$ around z, which has area $\pi((n+1)c_1 r)^2$, and both the expected time to reach the circle of $nc_1 r$ about z is less than c times this area, and the expected time to get back to z without reaching the larger circle is also less than c times this area. Thus $E_z^w \gamma = \Sigma E_z^w(\gamma|H_n)P(H_n) < cr^2 < C\delta(\Omega)$, proving (5.4) and finishing the proof of (1.6).

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