# Improved Confidence Sets for Spherically Symmetric Distributions *

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## Abstract

The usual confidence set for a multivariate mean vector can be improved upon by recentering the set at a Stein-type estimator: this fact is known to be true under many different distributional assumptions. Thus far, however, the case of unknown variance has not been dealt with analytically. In this paper we prove that recentered set estimators dominate the usual set estimator when sampling is from any of a class of spherically symmetric distributions with unknown variance.

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## I. Introduction.

The problem of estimating the mean of a spherically symmetric distribution (s.s.d.) has begun to receive much attention recently. In particular, the work of Hwang and Chen (1986), in set estimation, and Cellier, Fourdrinier and Robert (CFR) (1988), in point estimation, has greatly added to our knowledge of the problem. This paper is, in a sense, a synthesis of the previously mentioned ones. We adapt the techniques of Hwang and Chen (1986) (which, themselves, are adaptations of the techniques of Hwang and Casella, 1984), to the more general case of s.s.d.'s with unknown variance, as considered by CFR in the point estimation case.

Obtaining results that are valid for a class of s.s.d.'s containing the multivariate-$t$ distribution (but not the normal distribution) has important practical implications. Zellner (1976) has shown that a Student-$t$ model leaves more freedom to the experimenter (through the choice of the number of degrees of freedom) and, still, gives a good approximation of the normal model.

The general case we consider is that of $z = (x', y')'$, an observation from a $p + \nu$ dimensional arbitrary s.s.d. with location parameter $(\theta', 0')'$ and dispersion matrix $\sigma^2 I_{p+\nu}$. The dimensionality of $X$ and $\theta$ are both $p$ (assumed $\geq 3$). For the point estimation problem with $\sigma^2$ unknown, CFR have shown that a rather large class of shrinkage estimators, generalizing those of Judge and Bock (1978), was minimax for every s.s.d. In particular, the positive-part James-Stein estimator

$$\delta_a^+(z) = \left(1 - \frac{a||y||^2}{||x||^2}\right)^+ x \tag{1.1}$$

is minimax for every s.s.d. if $a \leq 2\ (p-2)$.

It is then tempting to try to show that this robustness of shrinkage estimators carries over to the case of confidence sets. Although Casella and Hwang (1983, 1987), and Hwang and Casella (1982, 1984) have very general results in the normal case with $\sigma^2$ known, they only have numerical evidence that the usual confidence interval can be dominated in the unknown variance case (see Casella and Hwang (1987)). For a large class of s.s.d.'s, including the usual ones (normal, multivariate $t$, double exponential), Hwang and Chen (1986) have established some sufficient conditions on $a$ (in 1.1) for the domination of the usual set estimator in the known variance case. (These conditions depend on the s.s.d.)

In this paper, our aim is therefore more modest: we establish that, for a given class of s.s.d.'s (which includes the multivariate-$t$), there exists $a_\circ$ such that, for $a \in (0, a_\circ]$, the usual confidence set can be dominated, no matter which s.s.d. in the class is sampled. Unfortunately, this class does not contain the normal distribution with unknown variance, a point that we discuss in more detail in the last section of the paper.

## II. Domination in the Class of Spherically Symmetric Distributions.

When sampling from a s.s.d., the usual confidence set, based on the $F$-distribution, maintains its coverage probability no matter what s.s.d. is sampled. A direct consequence of the work of Kelker (1970) on the characterizations of s.s.d.'s is the following result:

**Proposition 1.** *Let $F_{\alpha,p,\nu}$ be the $(1-\alpha)$-quantile of an $F$-distribution with $p$ and $\nu$ degrees of freedom. Let $Z = (X', Y')'$ be a $(p+\nu)$ vector following an arbitrary s.s.d. with location* *parameter* $(\theta', 0')'$ *and dispersion matrix* $\sigma^2 I_{p+\nu}$. *Then*

$$C^\circ = \left\{\theta : ||\theta - X||^2 \le \frac{p}{\nu} F_{\alpha,p,\nu} ||Y||^2\right\} \tag{2.1}$$

*is a* $(1-\alpha)$*-confidence interval.*

**Proof.** From Theorem 11 of Kelker (1970), it follows that $\frac{||\theta-X||^2/p}{||Y||^2/\nu}$ has an $F$-distribution with parameters $p$ and $\nu$, independently of the s.s.d. □

Note that this result is true whether or not $\sigma^2$ is known. In the rest of the paper, we will suppose $\sigma^2$ unknown.

Using (1.1), we construct the recentered set estimator

$$C_a^+ = \left\{\theta : ||\theta - \delta_a^+(Z)||^2 \le \frac{p}{\nu} F_{\alpha,p,\nu} ||Y||^2\right\}. \tag{2.2}$$

It is clear, by construction, that Volume $(C_a^+) =$ Volume $(C^\circ)$. Thus, in order to establish dominance, we need to show dominance in coverage probability; that is, $P_{\theta,\sigma^2}(\theta \in C_a^+) \ge P_{\theta,\sigma^2}(\theta \in C^\circ)$ for all $\theta$, $\sigma^2$.

We have

$$P_{\theta,\sigma^2}(\theta \in C_a^+) = \int_{C_a^+} f(||x-\theta||^2 + ||y||^2) dz,$$

if the density of $Z$ factors through $f$. Thus, defining $k^2 = \frac{p}{\nu} F_{\alpha,p,\nu}$, we can write

$$P_{\theta,\sigma^2}(\theta \in C_a^+) = \int_{\mathbb{R}^\nu} \int_{\{x \in \mathbb{R}^p : ||\theta - \delta_a^+(z)||^2 \le k^2 ||y||^2\}} f(||x-\theta||^2 + ||y||^2) dx\ dy.$$

Following the argument in Theorem 2.1 of Hwang and Casella (1982), we have

$$\begin{aligned}
&\int_{\frac{||\theta||}{k}}^{+\infty} (2\pi)^{\nu-1} s^{\nu-1} \int_{\{x : ||\theta - \delta_a^+(x,s^2)||^2 \le k^2 s^2\}} f(||x-\theta||^2 + s^2) dx\ ds \\
&= \int_{\frac{||\theta||}{k}}^{+\infty} (2\pi)^{\nu-1} P_{\theta,\sigma^2}(||\theta - \delta_a^+(x,s^2)||^2 \le k^2 s^2 | s) s^{\nu-1} \phi(s) ds \\
&\ge \int_{\frac{||\theta||}{k}}^{+\infty} (2\pi)^{\nu-1} P_{\theta,\sigma^2}(||\theta - x||^2 \le k^2 s^2 | s) s^{\nu-1} \phi(s) ds, \qquad (2.3)
\end{aligned}$$

where $S = ||Y||$ and $\phi$ is its density function. In fact, due to the convexity of $C^\circ$, $\{x : ||\theta - \delta_a^+(x,s^2)||^2 \le k^2 s^2\}$ contains $\{x : ||\theta - x||^2 \le k^2 s^2\}$ if $||\theta||/s < k$. Integration over $s^2$ then gives inequality (2.3).

Therefore, we only need to consider the integral

$$I(a) = \int_o^{\frac{||\theta||}{k}} (2\pi)^{\nu-1} s^{\nu-1} \int_{\{x : ||\theta - \delta_a^+(x,s^2)||^2 \le k^2 s^2\}} f(||x-\theta||^2 + s^2) dx\ ds, \tag{2.4}$$

and establish conditions under which it is an increasing function of $a$. At $a = 0$, $I(a)$ is equal to the integral obtained for the usual set estimator (2.1). Thus, if we establish a range of $a$ for which $I(a) > I(0)$ for every $\theta$ and every $\sigma^2$, this, together with (2.3), will establish dominance of $C_a^+$ over $C^\circ$.

Using the notation

$$\begin{aligned}
\alpha(r) &= r^2 - 2||\theta||r \cos\beta + ||\theta||^2, \\
r_\pm^\circ &= ||\theta|| \cos\beta \pm \sqrt{k^2s^2 - ||\theta||^2 \sin^2\beta}, \\
r_\pm &= (r_\pm^\circ + \sqrt{(r_\pm^\circ)^2 + 4as^2}\ )/2, \\
\sin\beta_\circ &= \frac{ks}{||\theta||} \quad \text{and} \quad 0 \le \beta_\circ < \frac{\pi}{2},
\end{aligned} \tag{2.5}$$

we can write

$$I(a) = \Omega \int_\circ^{\frac{||\theta||}{k}} (2\pi)^{\nu-1} s^{\nu-1} \int_\circ^{\beta_\circ} \int_{r_-}^{r_+} r^{p-1} \sin^{p-2}(\beta) f(\alpha(r) + s^2) dr d\beta ds,$$

where $\Omega$ is a positive constant, and can be ignored. Differentiating $I(a)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial a}(I(a)) \ \propto\ & \int_\circ^{\frac{||\theta||}{k}} s^{\nu-1} \int_\circ^{\beta_\circ} \sin^{p-2}\beta \Big[ r_+^{p-1} f(\alpha(r_+) + s^2) \frac{\partial}{\partial a}(r_+) \\
& \qquad - r_-^{p-1} f(\alpha(r_-) + s^2) \frac{\partial}{\partial a}(r_-) \Big] d\beta ds \\
= & \int_\circ^{\frac{||\theta||}{k}} s^{\nu+1} \int_\circ^{\beta_\circ} \sin^{p-2}\beta \left[ \frac{r_+^p f(\alpha(r_+) + s^2)}{r_+^2 + as^2} - \frac{r_-^p f(\alpha(r_-) + s^2)}{r_-^2 + as^2} \right] d\beta ds
\end{aligned} \tag{2.6}$$

by simple algebraic manipulations.

Define $f_1(\cdot)$ by $f(t) = f_1(\frac{t}{\sigma^2})\ \ \forall\ t$. Using the new notation

$$\xi = \frac{\theta}{\sigma} \quad , \quad \omega = \frac{s}{\sigma}$$

and modifying the old ones accordingly as

$$\begin{aligned}
r_\pm^\circ &= ||\xi|| \cos\beta \pm \sqrt{k^2\omega^2 - ||\xi||^2 \sin^2\beta}, \\
r_\pm &= (r_\pm^\circ + \sqrt{r_\pm^{\circ 2} + 4a\omega^2})/2, \\
\alpha(r) &= r^2 - 2||\xi||r \cos\beta + ||\xi||^2, \quad \text{and} \quad \sin\beta_\circ = \frac{k\omega}{||\xi||},
\end{aligned}$$

we see that $\frac{\partial}{\partial a} I(a)$ is proportional to

$$\int_\circ^{||\xi||/k} \omega^{\nu+1} \int_\circ^{\beta_\circ} \sin^{p-2}\beta \left[ \frac{r_+^p f_1(\alpha(r_+) + \omega^2)}{r_+^2 + a\omega^2} - \frac{r_-^p f_1(\alpha(r_-) + \omega^2)}{r_-^2 + a\omega^2} \right] d\beta d\omega.$$

As we want a condition uniform in $||\theta||$, we have only to consider the case $\sigma^2 = 1$, even if $\sigma^2$ is unknown.

To show that

$$\frac{r_+^p f_1(\alpha(r_+) + \omega^2)}{r_+^2 + a\omega^2} > \frac{r_-^p f_1(\alpha(r_-) + \omega^2)}{r_-^2 + a\omega^2}, \tag{2.7}$$

for each $\beta$ and $\omega^2$, and hence that dominance can be attained, we can apply the result of Hwang and Chen (1986). A sufficient condition for (2.7) to hold is

$$\inf_{\alpha_0 \le v \le \alpha_1} \frac{f_1'(v + \omega^2)}{f_1(v + \omega^2)} \ge -\frac{p-2}{2\omega^2 k\sqrt{a}} \ \ell n \frac{k + \sqrt{k^2 + a}}{\sqrt{a}} \ , \tag{2.8}$$

where

$$\begin{cases} \alpha_0 = (k - \sqrt{a})^2 \omega^2, \\ \alpha_1 = (k^2 + a)\omega^2. \end{cases}$$

Therefore, we have established

**Theorem 2.1.** *If a is such that*

$$\inf_{\omega} \left\{ \omega^2 \inf_{\alpha_0 \le v \le \alpha_1} \frac{f_1'(v + \omega^2)}{f_1(v + \omega^2)} \right\} \ge -\frac{p-2}{2k\sqrt{a}} \ \ell n (\frac{k + \sqrt{k^2 + a}}{\sqrt{a}}),$$

*then $C_a^+$ dominates $C^\circ$ for every $\theta$ and every $\sigma^2$.*

## III. Discussion.

It is clear from expression (2.8) that there is a restriction on the s.s.d.'s to which Theorem 2.1 applies. In particular, the theorem can be applied only if the first term of the inequality is finite, i.e. if

$$\omega^2 \inf_{\alpha_o \le v \le \alpha_1} \frac{f_1'(v + \omega^2)}{f_1(v + \omega^2)} \tag{3.1}$$

is bounded from below for every $\omega^2$. In the normal case, $\frac{f_1'(t)}{f_1(t)} = -\frac{1}{2}$; therefore we cannot apply Theorem 2.1 to show dominance in this case.

For the double-exponential distribution, we have

$$f(z|\theta, \sigma^2) \propto \ \exp\{-\frac{||z - \theta||}{\sigma}\}$$

and thus $f_1(t) \propto \ \exp(-\sqrt{t})$. Therefore, we get

$$\frac{f_1'(t)}{f_1(t)} = -\frac{1}{2\sqrt{t}} \text{ and } \frac{f_1'(v + \omega^2)}{f_1(v + \omega^2)} = -\frac{1}{2} \frac{1}{\sqrt{(v + \omega^2)}}$$

Thus (3.1) cannot be bounded from below. Similar to the normal distribution, the double exponential has tails that are not flat enough to satisfy this condition. We will see below that the multivariate-$t$ is a limiting case with respect to this criterion.

Consider now the multivariate-$t$ distribution with parameters $\theta, \sigma^2$ and $N$ (degrees of freedom):

$$f(z|\theta,\sigma^2,N) \propto \left\{1+\frac{1}{N\sigma^2}(||x-\theta||^2+||y||^2)\right\}^{-(N+p+\nu)/2} \tag{3.2}$$

For this distribution, we have

$$\frac{f_1'(t)}{f_1(t)} = -\frac{N+p+\nu}{2(N+t)} \quad ,$$

an increasing function of $t$. Thus

$$\inf_{\alpha_\circ \le v \le \alpha_1} \frac{f'(v+\omega^2)}{f(v+\omega^2)} = -\frac{N+p+\nu}{2(N+\alpha_\circ+\omega^2)} = -\frac{N+p+\nu}{2(N+(1+(k-\sqrt{a})^2)\omega^2)},$$

and

$$\inf_\omega \left\{-\frac{\omega^2(N+p+\nu)/2}{N+(1+(k-\sqrt{a})^2)\omega^2)}\right\} = -\frac{N+p+\nu}{2(1+(k-\sqrt{a})^2)}.$$

We now just have to solve the equation

$$\frac{N+p+\nu}{1+(k-\sqrt{a})^2} = \frac{p-2}{k\sqrt{a}}\ell n\left(\frac{k+\sqrt{k^2+a}}{\sqrt{a}}\right) \tag{3.3}$$

to get an upper bound $a_\circ$. And, applying Theorem 2.1, for every $a \in [0, a_\circ]$, $C_a^+$ dominates $C^\circ$.

Note that, as $N \to \infty$, the solution to (3.3) goes to zero, again showing that we cannot deduce anything for the normal distribution with unknown variance. The reason why the normal case cannot be covered by Theorem 2.1 is that it cannot be handled by any technique that operates conditionally on $S^2$. That is, in order to prove dominance in the normal case one cannot work with the square-bracketed expression in (2.6), but rather with the entire integrand. This presents an analytical problem of great difficulty, as the entire integrand is extremely unwieldy.

The fact that the proof conditional on $S^2$ would not work for the normal distribution can be deduced from Hwang and Casella (1984), since a reparametrization shows that the domination of $C^\circ$ by $C_a^+$ is equivalent to the domination of the set

$$\left\{\theta : ||\mu - T||^2 \le c\right\}$$

by the recentered set

$$C_a^+ = \left\{\theta : ||\mu - \left(1-\frac{a}{||T||^2}\right)^+ T||^2 \le c\right\},$$

where $T \sim \mathcal{N}(\mu, \tau^2)$, for every $\mu$ and $\tau$. As $a$ is fixed, this is not possible (Hwang and Casella, 1984).

This surprising difference between normal and multivariate-$t$ cases can also be found in Hwang (1985), which is concerned with *stochastic domination* of point estimators. Informally speaking, one estimator will stochastically dominate another only if its confidence sets dominate for all confidence levels. Hwang proved that the mle can be stochastically dominated in the multivariate-$t$ case but not in the normal case. (See also Brown and Hwang (1989).) As $\delta_a^+(z)$ also depends on $S^2$, we cannot apply his result to the sets (2.2). However, the similarity between the two problems suggests a possible relationship between stochastic domination and set domination.

The difficulty of the normal case is also illustrated by Kim (1987). The estimators considered in this thesis are more complex than (1.1), as the constant $a$ is replaced by a function $a(s)$. However, the author still needs a lower bound on $\sigma$ to get a domination result.

Table 1 gives some values of $a_\circ$ for $\alpha = 0.05$. As one can see, the obtained values are far from

$$a^* = \frac{\nu}{\nu+2}(p-2), \tag{3.4}$$

'optimal choice' for the point estimation problem. The bounds given in Table 1 are logically decreasing functions of $\nu$ and $N$ since, as $\nu$ or $N$ goes to infinity, the multivariate$-t$ converges toward a normal distribution. Although the decrease is not very rapid in $N$, if we compare our values with those of Hwang and Chen (Table 1, 1986), we can see that our values are much smaller: working conditionally on $S^2$ gives restrictive conditions.

Hwang and Chen (1986) also note that the general solution for $a_\circ$ yields a rather small upper bound, one that can be improved upon in special cases. It is clear that their result (Theorem 3.1) can be generalized in the following way.

**Proposition 3.1** *If*

*(a)* $g = \ell n f_1$ *is convex,*

*(b)* $\{t \geq 1; \frac{p-2}{t} + as^2 g'(s^2 a(t - t^{-1}) + (1 + (k - \sqrt{a})^2)s^2)(1 + t^2) \geq 0\}$ *is an interval (possibly degenerate),*

*(c)* $g(as^2(t^* - \frac{1}{t^*}) + (1 + (k - \sqrt{a})^2)s^2) - g((1 + (k - \sqrt{a})^2)s^2) \geq -(p-2)\ell n\, t^*$, *with* $t^* = \frac{k+\sqrt{k^2+a}}{\sqrt{a}}$, *for every* $s^2$,

*then* $C_a^+$ *dominates* $C^\circ$ *for all* $\theta$ *and all* $\sigma^2$.

In the *normal case*, we have $g(t) = -\frac{t}{2}$. Even if conditions $(a)$ and $(b)$ of Proposition 3.1 are satisfied, we have

$$g(as^2(t^* - t^{*^{-1}}) + (1 + (k - \sqrt{a})^2)s^2) - g((1 + (k - \sqrt{a})^2)s^2) = -as^2\frac{t^* - t^{*^{-1}}}{2}$$

which cannot be bounded from below.

For the *double-exponential distribution*, we get the same conclusion as before ($g(t) = \ell n M - \sqrt{t}$ where $M$ is a normalizing constant).

Consider now the *multivariate-t distribution*; we have (see (3.2))

$$g(t) = -\frac{N+p+\nu}{2}\ell n\left(1 + \frac{t}{N}\right)$$

It has been proved in Hwang and Chen (1986) that conditions ($a$) and ($b$) are satisfied for this function. We have

$$g((a(t^* - \frac{1}{t^*}) + 1 + (k - \sqrt{a})^2)s^2) - g((1 + (k - \sqrt{a})^2)s^2)$$
$$= -\frac{N+p+\nu}{2}\ell n\left\{1 + \sqrt{a}s^2 \frac{k + \sqrt{k^2+a})^2 - a}{(k + \sqrt{k^2+a})(N + (1 + (k - \sqrt{a})^2)s^2)}\right\}$$

As this expression is a decreasing function of $s^2$, we get the sufficient condition

$$\frac{N+p+\nu}{2(p-2)}\ell n\left(1 + \sqrt{a}\frac{(k+\sqrt{k^2+a})^2 - a}{(k+\sqrt{k^2+a})(1 + (k-\sqrt{a})^2)}\right) = \ell n\left(\frac{k + \sqrt{k^2+a}}{\sqrt{a}}\right) \qquad (3.5)$$

Therefore we have established

**Theorem 3.2.** *If $a \in (0, a_\circ)$ where $a_\circ$ is solution of (3.5), $C_a^{+}$ dominates $C^\circ$ for multivariate $-t$ distribution with $N$ degrees of freedom.*

The bounds obtained in Table 2 are larger than their counterparts in Table 1, as in the known variance case. As $p$ grows larger, the improvement increases. Table 3 gives the coverage probability of the recentered confidence interval at $\theta = 0$ using the values of Table 2, i.e. the solutions of (3.5). Despite the fact that the bounds are by far too small (see below), the gain at $\theta = 0$ (which is the maximum gain) is still quite substantial. One may also notice that, while the solutions of (3.5) are decreasing with $p$ and $\nu$, the improvement these bounds bring is increasing with $p$ and $\nu$.

Yet these bounds still remain significantly inferior to the "optimal" bound for point estimates, given in (3.4). In fact, Hwang and Ullah (1989) obtained some asymptotic bounds much larger than (3.4) which seem to insure uniform domination of $C^\circ$. This fact definitely shows the need for methods which do not work conditionally on $S^2$ but which, roughly speaking, stay "inside the integrals". Unfortunately, the proper method to attack the problem yet remains to be discovered.

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N = 2

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.50 | 0.19 | 0.09 | 0.07 |
| 10 | 1.40 | 0.49 | 0.23 | 0.19 |
| 20 | 3.21 | 1.06 | 0.52 | 0.39 |
| 30 | 5.02 | 1.60 | 0.70 | 0.56 |
| 60 | 10.42 | 3.40 | 1.31 | 1.03 |

N = 5

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.40 | 0.16 | 0.08 | 0.07 |
| 10 | 1.23 | 0.44 | 0.21 | 0.18 |
| 20 | 2.97 | 0.99 | 0.45 | 0.36 |
| 30 | 4.76 | 1.53 | 0.67 | 0.54 |
| 60 | 10.14 | 3.12 | 1.28 | 1.00 |

N = 10

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.3 | 0.12 | 0.07 | 0.06 |
| 10 | 1.02 | 0.37 | 0.19 | 0.16 |
| 20 | 2.65 | 0.89 | 0.41 | 0.34 |
| 30 | 4.37 | 1.41 | 0.63 | 0.50 |
| 60 | 9.68 | 2.98 | 1.22 | 0.96 |

**Table 1.** Solutions of (3.3) for $\alpha = 0.05$.

N = 2

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.74 | 0.25 | 0.11 | 0.09 |
| 10 | 2.41 | 0.79 | 0.33 | 0.29 |
| 20 | 6.00 | 1.91 | 0.8 | 0.62 |
| 30 | 9.68 | 3.02 | 1.24 | 0.96 |
| 60 | 20.80 | 6.32 | 2.52 | 1.94 |

N = 5

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.58 | 0.21 | 0.10 | 0.08 |
| 10 | 2.03 | 0.68 | 0.30 | 0.24 |
| 20 | 5.42 | 1.73 | 0.73 | 0.57 |
| 30 | 8.98 | 2.81 | 1.17 | 0.91 |
| 60 | 19.84 | 6.05 | 2.42 | 1.87 |

N = 10

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.41 | 0.16 | 0.08 | 0.07 |
| 10 | 1.60 | 0.56 | 0.26 | 0.21 |
| 20 | 4.66 | 1.51 | 0.65 | 0.52 |
| 30 | 8.03 | 2.56 | 1.06 | 0.83 |
| 60 | 18.64 | 5.72 | 2.32 | 1.80 |

**Table 2.** Solutions of (3.5) for $\alpha = 0.05$.

N = 2

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.969 | 0.975 | 0.981 | 0.983 |
| 10 | 0.977 | 0.986 | 0.992 | 0.995 |
| 20 | 0.981 | 0.990 | 0.996 | 0.998 |
| 30 | 0.983 | 0.992 | 0.997 | 0.998 |
| 60 | 0.984 | 0.993 | 0.998 | 0.999 |

N = 5

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.966 | 0.973 | 0.980 | 0.981 |
| 10 | 0.975 | 0.983 | 0.991 | 0.992 |
| 20 | 0.980 | 0.989 | 0.996 | 0.997 |
| 30 | 0.982 | 0.991 | 0.997 | 0.998 |
| 60 | 0.983 | 0.992 | 0.998 | 0.999 |

N = 10

| $p$ \ $\nu$ | 5 | 10 | 20 | 25 |
|---|---|---|---|---|
| 5 | 0.962 | 0.969 | 0.976 | 0.979 |
| 10 | 0.972 | 0.980 | 0.989 | 0.991 |
| 20 | 0.977 | 0.987 | 0.995 | 0.996 |
| 30 | 0.980 | 0.990 | 0.996 | 0.998 |
| 60 | 0.982 | 0.992 | 0.998 | 0.999 |

**Table 3**. Maximum coverage probability of the recentered sets (2.2).