Brownian motion and the distribution of orbits of polynomial mappings of the complex plane

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ABSTRACT

A theorem of Brolin states that if $Q(z)$ is a polynomial of degree ≥ 2 and if ξ is a randomly chosen solution of $Q^n(\xi) = z$ then as $n \to \infty$ the distribution of ξ converges to the equilibrium distribution on the Julia set of Q. A simple probabilistic proof of this theorem is given. Some new results about the distribution of the branches of Q^{-n} relative to Brownian paths are also obtained.

KEY WORDS AND PHRASES:
Brownian motion, conformal invariance, Julia set, strong mixing property.

1. Introduction

Let $Q(z)$ be a complex polynomial of degree ≥ 2 and let $Q^n(z), n = 0, 1, 2, \ldots$, be its iterates:

$$Q^o(z) = z,$$
$$Q^{n+1}(z) = Q(Q^n(z)), n = 0, 1, \ldots .$$

The *Julia set* J of Q is the set of complex numbers at which $\{Q^n\}_{n\geq 0}$ is *not* a normal family of analytic functions. It is well known (and easy to prove) that J is a nonempty, compact subset of the complex plane C (sec.2).

Fix $z \in C$; consider the set $Q^{-n}(z)$ of complex numbers ξ such that $Q^n(\xi) = z$. Observe that $Q^{-n}(z)$ has cardinality d^n where $d =$ degree of Q, provided multiple roots of $Q^n(\xi) = z$ are listed according to their multiplicities. Let μ_n^z be the uniform distribution on $Q^{-n}(z)$, i.e., μ_n^z is the probability measure that puts mass d^{-n} at each point of $Q^{-n}(z)$.

THEOREM (Brolin [2]): As $n \to \infty, \mu_n^z$ converges weakly to the equilibrium distribution μ on J, except for at most one point $z \in C$.

See [8] for the classical definition of the equilibrium distribution. Probabilists know [4] that the equilibrium distribution of a compact set J coincides with the hitting distribution for J by a Brownian motion started at ∞. Brolin's proof, which is based on results from classical potential theory, gives no probabilistic insight into why this limit distribution occurs. The purpose of this note is to give a simple probabilistic proof of Brolin's theorem that explains the occurrence of the equilibrium distribution. This proof is elementary, using only one result from (probabilistic) potential theory, namely, Lévy's theorem on the conformal invariance of Brownian motion (sec. 3).

The probabilistic arguments used here also give some information about how the various branches of Q^{-n} are distributed relative to Brownian motion Z_t started at ∞ and stopped at J. Let $\mathcal{F}_\infty$ be the connected component of $\mathcal{F} = J^c$ containing ∞ and let L be a closed subset of $\mathcal{F}_\infty$. Define $T =$ inf $\{t : Z_t \in J\}$. In sec. 3 we prove

COROLLARY 1: For any continuous $f : J \to C$,

$$\lim_{n\to\infty} Ef(Z_T)1\{Z \text{ hits } Q^{-n}(L)\} = Ef(Z_T)P\{Z \text{ hits } L\}$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{m=0}^{n-1} 1\{Z \text{ hits } Q^{-m}(L)\} = P\{Z \text{ hits } L\} \ a.s.$$

Brolin's theorem is not the only way to describe the distribution of points in the pre-orbit of a given $z \in C$ (nor even necessarily the most natural). Consider $\mathcal{O}^-(z) = \bigcup_{n\geq 0} Q^{-n}(z)$, and let μ_ϵ be the uniform distribution on $\{\xi \in \mathcal{O}^-(z) :$ dist $(\xi, J) \geq \epsilon\}$. If Q is expansive on J (i.e., there exists $n \geq 1$ such that $|(Q^n)'(\varsigma)| > 1$ for all $\varsigma \in J$) then it is apparently the case that as $\epsilon \to 0, \mu_\epsilon$ converges weakly to to normalized δ-dimensional Hausdorff measure H_δ on J, where $\delta =$ Hausdorff dimension of J. (See [7] for a somewhat weaker result in this direction, and [5] for similar results in the context of Kleinian groups.)

Brownian motion appears to play no natural role in this problem, so we shall not discuss it further in this paper.

No prior knowledge of complex analytic dynamics is necessary to follow the arguments of this paper. Some elementary results are collected in sec. 2; these may all be found (in some form) in [1].

2. Preliminaries

For our purposes it is usually more convenient to think of $Q(z)$ as a self-mapping of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ than as a mapping of the plane $\mathbf{C}$, since $\mathbf{C}$ is compact. Note that the point ∞ is an attractive fixed point of Q, because degree $(Q) \geq 2$. Thus there is a neighborhood $\mathcal{U}$ of ∞ such that $Q^n(z) \to \infty$ as $n \to \infty$ uniformly for $z \in \mathcal{U}$. Note also that the only solution of $Q^n(z) = \infty$ is $z = \infty$.

Of fundamental importance in complex analytic dynamics is the notion of a *normal family* of meromorphic functions [1]. A normal family is a set $\{f_\lambda\}$ of functions meromorphic in a domain $\mathcal{D}$ such that any sequence f_n has a subsequence that converges uniformly (with respect to the spherical metric on $\overline{\mathbf{C}}$) on each compact subset of $\mathcal{D}$. By the Arzela-Ascoli theorem, this is equivalent to the statement that $\{f_\lambda\}$ is uniformly equicontinuous in each compact subset of $\mathcal{D}$.

A family of meromorphic functions $\{f_\lambda\}$ is said to be normal at a point $z \in \overline{\mathbf{C}}$ if it is normal in some neighborhood of z. The *Fatou set* $\mathcal{F}$ of a polynomial $Q(z)$ is defined [1] to be the set of $z \in \overline{\mathbf{C}}$ at which $\{Q^n\}_{n \geq 0}$ is normal. The Fatou set $\mathcal{F}$ is clearly open, and if degree $(Q) \geq 2$ then $\infty \in \mathcal{F}$ because $Q^n \to \infty$ uniformly in a neighborhood of ∞. The *Julia set* J of Q is defined to be the complement of $\mathcal{F}$, hence J is a compact set of $\overline{\mathbf{C}}$. Clearly $Q(\mathcal{F}) = \mathcal{F}$ and $Q(J) = J$.

PROPOSITION 1: If degree $(Q) \geq 2$ *then* $J \neq \emptyset$.

PROOF: If $J = \emptyset$ then $\{Q^n\}_{n \geq 0}$ would be a normal family on $\overline{\mathbf{C}}$. But $Q^n(z) \to \infty$ uniformly for z in a neighborhood of ∞. It would then follow from normality that $Q^n(z) \to \infty$ uniformly for $z \in \overline{\mathbf{C}}$. But this is impossible, because each $Q^n : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is surjective. □

Henceforth we shall assume that $Q(z)$ is a polynomial of degree $d \geq 2$, so that $J \neq \emptyset$. Since $d \geq 2$ the inverse function of $Q^n, n \geq 1$, is multiple-valued, with d^n branches and branch points in $\mathcal{G}_n$, where

$$\mathcal{G}_o = \{z \in \mathbf{C} : \frac{d}{dz} Q(z) = 0\},$$
$$\mathcal{G}_n = \bigcup_{m=1}^{n} Q^m(\mathcal{G}_o),$$
$$\mathcal{G}_+ = \bigcup_{m=0}^{\infty} Q^m(\mathcal{G}_o).$$

The branches of the inverse function of Q^n will be denoted $Q_i^{-n}, i = 1, 2, \ldots, d^n$.

PROPOSITION 2: If $\mathcal{Q} = \{Q_i^{-n}\}_{n,i}$ *is a collection of certain branches of* Q^{-n} *such that each* $Q_i^{-n} \in \mathcal{Q}$ *is single-valued and analytic in the domain* $\mathcal{U}$, *then* $\mathcal{Q}$ *is a normal family in* $\mathcal{U}$.

PROOF: First notice that $\infty \notin \mathcal{U}$, because there is no branch of Q^{-n} analytic in a neighborhood of ∞ for any $n \geq 1$. Let $\xi \in \mathcal{U}$ be arbitrary; it suffices to show that $\mathcal{Q}$ is a normal family in a neighborhood of ξ. Since $Q(z)$ has degree $d \geq 2$ there exists $R < \infty$ such that if $|z| \geq R$ then $|Q(z)| > |z|$. It follows that for some neighborhood $\mathcal{N}$ of ξ, $\bigcup_{n=1}^{\infty} Q^{-n}(\mathcal{N})$ is disjoint from some neighborhood of ∞. Therefore $\mathcal{Q}$ is uniformly bounded on $\mathcal{N}$. It follows by standard arguments that $\mathcal{Q}$ is a normal family in $\mathcal{N}$ (uniform boundedness implies that the derivates are uniformly bounded on compact subsets of $\mathcal{N}$, by the Cauchy integral formula for derivatives, thus $\mathcal{Q}$ is uniformly equicontinuous on compact subsets of $\mathcal{N}$). □

Define $\mathcal{F}_\infty$ to be the path-connected component of ∞ in the Fatou set $\mathcal{F}$, i.e., $\mathcal{F}_\infty$ is the set of all points $z \in \mathcal{F}$ such that there is a continuous path in $\mathcal{F}$ beginning at ∞ and ending at z.

PROPOSITION 3: If $z \in \mathcal{F}_\infty$ then $\lim_{n\to\infty} Q^n(z) = \infty$.

PROOF: Let $\gamma(t), 0 \leq t \leq 1$, be a continuous path in $\mathcal{F}$ such that $\gamma(0) = \infty$ and $\gamma(1) = z$. Since $\{Q^n\}_{n\geq 1}$ is a normal family in $\mathcal{F}$ every subsequence of Q^n has a subsequence that converges uniformly to a meromorphic function in a neighborhood of $\gamma([0,1])$. But $Q^n(\varsigma) \to \infty$ uniformly for ς in a neighborhood of ∞, hence $Q^n \to \infty$ uniformly in a neighborhood of $\gamma([0,\epsilon])$ for some $\epsilon > 0$. It follows that any subsequence converging uniformly in a neighborhood of $\gamma([0,1])$ must in fact converge to ∞. Therefore, $Q^n(z) \to \infty$. □

Consider the set $\mathcal{G}_+ \cap \mathcal{F}_\infty$. Recall that $\mathcal{G}_+$ is the union of the forward orbits of the critical points $\mathcal{G}_o$. If $\xi \in \mathcal{G}_o$ and the forward orbit of ξ ever enters $\mathcal{F}_\infty$ then $Q^n(\xi) \to \infty$, by Proposition 3; since $\mathcal{G}_o$ is a finite set it follows that the only possible accumulation point of $\mathcal{G}_+ \cap \mathcal{F}_\infty$ in $\mathcal{F}_\infty$ is ∞. Consequently, each point of $\mathcal{F}_\infty$ not in $\mathcal{G}_+$ has a simply connected neighborhood disjoint from $\mathcal{G}_+$ in which *all* branches Q_i^{-n} are single-valued and analytic.

PROPOSITION 4: For each $n \geq 1, Q^{-n}(\mathcal{F}_\infty) \subset \mathcal{F}_\infty$.

PROOF: Fix $\xi \in \mathcal{F}_\infty$, and let $\gamma(t), 0 \leq t \leq 1$, be a smooth path in $\mathcal{F}$ such that $\gamma(0) = \infty, \gamma(1) = \xi$, and for each $0 < t < 1$ there is a neighborhood of $\gamma(t)$ in which all branches of Q^{-n} are single-valued and analytic. Let z be a point such that $Q^n(z) = \xi$. Then there is a continuous path $\tilde{\gamma}(t), 0 \leq t \leq 1$, such that $\tilde{\gamma}(1) = z$ and $Q^n(\tilde{\gamma}(t)) = \gamma(t)$ for each $0 \leq t \leq 1$. Clearly, $\tilde{\gamma}(0) = \infty$, because the only root of $Q^n(\varsigma) = \infty$ is $\varsigma = \infty$. Moreover, $\tilde{\gamma}$ lies entirely in $\mathcal{F}$, because $\mathcal{F}$ is Q-invariant. Thus $z \in \mathcal{F}_\infty$. □

PROPOSITION 5: Let $Q_{i(k)}^{-n(k)}$ be single-valued and analytic in $\mathcal{U}$, for each $k \geq 1$, where $\mathcal{U}$ is a connected open subset of $\mathcal{F}_\infty$. If $Q_{i(k)}^{-n(k)}$ converges uniformly on compact subsets of $\mathcal{U}$ then the limit is a constant function, and the constant is an element of the Julia set J.

PROOF: Call the limit function f. By Proposition 4, $f(\mathcal{U}) \subset \overline{\mathcal{F}}_\infty$. But on the other hand Proposition 3 implies that $\mathcal{F}_\infty \cap f(\mathcal{U}) = \emptyset$, because $\{Q^n(z)\}$ cannot accumulate at any point of $\mathcal{U}$ if $z \in \mathcal{F}_\infty$ (note that $\infty \notin \mathcal{U}$). Therefore, f is constant, say $f = \xi$. Since $\xi \in \overline{\mathcal{F}}_\infty$ and $\xi \notin \mathcal{F}_\infty, \xi \in J$. □

Finally, we introduce the notion of an *excluded value* of $Q(z)$. A point $\xi \in C$ is called an excluded value of $Q(z)$ iff Q has the form $Q(z) = C(z-\xi)^d + \xi$. This implies that ξ is a d-fold root of the equation $Q(z) = \xi$; consequently, there is no $z \neq \xi$ mapped onto ξ by any Q^n (hence the terminology). Observe that there is at most one excluded value for any polynomial $Q(z)$, because any excluded value is a (d-1)-fold root of $Q'(z) = 0$.

Suppose that ξ is not an excluded value of $Q(z)$. If $Q(z)$ is not of the form $Q(z) = C(z-\varsigma)^d + \xi$ then there are at least two distinct roots of $Q(z) = \xi$, and at least two distinct roots of $Q^2(z) = \xi$. On the other hand, if $Q(z) = C(z-\varsigma)^d + \xi$ there is only one root of $Q(z) = \xi$, namely $z = \varsigma$, but in this case the only root of $Q'(z) = 0$ is $z = \varsigma$, so there are at least two distinct roots of $Q^2(z) = \xi$. It follows that there are at least 2^n distinct roots of $Q^{2n}(z) = \xi$, also of $Q^{2n+1}(z) = \xi$. Thus the cardinality of $Q^{-n}(\xi) \to \infty$ as $n \to \infty$.

In section 4 we will show that the conclusion of Brolin's theorem holds for every $z \in C$ that is not an excluded value.

3. Brownian Motion in $\mathcal{F}_\infty$

According to a well-known theorem of Lévy [3], if Z_t is a Brownian motion in C started at z_o and if f is a nonconstant, entire, meromorphic function of z such that $f(z_o) \neq \infty$, then $f(Z_{\tau(t)})$ is a Brownian motion started at $f(z_o)$, where

$$\tau(t) = inf\{f : \int_o^r |f'(Z_s)|^2 ds \geq t\}. \tag{3.1}$$

Lévy's theorem is of a local character, as is apparent from Ito's formula, hence generalizes to meromorphic functions and Brownian motion on arbitrary Riemann surfaces. Thus, for example, if $f : \overline{C} \to \overline{C}$ is meromorphic and Z_t is a Brownian motion on $\overline{C}$ started at z_o then $f(Z_{\tau(t)})$ is a Brownian motion on $\overline{C}$ started at $f(z_o)$, where $\tau(t)$ is given by (3.1) but $|f'(z)|$ is interpreted as the local expansion factor for the mapping f relative to the spherical metric on $\overline{C}$. Also, if f is a function that admits an analytic continuation along every continuous path in $\overline{C} - F$, where F is a finite set of points, and if Z_t is a Brownian motion in $\overline{C}$ started at $z_o \in \overline{C} - F$, then $f(Z_{\tau(t)})$ is a Brownian motion in $\overline{C}$ started at $f(z_o)$. In all of these scenarios, $\tau(t)$ is almost surely a strictly increasing, continuous function of t satisfying $\tau(0) = 0$ and $\tau(t) \to \infty$ as $t \to \infty$.

Let Z_t be a Brownian motion on $\overline{C}$ started at $Z_0 = \infty$, and let $Q(z)$ be a polynomial of degree $d \geq 2$ with Julia set J. Define

$$T = inf\{t \geq 0 : Z_t \in J\}. \tag{3.2}$$

We will prove shortly that $T < \infty$ $w.p.1$. Observe that $Z_t, 0 \leq t < T$, is a stochastic process with continuous paths in $\mathcal{F}_\infty$ starting at $Z_o = \infty$ and either terminating at a point of J or avoiding J forever. The distribution of this process is a probability measure $P = P^{\infty}$ on the space $\Omega = \Omega^{\infty}$ of continuous paths in $\mathcal{F}_{\infty}$ starting at ∞ and terminating at J, where Ω is equipped with the σ-algebra $\mathcal{B}$ generated by sets of the form $\{Z_t \in A\}, A$ a Borel subset of $\overline{\mathbb{C}}$.

By Lévy's theorem, $\tilde{Z}_t = Q(Z_{\tau(t)})$ is a Brownian motion on $\overline{\mathbb{C}}$, where $\tau(t)$ is defined by (3.1) with $f' = Q'$. The transformed process $\tilde{Z}_t$ also has initial point $\tilde{Z}_o = \infty$, because $Q(\infty) = \infty$. Moreover, if $T < \infty$ then $\tau^{-1}(T) < \infty$, and

$$\tau^{-1}(T) = inf\{t \geq 0 : \tilde{Z}_t \in J\}$$

because $\mathcal{F}$ and J are Q-invariant sets. On the other hand, if $T = \infty$ then $\tau^{-1}(T) = \infty$, so $\tilde{Z}_t$ does not hit J in finite time. Thus, the transformed process $\tilde{Z}_t, 0 \leq t < \tau^{-1}(T)$, is a Brownian motion started at ∞ and terminated upon reaching J; in particular, the process $\tilde{Z}_t, 0 \leq t < \tau^{-1}(T)$, has the same distribution as $Z_t, 0 \leq t < T$. It follows that the sequence $\{Z_t : 0 \leq t < T\}, \{\tilde{Z}_t : 0 \leq t < \tau^{-1}(T)\}, \{\tilde{\tilde{Z}}_t : 0 \leq t < \tau^{-2}(T)\}, \ldots$, is a stationary process valued in Ω, where $\tilde{\tilde{Z}}_t = Q(\tilde{Z}_{\tilde{\tau}(t)})$, etc. In other words,

PROPOSITION 6: Q induces a measure-preserving transformation on the probability space $(\Omega, \mathcal{B}, P)$, *specifically, if* $z_t, 0 \leq t < T$ *is an element of* Ω *then* $(Qz)_t \triangleq Q(z_{\tau(t)})$ *where*

$$\tau(t) = inf\{r : \int_o^r |Q'(z_s)|^2 ds \geq t\}.$$

Next we will show that $P\{T < \infty\} = 1$. Since Brownian paths are continuous, the terminal point Z_T is a $\mathcal{B}$-measureable function of the path $Z_t, 0 \leq t < T$. Hence, by Proposition 6, the distribution of Z_T (the equilibrium distribution on J) is an invariant measure for the mapping $Q : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

LEMMA 1: Let X_t *be a Brownian motion in* $\mathbb{R}^2$, *and define* $T_R = inf\{t : |X_t| = R\}$. *If* $1 \leq r \leq R$ *then*

$$P\{T_R < T_1 \| X_o| = r\} = \frac{log\ r}{log\ R}.$$

This is well known: see [3], sec. 2.

LEMMA 2: For each $R < \infty$ *there exist* $R_1 > R_2 \geq R$ *and* $0 < p < \frac{1}{2}$ *such that*

$$p \leq \frac{log|Q^n(z)|}{log|Q^n(\xi)|} \leq 1 - p$$

for all $|z| = R_2, |\xi| = R_1$, *and* $n \geq 1$.

PROOF: This follows from the fact that $Q(z)$ looks like a monomial near ∞. Choose $\epsilon > 0$ small; then there exists $R < \infty$ such that for $|z| \geq R$,

$$|az|^d(1-\epsilon) \leq |Q(z)| \leq |az|^d(1+\epsilon)$$

for some constant $0 < a < \infty$. If R_2 is chosen large enough that $a^d R_2^d(1-\epsilon) > R_2$ then induction shows that for $|z| \geq R_2$,

$$|az|^{d^n}(1-\epsilon)^{1+d+\ldots+d^{n-1}} \leq |Q^n(z)| \leq |az|^{d^n}(1+\epsilon)^{1+d+\ldots+d^{n-1}}.$$

Now choose any $R_1 > R_2(\frac{1+\epsilon}{1-\epsilon})^{\Sigma_{i=1}^{\infty} d^{-i}}$. □

Let Z_t be a Brownian motion on $\overline{\mathbf{C}}$ started at ∞ and let T be defined by (3.2).

PROPOSITION 7: $P\{T < \infty\} = 1$.

PROOF: Let $\Gamma_i = \{z \in \mathbf{C} : |z| = R_i\}, i = 1,2,3,4$ where $R_1 > R_2$. We will not distinguish between Γ_i and its image in $\overline{\mathbf{C}}$ by stereographic projection. Assume that R_2 is sufficiently large that $\{z : |z| \geq R_2\}$ is entirely contained in $\mathcal{F}_\infty$.

A Brownian motion on $\overline{\mathbf{C}}$ started at ∞ or at any point in the circular neighborhood of ∞ bounded by Γ_1 will hit Γ_2 in finite time *w.p.*1. Consequently, to prove the proposition it suffices to show that for a Brownian motion started at any point of Γ_2 the probability of hitting J before Γ_1 is at least p for some $p > 0$, provided R_1, R_2 are suitably chosen. For this it suffices to show that for any $\epsilon > 0$ the probability of coming within ϵ of J before hitting Γ_1 is at least p.

Choose R_3 so that $\{z : |z| \geq R_3\} \subset \mathcal{F}_\infty$ and so that in some annulus $\mathcal{A} = \{z : R_3 - \delta < |z| < R_3 + \delta\}$ there is no point of $\mathcal{G}_+$ (recall that $\mathcal{G}_+$ is a countable set whose only accumulation point in $\mathcal{F}_\infty$ is ∞). Then in every open, simply connected subset of $\mathcal{A}$ all branches Q_i^{-n} are single-valued and analytic, so Proposition 2 implies that in each such subset $\{Q_i^{-n}\}$ is a normal family. Now Γ_3 may be covered by two simply connected, open neighborhoods $\mathcal{U}, \mathcal{V}$ for which $\overline{\mathcal{U}} \subset \mathcal{A}$ and $\overline{\mathcal{V}} \subset \mathcal{A}$. Proposition 5 implies that for each $\epsilon > 0$ there exists $n \geq 1$ such that if $z \in \overline{\mathcal{U}} \cup \overline{\mathcal{V}}$ then dist $(Q_i^{-n}(z), J) < \epsilon$ for each branch Q_i^{-n} of Q^{-n}. Choose $R_4 > R_3$ so that Γ_4 is contained in $\mathcal{U} \cup \mathcal{V}$; then by construction $Q^{-n}(\Gamma_4)$ lies entirely within a distance ϵ of J.

Now consider a Brownian motion Z_t started at a point $z_o \in \Gamma_2$. The probability that Z_t comes within a distance ϵ of J before hitting Γ_1 is at least the probability that it hits $Q^{-n}(\Gamma_4)$ before hitting Γ_1. By Lévy's theorem, this is no less than the probability that a Brownian motion $\tilde{Z}_t$ started at $Q^n(z_o)$ hits Γ_4 before $Q^n(\Gamma_1)$, which in turn is at least the probability that $\tilde{Z}_t$ hits Γ_3 before $Q^n(\Gamma_1)$, provided $|Q^n(z_o)| > R_4$. (If R_2 is chosen sufficiently large then this will hold for all $n \geq 1$.) By Lemmas 1 and 2 this probability is $\geq p > 0$, provided R_1 and R_2 are suitably chosen. □

We shall now consider in greater detail the action of Q on the measure space $(\Omega, \mathcal{B}, P)$. This action is d to 1, hence not invertible; if $(z_t)_{0 \leq t \leq T} \in \Omega$ then there are d distinct paths all mapped into $(z_t)_{0 \leq t \leq T}$ by Q. We will show how to permute these d paths in such a way that P is preserved, thus proving that the Wiener measure P is "equidistributed" among the d branches of Q^{-1}.

Consider first the special case where $Q(z) = z^d$. Let $\varsigma_1, \varsigma_2, \ldots, \varsigma_d$ be the d^{th} roots of unity. If Z_t is a Brownian motion started at ∞ then for any $i = 1,2,\ldots,d$, $\tilde{Z}_t = \varsigma_i Z_t$ is also a Brownian motion started at ∞, by Lévy's theorem (or more elementary arguments). Consequently,

$$(z_t)_{0\le t\le T} \longrightarrow (\varsigma_i z_t)_{0\le t\le T}$$

is a measure-preserving transformation of $(\Omega, \mathcal{B}, P)$ that permutes paths mapped into the same path by Q. (NOTE: T is the same for both z_t and $\varsigma_i z_t$ because for $Q(z) = z^d$, J is just the unit circle.) It follows that the distribution of $(Z_t)_{0\le t\le T}$ conditional on the value of $(Q(Z_t))_{0\le t\le T}$ is the uniform distribution on the d paths $(Q_i^{-1}(Q(Z_t)))_{0\le t\le T}, i = 1, 2, \ldots, d$.

In the general case the permutations of paths cannot be defined quite so easily. Let $\Gamma = \{z : |z| = R\}$ where $R < \infty$ is chosen so that $\mathcal{G}_+ \cap \Gamma = \emptyset$ and $\{z : |z| \ge R\} \subset \mathcal{F}_\infty$. Then each $z \in \Gamma$ has a neighborhood in which all branches of Q^{-1} are single-valued and analytic; consequently, if R is sufficiently large then $Q^{-1}(\Gamma)$ is a simple closed analytic curve, and $Q : Q^{-1}(\Gamma) \to \Gamma$ is a d to 1 covering projection (i.e., each $z \in Q^{-1}(\Gamma)$ has a neighborhood in which Q is a homeomorphism). Fix $z \in Q^{-1}(\Gamma)$; then in some neighborhood of z we may define analytic functions $F_i = Q_i^{-1} \circ Q, i = 1, \ldots, d$, where $Q_1^{-1}, \ldots, Q_d^{-1}$ are the distinct branches of Q^{-1} near $Q(z)$. Observe that each F_i has an analytic continuation along any curve that avoids the set $\mathcal{G}_o$ of critical points of Q, and, in particular, along $Q^{-1}(\Gamma)$. Note that $Q \circ F_i = Q$, that the d functions $\{F_i\}$ form a group under composition, and that each is a homeomorphism of $Q^{-1}(\Gamma)$ onto itself.

Now consider the Brownian motion Z_t started at ∞ and terminated at J. Since Z_t must hit Γ before it hits J, it must also hit $Q^{-1}(\Gamma)$ before J (recall that $Q(Z_t)$ is, after a time change, also a Brownian motion started at ∞). Let σ be the first time Z_t hits $Q^{-1}(\Gamma)$. With probability one, the paths

$$Z_t, 0 < t \le \sigma \qquad \text{and}$$
$$Z_t, \sigma \le t \le T$$

are continuous curves that avoid $\mathcal{G}_o$, so each F_i may be analytically continued from Z_σ both backwards and forwards in time, allowing us to define

$$Z_t^{(i)} = F_i(Z_{\sigma_i(t)}), 0 \le t \le \sigma_i^{-1}(T),$$
$$Z_0^{(i)} = \infty,$$

where

$$\sigma_i(t) = inf\{s : \int_0^s |F_i'(Z_r)|^2 dr \ge t\}$$

and $|F_i'|$ is the local expansion factor in the spherical metric. (NOTE: $F_i(z) \sim \alpha\varsigma_i z$ as $|z| \to \infty$ for some $\alpha \ne 0$ and one of the d^{th} roots of unity ς_i, so $|F_i'(Z_r)|$ is continuous and finite at $r = 0$.) By Lévy's theorem, each $Z_t^{(i)}$ is a Brownian motion started at ∞ and terminated at J. Moreover, $Z_t^{(1)}, Z_t^{(2)}, \ldots, Z_t^{(d)}$ are the d distinct Brownian paths mapped by Q into the path $(QZ)_t$ (after the appropriate time changes).

PROPOSITION 8: Conditional on the value of the path $(QZ)_t$, the distribution of the path Z_t is the uniform distribution on the d paths $Z_t^{(1)}, Z_t^{(2)}, \ldots, Z_t^{(d)}$.

PROOF: By the foregoing discussion, each of the processes $Z_t^{(i)}, i = 1,2,\dots,d$, is a Brownian motion started at ∞ and terminated at J. Consider the following method of generating a path $\tilde{Z}_t$: (1) generate Z_t according to P; (2) calculate the corresponding $Z_t^{(1)},\dots,Z_t^{(d)}$; (3) choose one of $Z_t^{(1)},\dots,Z_t^{(d)}$ at random (using the uniform distribution) and call it $\tilde{Z}_t$. Clearly, $\tilde{Z}_t$ is again a Brownian motion started at ∞ and terminated at J, so the processes Z_t and $\tilde{Z}_t$ have the same law. Given the value of the path $(Q\tilde{Z})_t$ the distribution of $\tilde{Z}_t$ is obviously the uniform distribution on the d paths $Z_t^{(1)}, Z_t^{(2)},\dots,Z_t^{(d)}$, which is the same as the uniform distribution on the d paths $\tilde{Z}_t^{(1)},\dots,\tilde{Z}_t^{(d)}$. □

PROPOSITION 9: The measure-preserving transformation of $(\Omega,\mathcal{B},P)$ *induced by* Q *is strongly mixing, i.e., for any events* $A,B\in\mathcal{B}$,

$$\lim_{n\to\infty} P(Q^{-n}A\cap B) = P(A)P(B) \tag{3.3}$$

The proof depends on a simple lemma. Let $U_n, n = 1,2,\dots$, be a decreasing sequence of neighborhoods of ∞ in $\overline{\mathbb{C}}$ such that $\cap U_n = \{\infty\}$, e.g., $U_n = \{z\in\mathbb{C} : |z| > n\}\cup\{\infty\}$. For a Brownian motion Z_t on $\overline{\mathbb{C}}$ started at ∞ let $\tau(n) = min\{t : Z_t \notin U_n\}$; note that $\tau(n)\downarrow 0$ a.s. Define $\mathcal{B}_n$ to be the σ-algebra generated by $Z_t, t\le\tau(n)$.

LEMMA 3: For any event $A\in\mathcal{B}$,

$$\lim_{n\to\infty} P(A|\mathcal{B}_n) = P(A) \ a.s.$$

PROOF: By the (backward) martingale convergence theorem, $P(A|\mathcal{B}_n)\longrightarrow P(A|\mathcal{B}_\infty)$ a.s., where $\mathcal{B}_\infty = \cap\mathcal{B}_n$. It follows from the Blumenthal 0 - 1 Law by an easy argument that $\mathcal{B}_\infty$ is a 0 - 1 σ-algebra, since $\tau(n)\downarrow 0$ a.s. □

PROOF of Proposition 9: First we will show that it suffices to prove (3.3) for a smaller class of events A, B. Let

$$\mathcal{U}_n = \{z : |z| > n\}\cup\{\infty\},$$
$$\mathcal{V}_n = \{z\in\mathcal{F}_\infty : dist(z,\mathcal{J}) < \frac{1}{n}\}.$$

Define $\mathcal{A}_n$ to be the σ-algebra of events $A\in\mathcal{B}$ such that 1_A depends only on the behavior of the path *after* it first exits U_n, and define $\mathcal{A}_n^*$ to be the σ-algebra of events $B\in\mathcal{B}$ such that 1_B depends only on the behavior of the path *before* it first enters $\mathcal{V}_n$. Any events $A,B\in\mathcal{B}$ may be arbitrarily well approximated by $\tilde{A}\in\cup_n\mathcal{A}_n$ and $\tilde{B}\in\cup_n\mathcal{A}_n^*$, i.e., for any $\epsilon > 0, \tilde{A},\tilde{B}$ may be chosen so that

$$P(A\triangle\tilde{A}) < \epsilon \text{ and } P(B\triangle\tilde{B}) < \epsilon.$$

Since Q is a measure-preserving transformation of $(\Omega,\mathcal{B},P)$ it therefore suffices to prove (3.3) for $A\in\cup_n\mathcal{A}_n$ and $B\in\cup_n\mathcal{A}_n^*$.

Let $Z_t, 0 \leq t < T$, be a random path with distribution P and let $\tilde{Z} = Q^n Z = (Q^n(Z_{\sigma(t)}))$ where $\sigma(t)$ is the appropriate time change. Thus $Z, \tilde{Z}$ are both Brownian motions started at ∞ and terminated at J. Then

$$P(Q^{-n}A \cap B) = P\{\tilde{Z} \in A; Z \in B\}.$$

If $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_m^*$ then the event $\{\tilde{Z} \in A\}$ depends only on the behavior of the path $\tilde{Z}$ after it exits $\mathcal{U}_m$, while $\{Z \in B\}$ depends only on the behavior of the path Z before it first enters $\mathcal{V}_m$. By Prop. 8 the conditional distribution of Z given $\tilde{Z}$ is the uniform distribution on the d^n paths mapped into $\tilde{Z}$ by Q^n; it follows that $P\{Z \in B|\tilde{Z}\}$ is a function only depending on the path $\tilde{Z}$ up to the time it first exits $Q^n(\mathcal{F}_\infty - \mathcal{V}_m)$. Now $\mathcal{F}_\infty - \mathcal{V}_m$ is a compact subset of $\mathcal{F}_\infty$ so, by Prop. 3, $Q^n(z) \to \infty$ uniformly for $z \in \mathcal{F}_\infty - \mathcal{V}_m$. Hence, for large $n, Q^n(\mathcal{F}_\infty - \mathcal{V}_m)$ is a small neighborhood of ∞ contained in $\mathcal{U}_m$, and as $n \to \infty, Q^n(\mathcal{F}_\infty - \mathcal{V}_m)$ shrinks to ∞. Let $\tilde{\mathcal{B}}_n$ be the σ-algebra of events depending only on the behavior of the path $\tilde{Z}$ up to the first time $\tilde{Z}$ exits $Q^n(\mathcal{F}_\infty - \mathcal{V}_m)$ and let $\mathcal{B}_n$ be the σ-algebra of events depending only on the behavior of Z up to the first time Z exits $Q^n(\mathcal{F}_\infty - \mathcal{V}_m)$; then $P\{\tilde{Z} \in A|\tilde{\mathcal{B}}_n\}$ and $P\{Z \in A|\mathcal{B}_n\}$ have the same distribution, so by Lemma 3

$$P\{\tilde{Z} \in A|\tilde{\mathcal{B}}_n\} \longrightarrow P\{Z \in A\}$$

in probability as $n \to \infty$. Therefore, for large n

$$\begin{aligned} P\{\tilde{Z} \in A; Z \in B\} &= E1\{\tilde{Z} \in A\}P\{Z \in B|\tilde{Z}\} \\ &= EP\{\tilde{Z} \in A|\tilde{\mathcal{B}}_n\}P\{Z \in B|\tilde{Z}\} \\ &\approx P\{Z \in A\}P\{Z \in B\}. \end{aligned}$$ □

COROLLARY 1: Let L be a closed subset of $\mathcal{F}_\infty$ and let $f : J \to \mathbb{C}$ be continuous. If Z is a Brownian motion started at ∞ and terminated at J then

$$\lim_{n\to\infty} Ef(Z_T)1\{Z \text{ hits } Q^{-n}(L)\} = Ef(Z_T)P\{Z \text{ hits } L\} \tag{3.4}$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{m=0}^{n-1} 1\{Z \text{ hits } Q^{-m}(L)\} = P\{Z \text{ hits } L\} a.s. \tag{3.5}$$

PROOF: (3.4) follows from the strong mixing property by a standard approximation argument, and (3.5) follows from Birkhoff's ergodic theorem, since strong mixing implies ergodicity. □

COROLLARY 2: The equilibrium distribution on J is a strongly mixing invariant measure for the transformation $Q : J \to J$.

PROOF: This follows immediately from (3.3) applied to events A, B that only depend on Z_T. □

4. Brolin's Theorem

Let D be a simply connected region contained in $\mathcal{F}_\infty$ such that $D \cap \mathcal{G}_+ = \emptyset$. Assume that the boundary ∂D is smooth. Then all branches $\{Q_i^{-n}\}$ of the inverse functions Q^{-n} are single-valued and analytic in D and, by Proposition 2, the collection $\{Q_i^{-n}\}$ is a normal family in D. Define

$$D_i^n = Q_i^{-n}(D) \; ; \; D^n = \bigcup_i D_i^n = Q^{-n}(D).$$

Then each D_i^n is a simply connected region with a smooth boundary, and $D_i^n \cap D_j^n = \emptyset$ if $i \neq j$. Proposition 5 implies that

$$\lim_{n\to\infty} \max_i \, diam(D_i^n) = 0, \tag{4.1}$$

$$\lim_{n\to\infty} \max_i \max_{z \in D_i^n} dist(z, J) = 0. \tag{4.2}$$

Let $\{\mu_n\}_{n\geq 1}$ be any sequence of Borel probability measures on $\overline{\mathbb{C}}$ such that $\mu_n(D_i^n) = d^{-n}$ for $i = 1, 2, \ldots, d^n$.

PROPOSITION 10: As $n \to \infty, \mu_n$ converges weakly to the equilibrium distribution on J.

PROOF: Let $Z_t, 0 \leq t \leq T$, be a Brownian motion started at ∞ and stopped at the first time T it hits J, and let f be an arbitrary continuous function on $\overline{\mathbb{C}}$. We must show that

$$\lim_{n\to\infty} \int f d\mu_n = Ef(Z_T).$$

Define $\tau_n = inf\{t : Z_t \in D^n\}$. For large n, if $\tau_n < \infty$ then τ_n is close to T, by (4.2), hence

$$\lim_{n\to\infty} E|f(Z_T) - f(Z_{\tau_n})| 1\{\tau_n < \infty\} = 0.$$

It now follows from Proposition 9 that for large n,

$$\begin{aligned} &Ef(Z_{\tau_n})1\{\tau_n < \infty\} \\ &\quad \sim Ef(Z_T)1\{\tau_n < \infty\} \\ &\quad = Ef(Z_T)1\{Z_t \text{ hits } Q^{-n}(D)\} \\ &\quad \sim Ef(Z_T)P\{Z_t \text{ hits } D\}. \end{aligned}$$

It remains to show that

$$\lim_{n\to\infty} |Ef(Z_{\tau_n})1\{\tau_n < \infty\} - (\int f d\mu_n) P\{\tau_n < \infty\}| = 0.$$

The event $\{\tau_n < \infty\} = \{Q^n Z_t \text{ hits } D\}$ depends only on the path $Q^n Z$. By Proposition 8, conditional on the value of the path $Q^n Z$, the distribution of Z is the uniform distribution on the d^n paths $\tilde{Z}^{(i)} = Q_i^{-n}(Q^n Z)$; consequently

$$\begin{aligned} &Ef(Z_{\tau_n})1\{\tau_n < \infty\} \\ &\quad = E(d^{-n} \sum_{i=1}^{d^n} f(Q_i^{-n}(Q^n(Z_{\tau_n}))))1\{\tau_n < \infty\}. \end{aligned}$$

Since $Q^n(Z_{\tau_n}) \in D$ there is exactly one point among $Q_i^{-n}(Q^n(Z_{\tau_n}))$ in each D_i^n. It follows from (4.1) and the continuity of f that for large n,

$$d^{-n} \sum_{i=1}^{d^n} f(Q_i^{-n}(Q^n(Z_{\tau_n}))) \sim \int f d\mu_n$$

on the event $\{\tau_n < \infty\}$. □

COROLLARY 3: Let $z \in \mathcal{F}_\infty, z \notin \mathcal{G}_+$, and define μ_n^z to be the uniform distribution on the d^n points in $Q^{-n}(z)$. Then as $n \to \infty, \mu_n^z$ converges weakly to the equilibrium distribution on J. □

To complete the proof of Brolin's theorem we will show that μ_n^z, μ_n^ξ are eventually close in the weak topology for any $z, \xi \in \mathbf{C}$, provided neither z nor ξ is an excluded value of Q (recall that there is at most one excluded value). The argument involves no further use of Brownian motion or the results of section 3.

Suppose that both z and ξ are elements of a simply connected domain $D \subset \mathbf{C}$ such that $D \cap \mathcal{G}_+ = \emptyset$. Then all branches Q_i^{-n} of Q^{-n} are single-valued and analytic in D, and $D_i^n = Q_i^{-n}(D), i = 1, 2, \ldots, d^n$, are pairwise disjoint domains satisfying (4.1) and (4.2), and each containing precisely one root of $Q^n(\varsigma) = z$ and one root of $Q^n(\varsigma) = \xi$. Clearly, (4.1) implies that μ_n^z and μ_n^ξ are close in the weak topology.

Unfortunately, not all z, ξ are contained in a simply connected domain disjoint from $\mathcal{G}_+$, so not all z, ξ are contained in a connected domain in which all branches Q_i^{-n} are single-valued and analytic. Consequently, we must deal with domains that contain branch points of Q^{-n} for infinitely many n. For a simply connected domain $D \subset \mathbf{C}$ define $\beta_n(D)$ to be the number of branches of Q^{-n} that are single-valued and analytic in D. Recall that $\mathcal{G}_o$ is the set of critical points of Q and $\mathcal{G}_m = \bigcup_{i=1}^{m} Q^i(\mathcal{G}_o)$ is the set of finite branch points of Q^{-m}.

LEMMA 4: If $D \cap \mathcal{G}_m = \emptyset$ then for $n \geq 0$,

$$\beta_{m+n}(D) \geq d^{m+n} - d^{n+1} + d.$$

NOTE: see [6] for a similar result.

PROOF: By induction on n. By hypothesis D contains no branch points of Q^{-m}, so all branches of Q^{-m} are single-valued and analytic in D, whence $\beta_m(D) = d^m$, proving the case $n = 0$. Suppose now that the result holds for some $n \geq 0$. Each of the single-valued, analytic branches of $Q^{-(m+n)}$ maps D homeomorphically onto a simply connected domain D_i (since each branch of $Q^{-(m+n)}$ is obviously 1-1 in D), and the various regions $D_1, D_2, \ldots, D_r$ are pairwise disjoint. By the induction hypothesis, $r \geq d^{m+n} - d^{n+1} + d$. Now there are at most $(d-1)$ branch points of Q^{-1}, so at most $(d-1)$ of the regions $D_1, D_2, \ldots, D_r$ contain branch points of Q^{-1}; hence in at least $r - (d-1)$ of the regions D_i all d branches of Q^{-1} are single-valued and analytic. Therefore the number of single-valued, analytic branches of $Q^{-(m+n+1)}$ in D is at least

$$\begin{aligned} d(r - d + 1) &\geq d(d^{m+n} - d^{n+1} + d - d + 1) \\ &= d^{m+n+1} - d^{n+2} + d. \end{aligned}$$

□

Let $\xi \in \mathbf{C} - \mathcal{G}_+$; then ξ is not a branch point of any Q^{-n}. Fix $z \in \mathcal{F}_\infty - \mathcal{G}_+$. Then for any $m \geq 1$ there is a simply connected domain D containing both ξ and z and such that $D \cap \mathcal{G}_m = \emptyset$, because $\mathcal{G}_m$ is a finite set with neither ξ nor z as a member. Let $\{Q_i^{-n}\}$ be the set of all branches of some $Q^{-n}, n \geq 1$, that are single-valued and analytic in D. By Proposition 2, $\{Q_i^{-n}\}$ is a normal family in D. Define $D_i^n = Q_i^{-n}(D)$ for each Q_i^{-n} that is single-valued and analytic in D; then Proposition 5 implies that (4.1) and (4.2) hold.

Each D_i^n contains precisely one element of $Q^{-n}(z)$ and one element of $Q^{-n}(\xi)$. Pair off the elements of $Q^{-n}(\xi)$ and $Q^{-n}(z)$ in such a way that if $\xi' \in Q^{-n}(\xi)$ and $z' \in Q^{-n}(z)$ are in the same set D_i^n then they are paired together. Observe that for pairs (ξ', z') such that $\xi', z' \in D_i^n$ for some D_i^n the distance between ξ', z' is small if n is large, by (4.1). The number of such pairs is

$$\beta_n(D) \geq d^n(1 - d^{-(m-1)} - d^{-(n-1)}),$$

by Lemma 4, provided $n \geq m$. The total number of pairs is $d^n = |Q^{-n}(z)| = |Q^{-n}(\xi)|$. It now follows that if f is any continuous function on $\overline{\mathbf{C}}$ then

$$\limsup_{n \to \infty} \left| \int f d\mu_n^z - \int f d\mu_n^\xi \right| \leq 2\|f\|_\infty d^{-(m-1)}. \tag{4.3}$$

But $m \geq 1$ was arbitrary; consequently, as $n \to \infty$ the measures μ_n^z, μ_n^ξ become close in the weak topology. According to Corollary 3, μ_n^z converges weakly to the equilibrium distribution on $\mathcal{J}$. Therefore, for any $\xi \in \mathbf{C} - \mathcal{G}_+$, as $n \to \infty$ the measures μ_n^ξ converge weakly to the equilibrium distribution on $\mathcal{J}$.

It remains to consider points $\varsigma \in \mathcal{G}_+$. Assume that ς is not an excluded value (recall that there is at most one excluded value). Then as $n \to \infty$ the cardinality of $Q^{-n}(\varsigma) \to \infty$. It follows that for any $\epsilon > 0$ and each $m \geq 1$ the proportion of points in $Q^{-n}(\varsigma)$ that are in $\mathcal{G}_m$ is $< \epsilon$ for all $n \geq n(\epsilon, m) \geq m$.

Fix $z \in \mathcal{F}_\infty - \mathcal{G}_+$. For each $\xi \in Q^{-n}(\varsigma), \xi \notin \mathcal{G}_m$, there is a simply connected domain D containing both ξ and z such that $D \cap \mathcal{G}_m = \emptyset$. By the same argument as earlier, if f is any continuous function on $\overline{\mathbf{C}}$ then (4.3) holds. Now

$$\mu_{n+n(\epsilon,m)}^\varsigma = d^{-n(\epsilon,m)} \sum_{\xi \in Q^{-n(\epsilon,m)}(\varsigma)} \mu_n^\xi$$

and $\mu_n^z \to \mu$ weakly, where μ is the equilibrium distribution on $\mathcal{J}$; consequently, by (4.3),

$$\limsup_{n \to \infty} \left| \int f d\mu_n^\varsigma - \int f d\mu \right| \leq 2\|f\|_\infty (d^{-m+1} + \epsilon).$$

Since $\epsilon > 0$ and $m \geq 1$ are arbitrary, this proves that $\mu_n^\varsigma \to \mu$ weakly. This proves Brolin's theorem. □

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