Heat Kernel, Eigenfunctions, and Conditioned Brownian Motion in Planar Domains

by

Rodrigo Bañuelos* and Burgess Davis**
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Technical Report #87-41

Department of Statistics
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Abstract

Some recent results concerning uniform convergence of the shape of the heat kernel to that of the first eigenfunction, and the lifetime of h processes, in bounded Lipschitz domains in $\mathbb{R}^n$, are extended to all planar domains of finite area.

§1. **Introduction.** Let Ω be a planar domain of finite area and let $\lambda(\Omega) = \lambda$ be the first positive eigenvalue of half the Laplacian, $\frac{1}{2}\Delta$, in Ω. Let $\phi_\Omega = \phi$ be the corresponding first eigenfunction normalized so that $\int_\Omega \phi^2 = 1$ and let $P_t^\Omega(x,y) = P_t(x,y)$ be the fundamental solution for the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$ in $\Omega \times (0,\infty)$ with Dirichlet boundary conditions. For the basic properties of ϕ and $P_t(x,y)$ we refer the reader to [12]. In this paper we prove the following result.

Theorem 1. Let $x \in \Omega$. Then

$$\lim_{t\to\infty} \frac{e^{\lambda t} P_t(x,y)}{\phi(x)\ \phi(y)} = 1 \text{ uniformly in } \{y \in \Omega\}. \tag{1.1}$$

We will see that the following is an immediate consequence of Theorem 1.

Theorem 2. Let h be a positive superharmonic function in Ω. Then $\int_\Omega h(y)\ \phi(y) dy < \infty$, and if $\alpha(t) = \alpha(t, x, h, \Omega)$ denotes the probability that the lifetime of the h process in Ω, started at x, exceeds t, we have

$$\lim_{t\to\infty} e^{\lambda t}\ \alpha(t) = \frac{\phi(x)}{h(x)} \int_\Omega h(y)\ \phi(y)\ dy. \tag{1.2}$$

We recall that the h process associated with the function $h = 1$ on Ω is standard Brownian motion killed when it hits $\partial\Omega$. Thus the following theorem extends (in $\mathbb{R}^2$) a well known result for standard Brownian motion, which follows from Donsker and Varadhan [9], the two dimensional version of which is $\sup\limits_{\substack{x\in\Omega \\ t>0}} e^{\lambda t}\alpha(t,x,1,\Omega) < \infty$.

Theorem 3. Let $\Omega, \phi, \lambda, \alpha$ be as in Theorems 1 and 2. Let $\mathcal{H}$ be the collection of all functions h which are positive and superharmonic in Ω. Then

$$\sup_{\substack{h\in\mathcal{H} \\ x\in\Omega \\ t>0}} e^{\lambda t}\alpha(t, x, h, \Omega) < \infty. \tag{1.3}$$

An analog of Theorem 2 for bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, whose Lipschitz constant is sufficiently small was proved by R. D. De Blassie in [7] and [8]. More recently

C. Kenig and J. Pipher [11] have proved the analog for general Lipschitz domains (and NTA domains). Their paper uses the boundary Harnack inequalities for positive solutions of the heat equation in Lipschitz cylinders to control all the eigenfunctions of $\frac{1}{2}\Delta$ in Ω. We work with only the eigenfunction ϕ and since our domain is arbitrary we do not have the boundary Harnack principle available.

In [3] M. Cranston and T. McConnell give an example of a bounded domain D in $\mathbb{R}^3$ and a nonnegative superharmonic function h on D such that the expected lifetime of the h process, started at any point in D, is infinite. In the notation of Theorem 2, $\int_0^\infty \alpha(t,x,h,D)\,dt = \infty$ for all $x \in D$. Similar examples work in $\mathbb{R}^n$, $n > 3$. Thus the analog of Theorem 2, (and also Theorem 3), for general bounded domains in $\mathbb{R}^n$, $n \geq 3$, strongly fails. Furthermore, Theorem 1 cannot be extended to arbitrary bounded domains in $\mathbb{R}^n$, because, as will be seen, such an extension would imply that Theorem 2 could also be extended.

Theorem 1 comes very close to an analytic formulation of Theorem 2 and an analog for Lipschitz domains D follows from results in [11]. In fact, it is not hard to show using the methods of [11] that for bounded Lipschitz domains, the convergence in the analog of (1.1) is uniform in $(x,y) \in D \times D$. We will sketch an example of a bounded domain Ω in the plane, necessarily not Lipschitz, for which the convergence is not uniform in $(x,y) \in \Omega \times \Omega$. Theorems 1, 2, and 3 are proved in §3. In §4 we present the example and discuss some extensions of the theorems to uniformly elliptic equations.

§2. Notations and Preliminaries. We let Δ be a cementary or trap point defined to be at a distance 1 from all points of the plane and we shall use Z_t, $t \geq 0$, to designate a cadlag process taking values in $\mathbb{R}^2 \cup \Delta$. If D is a domain in $\mathbb{R}^2$ (from now on we will work exclusively in $\mathbb{R}^2$ and all our domains are assumed to have finite area) and h is a nonnegative superharmonic function in D, $P_x^{h,D}$ and $E_x^{h,D}$ will denote the probability and expectation associated with the h process in D started at x. No superscripts, as in P_x and E_x, will indicate standard (unconditioned) two dimensional Brownian motion. We refer the reader to Doob [10] for more information about h processes. Here we recall that if $\tau_D = \inf\{t \geq 0\colon\ Z_t \notin D\}$, the lifetime of the h process in D, then the paths are continuous until τ_D and form a strong Markov process with transition density

$$P_t^h(x,y) = \frac{1}{h(x)}\, P_t^D(x,y)\ h(y) \tag{2.1}$$

where $P_t^D(x,y)$ is the density of standard two dimensional Brownian motion started at x and killed when it leaves D. In the notation of the introduction, $P_t^D(x,y)$ is the Dirichlet heat kernel for $\frac{1}{2}\Delta$ in D. For $r \geq 0$ we define

$$G_r^D(x,y) = \int_r^\infty P_t^D(x,y)\ dt. \tag{2.2}$$

When $r = 0$, this is just the Green function for $\frac{1}{2}\Delta$ in D and in this case we simply write $G^D(x,y)$. If f is an integrable function in D we define the operator T_t^D by

$$T_t^D\ f(x) = \int_D P_t^D(x,y)\ f(y)\ dy \tag{2.3}$$

and observe that

$$T_t^D\ \phi(x) = e^{-\lambda t}\phi(x) \text{ and } T_t^D P_s^D(x,\cdot) = P_{s+t}^D(x,\cdot). \tag{2.4}$$

We will make extensive use of the following result of Cranston and McConnell [3].

Lemma 2.1. There is a constant C_1 ($C_1 = \frac{8}{\pi}$ will work) such that if D is a planar domain of finite area and h is a nonnegative superharmonic function on D then

$$E_x^{h,D}(\tau_D) \leq C_1 \text{ area } (D). \tag{2.5}$$

Now using (2.1) and (2.2) it follows easily that

$$E_x^{h,D}(\tau_D) = \frac{1}{h(x)} \int_D G^D(x,y)h(y)dy$$

and thus (2.5) has the equivalent formulation

$$\frac{1}{h(x)} \int_D G^D(x,y)h(y)dy \leq C_1 \text{ area } (D). \tag{2.6}$$

We should also remark here that Cranston and McConnell only proved (2.5)–(2.6) for h harmonic. It is known that this case easily implies the case when h is superharmonic. We briefly describe the argument. Fix $x_0 \in D$ and let $x \neq x_0$. Put $h(y) = G^D(x_0, y)$. Then h is harmonic in the domain $D' = D\backslash\{x_0\}$. Applying (2.6) to this function h in D' and using the fact that area $(D') =$ area (D) we get

$$\frac{1}{G^D(x_0,x)} \int_D G^D(x,y)G^D(x_0,y)dy \leq C_1 \text{ area } (D). \tag{2.7}$$

Now apply the Riesz decomposition theorem to the positive superharmonic function h, (2.7), and the case for harmonic functions of (2.6) to get the general case.

We also note that it is possible to recover Lemma 2.1, except for the value of C_1, from Theorem 3. For Theorem 3 implies

$$\sup_{x,h} E_x^{h,D}(\tau_D) < \infty \text{ if area } (D) < \infty. \tag{2.8}$$

If Lemma 2.1 did not hold for any value of C_1 then it is not hard to show that it would not hold even for bounded domains, and from this, together with translation and scaling, it is possible to find a sequence of disjoint domains Γ_i, together with x_i, h_i, such that area $(\Gamma_i) = 2^{-i}$ but $E_{x_i}^{h_i,\Gamma_i}(\tau_{\Gamma_i}) \to \infty$. If the Γ_i are connected with fine enough passages, and Γ is taken to be the union of the Γ_i together with the passages, then Γ will violate (2.1).

§3. **Proof of Theorem 1.** We may and do assume $\lambda(\Omega) = 1$. Furthermore we shorten $P_t^\Omega, G^\Omega, G_r^\Omega$ to P_t, G, and G_r. Let $\{Q_j\}$ be a Whitney decomposition of Ω. This is a decomposition of Ω into cubes with the following three properties:

$$Q_j^0 \cap Q_k^0 = \phi, \quad j \neq k \tag{1}$$

$$\frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_k)} \leq 4 \qquad \text{if } Q_j \cap Q_k \neq \phi \tag{2}$$

$$1 \leq \frac{d(Q_j, \partial\Omega)}{\ell(Q_j)} \leq 4\sqrt{2} \qquad \text{for all } j. \tag{3}$$

Here Q_j^0 denotes the interior of Q_i, $d(Q_j, \partial\Omega)$ is the Euclidean distance from Q_i to the boundary of Ω and $\ell(Q_i)$ is the edge length of Q_j. (See [13], p. 167 for the construction of this decomposition.) Let F be a finite union of the Q_j which satisfy

$$C_1 \text{ area } (\Omega - F) < 1/10 \tag{3.1}$$

where C_1 is the constant of Lemma 2.1, and let F' be the union of F together with all the Q_j which touch F. Notice that F' is also a finite union of Whitney cubes and that F is contained in the interior of F'.

The important point here is that we have two connected compact sets F and F' with F satisfying (3.1), $F \subset F'$, and $d(F, \partial F') > 0$. There are, of course, other ways to construct these sets without Whitney decompositions, for example, using level sets of the Green function. The analogue of the following lemma for bounded Lipschitz domains is proved in [11], (see also [5]).

Lemma 3.1. Let $x_0 \in \Omega$. There are constants $0 < c(x_0, \Omega) < C(x_0, \Omega)$ such that for all $y \in \Omega$,

$$c(x_0, \Omega) \min(1, G(x_0, y)) \leq \phi(y) \leq C(x_0, \Omega) G(x_0, y). \tag{3.2}$$

Proof: We may assume $x_0 \in F$. Let $G^{\Omega-F}(x, y)$ be the Green function for $\Omega - F$. Since ϕ is superharmonic in Ω it is also superharmonic in $\Omega - F$. Applying (2.6) we have

$$\frac{1}{\phi(y)} \int_{\Omega - F} G^{\Omega-F}(x, y)\phi(x)dx \leq C_1 \text{ area } (\Omega - F) < \frac{1}{10} \tag{3.3}$$

for $y \in \Omega - F$. Notice that since $G^{\Omega-F}(x, y) = 0$ if $y \notin \Omega - F$, we actually have (3.3) for all $y \in \Omega$.

Set $\psi(y) = \phi(y)/\int_\Omega \phi(y)dy$ and let Z_t be Brownian motion started with the initial distribution ψ. Let $\eta = \inf\{t \colon Z_t \in F\}$. Then

$$\int_{\Omega - F} G(y, x)\psi(x)dx = \int_{\Omega - F} G^{\Omega-F}(y, x)\psi(x)dx + \int_F G(y, x)d\mu(x) \tag{3.4}$$

where μ is the distribution of $Z_\eta I(Z_\eta \in F)$. Note $\eta = 0$ if $Z_0 \in F$.

Now since

$$\psi(y) = \int_\Omega G(y,x)\psi(x)dx = \int_{\Omega - F} G(y,x)\psi(x)dx + \int_F G(y,x)\psi(x)dx$$

it follows from (3.3) and (3.4) that

$$\frac{9}{10}\psi(y) \le \int_F G(y,x)d\mu(x) \tag{3.5}$$

Since $\mu(F) \le 1$, it follows from (3.5) that $G(y,z) \ge \frac{9}{10}\psi(y)$ for some $z \in F$. By the Harnack inequality $G(y,z) \ge \frac{9}{10}\psi(y)$ for $z \in F$, provided $y \in \Omega - F'$ and particularly $G(x_0,y) \ge C\psi(y)$. If $y \in F'$, $G(x_0,y) \ge C\psi(y)$ follows immediately since ψ is bounded above and $G(x_0,y)$ is bounded below on F'. Thus we have proved the right hand side of (3.2). The left hand side is easy to prove and since we do not use it, the proof is omitted.

Lemma 3.2. Let $x_0 \in \Omega$. There exists constants $c = c(x_0,\Omega)$ and $C = C(x_0,\Omega)$ such that

$$c\phi(y) \le P_{2c_0}(x_0,y) \le C\phi(y), \quad y \in \Omega, \tag{3.6}$$

where $c_0 = 3(C_1+1)$.

Proof: We may assume x_0 is in the interior of F and, by enlarging F if necessary, that it is at a positive distance δ to the boundary of F. We may also assume $y \in \Omega - F$. Let $B(x_0,\delta)$ be the ball centered at x_0 and radius δ, $B(x_0,\delta) \subset F$. Since $e^{-t}\phi(y) = \int_\Omega P_t(x,y)\phi(x)dx$ we have

$$\begin{aligned}
\int_{3c_0}^{4c_0} e^{-t}\phi(y)dt &= \int_{3c_0}^{4c_0}\int_\Omega P_t(x,y)\phi(x)dxdt \\
&\ge \int_{3c_0}^{4c_0}\int_{B(x_0,\delta/2)} P_t(x,y)\phi(x)dxdt \\
&\ge CP_{2c_0}(x_0,y)\int_{B(x_0,\delta/2)}\phi(x)dx \\
&= C(x_0,\Omega)P_{2c_0}(x_0,y)
\end{aligned}$$

where the last inequality follows from the parabolic Harnack inequality, (see [10], p. 272). We have proved the right hand side of (3.6).

To prove the left hand side of (3.6) first observe that

$$\begin{aligned}
G_1(x_0,y) &= \int_1^\infty P_t(x_0,y)dt \\
&= \int_0^\infty P_{t+1}(x_0,y)dt = \int_\Omega G(z,y)P_1(x_0,z)dz
\end{aligned} \tag{3.7}$$

by the definition of G and the semigroup property. Thus

$$\begin{aligned} G_1(x_0,y) &\geq \int_{B(x_0,\delta/2)} G(z,y)P_1(x_0,z)dz \\ &\geq \tilde{C} \int_{B(x_0,\delta/2)} G(z,y)dy \end{aligned}$$

where we may take $\tilde{C} = \inf_{z \in B(x_0,\delta/2)} P_1^{B(x_0,\delta)}(x_0,z)$. The previous inequality together with the harmonicity of $G(z,y)$ in $B(x_0,\delta)$ gives

$$G_1(x_0,y) \geq C(x_0,\Omega)G(x_0,y). \tag{3.8}$$

By another application of the Cranston-McConnell result, Lemma 2.1, we have

$$\int_1^{c_0} P_t(x_0,y)dt \geq \frac{1}{2}\, G_1(x_0,y). \tag{3.9}$$

Now the parabolic Harnack inequality together with (3.8), (3.9) and Lemma 3.1 give

$$\begin{aligned} P_{2c_0}(x_0,y) \geq C(x_0,\Omega) \int_1^{c_0} P_t(x_0,y)dt &\geq C(x_0,\Omega)G_1(x_0,y) \\ \geq C(x_0,\Omega)G(x_0,y) &\geq C(x_0,\Omega)\phi(y) \end{aligned}$$

completing the proof of Lemma 3.2.

Now fix x and let $g(t)$ and $f(t)$ be respectively the largest and smallest functions satisfying the following inequality:

$$g(t)e^{-t}\phi(y) \leq P_t(x,y) \leq f(t)e^{-t}\phi(y). \tag{3.10}$$

By (3.6), $0 < g(2c_0) \leq f(2c_0) < \infty$. Furthermore, upon applying the operator T_s^Ω to all sides of (3.10) and using (2.4) we see that g is not decreasing and f is not increasing.

Lemma 3.3. There is a constant $K(x)$ such that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = K(x)$.

Proof: The proof of Lemma 3.2 shows that $c_1\phi(y) \leq P_{2c_0}(x,y) \leq C_2\phi(y)$ with c_1 and C_2 independent of x if x is in the compact set F. Thus

$$\begin{aligned} \int_{\Omega-F} P_{2c_0}(x,y)\phi(x)dx &= \int_\Omega P_{2c_0}(x,y)\phi(x)dx - \int_F P_{2c_0}(x,y)\phi(x)dx \\ &= e^{-2c_0}\phi(y) - \int_F P_{2c_0}(x,y)\phi(x)dx \\ &\leq e^{-2c_0}\phi(y) - c_1\phi(y)\int_F \phi(x)dx \\ &= \delta e^{-2c_0}\phi(y) \end{aligned} \tag{3.11}$$

where $\delta < 1$.

Now set

$$\begin{aligned} P_t(x,y) &= [P_t(x,y) - g(t)\phi(y)e^{-t}] + g(t)\phi(y)e^{-t} \\ &= q_t(x,y) + g(t)\phi(y)e^{-t} \\ &= q_t(y) + g(t)\phi(y)e^{-t}. \end{aligned}$$

By the definition of f and g, $0 \le q_t(y) \le (f(t) - g(t))\phi(y)e^{-t}$. With this observation and (3.11) we have

$$\begin{aligned} T_{2c_0}q_t(y) &= \int_\Omega P_{2c_0}(y,z)q_t(z)dz = \int_{\Omega-F} P_{2c_0}(y,z)q_t(z)dz + \int_F P_{2c_0}(y,z)q_t(z)dz \\ &\le \int_{\Omega-F} P_{2c_0}(y,z)(f(t)-g(t))\phi(z)e^{-t}dz + \int_F P_{2c_0}(y,z)q_t(z)dz \\ &\le \delta(f(t)-g(t))e^{-(t+2c_0)}\phi(y) + \int_F P_{2c_0}(y,z)q_t(z)dz. \end{aligned} \tag{3.12}$$

Now suppose

$$\frac{\delta+1}{2}(f(t)-g(t))e^{-(t+2c_0)}\phi(y) < T_{2c_0}q_t(y).$$

Then (3.12) implies that

$$\left(\frac{\delta+1}{2} - \delta\right)(f(t)-g(t))e^{-(t+2c_0)}\phi(y) < \int_F P_{2c_0}(y,z)q_t(z)dz$$

and thus we have

$$\begin{cases} T_{2c_0}q_t(y) \le \frac{\delta+1}{2}(f(t)-g(t))e^{-(t+2c_0)}\phi(y) \\ \text{or} \\ \left(\frac{\delta+1}{2} - \delta\right)(f(t)-g(t))e^{-(t+2c_0)}\phi(y) < \int_F P_{2c_0}(y,z)q_t(z)dz. \end{cases} \tag{3.13}$$

We will now use (3.13) to show that there are constants $0 < \alpha < 1$ and $0 < \beta < 1$ such that for any t,

$$\begin{cases} f(2c_0+t) < \alpha(f(t)-g(t)) + g(t) \\ \text{or} \\ g(t) + \beta(f(t)-g(t)) < g(2c_0+t). \end{cases} \tag{3.14}$$

From here it follows that

$$f(2c_0+t) - g(2c_0+t) < (f(t)-g(t)) - \min(1-\alpha,\beta)(f(t)-g(t)),$$

which in turn, together with the fact that $f(t) - g(t)$ is a nonincreasing function, proves that $f(t) - g(t) \to 0$ as $t \to \infty$. Since f is nonincreasing and g is nondecreasing this shows that $f(t)$ and $g(t)$ approach a common positive limit which we call $K(x)$. Thus to complete the proof of the lemma we need to show that (3.13) implies (3.14). Suppose

$$T_{2c_0}q_t(y) \leq \frac{1+\delta}{2}(f(t)-g(t))e^{-(t+2c_0)}\phi(y).$$

Then, by the definition of q_t and both parts of (2.4),

$$\begin{aligned} P_{2c_0+t}(x,y) &= T_{2c_0}P_t(x,y) = T_{2c_0}q_t(y) + T_{2c_0}g(t)e^{-t}\phi(y) \\ &= T_{2c_0}q_t(y) + g(t)e^{-t}e^{-2c_0}\phi(y) \\ &\leq e^{-(t+2c_0)}\left\{\frac{1+\delta}{2}(f(t)-g(t)) + g(t)\right\}\phi(y). \end{aligned}$$

Now set $\alpha = \frac{1+\delta}{2}$ and use the definition of f conclude that the first alternative in (3.14) holds. Next suppose the second alternative in (3.13) holds. Then,

$$\begin{aligned} \left(\frac{1+\delta}{2}-\delta\right)(f(t)-g(t))e^{-(t+2c_0)}\phi(y) < \int_F P_{2c_0}(y,z)q_t(z)dz \leq T_{2c_0}q_t(y) \\ = P_{2c_0+t}(x,y) - g(t)\phi(y)e^{-(t+2c_0)} \end{aligned}$$

where we have again used both parts of (2.4). From here we have

$$e^{-(t+2c_0)}\left\{\left(\frac{1+\delta}{2}-\delta\right)(f(t)-g(t)) + g(t)\right\}\phi(y) < P_{2c_0+t}(x,y)$$

and by the definition of g we have the second alternative in (3.14). This completes the proof of the lemma.

To complete the proof of Theorem 1 we must show that $K(x) = \phi(x)$. From the inequality (3.10) and the first part of (2.4) we have,

$$g(t)e^{-t}\int_\Omega \phi^2(y)dy \leq e^{-t}\phi(x) \leq f(t)e^{-t}\int_\Omega \phi^2(y)dy.$$

This inequality together with Lemma 3.3 implies that $K(x) = \phi(x)$ and completes the proof of Theorem 1.

When D is a bounded Lipschitz domain (or NTA domain) the results in [11], (see the remark 3.5), show that if $\epsilon > 0$ then, for $t > \epsilon$,

$$\left|\frac{e^{\lambda_1 t}P_t(x,y)}{\phi(x)\phi(y)} - 1\right| \leq Ce^{-(\lambda_2-\lambda_1)t}$$

where λ_2 is the second eigenvalue and C is a constant. For our case of arbitrary planar domains of finite area our argument shows that for $t > \epsilon$,

$$\left|\frac{e^{\lambda\ t}P_t(x,y)}{\phi(x)\phi(y)} - 1\right| \leq C(x)e^{-ct},$$

where c is independent of x.

We now derive Theorem 2 from Theorem 1. Since $h(Z_t)$ under P_x is a supermartingale we have

$$\int_\Omega h(z)P_t(x,z)dz = E_x(h(Z_t)) \leq E_x(h(Z_0)) = h(x)$$

and so by letting x be any point in Ω satisfying $h(x) < \infty$ we see that

$$\int_\Omega h(z)P_t(x,z)dz < \infty$$

for each $t > 0$. The first assertion of Theorem 2 follows from this and Theorem 1. Next, we have

$$\alpha(t) = P_x^{h,\Omega}\{\tau_\Omega > t\} = \frac{1}{h(x)} \int_\Omega P_t(x,y)h(y)dy$$

and the second assertion follows immediately from this and again Theorem 1.

Next we prove Theorem 3. Let A be a compact subset of Ω such that C_1 area $(\Omega - A) \leq e^{-10}$. The arguments of this section show that

$$\sup_{\substack{h\in\mathcal{H}\\ x\in A\\ t>0}} P_x^h\{\tau_\Omega > t\}e^t = M < \infty. \tag{3.15}$$

Suppose $x \in \Omega - A$. Let $\eta = \inf\{t : Z_t \in A\}$.

Then under P_x^h, Z_t, $0 \leq t < \eta \wedge \tau_\Omega$, where $\wedge$ denotes the minimum, is still an h process in the region $\Omega - A$, up to its lifetime [10]. Thus, by Lemma 2.1,

$$P_x^h(\eta \wedge \tau_\Omega > 1) \leq E_x^h(\eta \wedge \tau_\Omega) \leq C_1 \text{ area } (\Omega - A) \leq e^{-10}.$$

Using the strong Markov property, it is easy to show that

$$P_x^h\{\eta \wedge \tau_\Omega > k\} \leq e^{-10k}, \quad k = 1, 2, \ldots \tag{3.16}$$

See the end of the first section of [6] for this argument. Another application of the strong Markov property and (3.15) give

$$P_x^h\{\tau_\Omega - \eta > t | \mathcal{F}_\eta\} \leq Me^{-t} \tag{3.17}$$

on $\{\eta < \tau_\Omega\}$. The theorem now follows easily from (3.16) and (3.17).

§4. **An example and further remarks.** We begin by briefly describing the example mentioned in the introduction. Let $D_i, i = 1, 2, \ldots$ be disjoint discs in the plane of finite total area. Let x_i be the center of D_i. Let D be the union of all the D_i together with thin connecting passages (tubes). We make the passages so small that for each i, $P_{x_i}\{Z_i \in$

$D_i, i < \tau_D\} \geq i\ P_{x_i}\{Z_i \notin D_i,\ i < \tau_D\}$ if i is even and $P_{x_i}\{Z_{i-1} \in D_i, i-1 < \tau\} \geq (i-1)P_{x_i}\{Z_{i-1} \notin D_{i-1}, i-1 < \tau_D\}$ for i odd, or what is equivalent,

$$\int_{D_i} P_i^D(x_i, y)dy \geq i \int_{D-D_i} P_i^D(x_i, y)dy,$$

if i is even, and a similar formula if i is odd. If i is even, $P_i(x_i, \cdot)$ is concentrated on D_i and $P_i(x_{i+1}, \cdot)$ is concentrated on D_{i+1}, so it is clear that even at time i these functions do not have the same shape at all, and thus clearly they cannot both be uniformly close to $e^{-i\lambda_D}\phi_D$. We leave the details to the reader.

As remarked in [11], the analog of Theorem 1 for bounded Lipschitz domains works equally well for uniformly elliptic operators of the form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) \tag{4.1}$$

where the coefficients a_{ij} are only assumed to be bounded measurable satisfying $a_{ij} = a_{ji}$ and

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

Our results above also remain valid for these type of operators. In fact, all we need to make our arguments work is the analog of the Cranston-McConnell result, Lemma 2.1, for these operators. Such a result is proved in [2] with constant $C_1 = 4/\lambda\pi$.

Several authors have investigated the rate of decay of the eigenfunction ϕ near the boundary for smooth and Lipschitz domains of $\mathbb{R}^n$. (See for example, Aizenman and Simon [1], Davies [4], Davies and Simon [5], as well as the references given there.) The estimate of Lemma 3.1 can also be used to obtain some of these results. For example, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ then the analog of Lemma 3.1 in [11] together with the well known estimates for the Green function give that $c_1(d(x, \partial\Omega))^{\alpha_1} \leq \phi(x) \leq c_2(d(x, \partial\Omega))^{\alpha_2}$ where α_1 and α_2 depend on the Lipschitz character of the domain. Back in the plane, and for bounded simply connected domains, Lemma 3.1 gives a similar result.

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