SOME RESULTS ON CONVOLUTIONS AND A STATISTICAL APPLICATION

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SOME RESULTS ON CONVOLUTIONS AND A STATISTICAL APPLICATION

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Classes of distributions, of both discrete and continuous type, are introduced for which the right tail of the distribution is nonincreasing. It is shown that these classes are closed under convolution, thus providing sufficient conditions for nonincreasing right tails to be preserved under convolution. A start is made on verifying a conjecture concerning the extension to the left of nondecreasing right tails under successive convolution. The results give properties of the distributions of random walks on the integers. A statistical application is the verification of a conjecture of Sobel and Huyett (1957) concerning the minimal probability of correct selection for the usual indifference zone procedure for selecting the Bernoulli population with the largest success probability.

1. Introduction. A well known result of Wintner (1938, pp. 30, 32) asserts that the class of symmetric (about 0) unimodal densities on the real line R is closed under convolution. The corresponding result for symmetric unimodal probability mass functions on the integers is proved by Gupta and Sobel (1960). Consequently, for symmetric distributions the property of having a nonincreasing right (or left) tail is preserved under convolution.

In the present paper, a larger class of distributions is introduced in which convolution preserves nonincreasing right tails. In Section 2, the following two theorems are proved.

Theorem 1. For any real number m, let $\mathcal{F}(m)$ be the class of density functions $f(\cdot)$ defined on the real line R which satisfy

(1.1) (i) $f(m-t) \geq f(m+t)$ for $t \geq 0$,

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$$\text{(ii)} \quad f(x) \geq f(y) \qquad \text{when } m \leq x \leq y.$$

Then, if $f_i(\cdot)$ belongs to $\mathcal{F}(m_i)$, $i = 1, 2$, the convolution $f_1 * f_2(\cdot)$ of $f_1(\cdot)$ and $f_2(\cdot)$ belongs to $\mathcal{F}(m_1 + m_2)$.

Theorem 2. For any integer m, let $\mathcal{P}(m)$ be the class of probability mass functions $p(\cdot)$ defined on the integers which satisfy

(1.2)
$$\text{(i)} \quad p(m-j) \geq p(m+j), \qquad j = 0, 1, 2, \ldots,$$
$$\text{(ii)} \quad p(j) \geq p(j+1), \qquad \text{for } j = m, m+1, \ldots .$$

Then if $p_i(\cdot)$ belongs to $\mathcal{P}(m_i)$, $i = 1, 2$, the convolution $p_1 * p_2(\cdot)$ of $p_1(\cdot)$ and $p_2(\cdot)$ belongs to $\mathcal{P}(m_1 + m_2)$.

Note that symmetric (about 0) unimodal densities belong to $\mathcal{F}(0)$, and that symmetric unimodal probability mass functions belong to $\mathcal{P}(0)$; for these distributions the inequalities (i) in (1.1) and (1.2) are actually equalities.

Suppose that a probability mass function $p(\cdot)$ belongs to $\mathcal{P}(0)$, and that $p(\cdot)$ is not symmetric about 0. Theorem 2 says that for every $n \geq 1$ the n-fold convolution $p_{(n)}(\cdot)$ of $p(\cdot)$ with itself satisfies

$$p_{(n)}(j) \geq p_{(n)}(j+1), \qquad \text{all } j = 0, 1, 2, \ldots .$$

Thus, $p_{(n)}(\cdot)$ has a nonincreasing right tail beginning with $j = 0$ for all $n \geq 1$. However, under these circumstances the mean (if it exists) of the distribution defined by $p(\cdot)$ is negative. Hence, the weak law of large numbers implies that the probability mass of $p_{(n)}(\cdot)$ moves to minus infinity as $n \to \infty$. Also, if the variance of $p(\cdot)$ exists, the Central Limit Theorem suggests that $p_{(n)}(j)$, $j = 0, \pm 1, \pm 2, \ldots$, becomes part of the right (decreasing) tail of the standard normal distribution as $n \to \infty$. These observations lead to the following conjecture.

Conjecture. There exists a nondecreasing sequence $\{n_i\colon\ i = 1, 2, \ldots\}$ of positive integers such that $p_{(n)}(-i) \geq p_{(n)}(-i+1)$, all $n \geq n_i$, for $i = 1, 2, \ldots$.

In Section 3, a special case of this conjecture is verified. Suppose that $p(\cdot)$ has support on the integers $-1, 0, 1$. That is

$$p(-1) > p(1), \qquad p(0) \geq p(1),$$

and $p(j) = 0$ for $j \neq -1, 0, 1$. In this case, it is shown that

$$p_{(n)}(-1) \geq p_{(n)}(0), \qquad n \geq \max\{3, \frac{1}{\alpha} - 1\}, \tag{1.3}$$

where $\alpha = p(-1) - p(1)$. The proof depends upon the following result, which is of independent interest.

Theorem 3. Let $q(\cdot)$ be the probability mass function of the uniform distribution on $\{-1, 0, 1\}$, and let $q_{(n)}(\cdot)$ be the n-fold convolution of $q(\cdot)$ with itself. Then $q_{(n)}(0) - q_{(n)}(1)$ is nonincreasing in n for all $n \geq 2$.

Both (1.3), in the case where $p(\cdot)$ has support $\{-1, 0, 1\}$, and also Theorem 3 give properties of the distribution of the n-th stage of a random walk on the integers.

Finally, in Section 4, the above probability results are applied to the problem of choosing the Bernoulli population with the largest probability of success when independent random samples of size n are chosen from each of two Bernoulli populations. If the two probabilities of success differ by at least an amount Δ, $0 < \Delta < 1$, it is shown that the probability of correct choice for the standard procedure (Sobel and Huyett, 1957) is, for all $n \geq \max\{4, \Delta^{-1}\}$, minimized when the smaller probability of success is $\frac{1}{2}(1 - \Delta)$ and the larger probability of success is $\frac{1}{2}(1 + \Delta)$. This verifies a conjecture of Sobel and Huyett (1957).

2. Proofs of Theorems 1 and 2.

Let X_i have density $f_i(\cdot)$ in $\mathcal{F}(m_i)$, or mass function $p_i(\cdot)$ in $\mathcal{P}(m_i)$, $i = 1, 2$. It is easily seen that $Y_i = X_i - m_i$ has density in $\mathcal{F}(0)$, or mass function in $\mathcal{P}(0)$. Further, $X_1 + X_2$ has density in $\mathcal{F}(m_1 + m_2)$, or mass function in $\mathcal{P}(m_1 + m_2)$, if and only if $Y_1 + Y_2$ has density in $\mathcal{F}(0)$, or mass function in $\mathcal{P}(0)$. Hence, Theorems 1 and 2 follow if it can be shown that $\mathcal{F}(0)$ and $\mathcal{P}(0)$ are closed under convolution.

Lemma 1. $\mathcal{F}(0)$ and $\mathcal{P}(0)$ are closed under convex linear combinations. That is, if $f_i(\cdot)$ belongs to $\mathcal{F}(0)$, $i = 1, \ldots, k$, and a_i, $1 \le i \le k$, are nonnegative constants satisfying $\sum_{i=1}^{k} a_i = 1$, then $\sum_{i=1}^{k} a_i f_i(\cdot)$ belongs to $\mathcal{F}(0)$. Similarly, if $p_i(\cdot)$ belongs to $\mathcal{P}(0)$, $i = 1, 2, \ldots, k$, then $\sum_{i=1}^{k} a_i p_i(\cdot)$ belongs to $\mathcal{P}(0)$.

Proof: Straightforward from the definitions of $\mathcal{F}(0)$ and $\mathcal{P}(0)$. □

Let $\mathcal{S}(0)$ be the collection of all symmetric (about 0) unimodal densities. Let $\mathcal{N}(0)$ be the collection of all densities on R having support $(-\infty, 0)$. It has already been noted that $\mathcal{S}(0)$ is a subcollection of $\mathcal{F}(0)$. It is easily seen that $\mathcal{N}(0)$ is a subcollection of $\mathcal{F}(0)$.

Lemma 2. Every density $f(\cdot)$ in $\mathcal{F}(0)$ is a convex linear combination of a density $s(\cdot)$ in $\mathcal{S}(0)$ and a density $n(\cdot)$ in $\mathcal{N}(0)$.

Proof: Let

$$\alpha = \int_{-\infty}^{0} [f(x) - f(-x)]dx.$$

Since $f(\cdot)$ belongs to $\mathcal{F}(0)$, $f(x) - f(-x) \ge 0$, all $x < 0$. Thus, $\alpha \ge 0$, and also $\alpha \le \int_{-\infty}^{0} f(x)dx \le 1$. Define

$$s(x) = \begin{cases} (1-\alpha)^{-1} f(-x), & x < 0, \\ (1-\alpha)^{-1} f(x), & x \ge 0, \end{cases}$$

if $\alpha < 1$, and $s(x) \equiv 0$ if $\alpha = 1$. Let

$$n(x) = \begin{cases} \alpha^{-1}(f(x) - f(-x)), & x < 0, \\ 0, & x \ge 0, \end{cases}$$

if $\alpha > 0$, and $n(x) \equiv 0$ if $\alpha = 0$. It is now straightforward to show that $f(x) = \alpha n(x) + (1-\alpha)s(x)$, that $s(\cdot)$ is a density in $\mathcal{S}(0)$ when $\alpha < 1$, and that $n(\cdot)$ is a density in $\mathcal{N}(0)$ when $\alpha > 0$. □

Lemma 2′. Let $\mathcal{S}(0)$ be the collection of all symmetric (about 0) unimodal mass functions on the integers, and let $\mathcal{N}(0)$ be the collection of all mass functions with support $\{-1,-2,\ldots\}$. Then $\mathcal{S}(0)$ and $\mathcal{N}(0)$ are subcollections of $\mathcal{P}(0)$, and every $p(\cdot)$ belonging to $\mathcal{P}(0)$ can be written in the form

$$p(\cdot) = \alpha n(\cdot) + (1-\alpha)s(\cdot),$$

where

$$\alpha = \sum_{i=1}^{\infty}(p(-i) - p(i)), \qquad 0 \le \alpha \le 1,$$

$n(\cdot)$ belongs to $\mathcal{N}(0)$, and $s(\cdot)$ belongs to $\mathcal{S}(0)$.

Proof: Similar to the proof of Lemma 2. (See also the discussion preceeding Lemma 2.)□

Proof of Theorem 1.

Let $f_1(\cdot)$ and $f_2(\cdot)$ belong to $\mathcal{F}(0)$. Then by Lemma 2,

$$f_i(\cdot) = \alpha_i n_i(\cdot) + (1-\alpha_i)s_i(\cdot), \qquad i = 1,2,$$

where $n_1(\cdot), n_2(\cdot)$ belong to $\mathcal{N}(0)$ and $s_1(\cdot), s_2(\cdot)$ belong to $\mathcal{S}(0)$, and where $0 \le \alpha_1,\ \alpha_2 \le 1$. Note that

(2.1) $$f_1 * f_2(\cdot) = \alpha_1\alpha_2[n_1 * n_2(\cdot)] + \alpha_1(1-\alpha_2)[n_1 * s_2(\cdot)] + (1-\alpha_1)\alpha_2[s_1 * n_2(\cdot)] + (1-\alpha_1)(1-\alpha_2)s_1 * s_2(\cdot).$$

From Wintner (1937),

(2.2) $$s_1 * s_2(\cdot) \in \mathcal{S}(0) \subset \mathcal{F}(0).$$

Also it is clear that

$$n_1 * n_2(\cdot) \in \mathcal{N}(0) \subset \mathcal{F}(0). \tag{2.3}$$

Lemma 3. Both $n_1 * s_2(\cdot)$ and $s_1 * n_2(\cdot) = n_2 * s_1(\cdot)$ belong to $\mathcal{F}(0)$.

Proof: We will show that $n_1 * s_2(\cdot) \in \mathcal{F}(0)$. The proof that $s_1 * n_2(\cdot) \in \mathcal{F}(0)$ is similar.

For $0 \le x \le y$,

$$\begin{aligned} n_1 * s_2(x) &= \int_{-\infty}^{\infty} n_1(t)s_2(x-t)\,dt = \int_{-\infty}^{0} n_1(t)s_2(x-t)\,dt \\ &\ge \int_{-\infty}^{0} n_1(t)s_2(y-t)\,dt = \int_{-\infty}^{\infty} n_1(t)s_2(y-t)\,dt \\ &= n_1 * s_2(y), \end{aligned}$$

since $n_1(t) = 0$ for $t \ge 0$ and $s_2(\cdot) \in \mathcal{S}(0)$. Also, for any $x \ge 0$,

$$\begin{aligned} n_1 * s_2(x) &= \int_{-\infty}^{\infty} n_1(t)s_2(x-t)\,dt = \int_{-\infty}^{0} n_1(t)s_2(x-t)\,dt \\ &= \int_{-\infty}^{-x} n_1(t)s_2(x-t)\,dt + \int_{-x}^{0} n_1(t)s_2(x-t)\,dt \\ &\le \int_{-\infty}^{-x} n_1(t)s_2(-x-t)\,dt + \int_{-x}^{0} n_1(t)s_2(x+t)\,dt \\ &= \int_{-\infty}^{-x} n_1(t)s_2(-x-t)\,dt + \int_{-x}^{0} n_1(t)s_2(-x-t)\,dt \\ &= \int_{-\infty}^{0} n_1(t)s_2(-x-t)\,dt = \int_{-\infty}^{\infty} n_1(t)s_2(-x-t)\,dt \\ &= n_1 * s_2(-x), \end{aligned}$$

again since $n_1(t) = 0$ for $t \ge 0$ and $s_2(\cdot) \in \mathcal{S}(0)$. This shows that $n_1 * s_2(\cdot)$ obeys the properties (ii) and (i) defining $\mathcal{F}(0)$. □

It now follows from (2.1), (2.2), (2.3), Lemma 3 and Lemma 1 that $f_1 * f_2(\cdot)$ belongs to $\mathcal{F}(0)$. This completes the proof of Theorem 1.

Proof of Theorem 2.

Let $p_1(\cdot)$ and $p_2(\cdot)$ belong to $\mathcal{P}(0)$. Then by Lemma 2′,

$$p_i(\cdot) = \alpha_i n_i(\cdot) + (1 - \alpha_i) s_i(\cdot), \qquad i = 1, 2,$$

where $n_1(\cdot)$, $n_2(\cdot)$ belong to $\mathcal{N}(0)$ and $s_1(\cdot)$, $s_2(\cdot)$ belong to $\mathcal{S}(0)$, and where $0 \le \alpha_1$, $\alpha_2 \le 1$. Again, $p_1 * p_2(\cdot)$ can be represented as a convex linear combination of $n_1 * n_2(\cdot)$, $n_1 * s_2(\cdot)$, $s_1 * n_2(\cdot)$, and $s_1 * s_2(\cdot)$. Further, it is clear that $n_1 * n_2(\cdot) \in \mathcal{N}(0) \subset \mathcal{P}(0)$, and it is shown by Gupta and Sobel (1960) that

$$s_1 * s_2(\cdot) \in \mathcal{S}(0) \subset \mathcal{P}(0).$$

From these facts and Lemma 1, it follows that $p_1 * p_2(\cdot)$ belongs to $\mathcal{P}(0)$ if $n_1 * s_2(\cdot)$ and $s_1 * n_2(\cdot)$ belong to $\mathcal{P}(0)$. Thus, the following lemma completes the proof of Theorem 2.

Lemma 3′. Both $n_1 * s_2(\cdot)$ and $s_1 * n_2(\cdot) = n_2 * s_1(\cdot)$ belong to $\mathcal{P}(0)$.

Proof: As in the proof of Lemma 3, we need only prove the assertion for $n_1 * s_2(\cdot)$. For $j \ge 0$,

$$\begin{aligned} n_1 * s_2(j) &= \sum_{i=-\infty}^{\infty} n_1(i) s_2(j-i) = \sum_{i=1}^{\infty} n_1(-i) s_2(j+i) \\ &\ge \sum_{i=1}^{\infty} n_1(-i) s_2(j+1+i) = \sum_{i=-\infty}^{\infty} n_1(i) s_2(j+1-i) \\ &= n_1 * s_2(j+1). \end{aligned}$$

Also, for $j \ge 1$,

$$\begin{aligned} n_1 * s_2(j) &= \sum_{i=j+1}^{\infty} n_1(-i) s_2(j+i) + \sum_{i=1}^{j} n_1(-i) s_2(j+i) \\ &\le \sum_{i=j+1}^{\infty} n_1(-i) s_2(-j+i) + \sum_{i=1}^{j} n_1(-i) s_2(j-i) \\ &= \sum_{i=j+1}^{\infty} n_1(-i) s_2(-j+i) + \sum_{i=1}^{j} n_1(-i) s_2(-j+i) \\ &= \sum_{i=-\infty}^{\infty} n_1(i) s_2(-j-i) = n_1 * s_2(-j). \end{aligned}$$

Thus, $n_1 * s_2(\cdot)$ obeys properties (ii) and (i), respectively, defining $\mathcal{P}(0)$. □

3. Proof of (1.3). Let $p(\cdot)$ be a probability mass function on the integers, with

$$p(-1) > p(1), \qquad p(0) \geq p(1), \tag{3.1}$$

and $p(j) = 0$ for $j \neq -1, 0, 1$. Thus, $p(\cdot)$ belongs to $\mathcal{P}(0)$, and by Theorem 2 the n-fold convolution,

$$p_{(n)}(\cdot) = p * p * \ldots * p(\cdot),$$

of $p(\cdot)$ with itself also belongs to $\mathcal{P}(0)$. The goal of the present section is to verify the conjecture (1.3) in this special case.

Let $\alpha = p(-1) - p(1)$,

$$s(j) = \begin{cases} (1-\alpha)^{-1} p(1), & \text{if } j = -1, 1, \\ (1-\alpha)^{-1} p(0), & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$n(j) = \begin{cases} 1, & \text{if } j = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$p(j) = (1-\alpha) s(j) + \alpha n(j),$$

and for any $m \geq 1$,

$$p_{(m)}(j) = \sum_{i=0}^{m} \binom{m}{i} (1-\alpha)^i \alpha^{m-i} s_{(i)} * n_{(m-i)}(j),$$

where $s_{(k)}(\cdot)$ and $n_{(k)}(\cdot)$ are, respectively, the k-fold convolutions of $s(\cdot)$, $n(\cdot)$ with themselves. (Define $s_{(0)}(\cdot)$ and $n_{(0)}(\cdot)$ to be mass functions placing probability 1 on $j = 0$.) It is easily seen that for $0 \leq i \leq m$,

$$n_{(m-i)}(j) = \begin{cases} 1, & \text{if } j = -(m-i), \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$s_{(i)} * n_{(m-i)}(j) = s_{(i)}(j + m - i).$$

Consequently for $m \geq 1$,

$$p_{(m)}(j) = \sum_{i=0}^{m} \binom{m}{i} (1-\alpha)^i \alpha^{m-i} s_{(i)}(m + j - i), \tag{3.2}$$

for $j = 0, \pm 1, \pm 2, \ldots$. Note that from (3.1), and from the definition of $s(\cdot)$, it follows that $s(\cdot) \in \mathcal{S}(0)$. Thus $s_{(i)}(\cdot) \in \mathcal{S}(0)$, $i = 2, 3, \ldots$.

Lemma 4. For all $m \geq 2$,

$$p_{(m)}(-1) - p_{(m)}(0) \geq (1-\alpha)^{m-1}\{m\alpha[s_{(m-1)}(0) - s_{(m-1)}(1)] - (1-\alpha)[s_{(m)}(0) - s_{(m)}(1)]\}.$$

Proof: It follows from (3.2) and the symmetry about 0 and unimodality of each $s_{(i)}(\cdot)$ that

$$\begin{aligned} &p_{(m)}(-1) - p_{(m)}(0) \\ &\quad = (1-\alpha)^m [s_{(m)}(-1) - s_{(m)}(0)] + m\alpha(1-\alpha)^{m-1}[s_{(m-1)}(0) - s_{(m-1)}(1)] \\ &\qquad + \sum_{i=0}^{m-2} \binom{m}{i} (1-\alpha)^i \alpha^{m-i} [s_{(i)}(m-1-i) - s_{(i)}(m-i)] \\ &\quad \geq -(1-\alpha)^m [s_{(m)}(0) - s_{(m)}(1)] + m\alpha(1-\alpha)^{m-1}[s_{(m-1)}(0) - s_{(m-1)}(1)], \end{aligned}$$

from which the stated inequality directly follows. □

Let

$$\beta = \frac{p(0) - p(1)}{1 - \alpha}, \quad r(j) = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$q(j) = \begin{cases} \frac{1}{3}, & j = -1, 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that

$$s(\cdot) = \beta r(\cdot) + (1 - \beta) q(\cdot),$$

and thus that

$$s_{(k)}(\cdot) = \sum_{i=0}^{k} \binom{k}{i} \beta^{k-i}(1-\beta)^i q_{(i)}(\cdot), \tag{3.3}$$

where $q_{(0)}(\cdot) \equiv r(\cdot)$. Define

$$\Delta(i) = \begin{cases} q_{(3)}(0) - q_{(3)}(1), & i = 0, 1, 2, \\ q_{(i)}(0) - q_{(i)}(1), & i \geq 3, \end{cases}$$

and for $i \geq 1$ let J_i denote a random variable having a binomial distribution with sample size i and probability of success $1 - \beta$.

If Theorem 3 is correct, $\Delta(i)$ is a nonincreasing function of i, all $i = 0, 1, 2, \ldots$. Since

$$q_{(0)}(0) - q_{(0)}(1) = 1, \qquad q_{(1)}(0) - q_{(1)}(1) = 0,$$
$$q_{(2)}(0) - q_{(2)}(1) = \frac{1}{9}, \qquad q_{(3)}(0) - q_{(3)}(1) = \frac{1}{27},$$

it follows from (3.3) that for $m \geq 2$,

$$\{s_{(m-1)}(0) - s_{(m-1)}(1)\} - \{s_{(m)}(0) - s_{(m)}(1)\}$$
$$= E\{\Delta(J_{m-1}) - \Delta(J_m)\} + R(m, \beta),$$

where

$$R(m,\beta) = \frac{26}{27}(P\{J_{m-1} = 0\} - P\{J_m = 0\}) - \frac{1}{27}(P\{J_{m-1} = 1\} - P(J_m = 1\})$$
$$+ \frac{2}{27}(P\{J_{m-1} = 2\} - P\{J_m = 2\}).$$

For fixed β, it is known that J_k is stochastically increasing in k. Thus, $\Delta(J_k)$ is stochastically nonincreasing in k, and

$$E\{\Delta(J_{m-1}) - \Delta(J_m)\} \geq 0.$$

Some algebra shows that

$$R(m,\beta) = \frac{\beta^{m-3}(1-\beta)}{27}\{(26+m^2)\left[\beta - \frac{(m-1)(2m-1)}{2(m^2+26)}\right]^2 + \frac{9(m-1)}{4(m^2+26)}(11m-23)\}$$
$$\geq 0$$

for $m \geq 3$. Thus, it follows from (3.5), assuming that Theorem 3 is true, that

$$s_{(m-1)}(0) - s_{(m-1)}(1) \geq s_{(m)}(0) - s_{(m)}(1) \tag{3.6}$$

for all $m \geq 3$.

Hence, if Theorem 3 is true, it follows from Lemma 4 and (3.6) that

$$p_{(m)}(-1) - p_{(m)}(0) \geq 0, \quad \text{all } m \geq \max\{3, \frac{1}{\alpha} - 1\},$$

which, since $\alpha = p(-1) - p(1)$, verifies (1.3). Note that, as one would intuitively expect, the nonincreasing right tail of $p_{(m)}(\cdot)$ moves one step to the left (from $j = 0$ to $j = -1$) at a rate depending inversely on the difference α in probability mass between the left tail and right tail of $p(\cdot)$.

It remains to prove Theorem 3.

Proof of Theorem 3.

The characteristic function of $q(\cdot)$ is

$$\phi(t) = \frac{1}{3}[1 + 2\cos(t)],$$

so

$$\phi^m(t) = \left(\frac{1}{3}\right)^m [1 + 2\cos(t)]^m$$

is the characteristic function of $q_{(m)}(\cdot)$. Using the Fourier inversion formula (see Feller, 1966, p. 484) and the fact that $\phi(t)$ is real-valued, we have

$$\begin{aligned} q_{(m)}(j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi^m(t) e^{-ijt} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{3}(1 + 2\cos(t))\right]^m \cos(jt) dt \end{aligned}$$

for $j = 0, \pm 1, \pm 2, \ldots$. Therefore,

$$\begin{aligned}\omega_m &= (q_{(m-1)}(0) - q_{(m-1)}(1)) - (q_{(m)}(0) - q_{(m)}(1)) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(\frac{1+2\cos(t)}{3}\right)^{m-1} - \left(\frac{1+2\cos(t)}{3}\right)^{m}\right](1-\cos(t))\,dt \\ &= \frac{4}{2\pi 3^m}\left[\int_0^{\pi}(1+2\cos(t))^{m-1}(1-\cos(t))^2 dt\right]\end{aligned}$$

which is obviously nonnegative when m is an odd integer ($m = 1, 3, 5, \ldots$). For $m = 4$, direct computation of the probabilities $q_{(3)}(0), q_{(3)}(1), q_{(4)}(0), q_{(4)}(1)$, or use of (3.7), yields

$$\omega_4 = 0.$$

Note that ω_m is nonnegative if and only if

$$\tau_m = \frac{3^m 2\pi}{4}\omega_m \tag{3.8}$$

is nonnegative. Also, $\tau_4 = 0$. We now show that τ_{2k} is nondecreasing in k, $k \geq 2$, and this will complete the proof that $\omega_m \geq 0$ for all even $m \geq 4$.

From (3.7), for $k \geq 2$,

$$\begin{aligned}\tau_{2k+2} - \tau_{2k} &= \int_0^{\pi}(1+2\cos(t))^{2k-1}(1-\cos(t))^2[(1+2\cos(t))^2 - 1]dt \\ &= 4\int_0^{\pi}(1+2\cos(t))^{2k-1}\sin^2(t)\cos(t)(1-\cos(t))\,dt \\ &\geq 4\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+2\cos(t))^{2k-1}\sin^2(t)\cos(t)(1-\cos(t))\,dt,\end{aligned}$$

since $1 + 2\cos(t)$ and $\cos(t)(1-\cos(t))$ have the same sign for t in $[0, \frac{1}{3}\pi] \bigcup [\frac{2}{3}\pi, \pi]$.

Let

$$H_k(t) = (1+2\cos(t))^{2k-1}\sin^2(t)\cos(t)(1-\cos(t)). \tag{3.10}$$

From (3.9),

$$(3.11) \qquad \tau_{2k+2} - \tau_{2k} \geq 4\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} H_k(t)dt = 4\int_0^{\frac{1}{6}\pi} [H_k(\frac{1}{2}\pi - u) + H_k(\frac{1}{2}\pi + u)]du.$$

$$= 4\int_0^{\frac{1}{6}\pi} \sin^2(\frac{1}{2}\pi + u)\sin(u)\{[1 + 2\sin(u)]^{2k-1}[1 - \sin(u)] - [1 - 2\sin(u)]^{2k-1}[1 + \sin(u)]\}du$$

since $\sin^2(\frac{1}{2}\pi - u) = \sin^2(\frac{1}{2}\pi + u)$, and

$$\cos(\frac{1}{2}\pi - u) = \sin(u) = -\cos(\frac{1}{2}\pi + u).$$

Noting that $0 \leq \sin(u) \leq \frac{1}{2}$ for $u \in [0, \frac{1}{6}\pi]$, and that for $x \in [0, \frac{1}{2}]$, $k \geq 2$,

$$(1 + 2x)^{2k-1}(1 - x) \geq (1 + 2x)(1 - x) \geq (1 - 2x)(1 + x) \geq (1 - 2x)^{2k-1}(1 + x),$$

it follows that the right-hand side of (3.11) is nonnegative, all $k \geq 2$. This completes the proof of Theorem 3, and verifies the result (1.3). □

Proof of the entire conjecture made in Section 1, even in the special case of $p(\cdot)$ considered in this section, appears to be extremely difficult. It is possible that the methods used to prove (1.3) can be extended, but such an approach appears cumbersome. A more promising attack on the problem may be through the characteristic function argument used to prove Theorem 3.

4. A Statistical Application In the indifference zone formulation for the problem of ranking Bernoulli parameters (Sobel and Huyett, 1957), independent random samples of size n are obtained from each of k Bernoulli populations. The goal is to choose the population with the largest probability of success, but there is concern about a correct choice only when the largest probability of success exceeds the second largest probability of success by at least Δ, $0 < \Delta < 1$, where Δ is a prespecified constant.

When $k = 2$ Bernoulli populations are being compared, the procedure usually recommended is to compare the observed numbers X_1, X_2 of successes in the two samples, and conclude that population 1 has the largest probability of success if $X_1 > X_2$ and population 2 has the largest probability of success if $X_2 > X_1$. If $X_1 = X_2$, a population is either randomly selected (without loss of generality by a mechanism that does not depend upon the common observed value of X_1 and X_2), or else the population believed *a'priori* to have the largest probability of success is chosen. Attention then concentrates on determining the smallest sample size n such that the probability of correctly choosing the population with the highest probability of success is no less than a prespecified constant γ, $0 < \gamma < 1$.

Let Y denote the number of successes in the sample obtained from the population with the largest probability of success, and let X denote the number of successes in the remaining sample. Under the given assumptions,

$$\begin{aligned} &X \text{ and } Y \text{ are statistically independent,} \\ &X \sim \text{ binomial } (n, p), \\ &Y \sim \text{ binomial } (n, p+d), \end{aligned} \tag{4.1}$$

where

$$0 \le p \le 1 - d, \qquad \Delta \le d \le 1,$$

and p, d are unknown.

Let θ be the (conditional) probability of selecting the population of Y when $Y = X$ ($0 \le \theta \le 1$). Note that $\theta = 1$ corresponds to always selecting Y when $X = Y$, while $\theta = 0$ corresponds to always selecting X in such a situation. Since selecting the population of Y is the correct choice, the probability of correct selection is

$$\begin{aligned} PCS(p, d, n) &= P\{Y > X\} + \theta P\{Y = X\} \\ &= \theta P\{Y - X \ge 0\} + (1 - \theta) P\{Y - X \ge 1\}. \end{aligned} \tag{4.2}$$

In order that PCS is never less than γ, n must be chosen so that

$$\inf_{\Delta \le d \le 1} \; \inf_{0 \le p \le 1-d} PCS(p, d, n) \ge \gamma.$$

However, Sobel and Huyett (1957) show that $PCS(p, d, n)$ is (strictly) decreasing in d for fixed p, n, θ. Thus, it can be assumed that $d = \Delta$, and n is determined to satisfy

$$\inf_{0 \leq p \leq 1-\Delta} PCS(p, \Delta, n) \geq \gamma. \tag{4.3}$$

Since $PCS(p, \Delta, n)$ is for fixed Δ, n, θ, a continuous function of p, and p takes values in a closed interval $[0, 1 - \Delta]$, the infimum in (4.3) is achieved. The value

$$p^* = p^*(\Delta, n, \theta)$$

which achieves the infimum (minimum) is said to be *least favorable*. In general, p^* depends upon n and θ, as well as on Δ. However, Sobel and Huyett (1957) use the large sample normal approximation to the distribution of $Y - X$ to show that for fixed θ, Δ,

$$\lim_{n \to \infty} p^*(\Delta, n, \theta) = \frac{1 - \Delta}{2}. \tag{4.4}$$

Using both normal approximations and exact calculations, they give a table of the smallest values of n needed to assure that

$$PCS(\frac{1 - \Delta}{2}, \Delta, n) \geq \gamma$$

when $\theta = \frac{1}{2}$. They remark that some exact calculations suggest that the limit in (4.4) is approached rapidly, so that their table gives a good approximation to an exact solution for determining the sample size n for the randomized selection rule with $\theta = \frac{1}{2}$. They also indicate how to adjust their table to find n when $\theta = 0$ or 1.

In this section, it is shown that

$$p^*(\Delta, n, 1) = \frac{1}{2}(1 - \Delta), \qquad \text{all } n \geq 1, \tag{4.5}$$

and that for $0 \leq \theta < 1$,

$$p^*(\Delta, n, \theta) = \frac{1}{2}(1 - \Delta), \qquad \text{all } n \geq \max\{4, \Delta^{-1}\}. \tag{4.6}$$

These results permit exact determination of the sample size n for both randomized ($0 < \theta < 1$) and nonrandomized selection rules.

Define

$$G(j;\ p, \Delta, n) = P\{X - Y \geq j\},$$

for $j = 0, \pm 1, \pm 2, \ldots, \pm n$. Note from (4.1) and the above discussion that

$$\begin{aligned} &\inf_{\Delta \leq d \leq 1} \inf_{0 \leq p \leq 1-d} PCS(p, d, n) \\ (4.7) \qquad &= \min_{0 \leq p \leq 1-\Delta} PCS(p, \Delta, n) \\ &= \min_{0 \leq p \leq 1-\Delta} [1 - (1-\theta)G(1;\ p, \Delta, n) - \theta G(0;\ p, \Delta, n)]. \end{aligned}$$

Theorem 4. Fix $\Delta, 0 < \Delta < 1$. For all $n \geq 1$, $j \geq 1$, $G(j;\ p, \Delta, n)$ is unimodal in p. Further, $G(0;\ p, \Delta, n)$ is unimodal in p for $n \geq \max\{4, \Delta^{-1}\}$. The mode in both cases is $p^* = \frac{1}{2}(1 - \Delta)$.

Proof: Let $v = p - \frac{1}{2}(1 - \Delta)$. Then from (4.1),

$$X \sim \text{ binomial } (n, v + \frac{1}{2}(1 - \Delta)),$$
$$Y \sim \text{ binomial } (n, v + \frac{1}{2}(1 + \Delta)),$$

and X, Y are independent. Further since $0 \leq p \leq 1 - \Delta$,

$$-\frac{1}{2}(1 - \Delta) \leq v \leq \frac{1}{2}(1 - \Delta).$$

Also

$$G(j;\ p, \Delta, n) = G(j;\ v + \frac{1}{2}(1 - \Delta), \Delta, n)$$

so that $G(j;\ p, \Delta, n)$ is unimodal in p, $0 \leq p \leq 1 - \Delta$, if and only if $G(j;\ v + \frac{1}{2}(1-\Delta), \Delta, n)$ is unimodal in v, $|v| \leq \frac{1}{2}(1 - \Delta)$.

Since X and Y are independent binomials,

$$X \sim \sum_{i=1}^{n} X_i, \qquad Y \sim \sum_{i=1}^{n} Y_i, \qquad X - Y = \sum_{i=1}^{n} Z_i$$

where $X_1, \ldots, X_n$, $Y_1, \ldots Y_n$ are independent Bernoulli variables with

$$X_i \sim \text{ Bernoulli } (v + \frac{1}{2}(1 - \Delta)), \qquad Y_i \sim \text{ Bernoulli } (v + \frac{1}{2}(1 + \Delta)).$$

Thus,

$$Z_i = X_i - Y_i, \qquad i = 1, \ldots, n,$$

are i.i.d. random variables with common mass function

$$\text{(4.8)} \qquad p(z) = \begin{cases} \frac{1}{4}(1 + \Delta)^2 - v^2, & \text{if } z = -1, \\ \frac{1}{2}(1 - \Delta^2) + 2v^2, & \text{if } z = 0, \\ \frac{1}{4}(1 - \Delta)^2 - v^2, & \text{if } z = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\text{(4.9)} \qquad G(j;\ v + \frac{1}{2}(1 - \Delta), \Delta, n) = P\{\sum_{i=1}^{n} Z_i \geq j\}$$

depends upon v only through v^2, and is thus an even function of v. Consequently, (4.9) is unimodal in v if and only if it is nonincreasing as a function of v^2, in which case the mode occurs at $v = 0$. [Note that $v = 0$ corresponds to $p = \frac{1}{2}(1 - \Delta)$.]

It is now convenient to change notation. Let $t = v^2$ and

$$\text{(4.10)} \qquad H(j;\ t, n) = P\{\sum_{i=1}^{n} Z_i \geq j\},$$

where our notation suppresses the dependence of this probablity on Δ. (Recall that Δ is held fixed.) Let

$$p(\ell; t) = P\{Z_i = \ell\}, \qquad \ell = -1, 0, 1,$$

and note from (4.8) that

$$\text{(4.11)} \qquad \begin{gathered} p(1; t) = \frac{1}{4}(1 - \Delta)^2 - t, \quad p(0; t) = \frac{1}{2}(1 - \Delta^2) + 2t, \\ p(-1; t) = \frac{1}{4}(1 + \Delta)^2 - t. \end{gathered}$$

Finally, if $p_{(n)}(\cdot;t)$ is the n-fold convolution of $p(\cdot;t)$ with itself, then

$$p_{(n)}(i;\ t) = P\{\sum_{i=1}^{n} Z_i = i\},$$

and

$$H(j;\ t, n) = \sum_{i=j}^{n} p_{(n)}(i;\ t).$$

In an appendix, it is shown that for all $i = 0,\ \pm 1,\ \pm 2, \ldots,$

$$\frac{d}{dt} p_{(n)}(i;t) = n[2p_{(n-1)}(i;t) - p_{(n-1)}(i-1;t) - p_{(n-1)}(i+1;t)].$$

Consequently,

$$\begin{aligned}\frac{d}{dt} H(j;t,n) &= \sum_{i=j}^{n} \frac{d}{dt} p_{(n)}(i;t) \\ &= n[2H(j;t,n-1) - H(j-1;t,n-1) - H(j+1;t,n-1)] \\ &= n[p_{(n-1)}(j;t) - p_{(n-1)}(j-1;t)],\end{aligned}$$

and $(d/dt)H(j;t,n)$ will be less than or equal to 0 for $0 \le t \le \frac{1}{4}(1-\Delta)^2$ if and only if

(4.12) $$p_{(n-1)}(j-1;t) \ge p_{(n-1)}(j;t), \text{ all } 0 \le t \le \frac{1}{4}(1-\Delta)^2.$$

Note from (4.8), or (4.11), that

$$p(-1;t) - p(1;t) = \frac{1}{4}(1+\Delta)^2 - \frac{1}{4}(1-\Delta)^2 = \Delta > 0,$$

$$p(0;t) - p(1;t) = \frac{1}{2}(1-\Delta^2) - \frac{1}{4}(1-\Delta)^2 + 3t = \frac{1}{4}(1-\Delta)(1+3\Delta) + 3t \ge 0.$$

for $0 \le t \le \frac{1}{4}(1-\Delta)^2$. Theorem 2 now applies to show that (4.12) holds for $j \ge 1$, all $n \ge 1$. Hence, $H(j;\ t, n)$ is nonincreasing in t for all $j \ge 1$, $n \ge 1$; and consequently, for all $n \ge 1$, $G(j;\ p, \Delta, n)$ is unimodal in p with mode at $p = \frac{1}{2}(1-\Delta)$.

For $j = 0$, the result (1.3) can be applied to show that (4.12) holds for

$$n - 1 \ge \max\{3,\ \Delta^{-1} - 1\}.$$

Thus, when $n \geq \max\{4, \Delta^{-1}\}$, $G(0;\ p, \Delta, n)$ is unimodal in p with mode at $p = \frac{1}{2}(1-\Delta)$.□

The asserted results (4.5) and (4.6) now follow immediately from (4.7) and Theorem 4.

Appendix

For functions $p(\cdot), q(\cdot)$ mapping the integers 0, ± 1, $\pm 2, \ldots$ into the real line, define the convolution $p * q(\cdot)$ by

$$p * q(j) = \sum_{i=-\infty}^{\infty} p(i) q(j-i),$$

provided the infinite sum exists. It is easily seen that

$$\begin{aligned} p * q(\cdot) &= q * p(\cdot), \\ p * (q * r)(\cdot) &= (p * q) * r(\cdot), \qquad \text{(A.1)} \\ (ap + bq) * r(\cdot) &= a[p * r(\cdot)] + b[q * r(\cdot)], \end{aligned}$$

for real constants a, b.

For each t in an interval (t_L, t_U), let $p(\cdot;\ t)$ and $q(\cdot;\ t)$ map the integers into the real line, and assume that for every integer j the derivatives

$$\frac{d}{dt} p(j; t), \quad \frac{d}{dt} q(j; t)$$

exist for all t in (t_L, t_U). If

$$p * q(j; t) = \sum_{i=-\infty}^{\infty} p(i; t) q(j - i; t) \qquad \text{(A.2)}$$

exists for all $j = 0, \pm 1, \pm 2, \ldots$, all t in (t_L, t_U), then under the usual conditions for interchange of summation and differentiation, we have

$$\left[\frac{d}{dt}\ (p * q)\right](\cdot; t) = \left[\left(\frac{d}{dt}\ p\right) * q\right](\cdot; t) + \left[p * \left(\frac{d}{dt}\ q\right)\right](\cdot; t). \qquad \text{(A.3)}$$

Lemma A.1. Let $p_{(n)}(\cdot;t)$ be the n-fold convolution of $p(\cdot;t)$, where $p(j;t)$ has a derivative with respect to t for all integers j, all t in (t_L, t_U). Then, assuming we can interchange summation and derivative, for all $n \geq 1$, all integers j,

$$\frac{d}{dt}\, p_{(n)}(\cdot;t) = n\left[p_{(n-1)} * \left(\frac{d}{dt}\, p\right)\right](j;t).$$

Proof. Using (A.1) and (A.3),

$$\begin{aligned}\frac{d}{dt}\, p_{(2)}(\cdot;t) &= \left[\left(\frac{d}{dt}\, p\right) * p\right](\cdot;t) + \left[p * \left(\frac{d}{dt}\, p\right)\right](\cdot;t)\\ &= 2\left[p * \left(\frac{d}{dt}\, p\right)\right](\cdot;t).\end{aligned}$$

The stated result now follows by use of (A.1), (A.3) and induction on n. □

An important application of Lemma A.1 is to the case where $p(\cdot;t)$ is linear in t. If

$$p(i;t) = a(i) + b(i)t$$

$i = 0,\ \pm 1, \pm 2, \ldots$, then $(d/dt)\ p(i;t) = b(i)$ and

$$\frac{d}{dt}\, p_{(n)}(j;t) = n \sum_{i=-\infty}^{\infty} p_{(n-1)}(i;t) b(j-i).$$

In particular, if $p(j;t)$ is given by (4.11), then

$$b(i) = \begin{cases} 2, & i = 0, \\ -1, & i = -1, 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\frac{d}{dt}\, p_{(n)}(i;t) = n[2p_{(n-1)}(i;t) - p_{(n-1)}(i-1;t) - p_{(n-1)}(i+1;t)].$$

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