CONTRIBUTIONS TO SELECTION AND RANKING THEORY WITH SPECIAL REFERENCE TO LOGISTIC POPULATIONS

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CONTRIBUTIONS TO SELECTION AND RANKING THEORY WITH SPECIAL REFERENCE TO LOGISTIC POPULATIONS [1]

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Abstract

Selection and ranking (more broadly multiple decision) problems arise in many practical situations where the so-called tests of homogeneity do not provide the answers the experimenter wants.

The logistic distribution has been applied in studies of population growth, of mental ability, of bio-assay, of life test data and of biochemical data, but the complete distribution of the sample means and variances of a logistic population has not been obtained yet.

In this paper we study the selection and ranking problems for logistic populations and an elimination type two-stage procedure for selecting the best population using a restricted subset selection rule in its first stage.

Chapter 2 deals with the selection and ranking procedures for logistic populations. An excellent approximation to the distribution of the sample means from a logistic population is derived by using the Edgeworth series expansions. Using this approximation, we propose and study a single-stage procedure using the indifference zone approach, two subset selection rules based on sample means and medians respectively for selecting the population with the largest mean from k logistic populations when the common variance is known.

Chapter 3 considers an elimination type two-stage procedure for selecting the population with the largest mean from k logistic populations when the common variance is known. A table of the constants needed to implement this procedure is provided and the efficiency of this procedure relative to the single-stage procedure is investigated.

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Chapter 4 deals with a single-stage restricted subset selection procedure for selecting the population with the largest mean from k logistic populations when the common variance is known. Some properties of this procedure such as monotonicity and consistency are investigated. Tables of required sample sizes for this procedure are provided. A new design criterion to get the needed sample size and the constant defining this procedure simultaneously is proposed and a table of these constants is given.

Chapter 5 deals with a more flexible two-stage procedure for selecting the best population, which uses a restricted subset selection rule in its first stage and the Bechhofer's (1954) natural decision procedure in the second stage, in terms of a set of consistent estimators of the real population parameters, whose distributions form a stochastically increasing family for a given sample size.

KEY WORDS: Selection and Ranking, Restricted Subset Selection Procedure, Two-Stage Procedure, Largest Mean, Subset Selection, Logistic Populations.

1 INTRODUCTION

It is not uncommon that we face a problem of making decisions regarding k given populations, for example, different varieties of wheat in an agricultural experiment, or different competing designs of engines to be used in an automobile plant, or different drugs for a certain ailment. Suppose $\theta_1, \ldots, \theta_k$ are the characteristics or parameters of the populations in which the experimenter is interested. The classical approach in the preceding problems has been to test the so-called homogeneity hypothesis $H_o : \theta_1 = \ldots = \theta_k$. However the experimenter's real goal often is to identify the best population (the variety with the largest average yield, the most reliable system and so on). Then the test of H_o is unrealistic for this problem. Attempts were made to overcome the shortcomings of the classical tests of homogeneity by formulating the problem in a more meaningful and realistic way. A partial answer was provided by Mosteller (1948) who tested homogeneity against slippage alternatives. Paulson (1949), Bahadur (1950) and Bahadur and Robbins (1950) are among the early investigators to recognize the shortcomings of the classical test of homogeneity hypothesis and to formulate the k-population problem as a multiple decision problem in the framework of what have now come to be known as selection and ranking procedures.

The two main approaches that have been used in formulating a selection and ranking problem are familiarly known as the indifference zone approach and the subset selection approach. The basic problem in the indifference zone approach, due to Bechhofer (1954), is to select one of the k populations with a guarantee that the probability of selecting the best population is at least a fixed probability $P^*(1/k < P^* < 1)$ whenever the unknown parameters lie outside some subspace of the parameter space, the so-called indifference zone (the complement of an indifference zone is called a preference zone). Here some knowledge of the parameter space is assumed known a priori, for example, the experimenter must be able to guarantee that the largest parameter is separated from all other ranking parameters by a distance not less than δ, say. Other contributions to this approach are Bechhofer and Sobel (1954), Bechhofer, Dunnett and Sobel (1954), Sobel and Huyett (1957), Sobel (1967), Bechhofer, Kiefer and Sobel

(1968), Mahamunulu (1967), Desu and Sobel (1968,1971) and Tamhane and Bechhofer (1977,1979) among others. There are several variations and generalizations of the basic goal discussed above. For details, reference can be made to Gupta and Panchapakesan (1979) and Dudewicz and Koo (1982).

In the subset selection approach known as "Gupta's formulation" for selecting the best population, the goal is to select a nonempty subset of the k populations so that the best population is included in the selected subset with a minimum guaranteed probability $P^*(1/k < P^* < 1)$ over the whole parameter space. Here the size of the selected subset is not determined in advance but depends on the data and hence it is a random variable. Among decision procedures which satisfy the basic probability requirement, one which yields the smallest expected size of the selected subset is considered in some ways to be the most desirable. Another performance criterion for comparing decision procedures is the expected number of the non-best populations in the selected subset. Some recent contributions in the subset selection formulation have been made by Gupta and Studden (1970), Gupta and Nagel (1971), Gupta and Panchapakesan (1972), Gupta and Santner (1973), Santner (1973,1975), Gupta and Huang (1975a,1975b), Gupta and Huang (1976), Bickel and Yahav (1977), Gupta and Singh (1980), Gupta and Hsiao (1983), Lorenzen and McDonald (1981) among others.

In the basic subset selection formulation we select a nonempty subset of the k given populations. When the parameters θ_i are all very close to one another, we are likely to select all the populations. So it is meaningful to put a restriction that the size of the selected subset will not exceed m $(1 < m < k)$. Even otherwise, one may want to select a nonempty subset of a random size to a maximum of m. Such a formulation is called a restricted subset selection formulation. The general theory was developed by Santner (1973,1975) and the normal means selection problem was investigated by Gupta and Santner (1973). An important feature of this formulation is that an indifference zone (preference zone) is introduced.

Besides being a goal in itself, selecting a subset containing the best can also serve as a first stage screening in a two-stage procedure designed to choose one population as the best. Some important contributions in this direction have made by Alam (1970), Tamhane and Bechhofer (1977,1979), Miescke (1982), Gupta and Miescke (1982,1984), Gupta and Kim (1984) and Lee and Choi (1985).

There are several other variations and generalizations of the basic subset selection formulation, for example, the decision-theoretic approach where some Bayes and empirical Bayes rules and several minimax and Γ-minimax rules have been studied by various authors, selection procedures for multivariate normal and multinomial distributions, nonparametric procedures, selection from restricted families, sequential procedures, isotonic procedures etc. For further developments in subset selection formulation, reference can be made to Gupta and Panchapakesan (1979), Gupta and Huang (1981), Dudewicz and Koo (1982), and Gupta and Panchapakesan (1985).

The main contributions of this paper are first, to propose and study new selection and ranking procedures for the logistic populations and second, to propose an elimination type two-stage procedure for selecting the best population using a restricted subset selection rule in its first stage and to apply this procedure to specific problems.

Chapter 2 deals with the basic selection and ranking procedures for logistic populations. The range of application of the logistic distribution as a probability model to describe random phenomenon covers such areas as population growth, bioassay, life test and physiochemical phenomena. The exact distribution of the mean of samples from a logistic populations has not been obtained completely yet though it is needed in the various studies about logistic distributions such as estimating, testing hypothesis and selection and ranking problems. An excellent approximation to the distribution of the sample means from a logistic population is derived by using the Edgeworth series expansion and it is compared to other approximations. Using this approximation we consider a single-stage indifference zone approach procedure $\mathcal{P}_1$ for selecting the best logistic population. We also propose two subset selection rules R_1 and R_2 based on sample means and medians respectively and compare them to each other by means of their performance characteristics.

Chapter 3 considers an elimination type two-stage procedure for selecting the population with the largest mean from k logistic populations. We propose a two-stage procedure $\mathcal{P}_2$ which is based on an optimization problem by using a minimax criterion.

Lower bounds for the infimum of the probability of a correct selection over the preference zone and the supremum over the whole parameter space of the expected total sample size needed for $\mathcal{P}_2$ are derived. A table of the constants needed to implement $\mathcal{P}_2$ is provided by solving the optimization problem and the efficiency of $\mathcal{P}_2$ relative to the single-stage procedure $\mathcal{P}_1$ is investigated.

Chapter 4 deals with a single-stage restricted subset selection procedure for logistic populations. We consider a restricted subset selection procedure R_3 based on the sample means for selecting the population with the largest mean from k logistic populations when the common variance is known. Formulas for the probability of a correct selection for any given set of parameters and for the infimum over the preference zone of the above probability are derived and some properties of this procedure such as monotonicity and consistency are studied. Tables of the bounds of the infimum of the probability of a correct selection over the preference zone, tables of the required sample sizes for the rules and tables of the expected number of selected populations are provided. A new design criterion to get the needed sample size (n) and the constant defining the rule (h) simultaneously is proposed and a table of the constants (n, h) is provided.

Chapter 5 deals with a more flexible two-stage procedure for selecting the best population. We propose an elimination type two-stage procedure $\mathcal{P}_2'$ in which a generalized restricted subset selection rule is used in the first stage and the Bechhofer's (1954) natural decision procedure in the second stage. This rule is based on a set of consistent estimators for the parameters, whose distributions are assumed to form a stochastically increasing family for a given sample size. We also propose a non-linear optimization problem using a minimax design criterion to find a set of design constants for $\mathcal{P}_2'$. A lower bound of the probability of a correct selection is derived and also a formula for the infimum of the lower bound over the preference zone is derived. An analytic expression for the expected total sample size, the conditions guaranteeing that the supremum over the whole parameter space of the expected total sample size occurs at some point where the parameters are all equal, and a general expression of the supremum of the expected total sample size under these conditions are derived. Finally we apply $\mathcal{P}_2'$ to the location parameter problem of univariate normal populations by providing the tables of the design constants to implement $\mathcal{P}_2'$ and of the values of the relative efficiency with respect to the single-stage procedure.

2 SELECTION AND RANKING OF THE LOGISTIC POPULATIONS

2.1 Introduction

The logistic distribution has been widely used by Berkson (1944,1951,1957) as a model for analyzing experiments involving quantal response. Pearl and Reed (1920) used this in studies connected with population growth. Plackett (1958,1959) has considered the use of this distribution with life test data. Gupta (1962) has studied this distribution as a model in life testing problems.

The shape of the logistic distribution is similar to the normal distribution. The simple explicit relationship between the logistic random variate, its probability density function (pdf) and its cumulative distribution function (cdf) render much of the analysis of the logistic distribution attractively simple and many authors, for example, Berkson (1951) prefer it to the normal distribution.

The importance of the logistic distribution in the modeling of stochastic phenomena has resulted in numerous other studies involving probabilistic and statistical aspects of the distribution. For example, Gumbel (1944), Gumbel and Keeney (1950) and Talacko (1956) show that it arises as a limiting distribution in various situations; Birnbaum and Dudman (1963), Gupta and Shah (1965) study its order statistics. Many other authors, for example, Antle, Klimko and Harkness (1970), Gupta and Gnanadesikan (1966) and Tarter and Clark (1965), investigate inference questions about its parameters.

As might be expected, because of the similarity between the logistic and the normal distributions, the sample mean and variance, the moment estimators of the logistic population parameters, are effective tools for statistical decisions involving the logistic distribution. Antle, Klimko and Harkness (1970) give a function of the sample mean as a confidence interval estimate of the population mean when the population variance is known. Schafer and Sheffield (1973) show that in terms of the mean squared error the moment estimators of the logistic population parameters are as good as their maximum likelihood estimators. The fact that the distribution of a sample mean has monotone likelihood ratio (MLR) with respect to the population mean when the variance is known is used by Goel (1975) to obtain a uniformly most accurate confidence interval for the population mean and a uniformly most powerful test for one-sided hypotheses involving the population mean. The sampling distribution of the mean is a primary requirement for these statistical purposes. The papers by Antle, Klimko and Harkness (1970) and Tarter and Clark (1965) used a Monte Carlo method for this distribution.

Goel (1975) obtains an expression for the distribution function of the sum of independent and identically distributed (*iid*) logistic variates by using the Laplace transform inverse method for convolutions of Pólya type functions, a technique developed by Schoenberg (1953) and Hirschman and Widder (1955). He provides a table of the cdf of the sum of *iid* logistic variates for the sample size $n = 2(1)12$, $x = 0(0.01)3.99$ and $n = 13(1)15$, $x = 1.20(0.01)3.99$. He also gives a table of the quantiles for $n = 2(1)15$, $\alpha = 0.90, 0.95, 0.975, 0.99, 0.995$. George and Mudholkar (1983) obtain an expression for the distribution of a convolution of the *iid* logistic variables by directly inverting the characteristic function. However, since both formulas of Goel (1975) and George and Mudholkar (1983) contain a term $(1-e^x)^{-k}, k = 1, \ldots, n$, a problem of precision of the computation at the values of x near zero arises when n is large. George and Mudholkar (1983) also show that a standardized Student's t distribution provides a very good approximation for the distribution of a convolution of the *iid* logistic random variables.

This chapter considers approximation problems to the distribution and quantiles of a standardized mean of samples from a logistic population by using Edgeworth and Cornish-Fisher series expansions respectively. Tables of the cdf and quantiles are provided and it is shown that these are far better approximations than the Student's t distributions as suggested in Goerge and Mudholkar (1983) and hence these approximations will be used henceforth.

In the rest of this chapter a single-stage procedure $\mathcal{P}_1$ using an indifference zone formulation for selecting the best among several logistic populations with unknown means and a common known variance based on sample means is proposed and studied. A table of the smallest sample size n needed to implement $\mathcal{P}_1$ subject to the basic probability requirement is provided.

Two subset selection rules R_1 and R_2 based on sample means and sample medians respectively for selecting the best among several logistic populations are proposed and tables of constants to implement the rules are provided. We also compare the two rules with respect to their performance characteristics.

2.2 Distribution of logistic sample means

2.2.1 Logistic distribution

A random variable X has the logistic distribution with mean μ and variance σ^2, sometimes denoted by $L(\mu, \sigma^2)$, if the pdf of X is given by

$$f(x) = (g/\sigma)[\exp\{-g(x-\mu)/\sigma\}][1+\exp\{-g(x-\mu)/\sigma\}]^{-2} \tag{1}$$

and the cdf of X is defined by

$$F(x) = [1+\exp\{-g(x-\mu)/\sigma\}]^{-1}, \tag{2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ and $g = \pi/\sqrt{3}$. This distribution is symmetrical about the mean μ.

The standard logistic distribution with mean zero and variance unity, denoted by $L(0,1)$, has the pdf and cdf defined as

$$f(x) = g[\exp\{-gx\}][1+\exp\{-gx\}]^{-2} \tag{3}$$

and

$$F(x) = [1+\exp\{-gx\}]^{-1} \tag{4}$$

respectively, where $-\infty < x < \infty$. The standard logistic distribution has a shape similar to the standard normal distribution. The curve of the logistic distribution crosses the density curve of the normal between 0.68 and 0.69. The inflection points of the pdf of the standard logistic distribution are ± 0.53 (approx.).

Letting $Y = (X-\mu)g/\sigma$, the random variable Y has the logistic distribution with mean zero and variance $\pi^2/3$. The pdf and cdf of the random variable Y are given by

$$f(y) = [\exp\{-y\}][1+\exp\{-y\}]^{-2} \tag{5}$$

and

$$F(y) = [1 + \exp\{-y\}]^{-1} \tag{6}$$

respectively, where $-\infty < y < \infty$. (5) may be written in terms of $F(y)$ as

$$f(y) = F(y)(1 - F(y)). \tag{7}$$

The moment generating function (mgf) of Y is given by

$$\begin{aligned} M_Y(t) &= \Gamma(1+t)\Gamma(1-t) \\ &= \pi t / \sin \pi t, \ |t| < 1. \end{aligned} \tag{8}$$

We can also express (8) as

$$M_Y(t) = \sum_{j=0}^{\infty} (-1)^{j-1} [2(2^{2j-1} - 1)/(2j)!] B_{2j} (\pi t)^{2j}, \tag{9}$$

where B_ν's are Bernoulli numbers defined as

$$x/(\exp(x) - 1) = \sum_{\nu=0}^{\infty} B_\nu x^\nu / (\nu!). \tag{10}$$

The relationships between B_ν's are given by

$$\begin{aligned} B_0 &= 1 \\ 1 + \binom{k}{1} B_1 + \binom{k}{2} B_2 + &\cdots + \binom{k}{k-1} B_{k-1} = 0, \ k = 1, \ldots, \end{aligned} \tag{11}$$

and hence the first few values of B_ν are

$$\begin{aligned} B_0 &= 1, \\ B_1 &= -1/2, \\ B_2 &= 1/6, \\ B_4 &= -1/30, \\ B_6 &= 1/42, \\ B_8 &= -1/30, \\ B_{10} &= 5/66, \\ &\vdots \\ B_{2m+1} &= 0, m = 1, 2, \ldots. \end{aligned} \tag{12}$$

The ν^{th} central moments of Y, denoted by $\mu_\nu(y)$, can be obtained as

$$\begin{aligned}
\mu_\nu(y) &= E(Y^\nu) \\
&= \begin{cases} (-1)^{\nu/2-1}[2(2^{\nu-1}-1)]B_\nu\pi^\nu; & \text{if } \nu = 2j, j = 1, 2, \ldots, \\ 0; & \text{otherwise,} \end{cases}
\end{aligned}$$

by using (9).

Then the ν^{th} central moments of X, denoted by $\mu_\nu(x)$, are given by

$$\begin{aligned}
\mu_\nu(x) &= E(X-\mu)^\nu \\
&= (\sigma/g)^\nu E(Y^\nu) \\
&= \begin{cases} (-1)^{\nu/2-1}(\sqrt{3}\sigma)^\nu[2(2^{\nu-1}-1)]B_\nu; & \text{if } \nu = 2j, j = 1, 2, \ldots, \\ 0; & \text{otherwise.} \end{cases}
\end{aligned}$$

In terms of the central moments $\mu_\nu(x)$ of X, first few of the ν^{th} cumulants of X, denoted by $K_\nu(x),\ \nu = 1, 2, \ldots$, which are defined by

$$\log \varphi_X(t) = \sum_{\nu=1}^{\infty} K_\nu(x)(it)^\nu/(\nu!),$$

where $\varphi_X(t)$ is the characteristic function of the random variable X, are given by

$$\begin{aligned}
K_1(x) &= \mu_1(x) = \mu, \\
K_2(x) &= \mu_2(x) = \sigma^2, \\
K_4(x) &= \mu_4(x) - 3(\mu_2(x))^2 = \frac{6}{5}\sigma^4, \\
K_6(x) &= \mu_6(x) - 15\mu_2(x)\mu_4(x) + 30(\mu_2(x))^3 = \frac{48}{7}\sigma^6, \\
K_8(x) &= \mu_8(x) - 28\mu_2(x)\mu_6(x) - 35(\mu_4(x))^2 + 420(\mu_2(x))^2\mu_4(x) \\
&\quad - 630(\mu_2(x))^4 = \frac{432}{5}\sigma^8, \\
K_{10}(x) &= \mu_{10}(x) - 45\mu_2(x)\mu_8(x) - 210\mu_4(x)\mu_6(x) \\
&\quad + 1260(\mu_2(x))^2\mu_6(x) + 3150\mu_2(x)(\mu_4(x))^2 \\
&\quad - 18900(\mu_2(x))^3\mu_4(x) + 22680(\mu_2(x))^5 = \frac{145152}{77}\sigma^{10}, \\
&\vdots \\
K_{2j+1}(x) &= 0, \quad j = 1, 2, \ldots.
\end{aligned}$$

The first few relative cumulants of X, $\lambda_\nu(x)$, where $\lambda_\nu(x)$ is defined as

$$\lambda_\nu(x) = K_\nu(x)(K_2(x))^{-\nu/2},$$

are given by

$$\begin{aligned}
\lambda_1(x) &= \mu/\sigma, \\
\lambda_2(x) &= 1, \\
\lambda_4(x) &= 6/5, \\
\lambda_6(x) &= 48/7, \\
\lambda_8(x) &= 432/5, \\
\lambda_{10}(x) &= 145152/77, \\
&\vdots \\
\lambda_{2j+1}(x) &= 0, \quad j = 1, 2, \ldots. \qquad (13)
\end{aligned}$$

2.2.2 Edgeworth series expansions for the distribution of the mean of samples from a logistic population

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a logistic population $L(\mu, \sigma^2)$ with mean μ and variance σ^2 whose cdf and pdf are given in (1) and (2) respectively. Define a standardized mean of samples of size n from $L(\mu, \sigma^2)$, Z say, as

$$\begin{aligned}
Z &= \frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}(X_i - \mu) \\
&= \frac{\sqrt{n}}{\sigma}(\overline{X} - \mu), \qquad (14)
\end{aligned}$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean.

Let $f_n(z)$ and $F_n(z)$ denote the pdf and cdf of the standardized mean of samples of size n from $L(\mu, \sigma^2)$. Then the Edgeworth series expansions of the $f_n(z)$ and $F_n(z)$ are given symbolically as

$$f_n(z) = \phi(z) + \phi(z)\sum_{j=1}^{\nu} p_j(z) n^{-j/2} + O(n^{-(\nu+1)/2})$$

and

$$F_n(z) = \Phi(z) - \phi(z) \sum_{j=1}^{\nu} P_j(z) n^{-j/2} + O(n^{-(\nu+1)/2})$$

respectively, where $\phi(z)$ and $\Phi(z)$ are the standard normal pdf and cdf respectively and $p_j(z)$ and $P_j(z)$ are polynomials in z, which are obtained up to $\nu = 10$ in Draper and Tierney (1973).

Using $p_j(z)$ and $P_j(z)$ from TABLE II of Draper and Tierney (1973) and the relative cumulants of X given in (13), the Edgeworth series expansions of the $f_n(z)$ and $F_n(z)$ correct to order $n^{-\nu/2}$, $\nu = 4, 6, 8$, are given by

$$\begin{aligned} & f_n(z, \nu = 4) \\ = \; & \phi(z)\{1 + [(\frac{1}{4!})(\frac{6}{5})H_4(z)]n^{-1} \\ & + [(\frac{1}{6!})(\frac{48}{7})H_6(z) + (\frac{35}{8!})(\frac{6}{5})^2 H_8(z)]n^{-2}\} + O(n^{-5/2}), \end{aligned}$$

$$\begin{aligned} & F_n(z, \nu = 4) \\ = \; & \Phi(z) - \phi(z)\{[(\frac{1}{4!})(\frac{6}{5})H_3(z)]n^{-1} \\ & + [(\frac{1}{6!})(\frac{48}{7})H_5(z) + (\frac{35}{8!})(\frac{6}{5})^2 H_7(z)]n^{-2}\} + O(n^{-5/2}), \end{aligned}$$

$$\begin{aligned} & f_n(z, \nu = 6) \\ = \; & f_n(z, \nu = 4) + \phi(z)[(\frac{1}{8!})(\frac{432}{5})H_8(z) \\ & + (\frac{210}{10!})(\frac{48}{7})(\frac{6}{5})H_{10}(z) + (\frac{5775}{12!})(\frac{6}{5})^3 H_{12}(z)]n^{-3} + O(n^{-7/2}), \end{aligned} \tag{15}$$

$$\begin{aligned} & F_n(z, \nu = 6) \\ = \; & F_n(z, \nu = 4) - \phi(z)[(\frac{1}{8!})(\frac{432}{5})H_7(z) \\ & + (\frac{210}{10!})(\frac{48}{7})(\frac{6}{5})H_9(z) + (\frac{5775}{12!})(\frac{6}{5})^3 H_{11}(z)]n^{-3} + O(n^{-7/2}), \end{aligned} \tag{16}$$

$$\begin{aligned} & f_n(z, \nu = 8) \\ = \; & f_n(z, \nu = 6) + \phi(z)[(\frac{1}{10!})(\frac{145152}{77})H_{10}(z) \\ & + (\frac{495}{12!})(\frac{432}{5})(\frac{6}{5})H_{12}(z) + (\frac{462}{12!})(\frac{48}{7})^2 H_{12}(z) \\ & + (\frac{105105}{14!})(\frac{6}{5})^2(\frac{48}{7})H_{14}(z) + (\frac{2627625}{16!})(\frac{6}{5})^4 H_{16}(z)]n^{-4} + O(n^{-9/2}) \end{aligned}$$

and

$$
\begin{aligned}
&F_n(z,\nu=8) \\
&= F_n(z,\nu=6)-\phi(z)[(\frac{1}{10!})(\frac{145152}{77})H_9(z) \\
&+(\frac{495}{12!})(\frac{432}{5})(\frac{6}{5})H_{11}(z)+(\frac{462}{12!})(\frac{48}{7})^2H_{11}(z) \\
&+(\frac{105105}{14!})(\frac{6}{5})^2(\frac{48}{7})H_{13}(z)+(\frac{2627625}{16!})(\frac{6}{5})^4H_{15}(z)]n^{-4}+O(n^{-9/2}),
\end{aligned}
$$

where $H_j(x)$'s are the Hermite polynomials of degree j, which are defined by

$$(\frac{d}{dx})^j \exp(-x^2/2) = (-1)^j H_j(x)\exp(-x^2/2), \quad j=0,1,\ldots.$$

The first thirty Hermite polynomials which follow the recurrence relation

$$H_j(x) = xH_{j-1}(x) - (j-1)H_{j-2}(x), \quad j=2,3,\ldots,$$

are given in TABLE III in Draper and Tierney (1973).

Table 1, Table 2 and Table 3 contain the values of the cdf of the standardized mean of samples of size n from a logistic population with mean μ and variance σ^2 for $n=3,10,15$ and $z=0.00(0.01)3.99$ using the Edgeworth series expansion correct to order n^{-3} given in (16). Entries in the tables were computed by using double-precision arithmetic on a Vax-11/780.

2.2.3 Cornish-Fisher series expansions for the quantiles of the mean of samples from a logistic population

The representation of a quantile of one distribution in terms of the corresponding quantile of another is widely used as a technique for obtaining approximations for the percentage points. One of the most popular of such quantile representations was introduced by Cornish and Fisher (1937) and later reformulated by Fisher and Cornish (1960) and is referred to as the Cornish-Fisher expansion.

By means of formal substitutions, Taylor expansions and identification of powers of n, the Cornish-Fisher expansion of a quantile z of $F_n(z)$ which is the cdf of the standardized mean of samples of size n from $L(\mu, \sigma^2)$, in terms of the corresponding normal quantile y, is of the form

$$z = y + \sum_{j=1}^{\nu} Q_j(y) n^{-j/2} + O(n^{-(\nu+1)/2}),$$

where $Q_j(y)$'s are polynomials of y, which are obtained up to $\nu = 8$ in Draper and Tierney (1973).

Using $Q_j(y)$ from TABLE VII of Draper and Tierney (1973) and $\lambda_j(x)$ in (13), we obtain the Cornish-Fisher series expansions for the quantiles z of $F_n(z)$ up to order $\nu = 4, 6, 8$ as follows:

$$\begin{aligned} z(\nu = 4) &= y + [(\frac{1}{4!})(\frac{6}{5})(y^3 - 3y)]n^{-1} \\ &\quad + [(\frac{1}{6!})(\frac{48}{7})(y^5 - 10y^3 + 15y) \\ &\quad + (\frac{35}{8!})(\frac{6}{5})^2(-9y^5 + 72y^3 - 87y)]n^{-2} + O(n^{-5/2}), \end{aligned}$$

$$\begin{aligned} z(\nu = 6) &= z(\nu = 4) + [(\frac{1}{8!})(\frac{432}{5})(y^7 - 21y^5 + 105y^3 - 105y) \\ &\quad + (\frac{210}{10!})(\frac{48}{7})(\frac{6}{5})(-15y^7 + 255y^5 - 1035y^3 + 855y) \\ &\quad + (\frac{5775}{12!})(\frac{6}{5})^3(243y^7 - 3537y^5 + 12177y^3 - 8667y)]n^{-3} \\ &\quad + O(n^{-7/2}) \end{aligned} \quad (17)$$

and

$$\begin{aligned} & z(\nu = 8) \\ &= z(\nu = 6) + [(\frac{1}{10!})(\frac{145152}{77})(y^9 - 36y^7 + 378y^5 - 1260y^3 + 945y) \\ &\quad + (\frac{495}{12!})(\frac{432}{5})(\frac{6}{5})(-21y^9 + 630y^7 - 5502y^5 + 15330y^3 - 9765y) \\ &\quad + (\frac{462}{12!})(\frac{48}{7})^2(-25y^9 + 700y^7 - 5850y^5 + 15900y^3 - 9945y) \\ &\quad + (\frac{105105}{14!})(\frac{6}{5})^2(\frac{48}{7})(495y^9 - 12510y^7 + 92370y^5 \\ &\quad - 219810y^3 + 121455y) \\ &\quad + (\frac{2627625}{16!})(\frac{6}{5})^4(-11583y^9 + 259848y^7 - 1686906y^5 \end{aligned}$$

$$+ 3539376y^3 - 1743471y)]n^{-4} + O(n^{-9/2}).$$

Table 4 provides the quantiles of the distribution of the standardized mean of samples from the logistic population for sample sizes $n = 3(1)10(5)30$ and probability levels $\alpha = 0.900, 0.950, 0.975, 0.990, 0.995$ using the Cornish-Fisher series expansions correct to order $\nu = 4, 6$ and 8 respectively. Entries of the table were calculated by using double-precision arithmetic on a Vax-11/780.

2.2.4 Legitimacy of using the Edgeworth and Cornish-Fisher series expansions

Noting the similarity of the distribution of Z in (14), the standardized mean of samples from $L(\mu, \sigma^2)$, to the normal distribution in shape except its relatively longer tails, George and Mudholkar (1983) compare the three approximations, that is, the standard normal distribution, the Edgeworth series expansion correct to order n^{-1} and the standardized Student's t distribution to the exact distribution of Z. In using the standardized Student's t distribution, they use the degree of freedom $\xi = 5n + 4$ which can be obtained by equating the coefficients of kurtosis. They show that the Student's t distribution provides a very good approximation.

We show here that the Edgeworth and Cornish-Fisher series expansions correct to order n^{-3}, which are given in the (16) and (17) respectively, are far better approximations than even the Student's t distribution in George and Mudholkar (1983).

Table 5, Table 6 and Table 7 illustrate the quality of the four approximations. In Table 5 the four approximations, that is, the standard normal, the Edgeworth series expansion correct to n^{-1}, the standardized Student's t and the Edgeworth series expansion correct to order n^{-3} are compared to the exact distribution given in Goel (1975). The approximation using the Edgeworth series expansion correct to order n^{-3} appears to be superior to the other three by noting that the maximum error is about 0.0001 as shown in the last column of Table 5. In Table 6, the exact values of the distribution function for $n = 10$ tabled by Goel (1975) are compared with the values obtained from the approximations using Student' t and the Cornish-Fisher series expansion correct to order n^{-3}. In Table 7, the exact quantiles for $n = 2, 3, \ldots, 15$ tabled by Goel (1975) are compared with the corresponding approximations obtained from the Student' t distribution and the Cornish-Fisher series expansion correct to order n^{-3}. From these tables, it is clear that for sample size 7 or more the Edgeworth series expansion correct to order n^{-3} provides an excellent approximation for the standardized mean of samples from the logistic distribution. Consequently, we will use the Edgeworth series expansion correct to order n^{-3} as an approximation to the distribution of the standardized mean of the samples from the logistic distribution henceforth.

2.3 A single-stage procedure $\mathcal{P}_1$ for selecting the population with the largest mean from k logistic populations

Bechhofer (1954), in introducing the indifference zone formulation, considered the problem of ranking means of normal populations with a common known variance. Here we consider a single-stage procedure using an indifference zone approach for selecting the population with the largest mean from k logistic populations when they have a common known variance.

2.3.1 Statement of the problem

Let $\pi_1, \cdots, \pi_k$ be k independent logistic populations with unknown means μ_i and a common known variance σ^2. Let $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$ be the ranked μ_i. We assume that it is not known which population is associated with $\mu_{[i]}$, $i = 1, \ldots, k$. We further assume that a population is characterized by its population mean and the 'best' population is the one having the largest mean.

Our procedure will be based on the sample means. Let $\overline{X}_i$, $i = 1, \ldots, k$, denote the means of independent samples of size n from i^{th} population. The sample mean associated with population having population mean $\mu_{[i]}$ will be denoted by $\overline{X}_{(i)}$, that is, the expected value of $\overline{X}_{(i)}$ is $\mu_{[i]}$. Let $\overline{X}_{[1]} \leq \cdots \leq \overline{X}_{[k]}$ be ranked $\overline{X}_i$. If $\overline{X}_i = \overline{X}_j$ for $i \neq j$, due to the limitations of the measuring instrument, the tied means should be 'ranked' using a randomized procedure which assigns equal probability to each ordering.

Assuming that the goal of the experimenter is to select the best among the k populations, we propose a single-stage procedure $\mathcal{P}_1$ as follows.

Procedure $\mathcal{P}_1$; Take n observations from the i^{th} population for each $i = 1, \ldots, k$. Compute the k sample means $\overline{X}_1, \ldots, \overline{X}_k$. Select the population associated with $\overline{X}_{[k]}$ as the best one.

Defining the event of the experimenter's selection of the best population with $\mathcal{P}_1$ as $[CS|\mathcal{P}_1]$, the probability of a correct selection with the procedure $\mathcal{P}_1$, $P\{CS|\mathcal{P}_1\}$ can be written as

$$\begin{aligned}
P\{CS|\mathcal{P}_1\} &= P_{\vec{\mu}}[\textit{the best population is selected}] \\
&= P_{\vec{\mu}}[\overline{X}_{(k)} \geq \max_{1 \leq j \leq k} \overline{X}_{(j)}] \\
&= P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(j)}, j = 1, \ldots, k-1\ ] \\
&= P_{\vec{\mu}}[(\sqrt{n}/\sigma)(\overline{X}_{(j)} - \mu_{[j]}) \leq (\sqrt{n}/\sigma)(\overline{X}_{(k)} - \mu_{[k]}) \\
&\quad + (\sqrt{n}/\sigma)(\mu_{[k]} - \mu_{[j]}), j = 1, \ldots, k-1] \\
&= \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} F_n(z + (\sqrt{n}/\sigma)(\mu_{[k]} - \mu_{[j]}))dF_n(z), \qquad (18)
\end{aligned}$$

where $F_n(z)$ is the cdf of the standardized mean of samples from a logistic population.

For the fixed values of the μ_i and σ^2 the probability of a correct selection will depend only on the sample size n. We propose to design the experiment in such a way, that is, choose the n in such a way that under specified conditions the probability of a correct selection with procedure $\mathcal{P}_1$ will be equal to or greater than some preassigned value P^*.

2.3.2 Determination of the sample sizes

Now for the problem to be meaningful P^* lies between $1/k$ and 1. Since the true values of the μ_i are not known, we need the probability of a correct selection to be at least P^* whatever be the values of the μ_i. Thus we are interested in the configuration of the μ_i for which the probability in (18) is a minimum. Such a configuration will be called a least favorable configuration (LFC). It is obvious that the LFC is given by

$\mu_{[1]} = \cdots = \mu_{[k]}$. But unfortunately the minimum value of the probability in this LFC case is $1/k$. So we cannot achieve the probability requirement whatever be the sample size unless some modification is made in the probability requirement.

A natural modification is to insist on the minimum probability P^* of selecting the best population whenever the best is sufficiently far apart from the next best. In other words, the experimenter specifies a positive constant δ and requires that the probability of selecting the best population is at least P^* whenever $(\mu_{[k]} - \mu_{[k-1]}) \geq \delta$. The specification of δ provides a partition of the parameter space Ω where

$$\Omega = \{\vec{\mu} = (\mu_1, \cdots, \mu_k); -\infty < \mu_i < \infty, i = 1, \ldots, k\} \tag{19}$$

into two parts, namely $\Omega(\delta)$ where

$$\Omega(\delta) = \{\vec{\mu} \in \Omega |\ (\mu_{[k]} - \mu_{[k-1]}) \geq \delta\} \tag{20}$$

and the compliment $\Omega^c(\delta)$ of $\Omega(\delta)$. The minimization of the probability of selecting the best population is over $\Omega(\delta)$. For an obvious reason, $\Omega^c(\delta)$ was called the indifference zone by Bechhofer (1954). Subsequent authors have termed $\Omega(\delta)$ the preference zone.

It is now easy to see that the LFC in $\Omega(\delta)$ is given by

$$\Omega^0(\delta) = \{\vec{\mu} \in \Omega(\delta) |\ \mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta\} \tag{21}$$

and the minimum sample size required is the smallest integer n for which

$$\inf_{\vec{\mu} \in \Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_1] = \int_{-\infty}^{\infty} (F_n(z + (\sqrt{n}/\sigma)\delta))^{k-1} dF_n(z) \ \geq P^*. \tag{22}$$

A table has been prepared to assist the experimenter in designing the experiments to meet the above goal. Table 8 is to be used for designing experiments involving k logistic populations to decide which has the largest population mean. The table provides the estimates $\hat{n}$ of the values of minimum sample size n associated with the probability $P^* = 0.75, 0.90, 0.95, 0.99$ for $k = 2, 3, 4, 5, 10, 15$, and $\delta/\sigma = 0.1, 0.5, 1.0, 2.0, 4.0$. These were computed by setting the left hand side of (22) equal to P^*. The minimum sample size n can be obtained by $n = [\hat{n} + 1]$ where $[t]$ denotes the greatest integer which is less than t. All computations were carried out in double-precision arithmetic on a Vax-11/780.

2.4 Subset selection procedures

Gupta (1956) introduced a subset selection formulation as a multiple decision problem, where the investigation was carried out for the case of normal means. Here we consider the subset selection rules for selecting the population with the largest mean from k logistic populations. We propose two subset selection rules R_1 and R_2 based on sample means and sample medians respectively, provide tables for implementing these rules, consider the performance characteristics of each rule, and we compare the two rules to each other.

2.4.1 Statement of the problem

Let π_i, $i = 1, \ldots, k$, be k independent logistic populations with unknown means μ_i and a common known variance σ^2. Let $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$ be ranked μ_i and $\pi_{(i)}$ be the population with mean $\mu_{[i]}$. We assume that it is not known which population is associated with $\mu_{[i]}$, $i = 1, \ldots, k$. We further assume that a population is characterized by its population mean and the 'best' population is the one having the largest mean, that is, $\pi_{(k)}$.

Let X_{ij}, $j = 1, \ldots, n$, denote a random sample from π_i, $i = 1, \ldots, k$, where the observations within and between populations are all independent. Let $\overline{X}_i$ and $X_{i:l}$, $i = 1, \ldots, k$, $n = 2l - 1$, denote the means and medians of samples of size n from π_i respectively. The sample mean and the sample median associated with the population having population mean $\mu_{[i]}$ will be denoted by $\overline{X}_{(i)}$ and $X_{(i):l}$, $i = 1, \ldots, k$, respectively. Let $\overline{X}_{[1]} \leq \cdots \leq \overline{X}_{[k]}$ and $X_{[1]:l} \leq \cdots \leq X_{[k]:l}$ be ranked $\overline{X}_i$ and $X_{i:l}$ respectively.

The goal is to select a small but non-empty subset S of the k populations so that the selected subset includes with a high probability P^* the 'best' population. The size of the selected subset S is an integer-valued random variable taking on values $1, \ldots, k$.

Let us define the two subset selection rules R_1 and R_2 based on the sample means and sample medians, respectively, as follows;

<u>*Rule R_1*</u> : *select π_i iff*

$$\overline{X}_i \geq \max_{1 \leq j \leq k} \overline{X}_j - h_1 \sigma / \sqrt{n}, \ i = 1, \ldots, k, \tag{23}$$

and

$$\underline{\text{Rule } R_2} \quad : \quad \textit{select } \pi_i \textit{ iff}$$

$$X_{i:l} \geq \max_{1 \leq j \leq k} X_{j:l} - h_2\sigma/\sqrt{n}, i = 1, \ldots, k, \tag{24}$$

where h_1 and h_2 are nonnegative constants.

By defining the events $[CS|R_i], i = 1, 2$, as selections of any non-empty subset of k populations which includes the best population using $R_i, i = 1, 2$, respectively, it is required that for any $\vec{\mu} \in \Omega$

$$P_{\vec{\mu}}[CS|R_i] \geq P^*, \tag{25}$$

where $P^* \in (1/k, 1)$ and Ω is the parameter space given by (19).

The requirement of (25) will be called as the basic probability requirement or the P^*-condition.

Remark 2.1 *Lorenzen and McDonald (1981) used a subset selection rule R based on sample medians defined as*

$$\textit{Rule } R \quad : \quad \textit{select } \pi_i \textit{ iff}$$

$$X_{i:l} \geq \max_{1 \leq j \leq k} X_{j:l} - d \ , \ d \geq 0, \ i = 1, \ldots, k,$$

where $X_{i:l}$ is defined as above. Here we use R_2 instead of R only for the purpose of comparing R_1 to R_2 easily. Basically the rule R_2 is the same as Lorenzen and McDonald's rule R.

2.4.2 Probability of a correct selection

- Probability of a correct selection for the means rule R_1

Using (23) we can write the probability of a correct selection for the rule R_1 as follows. For $\vec{\mu} \in \Omega$,

$$\begin{aligned} & P_{\vec{\mu}}[CS|R_1] \\ = \ & P_{\vec{\mu}}[\overline{X}_{(k)} \geq \max\nolimits_{1 \leq j \leq k} \overline{X}_j - h_1\sigma/\sqrt{n} \ , h_1 \geq 0] \\ = \ & P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(j)} - h_1\sigma/\sqrt{n} \ , \forall j = 1, \ldots, k-1 \ ] \end{aligned}$$

$$= P_{\vec{\mu}}[(\sqrt{n}/\sigma)(\overline{X}_{(j)} - \mu_{[j]}) \le (\sqrt{n}/\sigma)(\overline{X}_{(k)} - \mu_{[k]}) + h_1 + (\sqrt{n}/\sigma)(\mu_{[k]} - \mu_{[j]}), \forall j = 1, \dots, k-1 \,]$$

$$= \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} F_n(z + h_1 + (\sqrt{n}/\sigma)(\mu_{[k]} - \mu_{[j]}))dF_n(z), \tag{26}$$

where $F_n(z)$ is the cdf of the standardized mean of samples from a logistic distribution.

We see from (26) that the infimum over the parameter space of the probability of a correct selection for the rule R_1 takes place when $\mu_1 = \cdots = \mu_k$ and so

$$\inf_{\vec{\mu}\in\Omega} P_{\vec{\mu}}[CS|R_1] = \int_{-\infty}^{\infty} \{F_n(z+h_1)\}^{k-1} dF_n(z). \tag{27}$$

That is, the LFC for the rule R_1 is Ω^0 where

$$\Omega^0 = \{\vec{\mu} \in \Omega |\ \mu_1 = \cdots = \mu_k = \mu\} \tag{28}$$

and the $P_{\vec{\mu}}[CS|R_1]$ in the LFC does not depend on this common μ. Hence, if we choose h_1 to satisfy

$$\int_{-\infty}^{\infty} \{F_n(z+h_1)\}^{k-1} dF_n(z) \ = P^* \tag{29}$$

then we have determined the smallest h_1 for which

$$\inf_{\vec{\mu}\in\Omega} P_{\vec{\mu}}[CS|R_1] \ge P^*. \tag{30}$$

It should be noted that $h_1 = h_1(n, k, P^*)$ depends on n as well as k and P^* unlike the normal populations problem.

Table 9 and Table 10 give the values of $h_1 = h_1(n, k, P^*)$ which satisfy (29) for $n = 1(1)10$, $k = 2(1)10$ and $P^* = 0.75, 0.90, 0.95, 0.975, 0.99$. We use the Edgeworth series expansions correct to order n^{-3} for $F_n(x)$ and $f_n(x)$, the Gauss-Hermite quadrature algorithm with sixty nodes for the evaluation of the integrals and the modified regular falsi algorithm for solving the non-linear equation. The entries were calculated by using double-precision arithmetic on a Vax-11/780.

• Probability of a correct selection for the medians rule R_2

Let $Z_{i:1}, \dots, Z_{i:n}$ be a random sample of size n, where n is an odd integer, drawn from the i^{th} standard logistic population. Then it is well known that the sample median, denoted by $Z_{i:l}$, $(n = 2l - 1)$, has the pdf

$$g_n(z) = \frac{\Gamma(2l)}{\Gamma(l)^2}[F(z)]^{l-1}[1 - F(z)]^{l-1}f(z)$$

and the cdf

$$G_n(z) = I\{F(z); l, l\}, \tag{31}$$

where $f(z)$ and $F(z)$ are the pdf and cdf of the standard logistic population given by (3) and (4) respectively and $I\{y; a, b\}$ is the incomplete beta function with parameters a and b, which is given by

$$I\{y; a, b\} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^y w^{a-1}(1-w)^{b-1}dw. \tag{32}$$

Now the probability of a correct selection for the medians rule R_2 can be written as follows. For $\vec{\mu} \in \Omega$

$$\begin{aligned} P_{\vec{\mu}}[CS|R_2] &= P_{\vec{\mu}}[X_{(k):l} \geq \max_{1 \leq j \leq k} X_{(j):l} - h_2\sigma/\sqrt{n}\ , h_2 \geq 0] \\ &= P_{\vec{\mu}}[X_{(k):l} \geq X_{(j):l} - h_2\sigma/\sqrt{n}\ , \forall j = 1, \ldots, k-1\ ] \\ &= P_{\vec{\mu}}[Z_{(j):l} \leq Z_{(k):l} + h_2/\sqrt{n} \\ &\quad + (\mu_{[k]} - \mu_{[j]})/\sigma, \forall j = 1, \ldots, k-1\ ] \\ &= \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} G_n(t + h_2/\sqrt{n} + (\mu_{[k]} - \mu_{[j]})/\sigma) dG_n(t), \end{aligned} \tag{33}$$

where G_n is given by (31).

We see that the infimum over Ω of the probability of a correct selection for the rule R_2 takes place when $\mu_1 = \cdots = \mu_k = \mu$ and so

$$\inf_{\vec{\mu} \in \Omega} P_{\vec{\mu}}[CS|R_2] = \int_{-\infty}^{\infty} \{G_n(t + h_2/\sqrt{n})\}^{k-1} dG_n(t). \tag{34}$$

Hence, if we choose h_2 to satisfy

$$\int_{-\infty}^{\infty} \{G_n(t + h_2/\sqrt{n})\}^{k-1} dG_n(t) \ = P^*, \tag{35}$$

then we have determined the smallest h_2 for which

$$\inf_{\vec{\mu} \in \Omega} P_{\vec{\mu}}[CS|R_2] \geq P^*. \tag{36}$$

The values of $h_2/\sqrt{n} = h_2(n,k,P^*)/\sqrt{n}$ which satisfy (35) for $n = 1(2)19$, $k = 2(1)10$ and $P^* = 0.75, 0.90, 0.95, 0.975, 0.99$ were given in TABLE I of Lorenzen and McDonald (1981).

2.4.3 Performance characteristics

In this section some performance characteristics of the subset selection procedures R_1 and R_2 are studied.

Let $P_{\vec{\mu}}[\pi_{(i)}|R_j]$, $i = 1,\ldots,k$, $j = 1,2$, denote the probabilities of including in the subset the population $\pi_{(i)}$, that is, the i^{th} ranked population, using the rule R_j for the $\vec{\mu} \in \Omega$, then for $i = 1,\ldots,k$,

$$\begin{aligned} P_{\vec{\mu}}[\pi_{(i)}|R_1] &= P_{\vec{\mu}}[\overline{X}_{(i)} \geq \max_{1\leq j\leq k} \overline{X}_j - h_1\sigma/\sqrt{n}\ , h_1 \geq 0] \\ &= \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{k} F_n(t + h_1 + (\sqrt{n}/\sigma)(\mu_{[i]} - \mu_{[j]}))dF_n(t), \end{aligned} \tag{37}$$

where $F_n(t)$ is the cdf of the standardized mean of samples of size n from a logistic distribution and

$$\begin{aligned} P_{\vec{\mu}}[\pi_{(i)}|R_2] &= P_{\vec{\mu}}[X_{(i):l} \geq \max_{1\leq j\leq k} X_{(j):l} - h_2\sigma/\sqrt{n}\ , h_2 \geq 0] \\ &= \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{k} G_n(t + h_2/\sqrt{n} + (\mu_{[i]} - \mu_{[j]})/\sigma)dG_n(t), \end{aligned} \tag{38}$$

where $G_n(t)$, given in (31), is the cdf of the median of samples of size n, where n is an odd integer, from the standard logistic population.

It is easy to see that the expected sizes of the selected subset using the rule R_j for $\vec{\mu} \in \Omega$, denoted by $E_{\vec{\mu}}[S|R_j]$, $j = 1,2$, are given as follows:

$$E_{\vec{\mu}}[S|R_j] = \sum_{i=1}^{k} P_{\vec{\mu}}[\pi_{(i)}|R_j]. \tag{39}$$

Consistent with the basic probability requirement, we would like the size of the selected subset to be small.

The expected numbers of non-best populations selected by rule R_j for $\vec{\mu} \in \Omega$, denoted by $E_{\vec{\mu}}[S^*|R_j]$, $j = 1, 2$, are defined as

$$E_{\vec{\mu}}[S^*|R_j] = \sum_{i=1}^{k-1} P_{\vec{\mu}}[\pi_{(i)}|R_j] \tag{40}$$

and also we would like the value of the $E_{\vec{\mu}}[S^*|R_j]$ to be small.

In using the rule R_j, $j = 1, 2$, the ranks of the selected populations are random variables and one may want to evaluate the expected sum of ranks of the selected populations. Let the population with parameter $\mu_{[i]}$ be assigned rank i, $i = 1, \ldots, k$. Then the expected sums of ranks of the selected populations by rule R_j for $\vec{\mu} \in \Omega$, denoted by $E_{\vec{\mu}}[SR|R_j]$, $j = 1, 2$, are

$$E_{\vec{\mu}}[SR|R_j] = \sum_{i=1}^{k} i P_{\vec{\mu}}[\pi_{(i)}|R_j]. \tag{41}$$

For given $\vec{\mu} \in \Omega$, the expected proportions of the selected populations by the rule R_j, denoted by $E_{\vec{\mu}}[P|R_j]$, $j = 1, 2$, are given by

$$E_{\vec{\mu}}[P|R_j] = E_{\vec{\mu}}[S|R_j]/k. \tag{42}$$

Since the values of $P_{\vec{\mu}}[\pi_{(i)}|R_j]$, $j = 1, 2$, depend on $\vec{\mu} \in \Omega$, we consider them for the two special cases, namely the equally spaced configuration and the slippage configuration.

First, for the equally spaced configuration, we assume that the unknown means of the k populations are $\mu, \mu + \delta\sigma, \cdots, \mu + (k-1)\delta\sigma$ which have ranks $1, \ldots, k$ respectively. Then the probabilities of including in the subset the population $\pi_{(i)}$ using the rule R_j for this configuration, denoted by $P_{eq}[\pi_{(i)}|R_j]$, $j = 1, 2$, are given by

$$P_{eq}[\pi_{(i)}|R_1] = \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{k} F_n(t + h_1 + (i-j)\delta\sqrt{n})dF_n(t) \tag{43}$$

and

$$P_{eq}[\pi_{(i)}|R_2] = \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{k} G_n(t + h_2/\sqrt{n} + (i-j)\delta)dG_n(t) \tag{44}$$

respectively.

Next, for the slippage configuration, we assume that the unknown means of k populations are $\mu_{[j]} = \mu$, $j = 1, \ldots, k-1$, and $\mu_{[k]} = \mu + \delta\sigma$ for $\delta > 0$. Then the probabilities of including in the selected subset the population $\pi_{(i)}$ using the rule R_j, denoted by $P_{sp}[\pi_{(i)}|R_j]$, $j = 1, 2$, are given by

$$P_{sp}[\pi_{(i)}|R_1] = \int_{-\infty}^{\infty} \{F_n(t+h_1)\}^{k-2} F_n(t+h_1-\delta\sqrt{n}) dF_n(t) \ , i = 1, \ldots, k-1,$$

$$P_{sp}[\pi_{(k)}|R_1] = \int_{-\infty}^{\infty} \{F_n(t+h_1+\delta\sqrt{n})\}^{k-1} dF_n(t),$$

$$P_{sp}[\pi_{(i)}|R_2] = \int_{-\infty}^{\infty} \{G_n(t+h_2/\sqrt{n})\}^{k-2} G_n(t+h_1/\sqrt{n}-\delta) dG_n(t), i = 1, \ldots, k-1,$$

and

$$P_{sp}[\pi_{(k)}|R_2] = \int_{-\infty}^{\infty} \{G_n(t+h_2/\sqrt{n}+\delta)\}^{k-1} dG_n(t).$$

Now we can compute the performance characteristics $E_{\vec{\mu}}[S|R_j]$, $E_{\vec{\mu}}[S^*|R_j]$, $E_{\vec{\mu}}[SR|R_j]$ and $E_{\vec{\mu}}[P|R_j]$ for two special configurations by substituting $P_{eq}[\pi_{(i)}|R_j]$ and $P_{sp}[\pi_{(i)}|R_j]$ for $P_{\vec{\mu}}[\pi_{(i)}|R_j]$ in (39), (40), (41) and (42) respectively.

Table 11 and Table 12 give the values of the performance characteristics of the means rule R_1 and Table 13 and Table 14 give the same values of the medians rule R_2 for the equally spaced and the slippage configurations respectively for the given values of $k = 2, 3, 4, 5, 10$, $P^* = 0.90$, $n = 3$ and $\sqrt{n}\delta = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0$.

For instance, from Table 11 for $P^* = 0.90$, $n = 3$, $k = 5$ and $\delta\sqrt{n} = 1.5$, the probability of a correct selection by using the means rule R_1 is 0.997. The expected size of the selected subset is 2.208 and the expected number of the non-best populations selected is 1.211. The expected sum of the ranks in the selected subset is 9.330 and the expected proportion of the selected population is 0.442. It should be noted that the expected sum of ranks by itself is not a good criterion of the performance of a selection rule. It should be looked at together with the expected values of S and S^* to make a more meaningful performance characteristic.

The entries in all tables were calculated by using double-precision arithmetic on a Vax-11/780.

Note that, for both rules R_1 and R_2 and for the fixed values of P^*, n, k and $i = 1, 2, \ldots, k-1(k)$, the probability of selecting the i^{th} ranked population in the slippage configuration can be proved to be monotonically decreasing(increasing) with $\delta\sqrt{n}$ and hence monotonically decreasing(increasing) with δ and n separately. Also for $i = 1(k)$, the probability of selecting the i^{th} ranked population in the equally spaced configuration can be proved to be monotonically decreasing(increasing) with $\delta\sqrt{n}$. A look at the table values seems to indicate that, for both rules R_1 and R_2 and for the fixed values of P^*, n, k and $i = 2, \ldots, k-1$, the probability of selecting the i^{th} ranked population in the equally spaced configuration is also monotonically decreasing with $\delta\sqrt{n}$. For fixed P^*, i, n and $\delta\sqrt{n}$, the probability of selecting the i^{th} ranked population is monotonically decreasing with the values of k for all i, $i = 1, \ldots, k$.

2.4.4 Comparison between the means rule R_1 and the medians rule R_2

In this section we compare the efficiency of the means rule R_1 to that of the medians rule R_2. Lorenzen and McDonald (1981) have studied the problem of large sample comparisons between the two rules R_1 and R_2. They computed the asymptotic relative efficiency (ARE) of R_1 relative to R_2 defined by, for $\epsilon \in (0, 1)$ and $\vec{\mu} \in \Omega$,

$$ARE(R_1, R_2; \vec{\mu}) = \lim_{\epsilon \downarrow 0} \frac{N_{R_1}}{N_{R_2}},$$

where N_{R_j}, $j = 1, 2$, are the numbers of observations needed so that

$$\inf P_{\vec{\mu}}[CS|R_j] = P^*$$

and

$$E_{\vec{\mu}}[S^*|R_j] = \epsilon$$

by assuming a slippage configuration, that is,

$$\mu_{[1]} = \cdots = \mu_{[k-1]} = 0, \;\; \mu_{[k]} = \delta > 0.$$

Their value of the $ARE(R_1, R_2; \vec{\mu})$ is 0.822. Thus, under a slippage configuration, asymptotically the means procedure requires about 82% of the sample size required by the medians rule to achieve the same expected number of non-best populations in the selected subset.

Now we consider the small sample comparisons between the rules R_1 and R_2 by using the performance characteristics of each rule given in the previous section. In Table 15, we compute the values of the probability of a correct selection ($P(CS)$), the expected sizes of the selected subset ($E(S)$), the expected numbers of non-best populations in the selected subset ($E(S^*)$), the expected sums of the ranks of the populations selected in the subset ($E(SR)$) and the expected proportions of the populations selected in the subset ($E(P)$) for each rule R_1 and R_2 and the ratio of those values of the rules when the unknown means are equally spaced for the selected values of $P^* = 0.90, 0.95$, $n = 3, 5$, $k = 4$, and $\delta\sqrt{n} = 1.5, 3.0$. The same values for the slippage configurations are given in Table 16.

For both of the configurations;

- Since $P(CS|R_1)/P(CS|R_2) \geq 0.991$ for all cases, the values of $P(CS)$'s are not much different for all cases.

- Since, for example, $E(S|R_1)/E(S|R_2) \leq 1$ for all cases, the values of $E(S)$, $E(S^*)$, $E(SR)$ and $E(P)$ for the rule R_1 are less than or equal to the same values for the rule R_2 for all cases.

- The values of the ratio of the rules R_1 and R_2 for all characteristics are decreasing as the values of n are increasing.

Hence, as expected, the means rule R_1 is definitely better than the medians rule R_2 in the sense of their performance characteristics and the performance of the rule R_1 relative to the rule R_2 improves as sample sizes are increasing for both configurations.

Table 1: Approximate cdf of the standardized mean of samples from a logistic population: Sample size $n = 3$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.	0.5000	0.5042	0.5084	0.5125	0.5167	0.5209	0.5251	0.5292	0.5334	0.5376
0.10	0.5417	0.5459	0.5500	0.5542	0.5583	0.5625	0.5666	0.5707	0.5748	0.5789
0.20	0.5830	0.5871	0.5911	0.5952	0.5992	0.6033	0.6073	0.6113	0.6153	0.6193
0.30	0.6232	0.6272	0.6311	0.6351	0.6390	0.6429	0.6467	0.6506	0.6544	0.6583
0.40	0.6621	0.6659	0.6696	0.6734	0.6771	0.6808	0.6845	0.6882	0.6919	0.6955
0.50	0.6991	0.7027	0.7063	0.7098	0.7134	0.7169	0.7203	0.7238	0.7272	0.7306
0.60	0.7340	0.7374	0.7407	0.7440	0.7473	0.7506	0.7538	0.7571	0.7603	0.7634
0.70	0.7666	0.7697	0.7728	0.7758	0.7789	0.7819	0.7849	0.7878	0.7908	0.7937
0.80	0.7966	0.7994	0.8022	0.8051	0.8078	0.8106	0.8133	0.8160	0.8187	0.8213
0.90	0.8239	0.8265	0.8291	0.8316	0.8341	0.8366	0.8391	0.8415	0.8439	0.8463
1.00	0.8486	0.8510	0.8533	0.8555	0.8578	0.8600	0.8622	0.8644	0.8665	0.8686
1.10	0.8707	0.8728	0.8748	0.8769	0.8789	0.8808	0.8828	0.8847	0.8866	0.8884
1.20	0.8903	0.8921	0.8939	0.8957	0.8974	0.8992	0.9009	0.9026	0.9042	0.9059
1.30	0.9075	0.9091	0.9106	0.9122	0.9137	0.9152	0.9167	0.9182	0.9196	0.9210
1.40	0.9224	0.9238	0.9251	0.9265	0.9278	0.9291	0.9304	0.9316	0.9329	0.9341
1.50	0.9353	0.9365	0.9377	0.9388	0.9399	0.9410	0.9421	0.9432	0.9443	0.9453
1.60	0.9463	0.9474	0.9483	0.9493	0.9503	0.9512	0.9522	0.9531	0.9540	0.9549
1.70	0.9557	0.9566	0.9574	0.9582	0.9591	0.9598	0.9606	0.9614	0.9622	0.9629
1.80	0.9636	0.9644	0.9651	0.9657	0.9664	0.9671	0.9678	0.9684	0.9690	0.9697
1.90	0.9703	0.9709	0.9715	0.9720	0.9726	0.9732	0.9737	0.9742	0.9748	0.9753
2.00	0.9758	0.9763	0.9768	0.9772	0.9777	0.9782	0.9786	0.9791	0.9795	0.9799
2.10	0.9804	0.9808	0.9812	0.9816	0.9820	0.9823	0.9827	0.9831	0.9834	0.9838
2.20	0.9841	0.9845	0.9848	0.9851	0.9854	0.9857	0.9861	0.9864	0.9866	0.9869
2.30	0.9872	0.9875	0.9878	0.9880	0.9883	0.9885	0.9888	0.9890	0.9893	0.9895
2.40	0.9897	0.9900	0.9902	0.9904	0.9906	0.9908	0.9910	0.9912	0.9914	0.9916
2.50	0.9918	0.9920	0.9922	0.9923	0.9925	0.9927	0.9928	0.9930	0.9931	0.9933
2.60	0.9935	0.9936	0.9937	0.9939	0.9940	0.9942	0.9943	0.9944	0.9946	0.9947
2.70	0.9948	0.9949	0.9950	0.9951	0.9953	0.9954	0.9955	0.9956	0.9957	0.9958
2.80	0.9959	0.9960	0.9961	0.9962	0.9963	0.9963	0.9964	0.9965	0.9966	0.9967
2.90	0.9968	0.9968	0.9969	0.9970	0.9971	0.9971	0.9972	0.9973	0.9973	0.9974
3.00	0.9975	0.9975	0.9976	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980
3.10	0.9980	0.9981	0.9981	0.9981	0.9982	0.9982	0.9983	0.9983	0.9984	0.9984
3.20	0.9984	0.9985	0.9985	0.9986	0.9986	0.9986	0.9987	0.9987	0.9987	0.9988
3.30	0.9988	0.9988	0.9989	0.9989	0.9989	0.9989	0.9990	0.9990	0.9990	0.9990
3.40	0.9991	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9992	0.9993
3.50	0.9993	0.9993	0.9993	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994
3.60	0.9994	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995	0.9996
3.70	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.80	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
3.90	0.9997	0.9997	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

Table 2: Approximate cdf of the standardized mean of samples from a logistic population: Sample size $n = 10$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.	0.5000	0.5040	0.5081	0.5121	0.5162	0.5202	0.5243	0.5283	0.5324	0.5364
0.10	0.5404	0.5444	0.5485	0.5525	0.5565	0.5605	0.5645	0.5685	0.5725	0.5764
0.20	0.5804	0.5844	0.5883	0.5923	0.5962	0.6001	0.6040	0.6079	0.6118	0.6157
0.30	0.6196	0.6234	0.6273	0.6311	0.6349	0.6387	0.6425	0.6463	0.6500	0.6538
0.40	0.6575	0.6612	0.6649	0.6686	0.6722	0.6759	0.6795	0.6831	0.6867	0.6903
0.50	0.6938	0.6974	0.7009	0.7044	0.7079	0.7113	0.7148	0.7182	0.7216	0.7250
0.60	0.7283	0.7317	0.7350	0.7383	0.7416	0.7448	0.7480	0.7512	0.7544	0.7576
0.70	0.7607	0.7638	0.7669	0.7700	0.7730	0.7761	0.7791	0.7820	0.7850	0.7879
0.80	0.7908	0.7937	0.7965	0.7994	0.8022	0.8050	0.8077	0.8104	0.8132	0.8158
0.90	0.8185	0.8211	0.8237	0.8263	0.8289	0.8314	0.8339	0.8364	0.8388	0.8413
1.00	0.8437	0.8461	0.8484	0.8508	0.8531	0.8554	0.8576	0.8598	0.8621	0.8642
1.10	0.8664	0.8685	0.8707	0.8727	0.8748	0.8768	0.8789	0.8808	0.8828	0.8848
1.20	0.8867	0.8886	0.8904	0.8923	0.8941	0.8959	0.8977	0.8995	0.9012	0.9029
1.30	0.9046	0.9063	0.9079	0.9095	0.9111	0.9127	0.9143	0.9158	0.9173	0.9188
1.40	0.9203	0.9217	0.9232	0.9246	0.9260	0.9273	0.9287	0.9300	0.9313	0.9326
1.50	0.9339	0.9351	0.9364	0.9376	0.9388	0.9400	0.9411	0.9423	0.9434	0.9445
1.60	0.9456	0.9466	0.9477	0.9487	0.9497	0.9507	0.9517	0.9527	0.9537	0.9546
1.70	0.9555	0.9564	0.9573	0.9582	0.9590	0.9599	0.9607	0.9615	0.9623	0.9631
1.80	0.9639	0.9647	0.9654	0.9661	0.9669	0.9676	0.9683	0.9690	0.9696	0.9703
1.90	0.9709	0.9716	0.9722	0.9728	0.9734	0.9740	0.9745	0.9751	0.9757	0.9762
2.00	0.9767	0.9773	0.9778	0.9783	0.9788	0.9793	0.9797	0.9802	0.9806	0.9811
2.10	0.9815	0.9820	0.9824	0.9828	0.9832	0.9836	0.9840	0.9843	0.9847	0.9851
2.20	0.9854	0.9858	0.9861	0.9864	0.9868	0.9871	0.9874	0.9877	0.9880	0.9883
2.30	0.9886	0.9889	0.9891	0.9894	0.9897	0.9899	0.9902	0.9904	0.9906	0.9909
2.40	0.9911	0.9913	0.9915	0.9918	0.9920	0.9922	0.9924	0.9926	0.9928	0.9929
2.50	0.9931	0.9933	0.9935	0.9936	0.9938	0.9940	0.9941	0.9943	0.9944	0.9946
2.60	0.9947	0.9949	0.9950	0.9951	0.9953	0.9954	0.9955	0.9956	0.9957	0.9959
2.70	0.9960	0.9961	0.9962	0.9963	0.9964	0.9965	0.9966	0.9967	0.9968	0.9968
2.80	0.9969	0.9970	0.9971	0.9972	0.9973	0.9973	0.9974	0.9975	0.9976	0.9976
2.90	0.9977	0.9978	0.9978	0.9979	0.9979	0.9980	0.9981	0.9981	0.9982	0.9982
3.00	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9985	0.9986	0.9986	0.9987
3.10	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9989	0.9990	0.9990
3.20	0.9990	0.9991	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993
3.30	0.9993	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995
3.40	0.9995	0.9995	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996
3.50	0.9996	0.9996	0.9996	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
3.60	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.70	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.90	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Table 3: Approximate cdf of the standardized mean of samples from a logistic population: Sample size $n = 15$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.	0.5000	0.5040	0.5081	0.5121	0.5161	0.5201	0.5242	0.5282	0.5322	0.5362
0.10	0.5402	0.5442	0.5482	0.5522	0.5562	0.5602	0.5642	0.5682	0.5721	0.5761
0.20	0.5800	0.5840	0.5879	0.5918	0.5957	0.5996	0.6035	0.6074	0.6113	0.6152
0.30	0.6190	0.6229	0.6267	0.6305	0.6343	0.6381	0.6419	0.6456	0.6494	0.6531
0.40	0.6568	0.6605	0.6642	0.6679	0.6715	0.6751	0.6788	0.6824	0.6859	0.6895
0.50	0.6931	0.6966	0.7001	0.7036	0.7071	0.7105	0.7140	0.7174	0.7208	0.7241
0.60	0.7275	0.7308	0.7341	0.7374	0.7407	0.7439	0.7472	0.7504	0.7535	0.7567
0.70	0.7598	0.7630	0.7660	0.7691	0.7722	0.7752	0.7782	0.7812	0.7841	0.7870
0.80	0.7899	0.7928	0.7957	0.7985	0.8013	0.8041	0.8069	0.8096	0.8123	0.8150
0.90	0.8177	0.8203	0.8229	0.8255	0.8281	0.8306	0.8331	0.8356	0.8381	0.8405
1.00	0.8429	0.8453	0.8477	0.8500	0.8523	0.8546	0.8569	0.8591	0.8614	0.8636
1.10	0.8657	0.8679	0.8700	0.8721	0.8742	0.8762	0.8782	0.8802	0.8822	0.8842
1.20	0.8861	0.8880	0.8899	0.8918	0.8936	0.8954	0.8972	0.8990	0.9007	0.9024
1.30	0.9041	0.9058	0.9075	0.9091	0.9107	0.9123	0.9139	0.9154	0.9170	0.9185
1.40	0.9199	0.9214	0.9229	0.9243	0.9257	0.9271	0.9284	0.9298	0.9311	0.9324
1.50	0.9337	0.9349	0.9362	0.9374	0.9386	0.9398	0.9410	0.9421	0.9432	0.9444
1.60	0.9455	0.9465	0.9476	0.9486	0.9497	0.9507	0.9517	0.9527	0.9536	0.9546
1.70	0.9555	0.9564	0.9573	0.9582	0.9591	0.9599	0.9607	0.9616	0.9624	0.9632
1.80	0.9640	0.9647	0.9655	0.9662	0.9669	0.9677	0.9684	0.9691	0.9697	0.9704
1.90	0.9710	0.9717	0.9723	0.9729	0.9735	0.9741	0.9747	0.9753	0.9758	0.9764
2.00	0.9769	0.9774	0.9779	0.9785	0.9789	0.9794	0.9799	0.9804	0.9808	0.9813
2.10	0.9817	0.9822	0.9826	0.9830	0.9834	0.9838	0.9842	0.9845	0.9849	0.9853
2.20	0.9856	0.9860	0.9863	0.9867	0.9870	0.9873	0.9876	0.9879	0.9882	0.9885
2.30	0.9888	0.9891	0.9894	0.9896	0.9899	0.9901	0.9904	0.9906	0.9909	0.9911
2.40	0.9913	0.9916	0.9918	0.9920	0.9922	0.9924	0.9926	0.9928	0.9930	0.9932
2.50	0.9933	0.9935	0.9937	0.9939	0.9940	0.9942	0.9943	0.9945	0.9946	0.9948
2.60	0.9949	0.9951	0.9952	0.9953	0.9954	0.9956	0.9957	0.9958	0.9959	0.9960
2.70	0.9961	0.9963	0.9964	0.9965	0.9966	0.9967	0.9967	0.9968	0.9969	0.9970
2.80	0.9971	0.9972	0.9973	0.9973	0.9974	0.9975	0.9976	0.9976	0.9977	0.9978
2.90	0.9978	0.9979	0.9980	0.9980	0.9981	0.9981	0.9982	0.9982	0.9983	0.9983
3.00	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0.9987	0.9987	0.9987	0.9988
3.10	0.9988	0.9988	0.9989	0.9989	0.9990	0.9990	0.9990	0.9990	0.9991	0.9991
3.20	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	0.9993	0.9993	0.9993
3.30	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.9995	0.9995
3.40	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.50	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.60	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.70	0.9998	0.9998	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.90	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Table 4: Approximate quantiles of the standardized mean of samples from a logistic population using Cornish-Fisher series expansion.

n	ν	Probability level				
		0.900	0.950	0.975	0.990	0.995
	4	1.2555226	1.6381158	1.9849505	2.4082844	2.7099442
3	6	1.2548571	1.6371743	1.9843074	2.4095841	2.7143793
	8	1.2550232	1.6376558	1.9850174	2.4099828	2.7136348
	4	1.2614732	1.6395502	1.9791732	2.3899717	2.6805283
4	6	1.2611925	1.6391530	1.9789019	2.3905200	2.6823993
	8	1.2612450	1.6393053	1.9791265	2.3906461	2.6821637
	4	1.2652218	1.6404909	1.9755567	2.3782893	2.6615626
5	6	1.2650780	1.6402875	1.9754178	2.3785700	2.6625206
	8	1.2650995	1.6403499	1.9755098	2.3786217	2.6624241
	4	1.2677950	1.6411513	1.9730832	2.3702115	2.6483706
6	6	1.2677118	1.6410337	1.9730028	2.3703740	2.6489249
	8	1.2677222	1.6410637	1.9730472	2.3703989	2.6488784
	4	1.2696694	1.6416394	1.9712858	2.3642999	2.6386791
7	6	1.2696170	1.6415653	1.9712351	2.3644022	2.6390282
	8	1.2696226	1.6415816	1.9712591	2.3644156	2.6390031
	4	1.2710950	1.6420145	1.9699209	2.3597886	2.6312636
8	6	1.2710599	1.6419648	1.9698870	2.3598572	2.6314974
	8	1.2710632	1.6419743	1.9699010	2.3598650	2.6314827
	4	1.2722156	1.6423114	1.9688494	2.3562339	2.6254089
9	6	1.2721910	1.6422766	1.9688256	2.3562820	2.6255732
	8	1.2721930	1.6422825	1.9688344	2.3562870	2.6255640
	4	1.2731196	1.6425523	1.9679860	2.3533612	2.6206704
10	6	1.2731016	1.6425269	1.9679687	2.3533963	2.6207901
	8	1.2731029	1.6425308	1.9679744	2.3533995	2.6207841
	4	1.2758709	1.6432929	1.9653624	2.3445887	2.6061622
15	6	1.2758656	1.6432853	1.9653572	2.3445990	2.6061977
	8	1.2758658	1.6432861	1.9653584	2.3445997	2.6061965
	4	1.2772688	1.6436731	1.9640318	2.3401155	2.5987437
20	6	1.2772666	1.6436699	1.9640296	2.3401199	2.5987587
	8	1.2772667	1.6436702	1.9640300	2.3401201	2.5987583
	4	1.2781147	1.6439045	1.9632274	2.3374039	2.5942399
25	6	1.2781136	1.6439029	1.9632263	2.3374061	2.5942476
	8	1.2781136	1.6439030	1.9632265	2.3374062	2.5942474
	4	1.2786816	1.6440601	1.9626887	2.3355845	2.5912155
30	6	1.2786810	1.6440591	1.9626881	2.3355858	2.5912199
	8	1.2786810	1.6440592	1.9626881	2.3355859	2.5912198

Table 5: A comparison of four approximations for cdf of standardized mean of samples of size 3 from logistic populations.

x	$F_3^*(x)$	$F_3^*(x) - \Phi(x)$	$F_3^*(x) - G_3(x)$	$F_3^*(x) - T_3(x)$	$F_3^*(x) - G_3'(x)$
0.05	0.5209	0.0010	0.0000	0.0001	0.0000
0.15	0.5625	0.0029	0.0000	0.0003	0.0000
0.25	0.6033	0.0046	0.0008	0.0005	0.0000
0.45	0.6809	0.0073	-0.0017	0.0007	0.0001
0.65	0.7506	0.0084	-0.0006	0.0007	0.0000
0.85	0.8106	0.0083	-0.0007	0.0007	0.0000
1.00	0.8486	0.0073	-0.0008	0.0004	0.0000
1.20	0.8903	0.0054	-0.0007	0.0002	0.0000
1.45	0.9291	0.0026	-0.0004	0.0000	0.0000
1.75	0.9598	-0.0001	0.0001	-0.0002	0.0000
2.50	0.9918	-0.0020	0.0004	0.0002	0.0000
3.00	0.9975	-0.0012	0.0001	0.0001	0.0000

$F_3^*(x)$ = cdf of the standardized mean of 3 iid logistic r.v.'s.

$\Phi(x)$ = cdf of the standard normal r.v.

$G_3(x)$ = Edgeworth series expansion correct to order n^{-1}

$T_3(x)$ = cdf of the standardized Student's t r.v.'s with 19 d.f.

$G_3'(x)$ = Edgeworth series expansion correct to order n^{-3}

Table 6: An illustration of the Student's t and the Edgeworth series expansion approximation for $n = 10$.

x	$F_{10}^*(x)$	$F_{10}^*(x) - T_{10}(x)$	$F_{10}^*(x) - G_{10}'(x)$
0.10	0.540416	0.000021	0.000000
0.20	0.580406	0.000040	0.000000
0.40	0.657488	0.000070	0.000000
0.60	0.728341	0.000081	0.000000
0.80	0.790815	0.000073	0.000000
1.00	0.843689	0.000051	0.000000
1.20	0.886676	0.000023	0.000000
1.50	0.933882	-0.000014	0.000000
1.70	0.955515	-0.000028	0.000000
2.50	0.993123	-0.000014	0.000000
3.00	0.998265	0.000001	0.000000
3.50	0.999620	0.000004	0.000000

$F_{10}^*(x)$ = cdf of the standardized mean of 10 iid logistic r.v.'s.

$T_{10}(x)$ = cdf of the standardized Student's t r.v.'s with 54 d.f.

$G_{10}'(x)$ = Edgeworth series expansion correct to order n^{-3}

Table 7: Quantiles of the standardized mean of logistic variates.

Sample size	Probability(α)				
n	0.900	0.950	0.975	0.990	0.995
2	1.2452	1.6306	1.8757	2.4298	2.7560
	1.2432	1.6340	1.9951	2.4500	2.7718
	1.2425	1.6326	1.9934	2.4450	2.7755
4	1.2617	1.6381	1.9760	2.3861	2.6778
	1.2612	1.6393	1.9790	2.2906	2.6821
	1.2612	1.6392	1.9789	2.3905	2.6824
5	1.2654	1.6395	1.9734	2.3756	2.6597
	1.2651	1.6403	1.9755	2.3786	2.6642
	1.2651	1.6403	1.9754	2.3786	2.6625
7	1.2697	1.6411	1.9750	2.3628	2.6376
	1.2696	1.6416	1.9712	2.3644	2.6390
	1.2696	1.6416	1.9712	2.3644	2.6390
10	1.2731	1.6423	1.9674	2.3526	2.6201
	1.2731	1.6425	1.9680	2.3534	2.6208
	1.2731	1.6425	1.9680	2.3534	2.6208
12	1.2745	1.6427	1.9662	2.3484	2.6131
	1.2745	1.6426	1.9667	2.3491	2.6135
	1.2745	1.6429	1.9667	2.3490	2.6135
15	1.2759	1.6432	1.9651	2.3443	2.6059
	1.2758	1.6433	1.9654	2.3446	2.6062
	1.2759	1.6433	1.9654	2.3446	2.6062

Top element in each cell represents Student's t approximation.

Middle element in each cell represents the exact percentage point.

Bottom element in each cell represents the Cornish-Fisher series approximation (n^{-3}).

Table 8: Values of the estimate $\hat{n}$ of the minimum sample size n for the single-stage procedure.

k	δ/σ	P^*			
		0.75	0.90	0.95	0.99
2	4.00	0.05	0.19	0.33	0.75
	2.00	0.20	0.77	1.34	2.82
	1.00	0.81	3.22	5.40	10.94
	0.50	3.51	13.07	21.63	43.42
	0.10	90.86	328.42	541.09	1082.56
3	4.00	0.12	0.30	0.47	0.93
	2.00	0.46	1.21	1.85	3.42
	1.00	1.95	4.93	7.36	13.25
	0.50	8.11	19.86	29.40	52.49
	0.10	205.48	497.34	734.49	1308.66
4	4.00	0.16	0.37	0.55	1.05
	2.00	0.65	1.49	2.17	3.78
	1.00	2.74	5.99	8.55	14.61
	0.50	11.23	24.02	34.06	57.86
	0.10	282.89	601.00	850.48	1441.89
5	4.00	0.20	0.42	0.61	1.13
	2.00	0.80	1.69	2.39	4.04
	1.00	3.34	6.76	9.40	15.58
	0.50	13.56	27.04	37.40	61.67
	0.10	340.80	675.85	933.47	1536.53
10	4.00	0.32	0.59	0.81	1.32
	2.00	1.28	2.28	3.03	4.76
	1.00	5.12	8.96	11.80	18.29
	0.50	20.50	35.66	46.86	72.38
	0.10	512.42	889.88	1168.58	1802.84
15	4.00	0.39	0.68	0.92	1.44
	2.00	1.56	2.61	3.39	5.15
	1.00	6.14	10.17	13.12	19.77
	0.50	24.41	40.39	52.01	78.17
	0.10	609.05	1007.16	1296.45	1946.93

Table 9: Values of h_1 for the means rule R_1 for selecting the subset containing the largest logistic population mean: $n = 1, 2, 3, 4, 5$.

n	k	P^*				
		0.750	0.900	0.950	0.975	0.990
1	2	0.8981	1.7563	2.3165	2.8349	3.4592
	3	1.3604	2.1916	2.7434	3.2486	3.8504
	4	1.6119	2.4361	2.9827	3.4794	4.0715
	5	1.7855	2.6063	3.1481	3.6389	4.2272
	6	1.9185	2.7366	3.2738	3.7605	4.3484
	7	2.0265	2.8418	3.3750	3.8588	4.4482
	8	2.1174	2.9299	3.4594	3.9412	4.5335
	9	2.1960	3.0055	3.5317	4.0123	4.6083
	10	2.2652	3.0715	3.5950	4.0748	4.6751
2	2	0.6542	1.2612	1.6395	1.9797	2.3909
	3	0.9873	1.5623	1.9253	2.2539	2.6530
	4	1.1642	1.7265	2.0831	2.4066	2.8006
	5	1.2836	1.8388	2.1916	2.5121	2.9028
	6	1.3734	1.9239	2.2740	2.5924	2.9810
	7	1.4452	1.9921	2.3403	2.6569	3.0442
	8	1.5048	2.0490	2.3957	2.7113	3.0972
	9	1.5557	2.0978	2.4432	2.7578	3.1428
	10	1.6002	2.1404	2.4847	2.7985	3.1828
3	2	0.5395	1.0351	1.3400	1.6111	1.9355
	3	0.8131	1.2794	1.5696	1.8294	2.1422
	4	0.9572	1.4115	1.6953	1.9501	2.2573
	5	1.0538	1.5012	1.7813	2.0331	2.3377
	6	1.1261	1.5688	1.8463	2.0960	2.3984
	7	1.1835	1.6228	1.8984	2.1466	2.4473
	8	1.2311	1.6677	1.9418	2.1889	2.4882
	9	1.2716	1.7061	1.9790	2.2251	2.5233
	10	1.3068	1.7395	2.0115	2.2567	2.5541
4	2	0.4696	0.8988	1.1611	1.3930	1.6687
	3	0.7072	1.1097	1.3583	1.5795	1.8442
	4	0.8319	1.2232	1.4657	1.6822	1.9418
	5	0.9152	1.3000	1.5390	1.7527	2.0091
	6	0.9773	1.3577	1.5943	1.8059	2.0605
	7	1.0265	1.4038	1.6385	1.8486	2.1017
	8	1.0672	1.4420	1.6753	1.8843	2.1360
	9	1.1018	1.4745	1.7067	1.9148	2.1654
	10	1.1318	1.5029	1.7342	1.9414	2.1911
5	2	0.4213	0.8052	1.0388	1.2447	1.4884
	3	0.6342	0.9935	1.2143	1.4102	1.6434
	4	0.7457	1.0945	1.3097	1.5010	1.7295
	5	0.8200	1.1628	1.3746	1.5631	1.7886
	6	0.8752	1.2139	1.4234	1.6101	1.8336
	7	0.9190	1.2547	1.4625	1.6477	1.8695
	8	0.9551	1.2885	1.4949	1.6791	1.9000
	9	0.9858	1.3172	1.5226	1.7059	1.9258
	10	1.0123	1.3423	1.5468	1.7293	1.9483

Table 10: Values of h_1 for the means rule R_1 for selecting the subset containing the largest logistic population mean: $n = 6, 7, 8, 9, 10$.

n	k	P^*				
		0.750	0.900	0.950	0.975	0.990
6	2	0.3854	0.7358	0.9485	1.1355	1.3562
	3	0.5800	0.9075	1.1082	1.2857	1.4965
	4	0.6817	0.9994	1.1948	1.3680	1.5742
	5	0.7494	1.0614	1.2536	1.4242	1.6276
	6	0.7997	1.1079	1.2978	1.4666	1.6681
	7	0.8394	1.1448	1.3332	1.5006	1.7006
	8	0.8722	1.1754	1.3625	1.5289	1.7279
	9	0.9001	1.2015	1.3875	1.5531	1.7511
	10	0.9242	1.2241	1.4093	1.5742	1.7713
7	2	0.3573	0.6818	0.8783	1.0507	1.2539
	3	0.5377	0.8406	1.0257	1.1891	1.3829
	4	0.6318	0.9255	1.1056	1.2650	1.4543
	5	0.6944	0.9827	1.1598	1.3167	1.5034
	6	0.7408	1.0255	1.2005	1.3557	1.5405
	7	0.7775	1.0595	1.2330	1.3869	1.5703
	8	0.8078	1.0877	1.2600	1.4129	1.5950
	9	0.8334	1.1117	1.2830	1.4351	1.6166
	10	0.8556	1.1325	1.3030	1.4544	1.6351
8	2	0.3346	0.6381	0.8217	0.9825	1.1718
	3	0.5034	0.7866	0.9593	1.1116	1.2918
	4	0.5914	0.8658	1.0338	1.1823	1.3582
	5	0.6499	0.9192	1.0843	1.2303	1.4038
	6	0.6933	0.9591	1.1222	1.2666	1.4383
	7	0.7275	0.9908	1.1525	1.2957	1.4660
	8	0.7558	1.0171	1.1776	1.3198	1.4890
	9	0.7797	1.0394	1.1990	1.3404	1.5086
	10	0.8004	1.0588	1.2176	1.3584	1.5261
9	2	0.3157	0.6019	0.7748	0.9261	1.1039
	3	0.4750	0.7418	0.9043	1.0475	1.2166
	4	0.5580	0.8164	0.9744	1.1139	1.2789
	5	0.6130	0.8666	1.0219	1.1590	1.3217
	6	0.6539	0.9042	1.0575	1.1931	1.3540
	7	0.6861	0.9340	1.0859	1.2203	1.3800
	8	0.7126	0.9586	1.1095	1.2430	1.4016
	9	0.7351	0.9796	1.1296	1.2623	1.4199
	10	0.7546	0.9978	1.1470	1.2791	1.4363
10	2	0.2997	0.5712	0.7351	0.8783	1.0465
	3	0.4509	0.7039	0.8578	0.9933	1.1532
	4	0.5296	0.7746	0.9242	1.0560	1.2121
	5	0.5818	0.8221	0.9691	1.0988	1.2524
	6	0.6205	0.8577	1.0028	1.1310	1.2830
	7	0.6510	0.8859	1.0297	1.1568	1.3075
	8	0.6762	0.9092	1.0520	1.1782	1.3279
	9	0.6974	0.9291	1.0710	1.1964	1.3453
	10	0.7159	0.9463	1.0875	1.2123	1.3607

Table 11: Performance characteristics of the means rule R_1 under the equally spaced configuration.

$P^* = 0.90,\ n = 3$									
k	i	$\delta\sqrt{n}$							
		0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
2	1	0.824	0.717	0.584	0.440	0.305	0.192	0.059	0.013
	2	0.948	0.975	0.989	0.995	0.998	0.999	1.000	1.000
E(S)		1.772	1.692	1.573	1.436	1.303	1.192	1.058	1.013
E(S*)		0.824	0.717	0.584	0.440	0.305	0.192	0.059	0.013
E(SR)		2.720	2.667	2.562	2.431	2.301	2.191	2.058	2.013
E(P)		0.886	0.846	0.786	0.718	0.651	0.596	0.529	0.506
3	1	0.754	0.511	0.254	0.088	0.022	0.004	0.000	0.000
	2	0.876	0.805	0.697	0.562	0.418	0.286	0.101	0.025
	3	0.963	0.986	0.995	0.998	0.999	1.000	1.000	1.000
E(S)		2.592	2.302	1.945	1.648	1.439	1.289	1.101	1.025
E(S*)		1.630	1.316	0.951	0.650	0.440	0.289	0.101	0.025
E(SR)		5.394	5.080	4.631	4.206	3.856	3.574	3.202	3.050
E(P)		0.864	0.767	0.648	0.549	0.480	0.430	0.367	0.342
4	1	0.661	0.276	0.051	0.004	0.000	0.000	0.000	0.000
	2	0.799	0.580	0.313	0.118	0.031	0.006	0.000	0.000
	3	0.903	0.847	0.752	0.626	0.484	0.344	0.132	0.036
	4	0.972	0.990	0.996	0.999	1.000	1.000	1.000	1.000
E(S)		3.335	2.693	2.112	1.747	1.515	1.350	1.132	1.036
E(S*)		2.363	1.703	1.116	0.749	0.516	0.350	0.132	0.036
E(SR)		8.855	7.938	6.917	6.114	5.513	5.043	4.397	4.107
E(P)		0.834	0.673	0.528	0.437	0.379	0.337	0.283	0.259
5	1	0.547	0.103	0.005	0.000	0.000	0.000	0.000	0.000
	2	0.702	0.318	0.065	0.006	0.000	0.000	0.000	0.000
	3	0.829	0.626	0.356	0.143	0.040	0.008	0.000	0.000
	4	0.920	0.872	0.786	0.668	0.529	0.386	0.157	0.045
	5	0.978	0.993	0.997	0.999	1.000	1.000	1.000	1.000
E(S)		3.975	2.911	2.208	1.815	1.569	1.394	1.157	1.045
E(S*)		2.998	1.919	1.211	0.816	0.569	0.394	0.157	0.045
E(SR)		13.004	11.066	9.330	8.106	7.234	6.568	5.629	5.179
E(P)		0.795	0.582	0.442	0.363	0.314	0.279	0.231	0.209
10	1	0.053	0.000	0.000	0.000	0.	0.	0.	0.
	2	0.114	0.000	0.000	0.000	0.000	0.	0.	0.
	3	0.215	0.001	0.000	0.000	0.000	0.000	0.	0.
	4	0.354	0.007	0.000	0.000	0.000	0.000	0.	0.
	5	0.515	0.043	0.000	0.000	0.000	0.000	0.000	0.
	6	0.672	0.174	0.010	0.000	0.000	0.000	0.000	0.000
	7	0.802	0.441	0.115	0.013	0.001	0.000	0.000	0.000
	8	0.896	0.737	0.477	0.223	0.072	0.017	0.000	0.000
	9	0.955	0.923	0.861	0.767	0.645	0.504	0.238	0.078
	10	0.989	0.997	0.999	1.000	1.000	1.000	1.000	1.000
E(S)		5.564	3.321	2.462	2.003	1.718	1.520	1.239	1.078
E(S*)		4.576	2.324	1.463	1.004	0.718	0.520	0.239	0.078
E(SR)		40.208	28.535	22.418	18.778	16.386	14.667	12.147	10.705
E(P)		0.556	0.332	0.246	0.200	0.172	0.152	0.124	0.108

Table 12: Performance characteristics of the means rule R_1 under the slippage configuration.

		$P^* = 0.90,\ n = 3$							
k	i	$\delta\sqrt{n}$							
		0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
2	1	0.824	0.717	0.584	0.440	0.305	0.192	0.059	0.013
	2	0.948	0.975	0.989	0.995	0.998	0.999	1.000	1.000
E(S)		1.772	1.692	1.573	1.436	1.303	1.192	1.058	1.013
E(S*)		0.824	0.717	0.584	0.440	0.305	0.192	0.059	0.013
E(SR)		2.720	2.667	2.562	2.431	2.301	2.191	2.058	2.013
E(P)		0.886	0.846	0.786	0.718	0.651	0.596	0.529	0.506
3	1	0.856	0.784	0.681	0.552	0.413	0.283	0.101	0.025
	2	0.856	0.784	0.681	0.552	0.413	0.283	0.101	0.025
	3	0.950	0.977	0.990	0.996	0.999	0.999	1.000	1.000
E(S)		2.662	2.545	2.352	2.100	1.825	1.565	1.201	1.050
E(S*)		1.712	1.568	1.362	1.104	0.826	0.566	0.201	0.050
E(SR)		5.418	5.283	5.013	4.644	4.235	3.847	3.302	3.076
E(P)		0.887	0.848	0.784	0.700	0.608	0.522	0.400	0.350
4	1	0.869	0.813	0.726	0.610	0.475	0.339	0.132	0.036
	2	0.869	0.813	0.726	0.610	0.475	0.339	0.132	0.036
	3	0.869	0.813	0.726	0.610	0.475	0.339	0.132	0.036
	4	0.951	0.978	0.991	0.996	0.999	1.000	1.000	1.000
E(S)		3.557	3.416	3.169	2.826	2.423	2.018	1.395	1.107
E(S*)		2.606	2.438	2.179	1.830	1.425	1.018	0.395	0.107
E(SR)		9.014	8.787	8.320	7.645	6.844	6.034	4.789	4.213
E(P)		0.889	0.854	0.792	0.707	0.606	0.504	0.349	0.277
5	1	0.875	0.829	0.754	0.647	0.517	0.380	0.156	0.045
	2	0.875	0.829	0.754	0.647	0.517	0.380	0.156	0.045
	3	0.875	0.829	0.754	0.647	0.517	0.380	0.156	0.045
	4	0.875	0.829	0.754	0.647	0.517	0.380	0.156	0.045
	5	0.951	0.978	0.991	0.996	0.999	1.000	1.000	1.000
E(S)		4.453	4.295	4.006	3.585	3.067	2.519	1.624	1.178
E(S*)		3.502	3.317	3.015	2.588	2.068	1.520	0.624	0.178
E(SR)		13.510	13.183	12.492	11.453	10.163	8.797	6.561	5.446
E(P)		0.891	0.859	0.801	0.717	0.613	0.504	0.325	0.236
10	1	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	2	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	3	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	4	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	5	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	6	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	7	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	8	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	9	0.888	0.862	0.813	0.734	0.624	0.492	0.236	0.078
	10	0.952	0.979	0.992	0.997	0.999	1.000	1.000	1.000
E(S)		8.943	8.735	8.308	7.602	6.613	5.428	3.121	1.702
E(S*)		7.991	7.756	7.317	6.605	5.614	4.429	2.121	0.702
E(SR)		49.476	48.572	46.498	42.992	38.059	32.140	20.603	13.509
E(P)		0.894	0.874	0.831	0.760	0.661	0.543	0.312	0.170

Table 13: Performance characteristics of the medians rule R_2 under the equally spaced configuration.

		$P^* = 0.90,\ n = 3$							
k	i	$\delta\sqrt{n}$							
		0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
2	1	0.926	0.852	0.734	0.572	0.392	0.232	0.055	0.009
	2	0.985	0.994	0.997	0.999	1.000	1.000	1.000	1.000
E(S)		1.910	1.846	1.731	1.571	1.391	1.232	1.055	1.009
E(S*)		0.926	0.852	0.734	0.572	0.392	0.232	0.055	0.009
E(SR)		2.895	2.839	2.728	2.570	2.391	2.232	2.055	2.009
E(P)		0.955	0.923	0.866	0.785	0.696	0.616	0.527	0.504
3	1	0.935	0.780	0.462	0.155	0.029	0.003	0.000	0.000
	2	0.975	0.951	0.901	0.810	0.673	0.499	0.181	0.039
	3	0.995	0.998	0.999	1.000	1.000	1.000	1.000	1.000
E(S)		2.905	2.729	2.362	1.965	1.701	1.503	1.181	1.039
E(S*)		1.910	1.731	1.363	0.965	0.701	0.503	0.181	0.039
E(SR)		5.870	5.677	5.261	4.775	4.374	4.002	3.362	3.078
E(P)		0.968	0.910	0.787	0.655	0.567	0.501	0.394	0.346
4	1	0.921	0.586	0.114	0.005	0.000	0.000	0.000	0.000
	2	0.966	0.874	0.626	0.279	0.068	0.010	0.000	0.000
	3	0.988	0.976	0.947	0.892	0.795	0.652	0.302	0.080
	4	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000
E(S)		3.872	3.435	2.688	2.175	1.863	1.662	1.302	1.080
E(S*)		2.874	2.436	1.688	1.175	0.863	0.662	0.302	0.080
E(SR)		9.805	9.258	8.208	7.236	6.521	5.975	4.906	4.240
E(P)		0.968	0.859	0.672	0.544	0.466	0.415	0.326	0.270
5	1	0.889	0.300	0.006	0.000	0.000	0.000	0.000	0.000
	2	0.949	0.694	0.188	0.011	0.000	0.000	0.000	0.000
	3	0.979	0.918	0.724	0.383	0.113	0.019	0.000	0.000
	4	0.993	0.985	0.967	0.929	0.858	0.742	0.403	0.125
	5	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
E(S)		4.808	3.896	2.884	2.322	1.971	1.762	1.403	1.125
E(S*)		3.810	2.896	1.884	1.322	0.971	0.762	0.403	0.125
E(SR)		14.687	13.379	11.420	9.885	8.771	8.028	6.611	5.501
E(P)		0.962	0.779	0.577	0.464	0.394	0.352	0.281	0.225
10	1	0.286	0.000	0.000	0.	0.	0.	0.	0.
	2	0.510	0.000	0.000	0.000	0.	0.	0.	0.
	3	0.711	0.001	0.000	0.000	0.000	0.	0.	0.
	4	0.849	0.026	0.000	0.000	0.000	0.000	0.	0.
	5	0.928	0.226	0.000	0.000	0.000	0.000	0.000	0.
	6	0.968	0.621	0.050	0.000	0.000	0.000	0.000	0.000
	7	0.987	0.888	0.472	0.068	0.002	0.000	0.000	0.000
	8	0.995	0.977	0.899	0.673	0.324	0.087	0.002	0.000
	9	0.998	0.996	0.991	0.980	0.955	0.905	0.683	0.333
	10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
E(S)		8.232	4.735	3.413	2.721	2.281	1.992	1.684	1.333
E(S*)		7.232	3.735	2.413	1.721	1.281	0.992	0.684	0.333
E(SR)		51.129	37.959	29.720	24.682	21.198	18.844	16.157	12.999
E(P)		0.823	0.473	0.341	0.272	0.228	0.199	0.168	0.133

Table 14: Performance characteristics of the medians rule R_2 under the slippage configuration.

$P^* = 0.90,\ n = 3$									
k	i	$\delta\sqrt{n}$							
		0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
2	1	0.968	0.939	0.890	0.817	0.717	0.594	0.331	0.136
	2	0.992	0.997	0.999	0.999	1.000	1.000	1.000	1.000
E(S)		1.960	1.935	1.889	1.816	1.716	1.594	1.331	1.136
E(S*)		0.968	0.939	0.890	0.817	0.717	0.594	0.331	0.136
E(SR)		2.953	2.932	2.887	2.816	2.716	2.594	2.331	2.136
E(P)		0.980	0.968	0.944	0.908	0.858	0.797	0.666	0.568
3	1	0.986	0.976	0.955	0.920	0.862	0.779	0.540	0.284
	2	0.986	0.976	0.955	0.920	0.862	0.779	0.540	0.284
	3	0.996	0.998	0.999	1.000	1.000	1.000	1.000	1.000
E(S)		2.968	2.949	2.910	2.839	2.725	2.557	2.080	1.567
E(S*)		1.972	1.951	1.911	1.840	1.725	1.557	1.080	0.567
E(SR)		5.946	5.921	5.864	5.759	5.587	5.336	4.620	3.851
E(P)		0.989	0.983	0.970	0.946	0.908	0.852	0.693	0.522
4	1	0.991	0.985	0.974	0.952	0.914	0.853	0.654	0.390
	2	0.991	0.985	0.974	0.952	0.914	0.853	0.653	0.390
	3	0.991	0.985	0.974	0.952	0.914	0.853	0.653	0.390
	4	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000
E(S)		3.971	3.955	3.921	3.856	3.742	3.560	2.960	2.169
E(S*)		2.973	2.956	2.922	2.856	2.742	2.560	1.960	1.169
E(SR)		9.936	9.908	9.841	9.712	9.484	9.120	7.918	6.338
E(P)		0.993	0.989	0.980	0.964	0.935	0.890	0.740	0.542
5	1	0.993	0.990	0.982	0.967	0.939	0.893	0.723	0.468
	2	0.993	0.990	0.982	0.967	0.939	0.893	0.723	0.468
	3	0.993	0.990	0.982	0.967	0.939	0.893	0.723	0.468
	4	0.993	0.990	0.982	0.967	0.939	0.893	0.723	0.468
	5	0.998	0.999	1.000	1.000	1.000	1.000	1.000	1.000
E(S)		4.971	4.958	4.928	4.868	4.757	4.571	3.892	2.873
E(S*)		3.974	3.959	3.928	3.868	3.757	3.571	2.892	1.873
E(SR)		14.923	14.893	14.819	14.669	14.393	13.927	12.229	9.682
E(P)		0.994	0.992	0.986	0.974	0.951	0.914	0.778	0.575
10	1	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	2	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	3	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	4	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	5	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	6	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	7	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	8	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	9	0.997	0.996	0.994	0.989	0.978	0.959	0.868	0.676
	10	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
E(S)		9.971	9.963	9.942	9.897	9.804	9.627	8.809	7.088
E(S*)		8.972	8.963	8.943	8.897	8.804	8.627	7.809	6.088
E(SR)		54.851	54.812	54.711	54.485	54.021	53.136	49.045	40.440
E(P)		0.997	0.996	0.994	0.990	0.980	0.963	0.881	0.709

Table 15: Comparison of the rule R_1 to R_2: Equally spaced configuration.

$P^* = 0.90,\ n = 3,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.996	1.000	0.996	1.000	1.000	1.000
$E(S)$	2.112	2.688	0.792	1.350	1.662	0.812
$E(S^*)$	1.116	1.688	0.661	0.350	0.662	0.529
$E(SR)$	6.917	8.208	0.843	5.043	5.975	0.844
$E(P)$	0.528	0.672	0.786	0.337	0.415	0.812

$P^* = 0.90,\ n = 5,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.997	1.000	0.997	1.000	1.000	1.000
$E(S)$	2.114	3.371	0.627	1.352	2.000	0.676
$E(S^*)$	1.117	2.371	0.471	0.352	1.000	0.352
$E(SR)$	6.920	9.258	0.747	5.049	6.907	0.731
$E(P)$	0.528	0.843	0.626	0.338	0.500	0.676

$P^* = 0.95,\ n = 3,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.999	1.000	0.999	1.000	1.000	1.000
$E(S)$	2.405	3.276	0.734	1.469	1.951	0.767
$E(S^*)$	1.406	2.276	0.618	0.496	0.951	0.522
$E(SR)$	7.552	9.129	0.827	5.474	6.784	0.807
$E(P)$	0.601	0.819	0.734	0.374	0.488	0.766

$P^* = 0.95,\ n = 5,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.999	1.000	0.999	1.000	1.000	1.000
$E(S)$	2.400	3.836	0.626	1.494	2.396	0.624
$E(S^*)$	1.401	2.836	0.494	0.494	1.396	0.354
$E(SR)$	7.540	9.821	0.768	5.467	7.778	0.703
$E(P)$	0.600	0.959	0.626	0.373	0.599	0.623

Table 16: Comparison of the rule R_1 to R_2: Slippage configuration.

$P^* = 0.90,\ n = 3,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.991	1.000	0.991	1.000	1.000	1.000
$E(S)$	3.169	3.921	0.808	2.018	3.560	0.567
$E(S^*)$	2.179	2.922	0.746	1.018	2.560	0.398
$E(SR)$	8.320	9.841	0.845	6.034	9.120	0.662
$E(P)$	0.792	0.980	0.808	0.504	0.890	0.566

$P^* = 0.90,\ n = 5,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.991	1.000	0.991	1.000	1.000	1.000
$E(S)$	3.167	3.999	0.792	2.025	3.917	0.517
$E(S^*)$	2.176	2.992	0.727	1.025	2.917	0.351
$E(SR)$	8.317	9.983	0.833	6.048	9.834	0.615
$E(P)$	0.792	0.998	0.794	0.506	0.979	0.517

$P^* = 0.95,\ n = 3,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.996	1.000	0.996	1.000	1.000	1.000
$E(S)$	3.496	3.981	0.878	2.435	3.852	0.632
$E(S^*)$	2.500	2.981	0.839	1.435	2.852	0.501
$E(SR)$	8.985	9.962	0.902	6.869	9.703	0.689
$E(P)$	0.874	0.995	0.878	0.609	0.963	0.632

$P^* = 0.95,\ n = 5,\ k = 4$						
	$\delta\sqrt{n} = 1.5$			$\delta\sqrt{n} = 3.0$		
Perf.Char.	R_1	R_2	R_1/R_2	R_1	R_2	R_1/R_2
$P(CS)$	0.997	1.000	0.997	1.000	1.000	1.000
$E(S)$	3.489	3.999	0.827	2.429	3.987	0.609
$E(S^*)$	2.492	2.999	0.831	1.429	2.987	0.478
$E(SR)$	8.971	9.998	0.897	6.858	9.974	0.688
$E(P)$	0.872	1.000	0.872	0.607	0.997	0.609

3 AN ELIMINATION TYPE TWO-STAGE PROCEDURE FOR SELECTING THE POPULATION WITH THE LARGEST MEAN FROM k LOGISTIC POPULATIONS

3.1 Introduction

It is unrealistic to assume that we always have k populations with a common known variance. When the variances are unknown, it is not possible to predetermine the sample size for a single-stage procedure since the standard errors of the sample means are unknown. (See, for example, Dudewicz (1971)). Bechhofer, Dunnett and Sobel (1954) have considered a two-stage non-elimination type procedure in which the observations in the first stage are only used to obtain an estimate of the common unknown variance. Gupta and Kim (1984) considered an elimination type two-stage procedure for the case of common unknown variance and they showed that their procedure performs much better than the non-elimination type procedure of Bechhofer, Dunnett and Sobel (1954).

For selecting the population having the largest mean from normal populations with equal known variance σ^2, Cohen (1959), Alam (1970) and Tamhane and Bechhofer (1977, 1979) have all studied two-stage elimination type procedures, in which they used Gupta's (1956, 1965) subset selection procedure in the first stage to screen out non-contending populations and Bechhofer's (1954) indifference zone approach to all populations retained in the second stage.

Tamhane and Bechhofer (1977) studied in depth a two-stage elimination type procedure ($\mathcal{P}_2''$) for selecting the largest normal mean when the common variance is known. In order to determine a set of constants necessary to implement $\mathcal{P}_2''$, they proposed a criterion of minimizing the maximum over the entire parameter space of the expected total sample size required by $\mathcal{P}_2''$ subject to the procedure's guaranteeing a specified probability of a correct selection. As a consequence, $\mathcal{P}_2''$ based on this unrestricted minimax design criterion possesses the highly desirable property that the expected total sample size required by $\mathcal{P}_2''$ is always less than or equal to the total sample size required by the best competing single-stage procedure of Bechhofer (1954), regardless of the true configuration of the population means. Due to the difficulties of determining the LFC of the population means for $k > 3$, and of evaluating the probability of a correct selection associated with $\mathcal{P}_2''$ when the population means are in that configuration, they adopted a lower bound to the probability of a correct selection of $\mathcal{P}_2''$ and obtained a set of constants which provides a conservative solution to the problem.

In this chapter we consider an elimination type two-stage procedure for selecting the logistic population with the largest population mean when the populations have a common known variance.

We propose a two-stage elimination type procedure $\mathcal{P}_2$ and a non-linear optimization problem by using a minimax criterion to find a set of constants needed to implement $\mathcal{P}_2$. We derive lower bounds on the probability of a correct selection and the infimum over the preference zone of the lower bounds. We determine the supremum of the expected total sample size needed for $\mathcal{P}_2$ over the whole parameter space. We provide tables of constants to implement $\mathcal{P}_2$ and of the efficiency of $\mathcal{P}_2$ relative to the single-stage procedure $\mathcal{P}_1$ considered in the previous chapter for the two special cases of the equally spaced and slippage configurations.

3.2 Preliminaries

Let π_i, $i = 1, \ldots, k$, denote k logistic populations with unknown means μ_i and a common known variance σ^2, and let

$$\Omega = \{\vec{\mu} = (\mu_1, \cdots, \mu_k); -\infty < \mu_i < \infty, i = 1, \ldots, k\}$$

be the parameter space. Denote the ranked values of the μ_i by

$$\mu_{[1]} \leq \cdots \leq \mu_{[k]}$$

and let

$$\delta_{ij} = \mu_{[i]} - \mu_{[j]}.$$

We assume that the experimenter has no prior knowledge concerning the pairing of the π_i with the $\mu_{[j]}$, $i = 1, \ldots, k$, $j = 1, \ldots, k$. Let $\pi_{(j)}$ denote the population associated with $\mu_{[j]}$.

The goal of the experimenter is to select the 'best' population which is defined as the population with the largest mean. This event is referred to as a correct selection (CS). The experimenter restricts consideration to procedures ($\mathcal{P}$) which guarantee the basic probability requirement

$$P_{\vec{\mu}}[CS|\mathcal{P}] \geq P^*, \quad \forall\, \vec{\mu} \in \Omega(\delta), \tag{45}$$

where $\delta > 0$ and $1/k < P^* < 1$ are specified prior to the start of experimentation and

$$\Omega(\delta) = \{\vec{\mu} \in \Omega |\ (\mu_{[k]} - \mu_{[k-1]}) \geq \delta\}$$

which is defined as the preference zone for a correct selection.

Here we propose an elimination type two-stage procedure $\mathcal{P}_2$=$\mathcal{P}_2(n_1, n_2, h)$ which depends on non-negative integers n_1, n_2 and a real constant $h > 0$ which are determined prior to the start of experimentation. The constants (n_1, n_2, h) depend on k, δ and P^* and they are chosen so that $\mathcal{P}_2$ guarantees the basic probability requirement (45) and possess a certain minimax property.

Procedure $\mathcal{P}_2$;

Stage 1: Take n_1 independent observations

$$X_{ij}^{(1)}, \quad j = 1, \ldots, n_1,$$

from each π_i, $i = 1, \ldots, k$, and compute the k sample means

$$\overline{X}_i^{(1)} = \frac{1}{n_1} \sum_{j=1}^{n_1} X_{ij}^{(1)}, \quad i = 1, \ldots, k.$$

Let $\overline{X}_{[k]}^{(1)} = \max_{1 \leq j \leq k} \overline{X}_j^{(1)}$. Determine the subset $\mathbf{I}$ of $\{1, \ldots, k\}$ where

$$\mathbf{I} = \{i | \overline{X}_i^{(1)} \geq \overline{X}_{[k]}^{(1)} - h\sigma/\sqrt{n_1}\},$$

and let π_I denote the associated subset of $\{\pi_1, \ldots, \pi_k\}$.

1. If π_I consists of one population, stop sampling and assert that the population associated with $\overline{X}_{[k]}^{(1)}$ is best.

2. If π_I consists of more than one population, proceed to the second stage.

Stage 2: Take n_2 additional independent observations $X_{ij}^{(2)}$, $j = 1, \ldots, n_2$, from each population in π_I, and compute the cumulative sample means

$$\begin{aligned} \overline{X}_i &= \frac{1}{n_1+n_2}\left(\sum_{j=1}^{n_1} X_{ij}^{(1)} + \sum_{j=1}^{n_2} X_{ij}^{(2)}\right) \\ &= \frac{1}{n_1+n_2}\left(n_1\overline{X}_i^{(1)} + n_2\overline{X}_i^{(2)}\right) \end{aligned}$$

for $i \in \mathbf{I}$, where

$$\overline{X}_i^{(2)} = \frac{1}{n_2}\sum_{j=1}^{n_2} X_{ij}^{(2)}.$$

Assert that the population associated with $\max_{i\in\mathbf{I}} \overline{X}_i$ is the best.

Remark 3.1 *If $h = 0$ the two-stage procedure $\mathcal{P}_2$ reduces to the single-stage procedure $\mathcal{P}_1$ which was considered in Section 2.3 with single-stage sample size $n = n_1$ per population. Also the rule determining **I** in the first stage is of the type of the subset selection procedure considered in Section 2.4.*

There is an infinite number of combinations of (n_1, n_2, h) for given k, δ and P^*, which will exactly guarantee the basic probability requirement given by (45), and different design criteria lead to different choices. We will consider one of these criteria.

Let S' denote the cardinality of the set **I** in stage one and let

$$S = \begin{cases} 0; & \text{if } S' = 1 \\ S'; & \text{if } S' > 1. \end{cases} \tag{46}$$

Then the total sample size required by $\mathcal{P}_2$, TSS say, is

$$TSS = kn_1 + Sn_2.$$

Let $E_{\vec{\mu}}[TSS|\mathcal{P}_2]$ denote the expected total sample size for $\mathcal{P}_2$ under $\vec{\mu}$.

We adopt the following unrestricted minimax criterion to make a choice of (n_1, n_2, h) as well as to have the total sample size TSS small. For given k and specified δ and P^*, choose (n_1, n_2, h) to

$$\begin{array}{ll} \textit{minimize} & \sup_{\vec{\mu}\in\Omega} E_{\vec{\mu}}[TSS|\mathcal{P}_2] \\ \textit{subject to} & \inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2] \geq P^*, \end{array} \tag{47}$$

where (n_1, n_2) are non-negative integers and $h \geq 0$.

For any population whose sample mean has the MLR property, Bhandari and Chaudhuri (1987) proved that the least favorable configuration (LFC) of the two-stage population means problem is a slippage configuration. However, the problem of evaluating the exact probability of a correct selection in the LFC associated with $\mathcal{P}_2$ is complicated and still remains to be solved. Here we will consider lower bounds for $P_{\vec{\mu}}[CS|\mathcal{P}_2]$ and construct conservative two-stage procedures.

3.3 Lower bounds for the probability of a correct selection for $\mathcal{P}_2$

In this section we derive lower bounds for $P_{\vec{\mu}}[CS|\mathcal{P}_2]$. These lower bounds will prove to be particularly useful since we will prove that they achieve their infimum over $\Omega(\delta)$ at $\mu(\delta)$ which has components

$$\mu = \mu_{[1]} = \cdots = \mu_{[k-1]} = \mu_{[k]} - \delta, \ \delta \geq 0.$$

This result will permit us to construct a conservative two-stage procedure which guarantees the basic probability requirement (45).

The next theorem gives one of these lower bounds for $P_{\vec{\mu}}[CS|\mathcal{P}_2]$.

Theorem 3.1 *For any $\vec{\mu} \in \Omega$ we have*

$$\begin{aligned} & P_{\vec{\mu}}[CS|\mathcal{P}_2] \\ & \geq \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} F_{n_1}(x + \delta_{ki}\sqrt{n_1}/\sigma \ + h) dF_{n_1}(x) \\ & \quad + \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} F_{n_1+n_2}(x + \delta_{ki}\sqrt{n_1+n_2}/\sigma \ ) dF_{n_1+n_2}(x) - 1, \end{aligned} \tag{48}$$

where $F_n(x)$ is the cdf of the standardized sample means of size n from $L(\mu, \sigma^2)$.

Proof

For any $\vec{\mu} \in \Omega$ we have

$$
\begin{aligned}
&P_{\vec{\mu}}[CS|\mathcal{P}_2] \\
&= P_{\vec{\mu}}[\overline{X}_{(k)}^{(1)} \geq \overline{X}_{(i)}^{(1)} - h\sigma/\sqrt{n_1}\ , i \neq k,\ \overline{X}_{(k)} \geq \max_{i \in \mathbf{I}} \overline{X}_{(i)}] \\
&\geq P_{\vec{\mu}}[\overline{X}_{(k)}^{(1)} \geq \overline{X}_{(i)}^{(1)} - h\sigma/\sqrt{n_1}\ , \overline{X}_{(k)} \geq \overline{X}_{(i)},\ \forall i \neq k] \\
&\geq P_{\vec{\mu}}[\overline{X}_{(k)}^{(1)} \geq \overline{X}_{i}^{(1)} - h\sigma/\sqrt{n_1}\ ,\ \forall i \neq k] \\
&\quad + P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(i)},\ \forall i \neq k] - 1, \qquad (49)
\end{aligned}
$$

since $P(A \cap B) \geq P(A) + P(B) - 1$ for any two events A and B. Then a straightforward computation leads to the conclusion of this theorem. □

Corollary 3.1 *For all $\vec{\mu} \in \Omega(\delta)$ we have*

$$
\begin{aligned}
&\inf_{\vec{\mu} \in \Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2] \geq \qquad (50) \\
&\quad \int_{-\infty}^{\infty} \{F_{n_1}(x + \delta\sqrt{n_1}/\sigma\ + h)\}^{k-1} dF_{n_1}(x) \\
&\quad + \int_{-\infty}^{\infty} \{F_{n_1+n_2}(x + \delta\sqrt{n_1+n_2}/\sigma\ )\}^{k-1} dF_{n_1+n_2}(x)\ - 1. \qquad (51)
\end{aligned}
$$

Proof

The proof follows immediately on noting that the right hand side of (48) is non-decreasing in each δ_{ki} for $i = 1, \ldots, k-1$. □

Remark 3.2 *Since the right hand side of (51) is strictly increasing in each of n_1, $n_1 + n_2$ and h and tends to one as n_1 or, n_2 and h tend to ∞, we see that the basic probability requirement (45) can be guaranteed if one (or more) of these constants is chosen sufficiently large.*

Remark 3.3 *If we let $h \to \infty$ on the right hand side of (48) we obtain*

$$
\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} F_{n_1+n_2}(x + \delta_{ki}\sqrt{n_1+n_2}/\sigma\ ) dF_{n_1+n_2}(x)
$$

which is an expression for $P_{\vec{\mu}}[CS|\mathcal{P}_1]$ where $\mathcal{P}_1$ uses a common single-stage sample size $n = n_1 + n_2$ per population. Thus $\mathcal{P}_1$ is a special case of $\mathcal{P}_2$ based on a conservative lower bound and hence $E_{\vec{\mu}}[TSS|\mathcal{P}_2] \leq kn$ for all $\vec{\mu} \in \Omega$.

Remark 3.4 *The distribution of the mean of samples from logistic population has the monotone likelihood ratio (MLR) property with respect to the location parameter (Goel (1975)) and hence the distributions of the $\overline{X}_i^{(1)}$ and $\overline{X}_i^{(2)}$ are stochastically increasing (SI) families in μ_i, $i = 1, \ldots, k$.*

Remark 3.5 *The cumulative sample means*

$$\overline{X}_i = \frac{n_1}{n_1 + n_2}\overline{X}_i^{(1)} + \frac{n_2}{n_1 + n_2}\overline{X}_i^{(2)}$$

are strictly increasing in each $\overline{X}_i^{(j)}$, $j = 1, 2,\ i = 1, \ldots, k$.

We can now find another lower bound to the $P_{\vec{\mu}}[CS|\mathcal{P}_2]$ given in the following theorem by noting the facts mentioned in Remark 3.4 and Remark 3.5. This lower bound can be shown to be uniformly superior to the one given in Theorem 3.1. It is also straightforward to determine the LFC of the population means relative to this new lower bound.

Theorem 3.2 *For any $\vec{\mu} \in \Omega$ we have*

$$\begin{aligned}
&\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2] \\
&\geq \int_{-\infty}^{\infty} \{F_{n_1}(x + \delta\sqrt{n_1}/\sigma \ + h)\}^{k-1} dF_{n_1}(x) \\
&\quad \cdot \int_{-\infty}^{\infty} \{F_{n_1+n_2}(x + \delta\sqrt{n_1+n_2}/\sigma \ )\}^{k-1} dF_{n_1+n_2}(x), \qquad (52)
\end{aligned}$$

where $F_n(x)$ is the cdf of the standardized sample mean of size n from $L(\mu, \sigma^2)$.

Proof

Let $F(.|\mu_i)$ and $G(.|\mu_i)$ denote the cdf's of the $\overline{X}_i^{(1)}$ and $\overline{X}_i$ respectively and let $H(.,.|\mu_i)$ denote the joint cdf of the $\overline{X}_i^{(1)}$ and $\overline{X}_i$. Then $F(.|\mu_i)$, $G(.|\mu_i)$ and $H(.,.|\mu_i)$

are non-increasing in μ_i, $i = 1, \ldots, k$, from Remark 3.4 and Remark 3.5. Without loss of generality we may assume that $\mu_1 \leq \cdots \leq \mu_k$. Then for all $\vec{\mu} \in \Omega(\delta)$,

$$\begin{aligned}
&P_{\vec{\mu}}[CS|\mathcal{P}_2] \\
&= P_{\vec{\mu}}[\overline{X}^{(1)}_{(k)} \geq \max_{1\leq j\leq k} \overline{X}^{(1)}_{(j)} - h\sigma/\sqrt{n_1},\ \overline{X}_{(k)} = \max_{j\in \mathbf{I}} \overline{X}_{(j)}] \\
&\geq P_{\vec{\mu}}[\overline{X}^{(1)}_{(k)} \geq \overline{X}^{(1)}_{(j)} - h\sigma/\sqrt{n_1}, \overline{X}_{(k)} \geq \overline{X}_{(j)},\ \forall j = 1, \ldots, k-1\ ] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} H(x + h\sigma/\sqrt{n_1}, y|\mu_i) dH(x, y|\mu_k) \\
&\geq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} H(x + h\sigma/\sqrt{n_1}, y|\mu_k - \delta) dH(x, y|\mu_k) \\
&= E_{\mu_k}[H^{k-1}\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}, \overline{X}_{(k)}|\mu_k - \delta\}],
\end{aligned}$$

where the expectation is with respect to the joint distribution of $\overline{X}^{(1)}_{(k)}$ and $\overline{X}_{(k)}$. Hence

$$\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2] \geq \inf_{\vec{\mu}\in\Omega(\delta)} E_{\mu_k}[H^{k-1}\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}, \overline{X}_{(k)}|\mu_k - \delta\}]$$

and it is enough to show that for all $\vec{\mu} \in \Omega(\delta)$,

$$\begin{aligned}
&E_{\mu_k}[H^{k-1}\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}, \overline{X}_{(k)}|\mu_k - \delta\}] \\
&\geq E_{\mu_k}[F^{k-1}(\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}|\mu_k - \delta)] E_{\mu_k}[G^{k-1}(\overline{X}_{(k)}|\mu_k - \delta)].
\end{aligned}$$

By Remark 3.5, for all a, b and μ,

$$\begin{aligned}
&P_\mu\{\overline{X}^{(1)}_{(j)} \leq a, \overline{X}_{(j)} \leq b\} \\
&= P_\mu\{\overline{X}^{(1)}_{(j)} \leq a, \overline{X}^{(1)}_{(j)} \leq \tfrac{n_1+n_2}{n_1}(b - \tfrac{n_2}{n_1+n_2}\overline{X}^{(2)}_{(j)})\} \\
&= E_\mu[P_\mu\{\overline{X}^{(1)}_{(j)} \leq a, \overline{X}^{(1)}_{(j)} \leq \tfrac{n_1+n_2}{n_1}(b - \tfrac{n_2}{n_1+n_2}\overline{X}^{(2)}_{(j)})|\overline{X}^{(2)}_{(j)}\}] \\
&\geq E_\mu[P_\mu\{\overline{X}^{(1)}_{(j)} \leq a|\overline{X}^{(2)}_{(j)}\} \\
&\quad \cdot P_\mu\{\overline{X}^{(1)}_{(j)} \leq \tfrac{n_1+n_2}{n_1}(b - \tfrac{n_2}{n_1+n_2}\overline{X}^{(2)}_{(j)})|\overline{X}^{(2)}_{(j)}\}] \\
&= P_\mu\{\overline{X}^{(1)}_{(j)} \leq a\} P_\mu\{\overline{X}_{(j)} \leq b\}
\end{aligned}$$

and hence

$$E_{\mu_k}[H^{k-1}\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}, \overline{X}_{(k)}|\mu_k - \delta\}]$$

$$\geq E_{\mu_k}[F^{k-1}\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}|\mu_k - \delta\}$$
$$\cdot G^{k-1}\{\overline{X}_{(k)}|\mu_k - \delta\}]$$
$$\geq E_{\mu_k}[F^{k-1}\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}|\mu_k - \delta\}]$$
$$\cdot E_{\mu_k}[G^{k-1}\{\overline{X}_{(k)}|\mu_k - \delta\}]$$

by the Chebyshev inequality (Hardy, Littlewood and Pólya (1934)), since

$$F\{\overline{X}^{(1)}_{(k)} + h\sigma/\sqrt{n_1}|\mu_k - \delta\}$$

and

$$G\{\overline{X}_{(k)}|\mu_k - \delta\}$$

are non-decreasing in $\overline{X}^{(1)}_k$. □

Remark 3.6 *If we let*

$$a = \int_{-\infty}^{\infty} \{F_{n_1}(x + \delta\sqrt{n_1}/\sigma \ + h)\}^{k-1} dF_{n_1}(x)$$

and

$$b = \int_{-\infty}^{\infty} \{F_{n_1+n_2}(x + \delta\sqrt{n_1+n_2}/\sigma \ )\}^{k-1} dF_{n_1+n_2}(x) \ ,$$

then (51) states that

$$\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2] \geq a + b - 1$$

and (52) states that

$$\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2] \geq ab.$$

By noting that $a + b - 1 < ab$ for all $a, b \in (0, 1)$, the lower bound (52) is uniformly superior to the lower bound (51), and hence we will use the lower bound (52) henceforth.

3.4 Expected total sample size for $\mathcal{P}_2$

In order to solve the optimization problem (47) we first find an analytical expression for the $E_{\vec{\mu}}[TSS|\mathcal{P}_2]$ and then determine the $\sup_{\vec{\mu}\in\Omega} E_{\vec{\mu}}[TSS|\mathcal{P}_2]$ and the sets of μ_i-values at which this supremum occurs.

Theorem 3.3 *For any $\vec{\mu} \in \Omega$ we have*

$$\begin{aligned} E_{\vec{\mu}}[TSS|\mathcal{P}_2] &= kn_1 + n_2 \sum_{i=1}^{k} \int_{-\infty}^{\infty} \{\prod_{\substack{j=1 \\ j\neq i}}^{k} F_{n_1}(x + \delta_{ij}\sqrt{n_1}/\sigma \ + h) \\ &\quad - \prod_{\substack{j=1 \\ j\neq i}}^{k} F_{n_1}(x + \delta_{ij}\sqrt{n_1}/\sigma \ - h)\} dF_{n_1}(x), \end{aligned} \tag{53}$$

where $F_n(x)$ is the cdf of the standardized sample means of size n from $L(\mu, \sigma^2)$.

Proof

For any $\vec{\mu} \in \Omega$ we have

$$E_{\vec{\mu}}[TSS|\mathcal{P}_2] = kn_1 + n_2 E_{\vec{\mu}}[S|\mathcal{P}_2],$$

where S is defined as in (46). Now

$$\begin{aligned} E_{\vec{\mu}}[S|\mathcal{P}_2] &= E_{\vec{\mu}}[S'|\mathcal{P}_2] - P_{\vec{\mu}}[S' = 1|\mathcal{P}_2] \\ &= \sum_{i=1}^{k} P_{\vec{\mu}}[\overline{X}_{(i)}^{(1)} \geq \overline{X}_{(j)}^{(1)} - h\sigma/\sqrt{n_1}\ ,\ \forall j \neq i] \\ &\quad - \sum_{i=1}^{k} P_{\vec{\mu}}[\overline{X}_{(i)}^{(1)} \geq \overline{X}_{(j)}^{(1)} + h\sigma/\sqrt{n_1}\ ,\ \forall j \neq i] \end{aligned} \tag{54}$$

and hence Theorem 3.3 follows immediately. □

The following theorem summarizes the result concerning the supremum of the $E_{\vec{\mu}}[TSS|\mathcal{P}_2]$ for $\vec{\mu} \in \Omega$.

Theorem 3.4 *For any $\vec{\mu} \in \Omega$, fixed k and (n_1, n_2, h) we have*

$$\begin{aligned} &\sup_{\vec{\mu}\in\Omega} E_{\vec{\mu}}[TSS|\mathcal{P}_2] \\ &= kn_1 + n_2 \int_{-\infty}^{\infty} [\{F_{n_1}(x + h)\}^{k-1} - \{F_{n_1}(x - h)\}^{k-1}] dF_{n_1}(x) \end{aligned} \tag{55}$$

which occurs when $\mu_{[1]} = \cdots = \mu_{[k]}$, where $F_n(x)$ is the cdf of the standardized sample means of size n from $L(\mu, \sigma^2)$.

Proof

Noting Remark 3.4 and Remark 3.5 we can use the results of Gupta (1965) which show that $E_{\vec{\mu}}[S'|\mathcal{P}_2]$ achieves its supremum for $\vec{\mu} \in \Omega$ when $\mu_{[1]} = \cdots = \mu_{[k]}$. By the similar argument $P_{\vec{\mu}}[S' = 1|\mathcal{P}_2]$ achieves its infimum when $\mu_{[1]} = \cdots = \mu_{[k]}$. Hence the result follows immediately from Theorem 3.3. □

3.5 Optimization problem yielding conservative solutions

In this section we consider the optimization problem (47) which one must solve in order to determine the constants (n_1, n_2, h) which are necessary to implement $\mathcal{P}_2$. As we noted earlier, the problem of evaluating the exact probability of a correct selection in the LFC associated with $\mathcal{P}_2$ is very complicated. Thus we replace the exact $\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2]$ by the conservative lower bound given by the right hand side of (52), and consider the following optimization problem.

For the given k, δ and P^* choose the constants (n_1, n_2, h) to

$$\begin{aligned} &\textit{minimize} \quad && kn_1 + n_2 \int_{-\infty}^{\infty} [\{F_{n_1}(x+h)\}^{k-1} - \{F_{n_1}(x-h)\}^{k-1}] dF_{n_1}(x) \\ &\textit{subject to} \quad && \int_{-\infty}^{\infty} \{F_{n_1}(x + \delta\sqrt{n_1}/\sigma \ + h)\}^{k-1} dF_{n_1}(x) \\ & && \cdot \int_{-\infty}^{\infty} \{F_{n_1+n_2}(x + \delta\sqrt{n_1+n_2}/\sigma \ )\}^{k-1} dF_{n_1+n_2}(x) \geq P^*, \end{aligned} \tag{56}$$

where n_1 and n_2 are non-negative integers and $h \geq 0$.

Let us denote by $(\hat{n}_1, \hat{n}_2, \hat{h})$ the solution to the optimization problem (56). Then we can use the approximate design constants

$$n_1 = [\hat{n}_1 + 1],\ n_2 = [\hat{n}_2 + 1],\ h = \hat{h},$$

where $[z]$ denotes the greatest integer which is less than z, to implement $\mathcal{P}_2$.

Table 17, Table 18, Table 19 and Table 20 contain the constants $(\hat{n}_1, \hat{n}_2, \hat{h})$ necessary to approximate (n_1, n_2, h) and the values of the expected total sample size (ETSS) for $k = 2, 3, 4, 5, 10, 15$, $P^* = 0.75, 0.90, 0.95, 0.99$ and $\delta/\sigma = 0.1, 0.5, 1.0, 2.0, 4.0$. All computations were carried out in double-precision arithmetic on a Vax-11/780. The SUMT (Sequential Unconstrained Minimization Techniques: Fiacco and McCormick (1968)) algorithm is used to solve the non-linear optimization problem. A source program in Fortran for the SUMT algorithm is given by Kuester and Mize (1973).

3.6 The performance of the two-stage procedure relative to the single-stage procedure

As a measure of efficiency of the two-stage procedure $\mathcal{P}_2$ relative to that of the single-stage procedure $\mathcal{P}_1$ when both guarantee the same basic probability requirement (45), we consider the ratio termed relative efficiency (RE) $E_{\vec{\mu}}[TSS|\mathcal{P}_2]/k\hat{n}_s$ where $\hat{n}_s$ is the estimate of the minimum sample size n_s needed in the single-stage procedure $\mathcal{P}_1$. Clearly RE depends on $\vec{\mu}$, δ and P^*. Values of the RE less than unity favor $\mathcal{P}_2$ over $\mathcal{P}_1$.

Now the RE is given by

$$\begin{aligned} RE &= \frac{1}{k\hat{n}_s}[k\hat{n}_1 + \hat{n}_2 \sum_{i=1}^{k} \int_{-\infty}^{\infty} \{\prod_{\substack{j=1 \\ j\neq i}}^{k} F_{\hat{n}_1}(t + \delta_{ij}\sqrt{\hat{n}_1}/\sigma + \hat{h}) \\ &\quad - \prod_{\substack{j=1 \\ j\neq i}}^{k} F_{\hat{n}_1}(t + \delta_{ij}\sqrt{\hat{n}_1}/\sigma - \hat{h})\} dF_{\hat{n}_1}(t)]. \end{aligned} \tag{57}$$

where $\hat{n}_s$ is the solution of

$$\int_{-\infty}^{\infty} \{Fn_s(t + (\sqrt{n_s}/\sigma)\delta)\}^{k-1} dFn_s(t) = P^*. \tag{58}$$

We consider the relative efficiency for two special cases, namely, the equally spaced and the slippage configurations. First, for the equally spaced configuration, we assume that the unknown means of the k populations are $\mu,\ \mu+\delta, \ldots,\ \mu+(k-1)\delta$ which have ranks $1,\ 2, \ldots, k$, respectively. Let RE_{eq} denote the relative efficiency with respect to the above configuration. Then, since $\delta_{ij} = \mu_{[i]} - \mu_{[j]} = (i-j)\delta$,

$$\begin{aligned} RE_{eq} &= \frac{1}{k\hat{n}_s}[k\hat{n}_1 + \hat{n}_2 \sum_{i=1}^{k} \int_{-\infty}^{\infty} \{\prod_{\substack{j=1 \\ j\neq i}}^{k} F_{\hat{n}_1}(t + \sqrt{\hat{n}_1}(i-j)\delta/\sigma + \hat{h}) \\ &\quad - \prod_{\substack{j=1 \\ j\neq i}}^{k} F_{\hat{n}_1}(t + \sqrt{\hat{n}_1}(i-j)\delta/\sigma - \hat{h})\} dF_{\hat{n}_1}(t)]. \end{aligned} \tag{59}$$

Next, for the slippage configuration, we assume that the unknown means of the k populations are $\mu_{[j]} = \mu,\ j = 1, \ldots, k-1$, and $\mu_{[k]} = \mu + \delta,\ \delta \geq 0$. Then the relative efficiency with respect to the above configuration, RE_{sp}, is given by

$$RE_{sp} = \frac{1}{k\hat{n}_s}[k\hat{n}_1 + \hat{n}_2\{(k-1)\int_{-\infty}^{\infty}(F_{\hat{n}_1}(t+\hat{h}) - F_{\hat{n}_1}(t-\hat{h}))^{k-2} \cdot(F_{\hat{n}_1}(t-\sqrt{\hat{n}_1}\delta/\sigma+\hat{h}) - F_{\hat{n}_1}(t-\sqrt{\hat{n}_1}\delta/\sigma-\hat{h}))dF_{\hat{n}_1}(t) + \int_{-\infty}^{\infty}(F_{\hat{n}_1}(t+\sqrt{\hat{n}_1}\delta/\sigma+\hat{h}) - F_{\hat{n}_1}(t+\sqrt{\hat{n}_1}\delta/\sigma-\hat{h}))^{k-1}dF_{\hat{n}_1}(t)\}]. \quad (60)$$

Table 21 and Table 22 give the values of the RE_{eq} and RE_{sp} for given values of $P^* = 0.75, 0.90, 0.95, 0.99$, $k = 2, 3, 4, 5, 10, 15$ and $\delta/\sigma = 0.1, 0.5, 1.0, 2.0, 4.0$.

For any values of P^*, k and δ, $RE_{eq} \leq 1$ and $RE_{sp} \leq 1$ and hence the two-stage procedure is more efficient than the single-stage procedure in terms of the expected total sample sizes. Furthermore, the effectiveness of $\mathcal{P}_2$ appears to be increasing in k since the values of RE_{eq} and RE_{sp} are decreasing in k.

Table 17: Constants to implement the two-stage procedure $\mathcal{P}_2$ for selecting the largest logistic population: $P^* = 0.75$.

$P^* = 0.75$					
k	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
2	0.10	0.4548e+02	0.4539e+02	0.5530e+01	0.181720e+03
	0.50	0.2620e+01	0.8929e+00	0.7323e+01	0.702648e+01
	1.00	0.3121e+00	0.4983e+00	0.7097e+01	0.162056e+01
	2.00	0.9556e-01	0.1070e+00	0.6698e+01	0.405153e+00
	4.00	0.2834e-01	0.2232e-01	0.6127e+01	0.101300e+00
3	0.10	0.1018e+03	0.1044e+03	0.3688e+01	0.615938e+03
	0.50	0.4778e+01	0.3340e+01	0.4516e+01	0.243344e+02
	1.00	0.8971e+00	0.1050e+01	0.7309e+01	0.584085e+01
	2.00	0.1876e+00	0.2774e+00	0.6606e+01	0.139489e+01
	4.00	0.8039e-01	0.3586e-01	0.6112e+01	0.348738e+00
4	0.10	0.1392e+03	0.1515e+03	0.2751e+01	0.112241e+04
	0.50	0.5900e+01	0.5554e+01	0.2947e+01	0.447121e+02
	1.00	0.1711e+01	0.1037e+01	0.4159e+01	0.109639e+02
	2.00	0.3255e+00	0.3270e+00	0.5857e+01	0.260899e+01
	4.00	0.8061e-01	0.8252e-01	0.5737e+01	0.652252e+00
5	0.10	0.1631e+03	0.2013e+03	0.2278e+01	0.166485e+04
	0.50	0.6766e+01	0.7657e+01	0.2341e+01	0.666234e+02
	1.00	0.1826e+01	0.1630e+01	0.2746e+01	0.165961e+02
	2.00	0.3864e+00	0.4182e+00	0.4050e+01	0.399414e+01
	4.00	0.9599e-01	0.1051e+00	0.4087e+01	0.998538e+00
10	0.10	0.2357e+03	0.4304e+03	0.1494e+01	0.451824e+04
	0.50	0.9587e+01	0.1738e+02	0.1468e+01	0.181456e+03
	1.00	0.2504e+01	0.4455e+01	0.1398e+01	0.458878e+02
	2.00	0.6367e+00	0.1178e+01	0.1361e+01	0.117515e+02
	4.00	0.1582e+00	0.2758e+00	0.1470e+01	0.295260e+01
15	0.10	0.2714e+03	0.5855e+03	0.1369e+01	0.744887e+04
	0.50	0.1100e+02	0.2372e+02	0.1352e+01	0.299197e+03
	1.00	0.2858e+01	0.6119e+01	0.1308e+01	0.757369e+02
	2.00	0.7466e+00	0.1668e+01	0.1255e+01	0.195906e+02
	4.00	0.1906e+00	0.4032e+00	0.1313e+01	0.501004e+01

Table 18: Constants to implement the two-stage procedure $\mathcal{P}_2$ for selecting the largest logistic population: $P^* = 0.90$.

$P^* = 0.90$					
k	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
2	0.10	0.1668e+03	0.1728e+03	0.2446e+01	0.650194e+03
	0.50	0.7013e+01	0.6404e+01	0.2591e+01	0.259726e+02
	1.00	0.1932e+01	0.1311e+01	0.3369e+01	0.643201e+01
	2.00	0.4011e+00	0.3724e+00	0.5331e+01	0.154620e+01
	4.00	0.1044e+00	0.8907e-01	0.5026e+01	0.386564e+00
3	0.10	0.2745e+03	0.2513e+03	0.2017e+01	0.146152e+04
	0.50	0.1126e+02	0.9634e+01	0.2071e+01	0.585665e+02
	1.00	0.2971e+01	0.2135e+01	0.2332e+01	0.146860e+02
	2.00	0.6894e+00	0.5189e+00	0.5004e+01	0.362197e+01
	4.00	0.1693e+00	0.1310e+00	0.4955e+01	0.900049e+00
4	0.10	0.3298e+03	0.3318e+03	0.1713e+01	0.229940e+04
	0.50	0.1340e+02	0.1300e+02	0.1728e+01	0.922982e+02
	1.00	0.3489e+01	0.3048e+01	0.1796e+01	0.232917e+02
	2.00	0.8374e+00	0.7008e+00	0.2643e+01	0.592662e+01
	4.00	0.2090e+00	0.1704e+00	0.2831e+01	0.147462e+01
5	0.10	0.3664e+03	0.4034e+03	0.1556e+01	0.315013e+04
	0.50	0.1488e+02	0.1595e+02	0.1553e+01	0.126542e+03
	1.00	0.3863e+01	0.3858e+01	0.1559e+01	0.320185e+02
	2.00	0.9610e+00	0.9217e+00	0.1867e+01	0.829954e+01
	4.00	0.2403e+00	0.2184e+00	0.2071e+01	0.208230e+01
10	0.10	0.4549e+03	0.6465e+03	0.1367e+01	0.750100e+04
	0.50	0.1844e+02	0.2588e+02	0.1357e+01	0.301614e+03
	1.00	0.4784e+01	0.6497e+01	0.1328e+01	0.765662e+02
	2.00	0.1328e+01	0.1644e+01	0.1257e+01	0.201395e+02
	4.00	0.3335e+00	0.4262e+00	0.1362e+01	0.528481e+01
15	0.10	0.4934e+03	0.7911e+03	0.1366e+01	0.119540e+05
	0.50	0.1999e+02	0.3177e+02	0.1358e+01	0.480822e+03
	1.00	0.5180e+01	0.8022e+01	0.1335e+01	0.122187e+03
	2.00	0.1433e+01	0.2074e+01	0.1280e+01	0.322460e+02
	4.00	0.3751e+00	0.5593e+00	0.1328e+01	0.865274e+01

Table 19: Constants to implement the two-stage procedure $\mathcal{P}_2$ for selecting the largest logistic population: $P^* = 0.95$.

$P^* = 0.95$					
k	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
2	0.10	0.3008e+03	0.2827e+03	0.1781e+01	0.104953e+04
	0.50	0.1227e+02	0.1098e+02	0.1810e+01	0.421247e+02
	1.00	0.3215e+01	0.2504e+01	0.1958e+01	0.106222e+02
	2.00	0.7631e+00	0.5883e+00	0.3556e+01	0.268233e+01
	4.00	0.1899e+00	0.1457e+00	0.3785e+01	0.667647e+00
3	0.10	0.4362e+03	0.3657e+03	0.1574e+01	0.211419e+04
	0.50	0.1768e+02	0.1436e+02	0.1589e+01	0.849214e+02
	1.00	0.4579e+01	0.3388e+01	0.1654e+01	0.214801e+02
	2.00	0.1223e+01	0.6952e+00	0.2269e+01	0.553339e+01
	4.00	0.2858e+00	0.1853e+00	0.3237e+01	0.139794e+01
4	0.10	0.4991e+03	0.4519e+03	0.1452e+01	0.318364e+04
	0.50	0.2023e+02	0.1787e+02	0.1453e+01	0.127954e+03
	1.00	0.5232e+01	0.4325e+01	0.1464e+01	0.324315e+02
	2.00	0.1420e+01	0.9417e+00	0.1675e+01	0.846044e+01
	4.00	0.3393e+00	0.2423e+00	0.2163e+01	0.218343e+01
5	0.10	0.5381e+03	0.5259e+03	0.1392e+01	0.426098e+04
	0.50	0.2182e+02	0.2086e+02	0.1388e+01	0.171314e+03
	1.00	0.5649e+01	0.5112e+01	0.1379e+01	0.434710e+02
	2.00	0.1546e+01	0.1182e+01	0.1430e+01	0.114045e+02
	4.00	0.3809e+00	0.3002e+00	0.1751e+01	0.299628e+01
10	0.10	0.6279e+03	0.7682e+03	0.1349e+01	0.973702e+04
	0.50	0.2544e+02	0.3070e+02	0.1342e+01	0.391770e+03
	1.00	0.6592e+01	0.7667e+01	0.1321e+01	0.996400e+02
	2.00	0.1827e+01	0.1923e+01	0.1269e+01	0.263641e+02
	4.00	0.4897e+00	0.5216e+00	0.1344e+01	0.724983e+01
15	0.10	0.6674e+03	0.9126e+03	0.1377e+01	0.153152e+05
	0.50	0.2703e+02	0.3659e+02	0.1370e+01	0.616396e+03
	1.00	0.7002e+01	0.9178e+01	0.1354e+01	0.156917e+03
	2.00	0.1942e+01	0.2339e+01	0.1310e+01	0.416523e+02
	4.00	0.5293e+00	0.6784e+00	0.1300e+01	0.115109e+02

Table 20: Constants to implement the two-stage procedure $\mathcal{P}_2$ for selecting the largest logistic population: $P^* = 0.99$.

$P^* = 0.99$					
k	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
2	0.10	0.6892e+03	0.5071e+03	0.1295e+01	0.202774e+04
	0.50	0.2784e+02	0.2014e+02	0.1300e+01	0.815644e+02
	1.00	0.7189e+01	0.4895e+01	0.1309e+01	0.207286e+02
	2.00	0.1966e+01	0.1107e+01	0.1414e+01	0.546768e+01
	4.00	0.5013e+00	0.2771e+00	0.2143e+01	0.148773e+01
3	0.10	0.8588e+03	0.5804e+03	0.1248e+01	0.366047e+04
	0.50	0.3475e+02	0.2299e+02	0.1249e+01	0.147249e+03
	1.00	0.8954e+01	0.5614e+01	0.1254e+01	0.374304e+02
	2.00	0.2442e+01	0.1300e+01	0.1314e+01	0.988866e+01
	4.00	0.6518e+00	0.3209e+00	0.2052e+01	0.278376e+01
4	0.10	0.9268e+03	0.6663e+03	0.1254e+01	0.526885e+04
	0.50	0.3750e+02	0.2646e+02	0.1253e+01	0.211992e+03
	1.00	0.9668e+01	0.6501e+01	0.1250e+01	0.539214e+02
	2.00	0.2647e+01	0.1542e+01	0.1266e+01	0.142783e+02
	4.00	0.7350e+00	0.4650e+00	0.1230e+01	0.404965e+01
5	0.10	0.9639e+03	0.7432e+03	0.1271e+01	0.687580e+04
	0.50	0.3903e+02	0.2955e+02	0.1266e+01	0.276700e+03
	1.00	0.1008e+02	0.7283e+01	0.1258e+01	0.704195e+02
	2.00	0.2770e+01	0.1751e+01	0.1252e+01	0.186821e+02
	4.00	0.7668e+00	0.4714e+00	0.1451e+01	0.532902e+01
10	0.10	0.1049e+04	0.9971e+03	0.1343e+01	0.149575e+05
	0.50	0.4246e+02	0.3976e+02	0.1340e+01	0.602284e+03
	1.00	0.1099e+02	0.9879e+01	0.1327e+01	0.153541e+03
	2.00	0.3045e+01	0.2443e+01	0.1295e+01	0.409579e+02
	4.00	0.8789e+00	0.6787e+00	0.1394e+01	0.119728e+02
15	0.10	0.1088e+04	0.1147e+04	0.1400e+01	0.231194e+05
	0.50	0.4405e+02	0.4583e+02	0.1396e+01	0.931205e+03
	1.00	0.1140e+02	0.1143e+02	0.1384e+01	0.237593e+03
	2.00	0.3166e+01	0.2855e+01	0.1356e+01	0.635515e+02
	4.00	0.9277e+00	0.8022e+00	0.1455e+01	0.187987e+02

Table 21: Relative efficiency of the two-stage procedure $\mathcal{P}_2$: Equally spaced configuration.

Equally Spaced Configuration						
P^*	k	δ/σ				
		0.1	0.5	1.0	2.0	4.0
0.750	2	0.999	1.000	1.000	1.000	1.000
	3	0.973	0.991	1.000	0.999	0.999
	4	0.836	0.857	0.941	0.993	0.991
	5	0.720	0.732	0.776	0.858	0.861
	10	0.551	0.556	0.572	0.581	0.582
	15	0.503	0.507	0.520	0.530	0.536
0.900	2	0.922	0.935	0.975	0.998	0.998
	3	0.782	0.792	0.824	0.981	0.981
	4	0.689	0.695	0.713	0.762	0.776
	5	0.642	0.648	0.663	0.677	0.687
	10	0.554	0.559	0.573	0.612	0.603
	15	0.518	0.523	0.535	0.570	0.569
0.950	2	0.820	0.826	0.847	0.965	0.974
	3	0.716	0.721	0.736	0.795	0.852
	4	0.664	0.669	0.683	0.721	0.718
	5	0.634	0.639	0.653	0.690	0.678
	10	0.564	0.569	0.582	0.620	0.625
	15	0.533	0.537	0.550	0.586	0.585
0.990	2	0.716	0.719	0.729	0.758	0.782
	3	0.690	0.695	0.706	0.737	0.741
	4	0.666	0.670	0.682	0.715	0.713
	5	0.646	0.650	0.663	0.698	0.687
	10	0.591	0.596	0.609	0.646	0.668
	15	0.565	0.570	0.582	0.619	0.645

Table 22: Relative efficiency of the two-stage procedure $\mathcal{P}_2$: Slippage configuration.

Slippage Configuration						
P^*	k	δ/σ				
		0.1	0.5	1.0	2.0	4.0
0.750	2	0.999	1.000	1.000	1.000	1.000
	3	0.982	0.995	1.000	0.999	0.999
	4	0.907	0.930	0.988	0.999	0.998
	5	0.811	0.829	0.898	0.980	0.981
	10	0.516	0.520	0.531	0.540	0.557
	15	0.457	0.461	0.474	0.485	0.495
0.900	2	0.922	0.935	0.975	0.998	0.998
	3	0.796	0.809	0.852	0.994	0.993
	4	0.698	0.706	0.730	0.854	0.877
	5	0.636	0.642	0.658	0.711	0.745
	10	0.527	0.532	0.546	0.590	0.583
	15	0.494	0.499	0.513	0.551	0.551
0.950	2	0.820	0.826	0.847	0.965	0.974
	3	0.709	0.715	0.734	0.818	0.908
	4	0.651	0.656	0.671	0.722	0.754
	5	0.616	0.621	0.636	0.678	0.685
	10	0.545	0.550	0.564	0.606	0.612
	15	0.517	0.522	0.535	0.574	0.575
0.990	2	0.716	0.719	0.729	0.758	0.782
	3	0.678	0.683	0.695	0.729	0.741
	4	0.654	0.659	0.671	0.707	0.706
	5	0.634	0.640	0.653	0.690	0.682
	10	0.583	0.588	0.602	0.641	0.664
	15	0.559	0.564	0.577	0.615	0.643

4 A SINGLE-STAGE RESTRICTED SUBSET SELECTION PROCEDURE FOR SELECTING THE POPULATION WITH THE LARGEST MEAN FROM k LOGISTIC POPULATIONS

4.1 Introduction

In the subset selection formulation, if the data make the choice of the best population difficult (we would expect this to happen if the μ_i are all very close to one another), we are likely to select all the populations. In this case it is meaningful to put on an additional restriction that the size of the selected subset will not exceed m $(1 < m < k)$.

When we use an elimination type two-stage selection procedure to select the best population and we have only limited resources to use for the secondary exploration, we also need more flexible procedures which allow us to specify an upper bound m on the number of populations included in the selected subset. Any selection problem with such a restriction on the size of the subset is naturally called a restricted subset selection problem.

Gupta and Santner (1973) studied the restricted subset selection procedure for the normal means problem in terms of the sample means. They provided the tables of the required sample sizes and of the expected number of selected populations. Santner (1975) defined a general restricted subset selection procedure in terms of a set of consistent estimators for the parameters whose distributions form a stochastically increasing family for any given sample size. He proved that the infimum of the probability of a correct selection occurred at a point in the preference zone for which the parameters were as close together as possible. He also studied some properties of the rule and conditions which guaranteed that the supremum of the expected number of populations selected over the whole parameter space occurred at some point where the k populations were all the same.

In this chapter we consider a restricted subset selection procedure R_3, based on the sample means, for selecting the population with the largest mean from k logistic populations when the common variance is known.

Expressions for the probability of a correct selection for any configuration of the logistic means and for the infimum of the probability of a correct selection over the preference zone are derived and some properties of this procedure such as monotonicity and consistency are studied.

The restricted subset selection procedures are consistent with respect to the preference zone. However the infimum of the probability of a correct selection over the preference zone can not become arbitrarily close to the probability level P^* as the constant h, which defines the procedure, becomes infinitely large for the given values of k, m, δ and n. This is unlike the 'usual' subset selection procedures. A table of the bounds of the infimum of the probability of a correct selection over the preference zone is provided for given values of k, m, δ and n.

A table of the required sample sizes for the restricted subset selection procedure, the sample sizes for the corresponding fixed subset size procedure of Desu and Sobel (1968) and the ratio of the above two sample sizes is given for selected values of P^*, k, m and δ. The expected number of the selected populations for the two special configurations, namely the equally spaced and the slippage configurations, are considered.

Instead of designing the rule by choosing the required sample sizes for arbitrarily given values of h, we can make choice of the rule by controlling the supremum of the expected size of the populations selected over the whole parameter space as well as the probability level P^* simultaneously. Using this new design criterion a table of the design constants (n, h) for the restricted subset selection rule R_3 is provided.

4.2 Formulation of the problem

Let π_i, $i = 1, \ldots, k$, be k logistic populations with unknown means μ_i and a common known variance σ^2, which are denoted by $L(\mu_i, \sigma^2)$. Also let

$$\mu_{[1]} \leq \cdots \leq \mu_{[k]}$$

be the ordered means and $\pi_{(i)}$ the population with mean $\mu_{[i]}$, the best population being $\pi_{(k)}$. We assume that there is no a priori knowledge concerning the pairing of $\{\pi_{(i)}\}$ and $\{\pi_i\}$. Let $\delta > 0$ and

$$\Omega = \{\vec{\mu} = (\mu_1, \cdots, \mu_k); -\infty < \mu_i < \infty, i = 1, \ldots, k\}$$

$$\Omega(\delta) = \{\vec{\mu} \in \Omega |\ (\mu_{[k]} - \mu_{[k-1]}) \geq \delta\}$$

$$\Omega^0(\delta) = \{\vec{\mu} \in \Omega(\delta) |\ \mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta\}.$$

Each π_i yields *iid* observations X_{ij}, $j = 1, \ldots, n$, $i = 1, \ldots, k$, which are also independent between populations. We propose the following rule R_3 based on the means of samples of size n from the k populations. As usual, let $\overline{X}_i$ be the sample mean from π_i, $i = 1, \ldots, k$, and let

$$\overline{X}_{[1]} \leq \cdots \leq \overline{X}_{[k]}$$

denote the ordered sample means.

<u>*Rule R_3*</u> : *Select π_i iff*

$$\overline{X}_i \geq \max\{\overline{X}_{[k-m+1]}, \overline{X}_{[k]} - h_3\sigma/\sqrt{n}\},\ h_3 > 0. \tag{61}$$

<u>*Goal of the experimenter*</u> : Given P^*, δ and the rule R_3 which selects a subset of the populations not exceeding m in size, find the common sample size n necessary to achieve

$$P_{\vec{\mu}}[CS|R_3] \geq P^*\ \forall \vec{\mu} \in \Omega(\delta). \tag{62}$$

The event $[CS|R_3]$ occurs if and only if the selected subset contains $\pi_{(k)}$.

Remark 4.1 *Even though the emphasis in this chapter is on the case, $1 < m < k$, where the strict inequality $\delta > 0$ insures that the indifference zone does not vanish, it should be noted that the general theory formally reduces to give the results of Section 2.3 and Section 2.4 for the choices of $m = 1$ and $m = k$ respectively by allowing the weaker condition $\delta \geq 0$.*

Remark 4.2 *If $h_3 \longrightarrow \infty$, R_3 is the fixed size subset rule which is considered in Desu and Sobel (1968).*

4.3 Probability of a correct selection

We introduce the following notation. For every $l = 1, \ldots, k$ and for every $i = k - m, \ldots, k-1$, let

$$\{S_j^i(l), j = 1, \ldots, \binom{k-1}{i}\}$$

denote the collection of all subsets of size i from

$$u(l) = \{1, \ldots, k\} - \{l\}$$

and

$$\overline{S}_j^i(l) = u(l) - S_j^i(l).$$

Theorem 4.1 *For any $\vec{\mu} \in \Omega$, we have*

$$\begin{aligned} & P_{\vec{\mu}}[CS|R_3] \\ &= \sum_{i=k-m}^{k-1} \sum_{j=1}^{\binom{k-1}{i}} \int_{-\infty}^{\infty} \prod_{l \in S_j^i(k)} F_n(t + (\mu_{[k]} - \mu_{[l]})\sqrt{n}/\sigma\ ) \\ & \cdot \prod_{l \in \overline{S}_j^i(k)} \{F_n(t + h_3 + (\mu_{[k]} - \mu_{[l]})\sqrt{n}/\sigma\ ) \\ & -F_n(t + (\mu_{[k]} - \mu_{[l]})\sqrt{n}/\sigma\ )\} dF_n(t), \end{aligned} \tag{63}$$

where $F_n(t)$ is the cdf of the standardized mean of a sample of size n from $L(\mu_i, \sigma^2)$.

Proof

Let $\overline{X}_{(i)}$ denote the sample mean from the population $\pi_{(i)}$. Then,

$$\begin{aligned} & P_{\vec{\mu}}[CS|R_3] \\ &= P_{\vec{\mu}}[\overline{X}_{(k)} \geq \max\{\overline{X}_{[k-m+1]}, \overline{X}_{[k]} - h_3\sigma/\sqrt{n}\}] \\ &= Pr\{\overline{X}_{(k)} \geq \overline{X}_{(l)} - h_3\sigma/\sqrt{n} \ \ for\ l < k\ and \\ & \overline{X}_{(k)} \geq \ at\ least\ (k-m)\ \overline{X}_{(l)}{}'s\ with\ l \neq k\}. \end{aligned}$$

Now, for every $i = k - m, \ldots, k-1$ and $j = 1, \ldots, \binom{k-1}{i}$, let

$$A_j^i = [\overline{X}_{(k)} \geq \overline{X}_{(l)}\ \forall\, l \in S_j^i(k)\ and\ \overline{X}_{(k)} < \overline{X}_{(l)}\ \forall\, l \in \overline{S}_j^i(k)].$$

Then

$$
\begin{aligned}
& P_{\vec{\mu}}[CS|R_3] \\
&= P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(l)} - h_3\sigma/\sqrt{n} \ \ \forall\, l < k \ and \ \cup_{i=k-m}^{k-1} \cup_{j=1}^{\binom{k-1}{i}} A_j^i] \\
&= \sum_{i=k-m}^{k-1} \sum_{j=1}^{\binom{k-1}{i}} P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(l)} - h_3\sigma/\sqrt{n} \ \ \forall\, l < k \ and \ A_j^i].
\end{aligned}
$$

For fixed i and j,

$$
\begin{aligned}
& P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(l)} - h_3\sigma/\sqrt{n} \ \ \forall\, l < k \ and \ A_j^i] \\
&= P_{\vec{\mu}}[\overline{X}_{(k)} \geq \overline{X}_{(l)} \ \forall\, l \in S_j^i(k) \ and \ \overline{X}_{(k)} \leq \overline{X}_{(l)} \leq \overline{X}_{(k)} + h_3\sigma/\sqrt{n} \ \ \forall\, l \in \overline{S}_j^i(k)] \\
&= \int_{-\infty}^{\infty} \prod_{l \in S_j^i(k)} F_n(t + (\mu_{[k]} - \mu_{[l]})\sqrt{n}/\sigma\ ) \\
&\quad \cdot \prod_{l \in \overline{S}_j^i(k)} \{F_n(t + h_3 + (\mu_{[k]} - \mu_{[l]})\sqrt{n}/\sigma\ ) \\
&\quad - F_n(t + (\mu_{[k]} - \mu_{[l]})\sqrt{n}/\sigma\ )\} dF_n(t). \quad \square
\end{aligned}
$$

Remark 4.3 *An application of the dominated convergence theorem shows that*

$$
P_{\vec{\mu}}[CS|R_3] \to 1 \quad as \quad (\mu_{[k]} - \mu_{[k-1]}) \to \infty. \tag{64}
$$

Next we determine the infimum over $\Omega(\delta)$ of the probability of a correct selection in the following theorem.

Theorem 4.2 *For any $\vec{\mu} \in \Omega(\delta)$, we have*

$$
\begin{aligned}
& \inf_{\vec{\mu} \in \Omega(\delta)} P_{\vec{\mu}}[CS|R_3] = \inf_{\vec{\mu} \in \Omega^0(\delta)} P_{\vec{\mu}}[CS|R_3] \\
&= \sum_{i=k-m}^{k-1} \binom{k-1}{i} \int_{-\infty}^{\infty} \{F_n(t + \delta\sqrt{n}/\sigma\ )\}^i \\
&\quad \cdot \{F_n(t + h_3 + \delta\sqrt{n}/\sigma\ ) - F_n(t + \delta\sqrt{n}/\sigma\ )\}^{k-1-i} dF_n(t) \\
&= \int_{-\infty}^{\infty} \{F_n(t + h_3 + \delta\sqrt{n}/\sigma\ )\}^{k-1} \\
&\quad \cdot I\{\frac{F_n(t + \delta\sqrt{n}/\sigma\ )}{F_n(t + h_3 + \delta\sqrt{n}/\sigma\ )}; k-m, m\} dF_n(t),
\end{aligned} \tag{65}
$$

where $I\{y; a, b\} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^y w^{a-1}(1-w)^{b-1} dw$ denote the incomplete beta function with parameters a and b.

Proof

We use the following lemma due to Alam and Rizvi (1966) and also due to Mahamunulu (1966).

Lemma 4.1 *Let* $\mathbf{X} = (X_1, \ldots, X_k)$ *have* $k \geq 1$ *independent components such that for every* i, X_i *has cdf* $H(x_i|\theta_i)$. *Suppose that* $\{H(x|\theta)\}$ *form a stochastically increasing family. If* $\Psi(\mathbf{X})$ *is a monotone function of* X_i *when all other components of* $\mathbf{X}$ *are held fixed, then* $E_{\vec{\theta}}[\Psi(\mathbf{X})]$ *is monotone in* θ_i *in the same direction.*

Now, let

$$\Psi(\mathbf{X}) = \begin{cases} 1; & \text{if } \overline{X}_{(k)} \geq \max\{\overline{X}_{[k-m+1]}, \overline{X}_{[k]} - h_3\sigma/\sqrt{n}\} \\ 0; & \text{otherwise.} \end{cases}$$

We claim $\Psi(\mathbf{X})$ is non-increasing in $\overline{X}_{(i)}$ for $i = 1, \ldots, k-1$. Let

$$\overline{X}_{(i)} < \overline{X}'_{(i)},$$

$$\mathbf{X} = (\overline{X}_{(1)}, \ldots, \overline{X}_{(k)})$$

and

$$\mathbf{X}' = (\overline{X}_{(1)}, \ldots, \overline{X}_{(i-1)}, \overline{X}'_{(i)}, \overline{X}_{(i+1)}, \ldots, \overline{X}_{(k)}).$$

Then

$$\max\{\overline{X}_{[k-m+1]}, \overline{X}_{[k]} - h_3\sigma/\sqrt{n}\} \leq \max\{\overline{X}'_{[k-m+1]}, \overline{X}'_{[k]} - h_3\sigma/\sqrt{n}\}$$

where the primes denote the order statistics from $\mathbf{X}'$. So if $\Psi(\mathbf{X}) = 0$ then $\Psi(\mathbf{X}') = 0$. Hence

$$P_{\vec{\mu}}[CS|R_3] = E_{\vec{\mu}}(\Psi(\mathbf{X}))$$

is non-increasing in each of $\mu_{[1]}, \ldots, \mu_{[k-1]}$ when all other means are fixed. So

$$\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3] = \inf_{\vec{\mu}\in\Omega^0(\delta)} P_{\vec{\mu}}[CS|R_3]$$

and hence substituting the vector of means $(\mu_{[1]}, \ldots, \mu_{[1]}, \mu_{[1]} + \delta)$ gives the result. □

4.4 Properties of R_3

We consider next the properties of the restricted subset selection rule R_3 based on the sample means. To facilitate this study we let

$$P_{\vec{\mu}}(i|R) = P_{\vec{\mu}}\{rule\ R\ selects\ \pi_{(i)}\} \tag{66}$$

and recall the following definitions.

Definition 4.1 *R is a monotone procedure means that for all $\vec{\mu} \in \Omega$ and $i < j$,*

$$P_{\vec{\mu}}(i|R) \leq P_{\vec{\mu}}(j|R).$$

Definition 4.2 *R is an unbiased procedure means that for all $\vec{\mu} \in \Omega$ and $j < k$,*

$$P_{\vec{\mu}}[R\ does\ not\ select\ \pi_{(j)}] \geq P_{\vec{\mu}}[R\ does\ not\ select\ \pi_{(k)}].$$

Of course, R is monotone implies that R is unbiased. Other optimal properties are

Definition 4.3 *R is consistent with respect to Ω' means that*

$$\lim_{n\to\infty} \inf_{\vec{\mu}\in\Omega'} P_{\vec{\mu}}[CS|R] = 1.$$

Definition 4.4 *R is strongly monotone in $\pi_{(i)}$ means that*

$$P_{\vec{\mu}}(i|R) = \begin{cases} \uparrow & in\ \mu_{[i]}\ when\ all\ other\ components\ of\ \vec{\mu}\ are\ fixed \\ \downarrow & in\ \mu_{[j]}\ when\ all\ other\ components\ of\ \vec{\mu}\ are\ fixed\ (j \neq i). \end{cases}$$

Theorem 4.3 *For every $i = 1, \ldots, k$, R_3 is strongly monotone in $\pi_{(i)}$.*

Proof

We have already shown this result for $i = k$. Since for $i < k$ we have

$$P_{\vec{\mu}}(i|R_3) = E_{\vec{\mu}}\ [\eta(\mathbf{X})],$$

where

$$\eta(\mathbf{X}) = \begin{cases} 1; & \text{if } \overline{X}_{(i)} \geq \max\{\overline{X}_{[k-m+1]}, \overline{X}_{[k]} - h_3\sigma/\sqrt{n}\} \\ 0; & \text{if otherwise,} \end{cases}$$

the same argument applies to give the desired conclusion. □

Corollary 4.1 *All rules of the form (61) are monotone and unbiased.*

Proof

The proof follows from the definition of monotonicity and the property of being strongly monotone in $\pi_{(i)}$ for all i. □

Theorem 4.4 *R_3 is consistent with respect to $\Omega(\delta)$.*

Proof

We must show that

$$\sum_{i=k-m}^{k-1} \binom{k-1}{i} \lim_{n\to\infty} \int_{-\infty}^{\infty} \{F_n(t+\delta\sqrt{n}/\sigma)\}^i \cdot\{F_n(t+h_3+\delta\sqrt{n}/\sigma) - F_n(t+\delta\sqrt{n}/\sigma)\}^{k-1-i} dF_n(t) = 1. \qquad (67)$$

We note that each integrand is bounded with respect to the measure F_n and so the dominated convergence theorem applies. For every $i < (k-1)$ we have

$$\lim_{n\to\infty} \{F_n(t+\delta\sqrt{n}/\sigma)\}^i \{F_n(t+h_3+\delta\sqrt{n}/\sigma) - F_n(t+\delta\sqrt{n}/\sigma)\}^{k-1-i} = 0$$

and for $i = k-1$

$$\lim_{n\to\infty} \{F_n(t+\delta\sqrt{n}/\sigma)\}^{k-1} = 1.$$

Hence the result follows. □

This theorem says that no matter what probability level is required for a correct selection it can be met by choosing a sufficiently large sample, for any given k, m and δ.

Theorem 4.5 *For every n and rule R_3,*

$$\lim_{\delta\to\infty} \inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3] = 1.$$

For every n, $m < k$ and $\delta > 0$,

$$\begin{aligned} &\lim_{h_3\to\infty} \inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3] \\ &= (k-m)\binom{k-1}{k-m} \int_{-\infty}^{\infty} \{1 - F_n(t-\delta\sqrt{n}/\sigma)\} \\ &\quad \cdot\{F_n(t)\}^{k-m-1}\{1-F_n(t)\}^{m-1} dF_n(t) \\ &= \int_{-\infty}^{\infty} I\{F_n(t+\delta\sqrt{n}/\sigma); k-m, m\} dF_n(t). \end{aligned} \qquad (68)$$

Proof

Both results follow from the dominated convergence theorem. The second result follows the same theorem and

$$\begin{aligned}
&\lim_{h_3\to\infty} \inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3] \\
&= \sum_{i=k-m}^{k-1} \binom{k-1}{i} \int_{-\infty}^{\infty} \{F_n(t+\delta\sqrt{n}/\sigma)\}^i \\
&\quad \cdot\{1-F_n(t+\delta\sqrt{n}/\sigma)\}^{k-1-i} dF_n(t) \\
&= \int_{-\infty}^{\infty} (k-m)\binom{k-1}{k-m} \int_{(1-F_n(t+\delta\sqrt{n}/\sigma))}^{1} y^{m-1}\{1-y\}^{k-m-1} dy dF_n(t) \\
&= (k-m)\binom{k-1}{k-m} \int_{-\infty}^{\infty} \int_{w-\delta\sqrt{n}/\sigma}^{\infty} dF_n(t)\ \{1-F_n(w)\}^{m-1} \\
&\quad \cdot\{F_n(w)\}^{k-m-1} dF_n(w),
\end{aligned}$$

letting $w = F_n^{-1}(1-y)$ and changing the order of integration. □

Remark 4.4 *The first part states that by taking δ sufficiently large we can attain any P^* probability requirement for the rule R_3 based on any number of observations. The second result says that given a fixed $\delta \geq 0$ and a common sample size n, we cannot achieve all P^* values. We can only attain*

$$\begin{aligned}
P^* &\leq (k-m)\binom{k-1}{k-m} \int_{-\infty}^{\infty} \{1-F_n(t-\delta\sqrt{n}/\sigma)\} \\
&\quad \cdot\{F_n(t)\}^{k-m-1}\{1-F_n(t)\}^{m-1} dF_n(t) \\
&= \int_{-\infty}^{\infty} I\{F_n(t+\delta\sqrt{n}/\sigma); k-m, m\} dF_n(t) \ \leq 1. \qquad (69)
\end{aligned}$$

Remark 4.5 *Using the monotonicity of $\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3]$ we can obtain the following bounds. For $m < k$ and $\delta > 0$,*

$$\begin{aligned}
&\int_{-\infty}^{\infty} \{F_n(t+\delta\sqrt{n}/\sigma)\}^{k-1} dF_n(t) \\
&\leq \inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3] \\
&\leq \int_{-\infty}^{\infty} I\{F_n(t+\delta\sqrt{n}/\sigma); k-m, m\} dF_n(t). \qquad (70)
\end{aligned}$$

Table 23 contains the above lower and upper bounds of the infimum of the probability of a correct selection over $\Omega(\delta)$ for $k = 3, 5, 10$, $m = 2, 4, 5 < k$, $\delta/\sigma = 0.5, 1.0, 2.0$ and $n = 5, 15$. All computations were carried out in double-precision arithmetic on a Vax-11/780.

For the purpose of implementing the procedure R_3 and comparing R_3 to the fixed size subset rule, we have prepared Table 24 and Table 25. For $P^* = 0.90$, $k = 5, 10$, $m = 2, 3, 4, 5 < k$, $\delta/\sigma = 0.5, 1.0, 2.0$ and $h_3 = 0.4, 0.7, 1.3, 1.6$, the tables give the values of the minimum sample size $(n(h_3))$ which satisfies

$$\int_{-\infty}^{\infty} \{F_n(t + h_3 + \delta\sqrt{n}/\sigma)\}^{k-1} I\{\frac{F_n(t + \delta\sqrt{n}/\sigma)}{F_n(t + h_3 + \delta\sqrt{n}/\sigma)}; k - m, m\} dF_n(t) \geq P^*,$$

the values of the minimum sample size $(n(\infty))$ for the fixed size subset rule, which satisfies

$$\int_{-\infty}^{\infty} I\{F_n(t + \delta\sqrt{n}/\sigma); k - m, m\} dF_n(t) \geq P^*$$

and the ratio $(n(h_3)/n(\infty))$ of the sample size for the restricted subset selection rule R_3 to the sample size for the fixed size subset rule when both rules attain the same probability requirements. For large h_3 values this ratio is close to one, indicating that in many cases a slight additional cost will allow the use of a restricted subset selection procedure which meets the same probability requirement.

The expected number of selected populations depends, of course, on the underlying $\vec{\mu}$. Some exact comparisons for the equally spaced and slippage configurations will be considered in the next section.

4.5 Expected number of selected populations

As usual, we define

$$Y_i = \begin{cases} 1; & \text{if } \overline{X}_{(i)} \geq \max\{\overline{X}_{[k-m+1]}, \overline{X}_{[k]} - h_3\sigma/\sqrt{n}\} \\ 0; & \text{otherwise.} \end{cases}$$

This gives S, the number of populations selected, as

$$S = \sum_{i=1}^{k} Y_i.$$

Then the expected number of populations selected by R_3 is given by

$$E_{\vec{\mu}}[S|R_3] = \sum_{i=1}^{k} P_{\vec{\mu}}(i|R_3),$$

where $P_{\vec{\mu}}(i|R_3)$ is defined by (66).

Theorem 4.6 *For any $\vec{\mu} \in \Omega$, we have*

$$\begin{aligned} E_{\vec{\mu}}[S|R_3] &= \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{j=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{l \in S_j^p(i)} F_n(t + (\mu_{[i]} - \mu_{[l]})\sqrt{n}/\sigma) \\ &\cdot \prod_{l \in \overline{S}_j^p(i)} \{F_n(t + h_3 + (\mu_{[i]} - \mu_{[l]})\sqrt{n}/\sigma) \\ &-F_n(t + (\mu_{[i]} - \mu_{[l]})\sqrt{n}/\sigma)\} dF_n(t), \end{aligned} \quad (71)$$

where $F_n(t)$ is the cdf of the standardized mean of samples of size n from $L(\mu_i, \sigma^2)$.

Proof

From the above discussion, we see that it suffices to calculate $P_{\vec{\mu}}(i|R_3)$ for $i = 1, \ldots, k$. Using arguments similar to those in the proof of Theorem 4.1, we get

$$\begin{aligned} P_{\vec{\mu}}(i|R_3) &= \sum_{p=k-m}^{k-1} \sum_{j=1}^{\binom{k-1}{p}} P_r\{\overline{X}_{(i)} \geq \overline{X}_{(l)} \ \forall\, l \in S_j^p(i) \\ & \text{and } \overline{X}_{(i)} \leq \overline{X}_{(l)} \leq \overline{X}_{(i)} + h_3\sigma/\sqrt{n} \ \ \forall\, l \in \overline{S}_j^p(i)\} \\ &= \sum_{p=k-m}^{k-1} \sum_{j=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{l \in S_j^p(i)} F_n(t + (\mu_{[i]} - \mu_{[l]})\sqrt{n}/\sigma) \\ &\cdot \prod_{l \in \overline{S}_j^p(i)} \{F_n(t + h_3 + (\mu_{[i]} - \mu_{[l]})\sqrt{n}/\sigma) \\ &-F_n(t + (\mu_{[i]} - \mu_{[l]})\sqrt{n}/\sigma)\} dF_n(t). \quad \square \end{aligned}$$

Since $E_{\vec{\mu}}[S|R_3]$ is increasing in h_3 the experimenter may seek to use rules with small h_3. On the other hand, for fixed δ and P^*, the smaller h_3 is, the larger n must be to achieve the required probability condition (62). Hence, the experimenter must decide what trade off between n, h_3 and δ he is willing to accept. To investigate the interdependence in more detail, we have tabulated in Table 26 and Table 27 the values of

- $E(S) = E_{\vec{\mu}}[S|R_3]$,
- $E(SR) = \sum_{i=1}^{k} iP_{\vec{\mu}}(i|R_3)$; the expected sum of ranks of the selected populations and
- $E(S)/m$; the expected proportion of selected populations

under

1. Equally spaced means $\vec{\mu} = (\alpha, \alpha+\delta, \ldots, \alpha+(k-1)\delta)$ and
2. Slippage means $\vec{\mu} = (\alpha, \alpha, \ldots, \alpha, \alpha+\delta)$,

for

$(k,m) = (4,2),\ (5,3)$, $n = 2,3,4,5,10,15$, $h_3 = 0.4, 0.7$ and $\delta/\sigma = 0.1, 0.5, 1.0, 2.0$. All computations were carried out in double-precision arithmetic on a Vax-11/780.

4.6 Supremum of the expected number of selected populations and a new design criterion for R_3

Santner (1975) considered a general restricted subset selection procedure in terms of consistent estimators for the population parameters whose distributions form a stochastically increasing family for each given sample size. In particular he gave conditions which guarantee that the supremum of the expected number of populations selected over the whole parameter space occurs at some point where the k population parameters are all the same.

We can consider the means rule R_3 as a special case of a location parameter problem using Santner's general procedure. By noting that the distribution of the mean of samples from a logistic population has the MLR property with respect to the location parameter and hence forms a SI family, we can see that

Theorem 4.7 *For every $\vec{\mu} \in \Omega$, we have*

$$\sup_{\vec{\mu}\in\Omega} E_{\vec{\mu}}[S|R_3] = k\int_{-\infty}^{\infty}\{F_n(t+h_3)\}^{k-1} I\{\frac{F_n(t)}{F_n(t+h_3)}; k-m, m\}dF_n(t). \quad (72)$$

In Section 4.4 we determined the needed sample size n for the rule R_3 for the arbitrarily chosen values of h_3, k, m and δ because we could not determine the values of n and h_3 at the same time by controlling the basic probability requirement only. Since we desire to select smaller S, it is reasonable to make a choice of (n, h_3) by controlling the $\sup_{\vec{\mu}\in\Omega} E_{\vec{\mu}}[S|R_3]$ as well as $\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|R_3]$.

Using Theorem 4.2 and Theorem 4.7, we can choose a new set of design constants (n, h_3) to implement R_3 by solving the following equations simultaneously,

$$\int_{-\infty}^{\infty} \{F_n(t + h_3 + \delta\sqrt{n}/\sigma)\}^{k-1} \cdot I\{\frac{F_n(t + \delta\sqrt{n}/\sigma)}{F_n(t + h_3 + \delta\sqrt{n}/\sigma)}; k - m, m\} dF_n(t) = P^* \tag{73}$$

$$k \int_{-\infty}^{\infty} \{F_n(t + h_3)\}^{k-1} I\{\frac{F_n(t)}{F_n(t + h_3)}; k - m, m\} dF_n(t) = 1 + \epsilon \tag{74}$$

for the given values of P^*,k, m, δ and small $\epsilon > 0$.

Table 28 and Table 29 contain the estimates $(\hat{n}, \hat{h}_3)$ for the constants (n, h_3), which satisfy (73) and (74) simultaneously for given values of $P^* = 0.90, 0.975$, $k = 3, 4, 5, 10, 15$, $m = 2, 3, 4, 5$, $\delta/\sigma = 0.5, 1.0, 2.0$ and $\epsilon = 0.01$. All computations were carried out in single-precision arithmetic on a CDC-6500. The IMSL subroutine ZSCNT was used to solve the above system of non-linear equations and the f-norm in the tables indicates the accuracy of the computation, which is defined in the ZSCNT.

Table 23: Bounds on the infimum of the probability of a correct selection over the preference zone for rule R_3.

k	m	δ/σ	n	$l - bound$	$u - bound$
		0.50	5	0.671	0.905
			15	0.856	0.974
3	2	1.00	5	0.902	0.985
			15	0.994	1.000
		2.00	5	0.998	1.000
			15	1.000	1.000
		0.50	5	0.536	0.765
			15	0.774	0.921
	2	1.00	5	0.840	0.953
			15	0.988	0.999
		2.00	5	0.996	1.000
5			15	1.000	1.000
		0.50	5	0.536	0.962
			15	0.774	0.993
	4	1.00	5	0.840	0.996
			15	0.988	1.000
		2.00	5	0.996	1.000
			15	1.000	1.000
		0.50	5	0.379	0.574
			15	0.654	0.820
	2	1.00	5	0.743	0.885
			15	0.977	0.996
		2.00	5	0.992	0.999
			15	1.000	1.000
		0.50	5	0.379	0.789
			15	0.654	0.938
10	4	1.00	5	0.743	0.966
			15	0.977	0.999
		2.00	5	0.992	1.000
			15	1.000	1.000
		0.50	5	0.379	0.854
			15	0.654	0.964
	5	1.00	5	0.743	0.980
			15	0.977	1.000
		2.00	5	0.992	1.000
			15	1.000	1.000

Table 24: The minimum sample sizes needed for rule R_3 and the corresponding fixed size subset rule: $P^* = 0.90,\ k = 5$.

$P^* = 0.90,\ k = 5$					
m	h_3	δ/σ	$n(h_3)$	$n(\infty)$	$n(h_3)/n(\infty)$
2	0.40	0.50	22	13	1.692
		1.00	6	4	1.500
		2.00	2	1	2.000
	0.70	0.50	19	13	1.462
		1.00	5	4	1.250
		2.00	2	1	2.000
	1.30	0.50	18	13	1.385
		1.00	5	4	1.250
		2.00	2	1	2.000
	1.60	0.50	18	13	1.385
		1.00	5	4	1.250
		2.00	2	1	2.000
3	0.40	0.50	20	6	3.333
		1.00	5	2	2.500
		2.00	2	1	2.000
	0.70	0.50	16	6	2.667
		1.00	4	2	2.000
		2.00	1	1	1.000
	1.30	0.50	11	6	1.833
		1.00	3	2	1.500
		2.00	1	1	1.000
	1.60	0.50	10	6	1.667
		1.00	3	2	1.500
		2.00	1	1	1.000
4	0.40	0.50	20	2	10.000
		1.00	5	1	5.000
		2.00	2	1	2.000
	0.70	0.50	15	2	7.500
		1.00	4	1	4.000
		2.00	1	1	1.000
	1.30	0.50	8	2	4.000
		1.00	2	1	2.000
		2.00	1	1	1.000
	1.60	0.50	6	2	3.000
		1.00	2	1	2.000
		2.00	1	1	1.000

Table 25: The minimum sample sizes needed for rule R_3 and the corresponding fixed size subset rule: $P^* = 0.90,\ k = 10$.

$P^* = 0.90,\ k = 10$					
m	h_3	δ/σ	$n(h_3)$	$n(\infty)$	$n(h_3)/n(\infty)$
2	0.40	0.50	30	23	1.304
		1.00	8	6	1.333
		2.00	2	2	1.000
	0.70	0.50	28	23	1.217
		1.00	7	6	1.167
		2.00	2	2	1.000
	1.30	0.50	27	23	1.174
		1.00	7	6	1.167
		2.00	2	2	1.000
	1.60	0.50	27	23	1.174
		1.00	7	6	1.167
		2.00	2	2	1.000
3	0.40	0.50	28	16	1.750
		1.00	7	4	1.750
		2.00	2	1	2.000
	0.70	0.50	24	16	1.500
		1.00	6	4	1.500
		2.00	2	1	2.000
	1.30	0.50	20	16	1.250
		1.00	5	4	1.250
		2.00	2	1	2.000
	1.60	0.50	20	16	1.250
		1.00	5	4	1.250
		2.00	2	1	2.000
4	0.40	0.50	27	11	2.455
		1.00	7	3	2.333
		2.00	2	1	2.000
	0.70	0.50	22	11	2.000
		1.00	6	3	2.000
		2.00	2	1	2.000
	1.30	0.50	16	11	1.455
		1.00	4	3	1.333
		2.00	2	1	2.000
	1.60	0.50	15	11	1.364
		1.00	4	3	1.333
		2.00	1	1	1.000
5	0.40	0.50	27	8	3.375
		1.00	7	2	3.500
		2.00	2	1	2.000
	0.70	0.50	22	8	2.750
		1.00	6	2	3.000
		2.00	2	1	2.000
	1.30	0.50	14	8	1.750
		1.00	4	2	2.000
		2.00	1	1	1.000
	1.60	0.50	12	8	1.500
		1.00	3	2	1.500
		2.00	1	1	1.000

Table 26: Performance characteristics of rule R_3: $k = 4,\ m = 2$.

$k = 4,\ m = 2$								
n	h_3	δ/σ	Equally spaced conf.			Slippage conf.		
			$E(S)$	$E(SR)$	$E(S)/m$	$E(S)$	$E(SR)$	$E(S)/m$
2	0.400	0.100	1.366	3.125	0.683	1.371	3.337	0.685
		0.500	1.269	2.045	0.635	1.344	2.882	0.672
		1.000	1.147	1.426	0.573	1.259	2.221	0.630
		2.000	1.030	1.073	0.515	1.072	1.285	0.536
	0.700	0.100	1.572	3.622	0.786	1.578	3.850	0.789
		0.500	1.442	2.401	0.721	1.543	3.367	0.772
		1.000	1.256	1.626	0.628	1.424	2.630	0.712
		2.000	1.057	1.122	0.528	1.130	1.439	0.565
3	0.400	0.100	1.361	3.054	0.681	1.368	3.310	0.684
		0.500	1.238	1.848	0.619	1.329	2.741	0.664
		1.000	1.109	1.296	0.555	1.213	1.953	0.607
		2.000	1.012	1.028	0.506	1.031	1.116	0.516
	0.700	0.100	1.566	3.545	0.783	1.575	3.822	0.788
		0.500	1.396	2.162	0.698	1.523	3.213	0.761
		1.000	1.194	1.448	0.597	1.356	2.311	0.678
		2.000	1.024	1.050	0.512	1.059	1.191	0.529
4	0.400	0.100	1.357	2.995	0.679	1.366	3.289	0.683
		0.500	1.213	1.714	0.606	1.315	2.620	0.657
		1.000	1.084	1.217	0.542	1.175	1.749	0.587
		2.000	1.005	1.011	0.502	1.013	1.047	0.507
	0.700	0.100	1.562	3.481	0.781	1.574	3.801	0.787
		0.500	1.359	1.996	0.679	1.503	3.079	0.752
		1.000	1.151	1.337	0.575	1.296	2.058	0.648
		2.000	1.010	1.021	0.505	1.026	1.083	0.513
5	0.400	0.100	1.354	2.944	0.677	1.365	3.271	0.683
		0.500	1.193	1.616	0.596	1.301	2.512	0.651
		1.000	1.065	1.163	0.533	1.142	1.590	0.571
		2.000	1.002	1.004	0.501	1.006	1.019	0.503
	0.700	0.100	1.558	3.425	0.779	1.572	3.782	0.786
		0.500	1.328	1.872	0.664	1.484	2.959	0.742
		1.000	1.119	1.259	0.559	1.245	1.856	0.623
		2.000	1.004	1.009	0.502	1.011	1.035	0.506
10	0.400	0.100	1.341	2.749	0.670	1.361	3.202	0.681
		0.500	1.128	1.357	0.564	1.239	2.097	0.620
		1.000	1.019	1.045	0.510	1.049	1.181	0.524
		2.000	1.000	1.000	0.500	1.000	1.000	0.500
	0.700	0.100	1.541	3.210	0.770	1.568	3.709	0.784
		0.500	1.225	1.531	0.613	1.394	2.482	0.697
		1.000	1.037	1.078	0.519	1.090	1.293	0.545
		2.000	1.000	1.000	0.500	1.000	1.000	0.500
15	0.400	0.100	1.330	2.605	0.665	1.359	3.148	0.679
		0.500	1.091	1.237	0.546	1.187	1.808	0.594
		1.000	1.006	1.013	0.503	1.016	1.055	0.508
		2.000	1.000	1.000	0.500	1.000	1.000	0.500
	0.700	0.100	1.525	3.048	0.763	1.564	3.652	0.782
		0.500	1.163	1.365	0.581	1.315	2.132	0.658
		1.000	1.012	1.025	0.506	1.031	1.098	0.516
		2.000	1.000	1.000	0.500	1.000	1.000	0.500

Table 27: Performance characteristics of rule R_3: $k = 5,\ m = 3$.

			$k = 5,\ m = 3$					
			Equally spaced conf.			Slippage conf.		
n	h_3	δ/σ	$E(S)$	$E(SR)$	$E(S)/m$	$E(S)$	$E(SR)$	$E(S)/m$
2	0.400	0.100	1.503	3.992	0.501	1.508	4.423	0.503
		0.500	1.314	2.189	0.438	1.473	3.830	0.491
		1.000	1.153	1.440	0.384	1.357	2.885	0.452
		2.000	1.030	1.074	0.343	1.098	1.449	0.366
	0.700	0.100	1.882	5.076	0.627	1.884	5.551	0.628
		0.500	1.564	2.757	0.521	1.828	4.875	0.609
		1.000	1.275	1.672	0.425	1.637	3.721	0.546
		2.000	1.057	1.122	0.352	1.188	1.728	0.396
3	0.400	0.100	1.493	3.859	0.498	1.504	4.386	0.501
		0.500	1.267	1.929	0.422	1.452	3.632	0.484
		1.000	1.112	1.301	0.371	1.293	2.483	0.431
		2.000	1.012	1.028	0.337	1.042	1.183	0.347
	0.700	0.100	1.867	4.915	0.622	1.878	5.506	0.626
		0.500	1.480	2.387	0.493	1.794	4.638	0.598
		1.000	1.202	1.466	0.401	1.531	3.194	0.510
		2.000	1.024	1.050	0.341	1.084	1.314	0.361
4	0.400	0.100	1.485	3.750	0.495	1.502	4.358	0.501
		0.500	1.233	1.766	0.411	1.433	3.461	0.478
		1.000	1.085	1.219	0.362	1.240	2.172	0.413
		2.000	1.005	1.011	0.335	1.018	1.075	0.339
	0.700	0.100	1.855	4.784	0.618	1.874	5.474	0.625
		0.500	1.420	2.149	0.473	1.762	4.432	0.587
		1.000	1.154	1.344	0.385	1.439	2.774	0.480
		2.000	1.010	1.021	0.337	1.036	1.135	0.345
5	0.400	0.100	1.478	3.656	0.493	1.500	4.335	0.500
		0.500	1.208	1.651	0.403	1.414	3.307	0.471
		1.000	1.065	1.164	0.355	1.196	1.927	0.399
		2.000	1.002	1.004	0.334	1.007	1.030	0.336
	0.700	0.100	1.843	4.669	0.614	1.872	5.447	0.624
		0.500	1.373	1.983	0.458	1.732	4.244	0.577
		1.000	1.120	1.262	0.373	1.362	2.434	0.454
		2.000	1.004	1.009	0.335	1.016	1.057	0.339
10	0.400	0.100	1.450	3.302	0.483	1.495	4.246	0.498
		0.500	1.132	1.365	0.377	1.329	2.700	0.443
		1.000	1.019	1.045	0.340	1.066	1.287	0.355
		2.000	1.000	1.000	0.333	1.000	1.000	0.333
	0.700	0.100	1.796	4.231	0.599	1.864	5.345	0.621
		0.500	1.238	1.560	0.413	1.590	3.477	0.530
		1.000	1.037	1.078	0.346	1.129	1.487	0.376
		2.000	1.000	1.000	0.333	1.000	1.001	0.333
15	0.400	0.100	1.425	3.050	0.475	1.491	4.176	0.497
		0.500	1.092	1.239	0.364	1.258	2.263	0.419
		1.000	1.006	1.013	0.335	1.021	1.088	0.340
		2.000	1.000	1.000	0.333	1.000	1.000	0.333
	0.700	0.100	1.755	3.911	0.585	1.858	5.266	0.619
		0.500	1.167	1.374	0.389	1.469	2.899	0.490
		1.000	1.012	1.025	0.337	1.044	1.161	0.348
		2.000	1.000	1.000	0.333	1.000	1.000	0.333

Table 28: Estimates of constants for rule R_3: $P^* = 0.90$, $Sup(S) = 1.01$.

$P^* = 0.900$, $Sup(S) = 1.01$					
k	m	δ/σ	$\hat{n}$	$\hat{h}_3$	$f - norm$
3	2	.50	.1965e+02	.1179e-01	.6693e-15
		1.00	.4883e+01	.1166e-01	.4648e-22
		2.00	.1188e+01	.1112e-01	.8925e-15
4	2	.50	.2383e+02	.9717e-02	.2049e-17
		1.00	.5946e+01	.9648e-02	.2318e-18
		2.00	.1476e+01	.9384e-02	.1203e-13
	3	.50	.2383e+02	.9667e-02	.8419e-20
		1.00	.5946e+01	.9598e-02	.3176e-19
		2.00	.1476e+01	.9334e-02	.3359e-15
5	2	.50	.2686e+02	.8613e-02	.9771e-18
		1.00	.6715e+01	.8580e-02	.2066e-13
		2.00	.1680e+01	.8439e-02	.1658e-17
	3	.50	.2686e+02	.8563e-02	.4192e-16
		1.00	.6715e+01	.8528e-02	.3007e-17
		2.00	.1680e+01	.8386e-02	.5923e-18
	4	.50	.2686e+02	.8562e-02	.4433e-16
		1.00	.6715e+01	.8528e-02	.2910e-17
		2.00	.1680e+01	.8386e-02	.5918e-18
10	2	.50	.3550e+02	.6535e-02	.1496e-15
		1.00	.8922e+01	.6569e-02	.4289e-13
		2.00	.2273e+01	.6673e-02	.2864e-14
	3	.50	.3550e+02	.6488e-02	.2623e-15
		1.00	.8922e+01	.6521e-02	.2750e-16
		2.00	.2273e+01	.6623e-02	.7496e-14
	4	.50	.3550e+02	.6488e-02	.2639e-15
		1.00	.8922e+01	.6521e-02	.2893e-16
		2.00	.2273e+01	.6622e-02	.7455e-14
	5	.50	.3550e+02	.6488e-02	.2659e-15
		1.00	.8922e+01	.6521e-02	.2891e-16
		2.00	.2273e+01	.6622e-02	.7448e-14
15	2	.50	.4024e+02	.5803e-02	.2848e-15
		1.00	.1014e+02	.5861e-02	.1097e-16
		2.00	.2602e+01	.6062e-02	.4032e-15
	3	.50	.4024e+02	.5758e-02	.3480e-14
		1.00	.1014e+02	.5816e-02	.2559e-15
		2.00	.2602e+01	.6013e-02	.8833e-16
	4	.50	.4024e+02	.5758e-02	.1364e-13
		1.00	.1014e+02	.5816e-02	.2682e-15
		2.00	.2602e+01	.6012e-02	.8877e-16
	5	.50	.4024e+02	.5758e-02	.1381e-13
		1.00	.1014e+02	.5816e-02	.2684e-15
		2.00	.2602e+01	.6012e-02	.8877e-16

Table 29: Estimates of constants for rule R_3: $P^* = 0.975,\ Sup(S) = 1.01$.

$P^* = 0.975,\ Sup(S) = 1.01$					
k	m	δ/σ	$\hat{n}$	$\hat{h}_3$	$f-norm$
3	2	.50	.3894e+02	.1182e-01	.1243e-17
		1.00	.9794e+01	.1175e-01	.6086e-19
		2.00	.2504e+01	.1150e-01	.2001e-21
4	2	.50	.4399e+02	.9728e-02	.3195e-21
		1.00	.1108e+02	.9691e-02	.1435e-21
		2.00	.2841e+01	.9550e-02	.2667e-16
	3	.50	.4399e+02	.9677e-02	.1720e-17
		1.00	.1108e+02	.9640e-02	.1724e-17
		2.00	.2841e+01	.9499e-02	.3598e-18
5	2	.50	.4759e+02	.8618e-02	.7053e-16
		1.00	.1199e+02	.8600e-02	.1076e-15
		2.00	.3082e+01	.8525e-02	.3329e-16
	3	.50	.4759e+02	.8567e-02	.2190e-18
		1.00	.1199e+02	.8549e-02	.1444e-17
		2.00	.3082e+01	.8473e-02	.2198e-17
	4	.50	.4759e+02	.8567e-02	.3819e-22
		1.00	.1199e+02	.8548e-02	.7657e-23
		2.00	.3082e+01	.8473e-02	.3403e-23
10	2	.50	.5771e+02	.6530e-02	.9562e-16
		1.00	.1456e+02	.6552e-02	.1125e-15
		2.00	.3766e+01	.6623e-02	.1408e-15
	3	.50	.5772e+02	.6483e-02	.2547e-17
		1.00	.1457e+02	.6504e-02	.1364e-17
		2.00	.3766e+01	.6574e-02	.1772e-16
	4	.50	.5772e+02	.6483e-02	.2457e-22
		1.00	.1457e+02	.6504e-02	.1253e-22
		2.00	.3766e+01	.6574e-02	.3067e-22
	5	.50	.5772e+02	.6483e-02	.6563e-27
		1.00	.1457e+02	.6504e-02	.2524e-27
		2.00	.3766e+01	.6574e-02	.4089e-26
15	2	.50	.6320e+02	.5795e-02	.3436e-14
		1.00	.1596e+02	.5833e-02	.3331e-14
		2.00	.4139e+01	.5966e-02	.3695e-14
	3	.50	.6321e+02	.5751e-02	.8806e-17
		1.00	.1596e+02	.5788e-02	.1271e-17
		2.00	.4139e+01	.5919e-02	.8236e-17
	4	.50	.6321e+02	.5751e-02	.1326e-21
		1.00	.1596e+02	.5788e-02	.7942e-22
		2.00	.4139e+01	.5918e-02	.1707e-21
	5	.50	.6321e+02	.5751e-02	.2524e-27
		1.00	.1596e+02	.5788e-02	.9214e-27
		2.00	.4139e+01	.5918e-02	.5364e-26

5 AN ELIMINATION TYPE TWO-STAGE PROCEDURE USING RESTRICTED SUBSET SELECTION RULE IN ITS FIRST STAGE FOR SELECTING THE BEST POPULATION

5.1 Introduction

Tamhane and Bechhofer (1977, 1979) studied a two-stage elimination type procedure for selecting the largest normal mean and we considered in Chapter 3 an elimination type two-stage procedure $\mathcal{P}_2$ for selecting the largest among several logistic populations. It is well known that the above two-stage procedures are quite efficient relative to the corresponding single-stage procedures in terms of the required sample sizes. However, sometimes we may have only limited resources to use in the second stage. In those cases we need more flexible procedures which allow us to specify an upper bound m on the number of populations included in the selected subset in the first stage. Gupta and Santner (1973) studied the selection problem with such a restriction, which is called a restricted subset selection procedure, for selecting the largest normal mean and we considered restricted subset selection procedures for selecting the largest logistic mean in Chapter 4 in the framework of single-stage procedures.

Here we propose an elimination type two-stage procedure $\mathcal{P}_2'$ for selecting a population with the 'largest' real parameter, in which a generalized restricted subset selection procedure (Santner (1973, 1975)) is used in the first stage in terms of a set of consistent estimators for the population parameters whose distributions form a stochastically increasing family for a given sample size. We also propose an optimization problem using a minimax criterion to find a set of constants needed to implement $\mathcal{P}_2'$.

We derive a lower bound of the probability of a correct selection and a formula for the infimum of the lower bound over the preference zone.

We derive an analytical expression for the expected total sample size and study conditions guaranteeing that the supremum over the whole parameter space of the expected total sample size occurs at some point where all of the parameters are equal. We also derive a general expression for the supremum over the whole parameter space of the expected total sample size under these conditions.

A non-linear optimization problem which one must solve in order to determine the constants needed to implement $\mathcal{P}_2'$ for location and scale parameter problems and a relative efficiency of $\mathcal{P}_2'$ with respect to the corresponding single-stage procedure are defined.

We apply $\mathcal{P}_2'$ to the location parameter problem of univariate normal populations. Here we provide tables of constants to implement $\mathcal{P}_2'$ and of the relative efficiency for each case of the equally spaced and slippage configurations.

5.2 Preliminaries

Let π_i, $i = 1, \ldots, k$, be k populations which are characterized by unknown scalars $\lambda_i \in \Lambda$, where Λ is a known interval on the real line. Let $\lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$ be the ordered λ_i's,

$$\Omega = \{\vec{\lambda} = (\lambda_1, \cdots, \lambda_k) | \lambda_i \in \Lambda \ \ \forall\, i \ \}$$

the space of all possible underlying configurations of λ_i's and $\pi_{(i)}$ the (unknown) population with parameter $\lambda_{[i]}$. It is assumed that there is no a priori knowledge of the correct pairing of the elements in $\{\pi_i\}$ and $\{\pi_{(i)}\}$. The goal is to define a two-stage procedure $\mathcal{P}_2'$ to select the 'best' population where for sake of definiteness $\pi_{(k)}$ is taken to be the best population. In some cases $\pi_{(1)}$ might be the best population. Of course, if t $(2 \leq t \leq k)$ populations all have $\lambda_i = \lambda_{[k]}$, the selection of any of these tied populations accomplishes the goal.

Each π_i yields *iid* observations X_{ij}, $j = 1, \ldots, n$, which are also independent between populations. X_{ij} has cdf F_i with parameter λ_i. Furthermore it is assumed that there exists a sequence of Borel measurable functions $\{T_n\}$ so that T_n is defined on n dimensional sample space and

$$T_n(X_{i1}, \ldots, X_{in}) \equiv T_{in} \xrightarrow{\mathcal{P}} \lambda_i \quad as\ n \to \infty.$$

The assumptions concerning T_{in} are that its cdf $G_n(y|\lambda_i)$ with support $E_n^{\lambda_i}$ is absolutely continuous with respect to Lebesgue measure with pdf $g_n(y|\lambda_i)$ and $\{G_n(y|\lambda)|\lambda \in \Lambda\}$ forms a stochastically increasing family for each n.

A preference zone will be defined in Ω by means of a function

$$p : \Lambda \longrightarrow \Re,$$

where $\Re$ ia a real line, such that

1. $p(\cdot)$ is continuous and non-decreasing on Λ
2. $p(\lambda) < \lambda \ \ \forall \lambda \in \Lambda$
3. $p : \Lambda' \overset{onto}{\longrightarrow} \Lambda$, where $\Lambda' = \{\lambda \in \Lambda | p(\lambda) \in \Lambda\}$.

Define

$$\Omega(p) = \{\vec{\lambda} \in \Omega |\ \lambda_{[k-1]} \leq p(\lambda_{[k]})\}$$

and

$$\Omega^0(p) = \{\vec{\lambda} \in \Omega |\ \lambda_{[1]} = \lambda_{[k-1]} = p(\lambda_{[k]})\} \ .$$

Let $h_n(\cdot)$ be a sequence of functions such that for each n

$$h_n(\cdot) : \ E_n \longrightarrow \Re$$

where $\bigcup_{\lambda \in \Lambda} E_n^{\lambda} \subset E_n$, satisfying

1. For each n and x, $h_n(x) > x$,
2. For each n, $h_n(x)$ is continuous and strictly increasing in x.

Typical examples of $h_n(\cdot)$ are given by

$$h_n(x) = x + d_n \ \ (d_n > 0)$$

for the location-type procedures and

$$h_n(x) = c_n x \ \ (c_n > 1)$$

for the scale-type procedures.

The goal of the experimenter is to select a best population. The experimenter restricts consideration to procedures ($\mathcal{P}$) which guarantee the probability requirement

$$P_{\vec{\lambda}}[CS|\mathcal{P}] \geq P^* \quad \forall\, \vec{\lambda} \in \Omega(p) \tag{75}$$

where $p(\cdot)$ and P^* are specified prior to experimentation. The event $[CS]$ occurs if and only if the experimenter selects a best population.

Here we propose an elimination type two-stage procedure $\mathcal{P}_2'$ for selecting a best population using a restricted subset selection rule in its first stage and an indifference zone approach in its second stage.

Procedure $\mathcal{P}_2'$;

Stage 1: Take n_1 independent observations

$$X_{ij}^{(1)},\ j = 1, \ldots, n_1,$$

from each π_i, $i = 1, \ldots, k$, and compute the k estimates

$$T_{n_1}(X_{i1}^{(1)}, \ldots, X_{in_1}^{(1)}) \equiv T_{in_1}^{(1)},\ i = 1, \ldots, k.$$

Determine the subset I of $\{1, \ldots, k\}$ where

$$I = \{i | T_{in_1}^{(1)} \geq \max\{T_{[k-m+1]n_1}^{(1)}, h_{n_1}^{-1}(T_{[k]n_1}^{(1)})\}\ \},$$

and

$$T_{[1]n_1}^{(1)} \leq \cdots \leq T_{[k]n_1}^{(1)}$$

denotes the ordered $T_{in_1}^{(1)}$. Denote by π_I the associated subset of $\{\pi_1, \ldots \pi_k\}$.

1. If π_I consists of one population, stop sampling and assert that the population associated with $T_{[k]n_1}^{(1)}$ is best.
2. If π_I consists of more than one population, proceed to the second stage.

Stage 2: Take n_2 additional independent observations $X_{ij}^{(2)}$, $j = 1, \ldots, n_2$, from each population in π_I, and compute the cumulative estimates

$$T_n(X_{i1}^{(1)}, \ldots, X_{in_1}^{(1)}, X_{i1}^{(2)}, \ldots, X_{in_2}^{(2)}) \equiv T_{in} \quad for\ i \in I,$$

where $n = n_1 + n_2$. Assert that the population associated with $\max_{i \in I} T_{in}$ is best using randomization to break ties if necessary.

There is an infinite number of combinations of (n_1, n_2, h_{n_1}) for given k, m, P^* and $p(\cdot)$ guaranteeing the required probability condition (75), and different design criteria lead to different choices.

Let S' denote the cardinality of the set I in the first stage of procedure $\mathcal{P}_2'$ and let

$$S = \begin{cases} 0; & \text{if } S' = 1 \\ S'; & \text{if } S' > 1. \end{cases} \tag{76}$$

Then the total sample size required by $\mathcal{P}_2'$, TSS say, is

$$TSS = kn_1 + Sn_2.$$

Let $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ denote the expected total sample size for $\mathcal{P}_2'$ under $\vec{\lambda}$. To make a choice of (n_1, n_2, h_{n_1}) as well as to have the total sample size TSS small, we adopt the following minimax design criterion.

For given k, m, $p(\cdot)$ and P^* choose (n_1, n_2, h_{n_1}) to

$$\begin{array}{ll} \textit{minimize} & \sup_{\vec{\lambda} \in \Omega} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] \\ \textit{subject to} & \inf_{\vec{\lambda} \in \Omega(\delta)} P_{\vec{\lambda}}[CS|\mathcal{P}_2'] \geq P^*, \end{array} \tag{77}$$

where (n_1, n_2) are non-negative integers and h_{n_1} is a real function defined as before.

5.3 A lower bound on the probability of a correct selection

The so called LFC of the population parameters for general two-stage procedures has not been determined yet. Moreover, even if the LFC of the population parameters were known, the problem of evaluating the probability of a correct selection associated with $\mathcal{P}_2'$ when the population parameters are in that configuration would still remain.

However it is possible to determine a set of constants (n_1, n_2, h_{n_1}) (although not the best set) to implement $\mathcal{P}_2'$ if a lower bound to the probability of a correct selection can be found and the LFC of the population parameters can be determined for that lower bound. Such a set of constants provides a conservative solution to the problem since it overprotects the experimenter with respect to the probability requirement (75), this overprotection being purchased at the expense of an increase in $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$. In this section we consider the problem of a lower bound on the probability of a correct selection and the infimum of the lower bound.

First we derive a lower bound for $P_{\vec{\lambda}}[CS|\mathcal{P}_2']$ in Theorem 5.1. This lower bound is particularly useful since it achieves its infimum over $\Omega(p)$ at $\Omega^0(p)$. This result permits us to construct a conservative two-stage procedure which guarantees the probability requirement (75).

Lemma 5.1 *For any $\vec{\lambda} \in \Omega$, we have*

$$\begin{aligned} & P_{\vec{\lambda}}[T^{(1)}_{(k)n_1} \geq \max\{T^{(1)}_{[k-m+1]n_1}, h^{-1}_{n_1}(T^{(1)}_{[k]n_1})\}\ ] \\ &= \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \in S^p_\nu(k)} G^{(j)}_{n_1}(y) \\ &\cdot \prod_{j \in \overline{S^p_\nu}(k)} \{G^{(j)}_{n_1}(h_{n_1}(y)) - G^{(j)}_{n_1}(y)\} dG^{(k)}_{n_1}(y) \end{aligned} \tag{78}$$

and

$$P_{\vec{\lambda}}[T_{(k)n} \geq T_{(j)n},\ j < k] = \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} G^{(j)}_n(y) dG^{(k)}_n(y), \tag{79}$$

where $n = n_1 + n_2$,

$$\{S^i_j(l), \quad j = 1, \ldots, \binom{k-1}{i}\}$$

denote the collection of subsets of size i from

$$u(l) = \{1, \ldots, k\} - \{l\},$$

$$\overline{S}^i_j(l) = u(l) - S^i_j(l)$$

and

$$G^{(j)}_n(y) = G_n(y|\lambda_{[j]}).$$

Proof

The proof of Lemma 5.1 is in Santner (1973). □

Theorem 5.1 *For any $\vec{\lambda} \in \Omega$, we have*

$$P_{\vec{\lambda}}[CS|\mathcal{P}_2'] \geq \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \in S_\nu^p(k)} G_{n_1}^{(j)}(y) \cdot \prod_{j \in \overline{S_\nu^p}(k)} \{G_{n_1}^{(j)}(h_{n_1}(y)) - G_{n_1}^{(j)}(y)\} dG_{n_1}^{(k)}(y) + \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} G_n^{(j)}(y) dG_n^{(k)}(y) - 1. \qquad (80)$$

Proof

$$\begin{aligned} P_{\vec{\lambda}}[CS|\mathcal{P}_2'] &= P_{\vec{\lambda}}[T_{(k)n_1}^{(1)} \geq \max\{T_{[k-m+1]n_1}^{(1)}, h_{n_1}^{-1}(T_{[k]n_1}^{(1)})\}, \\ &\quad T_{(k)n} \geq \max_{j \in I} T_{(j)n}] \\ &\geq P_{\vec{\lambda}}[T_{(k)n_1}^{(1)} \geq \max\{T_{[k-m+1]n_1}^{(1)}, h_{n_1}^{-1}(T_{[k]n_1}^{(1)})\}, \\ &\quad T_{(k)n} \geq T_{(j)n},\ j < k] \\ &\geq P_{\vec{\lambda}}[T_{(k)n_1}^{(1)} \geq \max\{T_{[k-m+1]n_1}^{(1)}, h_{n_1}^{-1}(T_{[k]n_1}^{(1)})\}] \\ &\quad + P_{\vec{\lambda}}[T_{(k)n} \geq T_{(j)n},\ j < k] - 1, \qquad (81) \end{aligned}$$

and hence the result comes from Lemma 5.1. □

Next the infimum of the lower bound will be considered in Theorem 5.2. Lemma 4.1 due to Mahamunulu (1967) and Alam and Rizvi (1966) will be needed again.

Lemma 5.2 *For any $\vec{\lambda} \in \Omega(p)$, we have*

$$\inf_{\vec{\lambda} \in \Omega(p)} P_{\vec{\lambda}}[T_{(k)n_1}^{(1)} \geq \max\{T_{[k-m+1]n_1}^{(1)}, h_{n_1}^{-1}(T_{[k]n_1}^{(1)})\}] = \inf_{\lambda \in \Lambda'} \Psi_1(\lambda, n_1),$$

where

$$\Psi_1(\lambda, n_1) = \int_{-\infty}^{\infty} \{G_{n_1}(h_{n_1}(y)|p(\lambda))\}^{k-1} \cdot I\{\frac{G_{n_1}(y|p(\lambda))}{G_{n_1}(h_{n_1}(y)|p(\lambda))}; k-m, m\} dG_{n_1}(y|\lambda)$$

and

$$\inf_{\vec{\lambda} \in \Omega(p)} P_{\vec{\lambda}}[T_{(k)n} \geq T_{(j)n},\quad j < k] = \inf_{\lambda \in \Lambda'} \Psi_2(\lambda, n),$$

where

$$\Psi_2(\lambda, n) \;=\; \int_{-\infty}^{\infty} \{G_n(y|p(\lambda))\}^{k-1} dG_n(y|\lambda).$$

Proof

The first part of this lemma was proved in Santner (1975). To prove the second part, it suffices to show that for all $\vec{\lambda} \in \Omega(p)$,

$$P_{\vec{\lambda}}[T_{(k)n} \geq T_{(j)n}, \;\; j < k] \geq \inf_{\lambda \in \Lambda'} \Psi_2(\lambda, n).$$

Define

$$\eta(\mathbf{T}) = \begin{cases} 1; & \text{if } T_{(k)n} \geq T_{(j)n}, \;\; j < k \\ 0; & \text{otherwise.} \end{cases}$$

Then

$$P_{\vec{\lambda}}[T_{(k)n} \geq T_{(j)n}, \;\; j < k] = E_{\vec{\lambda}}[\eta(\mathbf{T})].$$

By Lemma 4.1, it suffices to show that $\eta(\mathbf{T})$ is non-increasing in $T_{(l)n}$ for all $l < k$. Let us define $\mathbf{T}'$ such that

$$T'_{(l)n} > T_{(l)n} \;\; and \;\; T'_{(j)n} = T_{(j)n} \;\; \forall\, j \neq l, \; j < k.$$

Then it suffices to show that $\eta(\mathbf{T}) = 0$ implies $\eta(\mathbf{T}') = 0$. Suppose that $\eta(\mathbf{T}) = 0$. However,

$$\eta(\mathbf{T}) = 0$$

if and only if

$$T_{(k)n} < T_{(j)n} \;\; for\ some \;\; j < k$$

and this implies that

$$T_{(k)n} < T_{(l)n} \;\; \Longrightarrow T_{(k)n} < T'_{(l)n}$$

or

$$T_{(k)n} < T_{(j)n}, \;\; j \neq l \;\; \Longrightarrow T_{(k)n} < T'_{(j)n}, \;\; j \neq l.$$

Hence, both cases imply $\eta(\mathbf{T}') = 0$. So we get

$$\begin{aligned} & P_{\vec{\lambda}}[T_{(k)n} \geq T_{(j)n}, \quad j < k] \\ \geq & \int_{-\infty}^{\infty} \{G_n(y|p(\lambda_{[k]}))\}^{k-1} dG_n(y|\lambda_{[k]}) \\ = & \Psi_2(\lambda_{[k]}, n) \\ \geq & \inf_{\lambda \in \Lambda'} \Psi_2(\lambda, n), \end{aligned}$$

which completes the proof. □

From Theorem 5.1 and Lemma 5.2, we can get the following result about the infimum of the lower bound.

Theorem 5.2 *For any $\vec{\lambda} \in \Omega(p)$, we have*

$$\inf_{\vec{\lambda} \in \Omega(p)} P_{\vec{\lambda}}[CS|\mathcal{P}_2'] \geq \inf_{\lambda \in \Lambda'} \Psi_1(\lambda, n_1) + \inf_{\lambda \in \Lambda'} \Psi_2(\lambda, n) - 1, \tag{82}$$

where $\Psi_1(\lambda, n_1)$ and $\Psi_2(\lambda, n)$ are defined in Lemma 5.2.

Remark 5.1 *For the special cases of location and scale parameter problems, the infimum of the lower bound is independent of λ.*

(1). Location parameter case; In this case,

$$G_n(x) = G_n(x - \lambda), \quad -\infty < x, \lambda < \infty,$$

the usual choice of $h_n(\cdot)$ is

$$h_n(x) = x + d_n, \quad d_n > 0,$$

and the preference zone is given by

$$p(\lambda) = \lambda - \delta, \quad \delta > 0,$$

that is,

$$\Omega(p) = \Omega(\delta) = \{\vec{\lambda} | \lambda_{[k]} - \lambda_{[k-1]} \geq \delta\}.$$

Then $\Psi_1(\lambda, n_1)$ and $\Psi_2(\lambda, n)$ are given by

$$\begin{aligned} \Psi_1(\lambda, n_1) = \Psi_1(\delta, d_{n_1}, n_1) &= \int_{-\infty}^{\infty} \{G_{n_1}(y + d_{n_1} + \delta)\}^{k-1} \\ & \cdot I\{\tfrac{G_{n_1}(y+\delta)}{G_{n_1}(y+d_{n_1}+\delta)}; k - m, m\} dG_{n_1}(y) \end{aligned} \tag{83}$$

and

$$\Psi_2(\lambda, n) = \Psi_2(\delta, n) = \int_{-\infty}^{\infty} \{G_n(y+\delta)\}^{k-1} dG_n(y), \tag{84}$$

where $G_n(\cdot)$ *is the cdf of* T_{in} *when* $\lambda_i = 0$.

(2). Scale parameter case; For this case,

$$G_n(x) = G_n(x/\lambda), \quad x \geq 0, \ \lambda \geq 0,$$

the usual choice of $h_n(\cdot)$ *is*

$$h_n(x) = c_n x, \quad c_n \geq 1,$$

and the preference zone is given by

$$p(\lambda) = \lambda/\delta, \quad \delta > 1,$$

that is,

$$\Omega(p) = \Omega(\delta) = \{\vec{\lambda} | \lambda_{[k]} \geq \delta \lambda_{[k-1]}\}.$$

Then $\Psi_1(\lambda, n_1)$ *and* $\Psi_2(\lambda, n)$ *are given by*

$$\begin{aligned} \Psi_1(\lambda, n_1) = \Psi_1(\delta, c_{n_1}, n_1) &= \int_{-\infty}^{\infty} \{G_{n_1}(c_{n_1}\delta y)\}^{k-1} \\ &\cdot I\{\tfrac{G_{n_1}(\delta y)}{G_{n_1}(c_{n_1}\delta y)}; k-m, m\} dG_{n_1}(y) \end{aligned} \tag{85}$$

and

$$\Psi_2(\lambda, n) = \Psi_2(\delta, n) = \int_{-\infty}^{\infty} \{G_n(\delta y)\}^{k-1} dG_n(y), \tag{86}$$

where $G_n(\cdot)$ *is the cdf of* T_{in} *when* $\lambda_i = 1$.

5.4 Expected total sample size for $\mathcal{P}_2'$

In order to solve the non-linear optimization problem (77) we will first find an analytical expression for the $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ and then determine $\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$. Note that the total sample size TSS can be written as

$$TSS = kn_1 + Sn_2$$

where S is defined in (76). The result concerning a general expression for the $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ is summarized in the following theorem.

Theorem 5.3 *For any $\vec{\lambda} \in \Omega$, we have*

$$\begin{aligned} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] &= kn_1 + n_2 \sum_{i=1}^{k} [\sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \in S_\nu^p(i)} G_{n_1}^{(j)}(y) \\ &\cdot \prod_{j \in \overline{S_\nu^p}(i)} \{G_{n_1}^{(j)}(h_{n_1}(y)) - G_{n_1}^{(j)}(y)\} dG_{n_1}^{(i)}(y) \\ &- \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) dG_{n_1}^{(i)}(y)]. \end{aligned} \tag{87}$$

Proof

$$\begin{aligned} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] &= E_{\vec{\lambda}}[(kn_1 + n_2 S)|\mathcal{P}_2'] \\ &= kn_1 + n_2 E_{\vec{\lambda}}[S|\mathcal{P}_2'] \\ &= kn_1 + n_2 \{E_{\vec{\lambda}}[S'|\mathcal{P}_2'] - P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']\}. \end{aligned} \tag{88}$$

Now for any $\vec{\lambda} \in \Omega$,

$$\begin{aligned} E_{\vec{\lambda}}[S'|\mathcal{P}_2'] &= \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \in S_\nu^p(i)} G_{n_1}^{(j)}(y) \\ &\cdot \prod_{j \in \overline{S_\nu^p}(i)} \{G_{n_1}^{(j)}(h_{n_1}(y)) - G_{n_1}^{(j)}(y)\} dG_{n_1}^{(i)}(y) \end{aligned} \tag{89}$$

from Theorem (5.1) in Santner (1975). Hence it suffices to show that

$$P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2'] = \sum_{i=1}^{k} \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) dG_{n_1}^{(i)}(y). \tag{90}$$

Now

$$\begin{aligned} P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2'] &= P_{\vec{\lambda}}[\textit{exactly one population is selected}|\mathcal{P}_2'] \\ &= \sum_{i=1}^{k} P_{\vec{\lambda}}[\pi_{(i)} \textit{ is the only one selected}]. \end{aligned}$$

However,

$$[\pi_{(i)} \textit{ is the only one selected}]$$

iff

$$[h_{n_1}^{-1}(T_{(i)n_1}^{(1)}) \geq T_{(j)n_1}^{(1)}, \ \forall j \neq i].$$

Hece the result in (90) holds. □

Next we will consider the maximum value of $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ over Ω. Conditions which guarantee that the supremum of $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ in Ω occurs at some point

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$$

for which $\lambda_{[1]} = \lambda_{[k]}$ are given in Theorem 5.4 and a general expression of the supremum of $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ in Ω under these conditions is given in Corollary 5.1.

The following regularity conditions will be assumed in some of the theorems that follow.

(i). $E_n^\lambda = E_n$ for all $\lambda \in \Lambda$.

(ii). For any $[\lambda_1, \lambda_2] \subset \Lambda$ there exists $e(y)$ possibly depending on λ_1 and λ_2 such that

$$|\partial G_n(y|\lambda)/\partial\lambda| \leq e(y) \quad \forall\, \lambda \in [\lambda_1, \lambda_2],$$

where

$$\{\int_{-\infty}^{\infty} e(y) dG_n(h_{n_1}(y)|\lambda')\}\{\int_{-\infty}^{\infty} e(y) dG_n(y|\lambda')\} < \infty \quad \forall\, \lambda' \geq \lambda_2. \tag{91}$$

Santner (1975) proved the following lemma in which conditions are given which guarantee that the supremum of $E_{\vec{\lambda}}[S'|\mathcal{P}_2']$ in Ω occurs at some point

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$$

for which $\lambda_{[1]} = \lambda_{[k]}$ where S' is the cardinality of the set I in the first stage of $\mathcal{P}_2'$.

Lemma 5.3 *If regularity conditions (91) are satisfied and for all λ_1, λ_2 in Ω with $\lambda_1 \leq \lambda_2$*

$$\begin{aligned}&\frac{\partial G_{n_1}(h_{n_1}(y)|\lambda_1)}{\partial\lambda_1} g_{n_1}(y|\lambda_2)\\ &- \frac{\partial G_{n_1}(y|\lambda_1)}{\partial\lambda_1} g_{n_1}(h_{n_1}(y)|\lambda_2)\frac{dh_{n_1}(y)}{dy} \geq 0 \quad a.e.\ in\ E_{n_1},\end{aligned} \tag{92}$$

then $E_{\vec{\lambda}}[S'|\mathcal{P}_2']$ is non-decreasing in $\lambda_{[1]}$ on

$$\Lambda(\lambda_{[2]}) = \{\lambda \in \Lambda | \lambda \leq \lambda_{[2]}\}$$

for any fixed $(\lambda_{[2]}, \ldots, \lambda_{[k]})$.

We will consider conditions which guarantee that the infimum of $P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']$ occurs at some point

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$$

for which $\lambda_{[1]} = \lambda_{[k]}$ in the following lemma.

Lemma 5.4 *If regularity conditions (91) are satisfied and for all λ_1, λ_2 in Ω with $\lambda_1 \leq \lambda_2$*

$$\begin{aligned} &\frac{\partial G_{n_1}(h_{n_1}^{-1}(y)|\lambda_1)}{\partial \lambda_1} g_{n_1}(y|\lambda_2) \\ - &\frac{\partial G_{n_1}(y|\lambda_1)}{\partial \lambda_1} g_{n_1}(h_{n_1}^{-1}(y)|\lambda_2) \frac{dh_{n_1}^{-1}(y)}{dy} \leq 0 \ \ a.e. \ in \ E_{n_1}, \end{aligned} \tag{93}$$

then $P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']$ is non-increasing in $\lambda_{[1]}$ on

$$\Lambda(\lambda_{[2]}) = \{\lambda \in \Lambda | \lambda \leq \lambda_{[2]}\}$$

for any fixed $(\lambda_{[2]}, \ldots, \lambda_{[k]})$.

Proof

Fix $\lambda_{[2]}, \ldots, \lambda_{[k]}$ for the following argument. Then

$$P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2'] = T_1(\vec{\lambda}) + T_2(\vec{\lambda}),$$

where

$$T_1(\vec{\lambda}) = \int_{E_{n_1}} \prod_{j=2}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) dG_{n_1}^{(1)}(y)$$

and

$$T_2(\vec{\lambda}) = \sum_{i=2}^{k} \int_{E_{n_1}} \prod_{\substack{j=1 \\ j \neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) dG_{n_1}^{(i)}(y).$$

Now $T_2(\vec{\lambda})$ can be rewritten as

$$\begin{aligned} T_2(\vec{\lambda}) &= \sum_{i=2}^{k} \int_{E_{n_1}} G_{n_1}^{(1)}(h_{n_1}^{-1}(y)) \prod_{\substack{j=2 \\ j \neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) dG_{n_1}^{(i)}(y) \\ &= \sum_{i=2}^{k} \int_{E_{n_1}} G_{n_1}^{(1)}(h_{n_1}^{-1}(y)) \prod_{\substack{j=2 \\ j \neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) g_{n_1}^{(i)}(y) dy. \end{aligned}$$

Next integrating $T_1(\vec{\lambda})$ by parts we obtain that

$$\begin{aligned}T_1(\vec{\lambda}) &= \textit{constant with respect to } \lambda_{[1]} \\ &\quad -\int_{E_{n_1}} G_{n_1}^{(1)}(y) \sum_{\substack{j=2\\j\neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) dG_{n_1}^{(i)}(h_{n_1}^{-1}(y)) \\ &= \textit{constant with respect to } \lambda_{[1]} \\ &\quad -\int_{E_{n_1}} G_{n_1}^{(1)}(y) \sum_{\substack{j=2\\j\neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) g_{n_1}^{(i)}(h_{n_1}^{-1}(y)) \frac{dh_{n_1}^{-1}(y)}{dy} dy.\end{aligned}$$

Hence combining terms it follows that

$$\begin{aligned}&P_{\vec{\lambda}}[S'=1|\mathcal{P}_2'] \\ &= \textit{constant with respect to } \lambda_{[1]} \\ &\quad +\sum_{i=2}^{k} \int_{E_{n_1}} \prod_{\substack{j=2\\j\neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) \\ &\quad \cdot\{G_{n_1}^{(1)}(h_{n_1}^{-1}(y)) g_{n_1}^{(i)}(y) - G_{n_1}^{(1)}(y) g_{n_1}^{(i)}(h_{n_1}^{-1}(y)) \frac{dh_{n_1}^{-1}(y)}{dy}\} dy.\end{aligned}$$

and finally

$$\begin{aligned}&\frac{dP_{\vec{\lambda}}[S'=1|\mathcal{P}_2']}{d\lambda_{[1]}} \\ &= \sum_{i=2}^{k} \int_{E_{n_1}} \prod_{\substack{j=2\\j\neq i}}^{k} G_{n_1}^{(j)}(h_{n_1}^{-1}(y)) \cdot \{\frac{\partial G_{n_1}^{(1)}(h_{n_1}^{-1}(y))}{\partial \lambda_{[1]}} g_{n_1}^{(i)}(y) \\ &\quad -\frac{\partial G_{n_1}^{(1)}(y)}{\partial \lambda_{[1]}} g_{n_1}^{(i)}(h_{n_1}^{-1}(y)) \frac{dh_{n_1}^{-1}(y)}{dy}\} dy. \qquad (94)\end{aligned}$$

But (93) gives, for every $i = 2, \ldots, k$,

$$\begin{aligned}&\frac{\partial G_{n_1}^{(1)}(h_{n_1}^{-1}(y))}{\partial \lambda_{[1]}} g_{n_1}^{(i)}(y) \\ &\quad -\frac{\partial G_{n_1}^{(1)}(y)}{\partial \lambda_{[1]}} g_{n_1}^{(i)}(h_{n_1}^{-1}(y)) \frac{dh_{n_1}^{-1}(y)}{dy} \leq 0 \quad \textit{a.e. in } E_{n_1},\end{aligned}$$

and hence (94) is non-positive and this completes the proof. □

Remark 5.2 *We can easily show that (92) and (93) are equivalent by using the transformation* $t = h_{n_1}^{-1}(y)$ *in (93). Hence the conditions*

$$\frac{dE_{\vec{\lambda}}[S'|\mathcal{P}_2']}{d\lambda_{[1]}} \geq 0$$

and

$$\frac{dP_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']}{d\lambda_{[1]}} \leq 0$$

are also equivalent.

From Lemma 5.3, Lemma 5.4 and Remark 5.2 we can get the following theorem in which conditions guaranteeing that the supremum of $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ in Ω occurs at some point

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$$

for which $\lambda_{[1]} = \lambda_{[k]}$ are given.

Theorem 5.4 *Suppose that the regularity conditions (91) are satisfied and for all* λ_1, λ_2 *in* Ω *with* $\lambda_1 \leq \lambda_2$

$$\begin{aligned} &\frac{\partial G_{n_1}(h_{n_1}(y)|\lambda_1)}{\partial \lambda_1} g_{n_1}(y|\lambda_2) \\ &- \frac{\partial G_{n_1}(y|\lambda_1)}{\partial \lambda_1} g_{n_1}(h_{n_1}(y)|\lambda_2) \frac{dh_{n_1}(y)}{dy} \geq 0 \quad a.e. \ in \ E_{n_1}. \end{aligned} \tag{95}$$

Then $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ *is non-decreasing in* $\lambda_{[1]}$ *on*

$$\Lambda(\lambda_{[2]}) = \{\lambda \in \Lambda | \lambda \leq \lambda_{[2]}\}$$

for any fixed $(\lambda_{[2]}, \ldots, \lambda_{[k]})$.

Proof

By noting that

$$E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] = kn_1 + n_2\{E_{\vec{\lambda}}[S'|\mathcal{P}_2'] - P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']\}$$

the result of the theorem is clear from Lemma 5.3, Lemma 5.4 and Remark 5.2. □

Remark 5.3 *Condition (95) reduces to the requirement of MLR in the location or scale parameter problems.*

A general expression of the supremum of the $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ in Ω under the condition (95) is given in the following corollary.

Corollary 5.1 *For every fixed* $\lambda_{[2]} \leq \ldots \leq \lambda_{[k]}$, *if*

$$\frac{dE_{\vec{\lambda}}[S'|\mathcal{P}_2']}{d\lambda_{[1]}} \geq 0$$

for $\lambda_{[1]}$ *in* $\Lambda[\lambda_{[2]}]$, *then*

$$\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] = \sup_{\lambda\in\Lambda} \gamma(\lambda, n_1),$$

where

$$\begin{aligned}
&\gamma(\lambda, n_1) \\
&= kn_1 + kn_2 \int_{-\infty}^{\infty} [\{G_{n_1}(h_{n_1}(y)|\lambda)\}^{k-1} I\{\frac{G_{n_1}(y|\lambda)}{G_{n_1}(h_{n_1}(y)|\lambda)}; k-m, m\} \\
&\quad -\{G_{n_1}(h_{n_1}^{-1}(y)|\lambda)\}^{k-1}] dG_{n_1}(y|\lambda). \qquad (96)
\end{aligned}$$

Furthermore, if the hypotheses of Theorem 5.4 hold for $\lambda_1 = \lambda_2$, *then* $\gamma(\lambda, n_1)$ *is non-decreasing in* λ *and hence if there is a greatest element* $\lambda_0 \in \Lambda$, *then*

$$\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] = \gamma(\lambda_0, n_1).$$

Proof

This corollary will be proved by using the following three lemmas and Remark 5.2.

□

Lemma 5.5 *For every fixed* $\lambda_{[2]} \leq \ldots \leq \lambda_{[k]}$, *if*

$$\frac{dE_{\vec{\lambda}}[S'|\mathcal{P}_2']}{d\lambda_{[1]}} \geq 0,$$

then

$$\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[S'|\mathcal{P}_2'] = \sup_{\lambda\in\Lambda} \gamma_1(\lambda, n_1),$$

where

$$\gamma_1(\lambda, n_1)$$
$$= k\int_{E_{n_1}^{\lambda}}\{G_{n_1}(h_{n_1}(y)|\lambda)\}^{k-1}I\{\frac{G_{n_1}(y|\lambda)}{G_{n_1}(h_{n_1}(y)|\lambda)}; k-m, m\}dG_{n_1}(y|\lambda). \qquad (97)$$

Furthermore if (95) holds for $\lambda_1 = \lambda_2$ then $\gamma_1(\lambda, n_1)$ is non-decreasing in λ and hence if there is a greatest element $\lambda_0 \in \Lambda$, then

$$\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[S'|\mathcal{P}_2'] = \gamma_1(\lambda_0, n_1).$$

Proof

Santner (1975) proved this lemma by using the following lemma due to Gupta and Panchapakesan (1972) which gives sufficient conditions for the monotone behavior of the $\gamma_1(\lambda, n_1)$ and hence the proof will be omitted. □

Lemma 5.6 *Let $\{F(\cdot|\lambda)|\lambda \in \Lambda\}$ be a family of absolutely continuous distributions on the real line with continuous density $f(\cdot|\lambda)$ and $\Psi(x, \lambda)$ a bounded real valued function possessing first partial derivatives Ψ_x and Ψ_λ with respect to x and λ respectively and satisfying regularity conditions (99). Then $E_\lambda[\Psi(x, \lambda)]$ is non-decreasing (non-increasing) in λ provided for all $\lambda \in \Lambda$*

$$f(x|\lambda)\frac{\partial\Psi(x,\lambda)}{\partial\lambda} - \frac{\partial F(x|\lambda)}{\partial\lambda}\frac{\partial\Psi(x,\lambda)}{\partial x} \geq (\leq) 0 \text{ for a.e. } x. \qquad (98)$$

The regularity conditions for Lemma 5.6 are given as follows.

(i). For all $\lambda \in \Lambda$, $\frac{\partial\Psi(x,\lambda)}{\partial x}$ is Lebesgue integrable on $\Re$.

(ii). For every $[\lambda_1, \lambda_2] \subset \Lambda$ and $\lambda_3 \in \Lambda$ there exists $h(y)$ depending only on λ_i, $i = 1, 2, 3$ such that

$$\left|f(x|\lambda_3)\frac{\partial\Psi(x,\lambda)}{\partial\lambda} - \frac{\partial F(x|\lambda)}{\partial\lambda}\frac{\partial\Psi(x,\lambda_3)}{\partial x}\right| \leq h(x)\ \forall\lambda \in [\lambda_1, \lambda_2] \qquad (99)$$

and $h(x)$ is Lebesgue integrable on $\Re$.

Lemma 5.7 *If for every fixed $\lambda_{[2]} \leq \ldots \leq \lambda_{[k]}$,*

$$\frac{dP_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']}{d\lambda_{[1]}} \leq 0$$

for $\lambda_{[1]}$ *in* $\Lambda[\lambda_{[2]}]$, *then*

$$\inf_{\vec{\lambda}\in\Omega} P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2'] = \inf_{\lambda\in\Lambda} \gamma_2(\lambda, n_1),$$

where

$$\gamma_2(\lambda, n_1) = k \int_{E_{n_1}^{\lambda}} \{G_{n_1}(h_{n_1}^{-1}(y)|\lambda)\}^{k-1} dG_{n_1}(y|\lambda). \tag{100}$$

Furthermore, if the hypotheses of Theorem 5.4 hold for $\lambda_1 = \lambda_2$, *then* $\gamma_2(\lambda, n_1)$ *is non-increasing in* λ *and hence if there is a greatest element* $\lambda_0 \in \Lambda$, *then*

$$\inf_{\vec{\lambda}\in\Omega} P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2'] = \gamma_2(\lambda_0, n_1).$$

Proof

It suffices to prove for all $q < k$ and fixed

$$\lambda_{[q+1]} \leq \cdots \leq \lambda_{[k]}$$

that

$$P_{\vec{\lambda}(q)}\{S' = 1|\mathcal{P}_2'\}$$

is non-increasing in λ on $\Lambda[\lambda_{[q+1]}]$ where

$$\vec{\lambda}(q) = (\lambda, \ldots, \lambda, \lambda_{[q+1]}, \ldots, \lambda_{[k]}).$$

Let

$$\vec{\lambda}' = (\lambda_{[1]}, \ldots, \lambda_{[k]})$$

and note from (90) that $P_{\vec{\lambda}(q)}\{S' = 1|\mathcal{P}_2'\}$ is invariant under permutations of the elements in $\vec{\lambda}'$. So

$$\begin{aligned}\frac{dP_{\vec{\lambda}(q)}\{S' = 1|\mathcal{P}_2'\}}{d\lambda} &= \sum_{i=1}^{q} \left.\frac{\partial P_{\vec{\lambda}'}\{S' = 1|\mathcal{P}_2'\}}{\partial \lambda_{[i]}}\right|_{\vec{\lambda}(q)} \\ &= q \left.\frac{\partial P_{\vec{\lambda}'}\{S' = 1|\mathcal{P}_2'\}}{\partial \lambda_{[1]}}\right|_{\vec{\lambda}(q)}.\end{aligned}$$

But from the proof of Lemma 5.4,

$$\left.\frac{\partial P_{\vec{\lambda}'}\{S' = 1|\mathcal{P}_2'\}}{\partial \lambda_{[1]}}\right|_{\vec{\lambda}(q)} \leq 0.$$

Hence the infimum over Ω of the $P_{\vec{\lambda}}[S' = 1|\mathcal{P}_2']$ occurs at some point where all the $\lambda_{[i]}$'s are equal. Since

$$\gamma_2(\lambda, n_1) = E[\Psi(y, \lambda)]$$

for

$$\Psi(y, \lambda) = k\{G_{n_1}(h_{n_1}^{-1}(y)|\lambda)\}^{k-1},$$

Lemma 5.6 can be applied and the sufficient condition (98) that $\gamma_2(\lambda, n_1)$ be non-increasing reduces to

$$\frac{\partial G_{n_1}(h_{n_1}^{-1}(y)|\lambda)}{\partial \lambda} g_{n_1}(y|\lambda) - \frac{\partial G_{n_1}(y|\lambda)}{\partial \lambda} g_{n_1}(h_{n_1}^{-1}(y)|\lambda) \frac{dh_{n_1}^{-1}(y)}{dy} \leq 0 \ \ \forall\, \lambda, \ a.e.\ y.$$

The final part of the result is obvious. □

Remark 5.4 *For the cases of location or scale parameter problems the supremum of the $E_{\vec{\lambda}}[TSS|\mathcal{P}_2']$ in Ω is independent of $\vec{\lambda}$ provided the conditions in Theorem 5.4 and Corollary 5.1. Under the same framework of Remark 5.1, we have*

(1). For the location parameter case:

$$\begin{aligned} &\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] \\ &= kn_1 + kn_2 \int_{-\infty}^{\infty} [\{G_{n_1}(y + d_{n_1})\}^{k-1} I\{\tfrac{G_{n_1}(y)}{G_{n_1}(y+d_{n_1})}; k-m, m\} \\ &\quad -\{G_{n_1}(y - d_{n_1})\}^{k-1}] dG_{n_1}(y), \end{aligned} \tag{101}$$

where $G_n(\cdot)$ is the cdf of the T_{in} when $\lambda_i = 0$.

(2). For the scale parameter case:

$$\begin{aligned} &\sup_{\vec{\lambda}\in\Omega} E_{\vec{\lambda}}[TSS|\mathcal{P}_2'] \\ &= kn_1 + kn_2 \int_{-\infty}^{\infty} [\{G_{n_1}(c_{n_1} y)\}^{k-1} I\{\tfrac{G_{n_1}(y)}{G_{n_1}(c_{n_1} y)}; k-m, m\} \\ &\quad -\{G_{n_1}(y/c_{n_1})\}^{k-1}] dG_{n_1}(y), \end{aligned} \tag{102}$$

where $G_n(\cdot)$ is the cdf of the T_{in} when $\lambda_i = 1$.

5.5 An optimization problem and the performance of $\mathcal{P}_2'$

In this section we first consider the optimization problem (77), which one must solve in order to determine (n_1, n_2, h_{n_1}) which is necessary to implement $\mathcal{P}_2'$ and then we consider the performance of $\mathcal{P}_2'$ relative to the corresponding single-stage procedure in terms of the total number of sample sizes needed.

As we noted in Section 5.3, the LFC of the parameter vector $\vec{\lambda}$ in $\Omega(p)$ has not been determined in the general case and hence we replace the exact $\inf_{\vec{\lambda}\in\Omega} P_{\vec{\lambda}}[CS|\mathcal{P}_2']$ by the conservative lower bound on that probability given by the right hand side of (82). For the special case of location parameter problems under some appropriate conditions the optimization problem (77) can be written as follows.

For given k, m, δ and P^* choose integers n_1 and n_2 and a real $d_{n_1} > 0$ to

$$\begin{aligned}
\textit{minimize} \quad & kn_1 + kn_2 \int_{-\infty}^{\infty} [\{G_{n_1}(y + d_{n_1})\}^{k-1} I\{\tfrac{G_{n_1}(y)}{G_{n_1}(y+d_{n_1})}; k-m, m\} \\
& -\{G_{n_1}(y - d_{n_1})\}^{k-1}] dG_{n_1}(y) \\
\textit{subject to} \quad & \int_{-\infty}^{\infty} \{G_{n_1}(y + d_{n_1} + \delta)\}^{k-1} \\
& \cdot I\{\tfrac{G_{n_1}(y+\delta)}{G_{n_1}(y+d_{n_1}+\delta)}; k-m, m\} dG_{n_1}(y) \\
& + \int_{-\infty}^{\infty} \{G_{(n_1+n_2)}(y+\delta)\}^{k-1} dG_{(n_1+n_2)}(y) - 1 \geq P^*, \qquad (103)
\end{aligned}$$

where $G_n(\cdot)$ is the cdf of the T_{in} when $\lambda_i = 0$.

For the case of scale parameter problems under some appropriate conditions the optimization problem (77) can be written as follows. For given k, m, δ and P^* choose integers n_1 and n_2 and a real $c_{n_1} > 1$ to

$$\begin{aligned}
\textit{minimize} \quad & kn_1 + kn_2 \int_{-\infty}^{\infty} [\{G_{n_1}(c_{n_1} y)\}^{k-1} I\{\tfrac{G_{n_1}(y)}{G_{n_1}(c_{n_1} y)}; k-m, m\} \\
& -\{G_{n_1}(y/c_{n_1})\}^{k-1}] dG_{n_1}(y) \\
\textit{subject to} \quad & \int_{-\infty}^{\infty} \{G_{n_1}(c_{n_1}\delta y)\}^{k-1} \\
& \cdot I\{\tfrac{G_{n_1}(\delta y)}{G_{n_1}(c_{n_1}\delta y)}; k-m, m\} dG_{n_1}(y) \\
& + \int_{-\infty}^{\infty} \{G_{(n_1+n_2)}(\delta y)\}^{k-1} dG_{(n_1+n_2)}(y) - 1 \geq P^*, \qquad (104)
\end{aligned}$$

where $G_n(\cdot)$ is the cdf of the T_{in} when $\lambda_i = 1$.

As a measure of the efficiency of $\mathcal{P}_2'$ relative to that of the corresponding single-stage procedure when both guarantee the same probability requirement (75), we consider the ratio, termed relative efficiency(RE),

$$\frac{E_{\vec{\lambda}}[TSS|\mathcal{P}_2']}{kn_s}$$

where n_s is the sample size needed in the single-stage procedure.

5.6 Applications

In this section we apply the results of previous sections to a problem of selecting the population with the largest mean from k univariate normal populations.

Suppose that

$$\pi_i \sim N(\mu_i, \sigma^2), \; i = 1, \ldots, k,$$

where the common variance σ^2 is known and the experimenter is interested in selecting the population having largest μ_i. We take

$$T_{in} = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \equiv \overline{X}_{in}, \quad \lambda_i = \mu_i.$$

Then

$$G_n(y|\lambda_i) = \Phi(\frac{\sqrt{n}(y - \mu_i)}{\sigma}),$$

where Φ is the cdf of a $N(0,1)$ random variable. Since this is a location parameter problem, we take

$$p(\mu) = \mu - \delta, \quad \delta > 0$$

and

$$h_n(x) = x + \frac{h\sigma}{\sqrt{n}}$$

so that

$$\Omega(p) = \{\vec{\mu}|\mu_{[k]} - \mu_{[k-1]} \geq \delta\}.$$

Noting that the distribution of the mean of a sample from a normal population has MLR with respect to the location parameter, and using Theorem 5.2 and Remark 5.1

it can be seen that

$$\inf_{\vec{\mu}\in\Omega(\delta)} P_{\vec{\mu}}[CS|\mathcal{P}_2']$$
$$\geq \int_{-\infty}^{\infty} \{\Phi(y+h+\delta\sqrt{n_1}/\sigma)\}^{k-1}$$
$$\cdot I\{\tfrac{\Phi(y+\delta\sqrt{n_1}/\sigma)}{\Phi(y+h+\delta\sqrt{n_1}/\sigma)}; k-m, m\} d\Phi(y)$$
$$+\int_{-\infty}^{\infty} \{\Phi(y+\delta\sqrt{n_1+n_2}/\sigma)\}^{k-1} d\Phi(y) - 1, \tag{105}$$

and using Corollary 5.1 and Remark 5.4,

$$\sup_{\vec{\mu}\in\Omega} E_{\vec{\mu}}[TSS|\mathcal{P}_2']$$
$$= kn_1 + kn_2 \int_{-\infty}^{\infty} [\{\Phi(y+h)\}^{k-1} I\{\tfrac{\Phi(y)}{\Phi(y+h)}; k-m, m\}$$
$$-\{\Phi(y-h)\}^{k-1}] d\Phi(y). \tag{106}$$

Hence the conservative non-linear optimization problem can be reduced to finding integers n_1, n_2 and a real number $h > 0$ to

$$\textit{minimize} \quad kn_1 + kn_2 \int_{-\infty}^{\infty} [\{\Phi(y+h)\}^{k-1} I\{\tfrac{\Phi(y)}{\Phi(y+h)}; k-m, m\}$$
$$-\{\Phi(y-h)\}^{k-1}] d\Phi(y) \tag{107}$$

$$\textit{subject to} \quad \int_{-\infty}^{\infty} \{\Phi(y+h+\delta\sqrt{n_1}/\sigma)\}^{k-1}$$
$$\cdot I\{\tfrac{\Phi(y+\delta\sqrt{n_1}/\sigma)}{\Phi(y+h+\delta\sqrt{n_1}/\sigma)}; k-m, m\} d\Phi(y)$$
$$+\int_{-\infty}^{\infty} \{\Phi(y+\delta\sqrt{n_1+n_2}/\sigma)\}^{k-1} d\Phi(y) - 1 \geq P^*, \tag{108}$$

for the given values of k, m, δ and P^*.

Table 30, Table 31, Table 32 and Table 33 contain the real valued solutions $(\hat{n}_1, \hat{n}_2, \hat{h})$ of the above optimization problem, which are necessary to approximate the constants (n_1, n_2, h) needed to implement $\mathcal{P}_2'$ for $P^* = 0.75, 0.90, 0.95, 0.99$, $k = 3, 4, 5$, $m = 2, 3, 4 < k$ and $\delta/\sigma = 0.1, 0.5, 1.0, 2.0, 4.0$. All computations were carried out in double-precision arithmetic on a Vax-11/780. The source program in Fortran for the SUMT algorithm given by Kuerter and Mize (1973) was used to solve the non-linear optimization problem and Gauss-Hermite quadrature with twenty nodes was used to compute the integrals.

Using above constants, we can define an elimination type two-stage procedure $\mathcal{P}_2'$ as follows;

Stage 1: Take n_1 independent observations

$$X_{ij}^{(1)},\ j = 1, \ldots, n_1,$$

from each π_i, $i = 1, \ldots, k$, and compute the k sample means

$$\overline{X}_{in_1}^{(1)} = \frac{1}{n_1} \sum_{j=1}^{n_1} X_{ij}^{(1)},\ i = 1, \ldots, k.$$

Determine the subset I of $\{1, \ldots, k\}$ where

$$I = \{i | \overline{X}_{in_1}^{(1)} \geq \max\{\overline{X}_{[k-m+1]n_1}^{(1)}, \overline{X}_{[k]n_1}^{(1)} - h\sigma/\sqrt{n_1}\}\},$$

where

$$\overline{X}_{[1]n_1}^{(1)} \leq \cdots \leq \overline{X}_{[k]n_1}^{(1)}$$

denotes the ordered $\overline{X}_{in_1}^{(1)}$. Denote by π_I the associated subset of $\{\pi_1, \ldots \pi_k\}$.

1. If π_I consists of one population, stop sampling and assert that the population associated with $\overline{X}_{[k]n_1}^{(1)}$ is best.

2. If π_I consists of more than one populations, proceed to the second stage.

Stage 2: Take n_2 additional independent observations $X_{ij}^{(2)}$, $j = 1, \ldots, n_2$, from each population in π_I, and compute the cumulative sample means

$$\overline{X}_{in} = \frac{1}{n}(\sum_{j=1}^{n_1} X_{ij}^{(1)} + \sum_{j=1}^{n_2} X_{ij}^{(2)})\ \forall i \in I,$$

where $n = n_1 + n_2$. Assert that the population associated with $\max_{i \in I} \overline{X}_{in}$ is best using randomization to break the ties if necessary.

The relative efficiency RE of the two-stage procedure $\mathcal{P}_2'$ relative to the corresponding single-stage procedure is given by

$$
\begin{aligned}
RE &= \frac{1}{k\hat{n}_s}[k\hat{n}_1 + \hat{n}_2 \sum_{i=1}^{k} \int_{-\infty}^{\infty} \{ \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \prod_{j \in S_\nu^p(i)} \Phi(y + \sqrt{\hat{n}_1}\delta_{ij}/\sigma) \\
&\quad \cdot \prod_{j \in \overline{S}_\nu^p(i)} \{\Phi(y + \hat{h} + \sqrt{\hat{n}_1}\delta_{ij}/\sigma) - \Phi(y + \sqrt{\hat{n}_1}\delta_{ij}/\sigma)\} \\
&\quad - \prod_{\substack{j=1 \\ j \neq i}}^{k} \Phi(y - \hat{h} + \sqrt{\hat{n}_1}\delta_{ij}/\sigma)\} d\Phi(y)],
\end{aligned}
\tag{109}
$$

where $(\hat{n}_1, \hat{n}_2, \hat{h})$ is the real valued solution of the non-linear optimization problem (107) and (108),

$$
\delta_{ij} = \mu_{[i]} - \mu_{[j]},
$$

$S_\nu^p(i)$ and $\overline{S}_\nu^p(i)$ are defined as in Lemma 5.1 and $\hat{n}_s$ is the real solution to

$$
\int_{-\infty}^{\infty} \{\Phi(z + \sqrt{\hat{n}_s}\delta/\sigma)\}^{k-1} d\Phi(z) = P^*.
$$

Of course, RE depends on δ and P^* as well as $\vec{\mu}$.

Table 34 and Table 35 contain the values of the relative efficiency for the two special cases, namely the equally spaced and slippage configurations, for $P^* = 0.75, 0.90, 0.95, 0.99$, $k = 3, 4, 5$, $m = 2, 3, 4 < k$ and $\delta/\sigma = 0.1, 0.5, 1.0, 2.0, 4.0$. All computations were carried out in double-precision arithmetic on a Vax-11/780 and Gauss-Hermite quadrature with twenty nodes was used to compute the integrals.

From Table 34 and Table 35, we see that for both configurations the values of RE are less than or equal to one except for some smaller values of k, m and P^*. Hence the two-stage procedure is more efficient than the single-stage procedure in terms of the expected total sample sizes. Furthermore, the effectiveness of the two-stage procedure appears to be increasing as each of k, m and P^* increases for fixed values of the others.

Table 30: Constants to implement the two-stage procedure $\mathcal{P}_2'$ for selecting the largest normal population: $P^* = 0.75$.

$P^* = 0.75$						
k	m	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
3	2	0.10	0.2135e+03	0.7261e+02	0.3291e+01	0.785194e+03
		0.50	0.8596e+01	0.2811e+01	0.4954e+01	0.314109e+02
		1.00	0.2150e+01	0.7019e+00	0.4997e+01	0.785274e+01
		2.00	0.5372e+00	0.1758e+00	0.4998e+01	0.196318e+01
		4.00	0.1343e+00	0.4397e-01	0.4999e+01	0.490796e+00
4	2	0.10	0.2445e+03	0.2163e+03	0.1883e+01	0.138729e+04
		0.50	0.9954e+01	0.7900e+01	0.4974e+01	0.556164e+02
		1.00	0.2490e+01	0.1973e+01	0.4998e+01	0.139041e+02
		2.00	0.6219e+00	0.4941e+00	0.4999e+01	0.347603e+01
		4.00	0.1555e+00	0.1234e+00	0.5001e+01	0.869007e+00
	3	0.10	0.2042e+03	0.1266e+03	0.3474e+01	0.119542e+04
		0.50	0.8118e+01	0.5118e+01	0.4964e+01	0.478226e+02
		1.00	0.2029e+01	0.1281e+01	0.4987e+01	0.119557e+02
		2.00	0.5072e+00	0.3200e+00	0.4996e+01	0.298891e+01
		4.00	0.1268e+00	0.7997e-01	0.4999e+01	0.747229e+00
5	2	0.10	0.2717e+03	0.3554e+03	0.1348e+01	0.198250e+04
		0.50	0.1097e+02	0.1242e+02	0.4995e+01	0.796946e+02
		1.00	0.2745e+01	0.3099e+01	0.4999e+01	0.199236e+02
		2.00	0.6861e+00	0.7752e+00	0.5002e+01	0.498091e+01
		4.00	0.1715e+00	0.1939e+00	0.5000e+01	0.124523e+01
	3	0.10	0.2268e+03	0.2159e+03	0.2957e+01	0.177798e+04
		0.50	0.9038e+01	0.8648e+01	0.4985e+01	0.711354e+02
		1.00	0.2259e+01	0.2164e+01	0.4996e+01	0.177839e+02
		2.00	0.5650e+00	0.5403e+00	0.5002e+01	0.444596e+01
		4.00	0.1413e+00	0.1350e+00	0.5001e+01	0.111149e+01
	4	0.10	0.1894e+03	0.1885e+03	0.3067e+01	0.169096e+04
		0.50	0.7827e+01	0.7222e+01	0.3071e+01	0.676307e+02
		1.00	0.1960e+01	0.1780e+01	0.4957e+01	0.169222e+02
		2.00	0.4906e+00	0.4444e+00	0.4989e+01	0.423056e+01
		4.00	0.1225e+00	0.1113e+00	0.4998e+01	0.105764e+01

Table 31: Constants to implement the two-stage procedure $\mathcal{P}_2'$ for selecting the largest normal population: $P^* = 0.90$.

$P^* = 0.90$						
k	m	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
3	2	0.10	0.3736e+03	0.2517e+03	0.1651e+01	0.155980e+04
		0.50	0.1501e+02	0.9920e+01	0.1664e+01	0.623911e+02
		1.00	0.3925e+01	0.1972e+01	0.4992e+01	0.157178e+02
		2.00	0.9815e+00	0.4925e+00	0.4999e+01	0.392945e+01
		4.00	0.2454e+00	0.1231e+00	0.4999e+01	0.982363e+00
4	2	0.10	0.4424e+03	0.3859e+03	0.1420e+01	0.243824e+04
		0.50	0.1774e+02	0.1531e+02	0.1426e+01	0.975289e+02
		1.00	0.4470e+01	0.3344e+01	0.4995e+01	0.245676e+02
		2.00	0.1117e+01	0.8361e+00	0.4997e+01	0.614189e+01
		4.00	0.2793e+00	0.2091e+00	0.5000e+01	0.153547e+01
	3	0.10	0.3468e+03	0.3290e+03	0.2055e+01	0.229674e+04
		0.50	0.1396e+02	0.1303e+02	0.2056e+01	0.918684e+02
		1.00	0.3465e+01	0.3087e+01	0.4166e+01	0.231167e+02
		2.00	0.8632e+00	0.7757e+00	0.4990e+01	0.577984e+01
		4.00	0.2156e+00	0.1942e+00	0.4999e+01	0.144496e+01
5	2	0.10	0.4923e+03	0.4830e+03	0.1419e+01	0.332487e+04
		0.50	0.1962e+02	0.1783e+02	0.4993e+01	0.133728e+03
		1.00	0.4905e+01	0.4453e+01	0.5001e+01	0.334320e+02
		2.00	0.1226e+01	0.1114e+01	0.5003e+01	0.835800e+01
		4.00	0.3064e+00	0.2787e+00	0.5001e+01	0.208950e+01
	3	0.10	0.3891e+03	0.4128e+03	0.2085e+01	0.312417e+04
		0.50	0.1542e+02	0.1609e+02	0.4959e+01	0.125383e+03
		1.00	0.3857e+01	0.4021e+01	0.4994e+01	0.313458e+02
		2.00	0.9643e+00	0.1005e+01	0.5000e+01	0.783644e+01
		4.00	0.2411e+00	0.2513e+00	0.5002e+01	0.195911e+01
	4	0.10	0.3443e+03	0.4035e+03	0.1957e+01	0.313107e+04
		0.50	0.1388e+02	0.1599e+02	0.1953e+01	0.125241e+03
		1.00	0.3472e+01	0.3996e+01	0.1953e+01	0.313102e+02
		2.00	0.8680e+00	0.9989e+00	0.1954e+01	0.782756e+01
		4.00	0.1998e+00	0.2460e+00	0.4995e+01	0.198308e+01

Table 32: Constants to implement the two-stage procedure $\mathcal{P}_2'$ for selecting the largest normal population: $P^* = 0.95$.

$P^* = 0.95$						
k	m	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
3	2	0.10	0.5268e+03	0.3670e+03	0.1390e+01	0.217158e+04
		0.50	0.2109e+02	0.1467e+02	0.1388e+01	0.868630e+02
		1.00	0.5409e+01	0.2915e+01	0.4739e+01	0.220558e+02
		2.00	0.1352e+01	0.7291e+00	0.4997e+01	0.551399e+01
		4.00	0.3381e+00	0.1821e+00	0.4999e+01	0.137850e+01
4	2	0.10	0.6098e+03	0.4848e+03	0.1364e+01	0.326506e+04
		0.50	0.2440e+02	0.1713e+02	0.4988e+01	0.131835e+03
		1.00	0.6099e+01	0.4281e+01	0.4995e+01	0.329587e+02
		2.00	0.1525e+01	0.1071e+01	0.4998e+01	0.823968e+01
		4.00	0.3812e+00	0.2676e+00	0.4999e+01	0.205992e+01
	3	0.10	0.4893e+03	0.4574e+03	0.1755e+01	0.314757e+04
		0.50	0.1955e+02	0.1833e+02	0.1755e+01	0.125903e+03
		1.00	0.4891e+01	0.4579e+01	0.1753e+01	0.314757e+02
		2.00	0.1223e+01	0.1144e+01	0.1756e+01	0.786892e+01
		4.00	0.2905e+00	0.2773e+00	0.4998e+01	0.199393e+01
5	2	0.10	0.6720e+03	0.5620e+03	0.1444e+01	0.437062e+04
		0.50	0.2660e+02	0.2140e+02	0.4981e+01	0.175772e+03
		1.00	0.6649e+01	0.5349e+01	0.4999e+01	0.439431e+02
		2.00	0.1662e+01	0.1338e+01	0.4999e+01	0.109858e+02
		4.00	0.4154e+00	0.3347e+00	0.4999e+01	0.274644e+01
	3	0.10	0.5321e+03	0.5408e+03	0.1890e+01	0.416541e+04
		0.50	0.2125e+02	0.2169e+02	0.1890e+01	0.166616e+03
		1.00	0.5191e+01	0.5312e+01	0.4993e+01	0.418897e+02
		2.00	0.1297e+01	0.1329e+01	0.5000e+01	0.104724e+02
		4.00	0.3243e+00	0.3321e+00	0.4999e+01	0.261811e+01
	4	0.10	0.4994e+03	0.5332e+03	0.1689e+01	0.422098e+04
		0.50	0.1989e+02	0.2144e+02	0.1691e+01	0.168839e+03
		1.00	0.4975e+01	0.5358e+01	0.1690e+01	0.422097e+02
		2.00	0.1244e+01	0.1339e+01	0.1691e+01	0.105524e+02
		4.00	0.2652e+00	0.3431e+00	0.4984e+01	0.269847e+01

Table 33: Constants to implement the two-stage procedure $\mathcal{P}_2'$ for selecting the largest normal population: $P^* = 0.99$.

$P^* = 0.99$						
k	m	δ/σ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
3	2	0.10	0.9227e+03	0.5789e+03	0.1267e+01	0.365524e+04
		0.50	0.3706e+02	0.2288e+02	0.1264e+01	0.146207e+03
		1.00	0.9266e+01	0.5720e+01	0.1265e+01	0.365518e+02
		2.00	0.2289e+01	0.1247e+01	0.4992e+01	0.936116e+01
		4.00	0.5723e+00	0.3117e+00	0.4999e+01	0.234029e+01
4	2	0.10	0.1024e+04	0.6701e+03	0.1388e+01	0.524630e+04
		0.50	0.4042e+02	0.2577e+02	0.1923e+01	0.210684e+03
		1.00	0.1013e+02	0.6256e+01	0.5003e+01	0.530281e+02
		2.00	0.2533e+01	0.1563e+01	0.5001e+01	0.132570e+02
		4.00	0.6332e+00	0.3908e+00	0.4998e+01	0.331426e+01
	3	0.10	0.8580e+03	0.7081e+03	0.1566e+01	0.517573e+04
		0.50	0.3423e+02	0.2841e+02	0.1573e+01	0.207029e+03
		1.00	0.8562e+01	0.7097e+01	0.1572e+01	0.517572e+02
		2.00	0.2141e+01	0.1775e+01	0.1570e+01	0.129393e+02
		4.00	0.4831e+00	0.4575e+00	0.4999e+01	0.330475e+01
5	2	0.10	0.1094e+04	0.7530e+03	0.1538e+01	0.685040e+04
		0.50	0.4353e+02	0.2889e+02	0.5027e+01	0.275422e+03
		1.00	0.1088e+02	0.7224e+01	0.5007e+01	0.688555e+02
		2.00	0.2720e+01	0.1807e+01	0.5001e+01	0.172139e+02
		4.00	0.6800e+00	0.4517e+00	0.4999e+01	0.430347e+01
	3	0.10	0.8880e+03	0.8004e+03	0.1814e+01	0.663844e+04
		0.50	0.3541e+02	0.3219e+02	0.1819e+01	0.265537e+03
		1.00	0.8545e+01	0.8060e+01	0.4998e+01	0.669026e+02
		2.00	0.2136e+01	0.2016e+01	0.4997e+01	0.167257e+02
		4.00	0.5340e+00	0.5038e+00	0.5000e+01	0.418141e+01
	4	0.10	0.8854e+03	0.7841e+03	0.1519e+01	0.680016e+04
		0.50	0.3536e+02	0.3144e+02	0.1521e+01	0.272006e+03
		1.00	0.8841e+01	0.7859e+01	0.1520e+01	0.680015e+02
		2.00	0.2210e+01	0.1964e+01	0.1522e+01	0.170004e+02
		4.00	0.5527e+00	0.4909e+00	0.1520e+01	0.425009e+01

Table 34: Relative efficiency of the two-stage procedure $\mathcal{P}_2'$ for selecting the largest normal population: Equally spaced configuration.

Equally Spaced Configuration							
P^*	k	m	δ/σ				
			0.1	0.5	1.0	2.0	4.0
0.750	3	2	1.25649	1.27244	1.27255	1.27255	1.27255
	4	2	1.09759	1.22689	1.22699	1.22698	1.22701
		3	1.00794	1.04971	1.04999	1.05007	1.05012
	5	2	0.97140	1.16714	1.16716	1.16717	1.16716
		3	0.94499	1.03442	1.03459	1.03468	1.03465
		4	0.83862	0.84180	0.95629	0.95751	0.95789
0.900	3	2	0.89503	0.89718	1.05021	1.05026	1.05026
	4	2	0.83675	0.83802	1.01745	1.01744	1.01745
		3	0.74452	0.74594	0.90249	0.93703	0.93729
	5	2	0.80967	0.98407	0.98415	0.98416	0.98415
		3	0.71673	0.89790	0.89903	0.89921	0.89928
		4	0.64800	0.64984	0.64995	0.64998	0.84687
0.950	3	2	0.80360	0.80374	0.99088	0.99442	0.99445
	4	2	0.77834	0.96054	0.96059	0.96067	0.96069
		3	0.67905	0.67871	0.67879	0.67900	0.89269
	5	2	0.76982	0.93225	0.93246	0.93250	0.93251
		3	0.66193	0.66138	0.84917	0.84946	0.84945
		4	0.61714	0.61578	0.61589	0.61597	0.79319
0.990	3	2	0.73595	0.73814	0.73814	0.93360	0.93380
	4	2	0.73343	0.74272	0.89843	0.89840	0.89822
		3	0.63625	0.63518	0.63545	0.63550	0.81908
	5	2	0.73263	0.87565	0.87525	0.87513	0.87507
		3	0.62080	0.61954	0.77992	0.77986	0.77998
		4	0.60718	0.60639	0.60651	0.60648	0.60666

Table 35: Relative efficiency of the two-stage procedure $\mathcal{P}_2'$ for selecting the largest normal population: Slippage configuration.

Slippage Configuration							
P^*	k	m	δ/σ				
			0.1	0.5	1.0	2.0	4.0
0.750	3	2	1.26655	1.27302	1.27307	1.27307	1.27307
	4	2	1.17637	1.22825	1.22829	1.22827	1.22829
		3	1.04978	1.05599	1.05603	1.05602	1.05604
	5	2	1.08908	1.16892	1.16893	1.16893	1.16893
		3	1.02243	1.03371	1.03371	1.03372	1.03373
		4	0.96786	0.96816	0.98471	0.98475	0.98474
0.900	3	2	0.94591	0.94740	1.05269	1.05270	1.05270
	4	2	0.90927	0.91006	1.02158	1.02155	1.02154
		3	0.86804	0.86831	0.95678	0.96090	0.96090
	5	2	0.88929	0.98914	0.98915	0.98915	0.98915
		3	0.84654	0.91784	0.91788	0.91788	0.91789
		4	0.81689	0.81689	0.81690	0.81695	0.92894
0.950	3	2	0.84638	0.84641	0.99831	0.99946	0.99947
	4	2	0.83358	0.96810	0.96807	0.96811	0.96812
		3	0.78521	0.78512	0.78500	0.78526	0.93630
	5	2	0.82922	0.94108	0.94105	0.94109	0.94110
		3	0.77670	0.77651	0.88763	0.88767	0.88768
		4	0.74590	0.74568	0.74563	0.74571	0.91389
0.990	3	2	0.75649	0.75825	0.75827	0.94765	0.94773
	4	2	0.76078	0.78319	0.91619	0.91619	0.91606
		3	0.69217	0.69180	0.69195	0.69191	0.91102
	5	2	0.76486	0.89414	0.89416	0.89418	0.89414
		3	0.69319	0.69278	0.85922	0.85925	0.85925
		4	0.66898	0.66852	0.66861	0.66868	0.66871

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19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Selection and ranking (more broadly multiple decision) problems arise in many practical situations where the so-called tests of homogeneity do not provide the answers the experimenter wants.

The logistic distribution has been applied in studies of population growth, of mental ability, of bio-assay, of life test data and of biochemical data, but the complete distribution of the sample means and variances of a logistic population has not been obtained yet.

In this paper we study the selection and ranking problems for logistic populations and an elimination type two-stage procedure for selecting the best population using a restricted subset selection rule in its first stage.

Chapter 2 deals with the selection and ranking procedures for logistic populations. An

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excellent approximation to the distribution of the sample means from a logistic population is derived by using the Edgeworth series expansions. Using this approximation, we propose and study a single-stage procedure using the indifference zone approach, two subset selection rules based on sample means and medians respectively for selecting the population with the largest mean from k logistic populations when the common variance is known.

Chapter 3 considers an elimination type two-stage procedure for selecting the population with the largest mean from k logistic populations when the common variance is known. A table of the constants needed to implement this procedure is provided and the efficiency of this procedure relative to the single-stage procedure is investigated.

Chapter 4 deals with a single-stage restricted subset selection procedure for selecting the population with the largest mean from k logistic populations when the common variance is known. Some properties of this procedure such as monotonicity and consistency are investigated. Tables of required sample sizes for this procedure are provided. A new design criterion to get the needed sample size and the constant defining this procedure simultaneously is proposed and a table of these constants is given.

Chapter 5 deals with a more flexible two-stage procedure for selecting the best population, which uses a restricted subset selection rule in its first stage and the Bechhofer's (1954) natural decision procedure in the second stage, in terms of a set of consistent estimators of the real population parameters, whose distributions form a stochastically increasing family for a given sample size.