# Trend–Free Run Orders of Mixed–Level Fractional Factorial Designs

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# Summary

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Coster and Cheng (1988) presented a Generalized Foldover Scheme for the construction of systematic run orders of fractional factorial designs, with all factors having the same prime power number of levels, for which all the main effects components of the factors are orthogonal to a polynomial trend present in every block of the design. We present here modifications to the foldover method that allow polynomial trend-free run orders to be constructed in the following more general settings: designs for which the number of levels of each factor is not a prime power; mixed-level factorial designs with factors at different numbers of levels; cases in which some or all two- and higher-factor interactions, not just the main effects, are required to be orthogonal to the polynomial trend.

**1. Introduction.** Suppose that the treatment combinations of a given fractional factorial plan are to be performed in a time (or space) sequence and that the experimenter has reason to believe that the observed yields will be influenced by a temporal (spatial) trend over the course of the experiment. In such cases, instead of the normally recommended randomized orders for the runs in each block of the design, the experimenter may prefer certain systematic run orders that improve or maximize the efficiency with which the main effects and certain multi-factor interactions are estimated in the presence of this nuisance trend. In this paper, we modify the Generalized Foldover Scheme (henceforth, GFS) of Coster and Cheng (1988) to achieve optimal efficiency for this estimation problem when the trend is modeled by a (typically, low-degree) polynomial over the equally spaced run positions of the observations in each block. We define optimal efficiency to be orthogonality between the factor effects of interest and the trend effects, in terms of the usual homoscedastic linear model, (see Section 2).

The principal extensions made in this paper to the results in Coster and Cheng (1988) involve the specification of sufficient conditions on the appearance of factors at non-zero levels in sequences of generators of a fractional factorial design such that two- and higher-factor interactions also achieve the trend orthogonality criterion previously applied only to the problem of main effects estimation. We further generalize the foldover approach to designs that need not have every factor with the same number of levels (mixed-level factorials) and the number of levels need not be a prime power.

Cox (1951) introduced systematic designs for replicated variety trials with the criterion of efficient estimation of the treatment effects in the presence of a smooth polynomial trend. Other early approaches to the problem of trend elimination are discussed in Draper and Stoneman (1968), Dickinson (1974), Cheng (1985) and John (1986). Much of their work is generalized in Coster and Cheng (1988). Cheng and Jacroux (1987) and Cheng (1988) discuss an alternative and elegant approach, first introduced by Daniel and Wilcoxon (1966), to the trend elimination problem for main effects and (some) two-factor interactions in unblocked two-level fractional factorial designs. For the $2^n$ series, they provide a

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construction technique with this approach that is essentially equivalent to the GFS generator sequence shown in Example 3 of Section 4. Cheng (1988) also discusses the correspondence between our foldover method and the Daniel and Wilcoxon scheme for designs for factors all with two levels. Numerous practical examples of fractional factorial designs for factors with two or three levels are available in two National Bureau of Standards publications, Applied Mathematics Series 48 (1957) and 54 (1959). Example 4 of Section 4 is taken from AMS 48 (1957).

The primary advantage of the GFS for achieving the trend orthogonality optimality criterion is the ease with which an experimenter can try various generator sequences and quickly verify, using the sufficient conditions detailed in Theorems 1, 2 and 3, whether trend elimination for the factor effects of interest has been achieved. Except in very small designs, it is not difficult to achieve orthogonality for the main effects components. Interactions present a greater challenge, in particular when we model non-prime-leveled factors with prime-leveled pseudofactors and *require* interactions among the pseudofactors belonging to each real factor to be trend free. The examples of Section 4 fall mostly into this latter category.

In Section 2, we summarize the definition of the mixed-level fractional factorial designs to which the modified GFS is applied in Section 3 to obtain optimal, trend-free run orders. All proofs of construction results are left until the Appendix. Some applications are shown in Section 4.

**2. Mixed-Level Fractional Factorial Model.** Let $G$ denote a fractional factorial design involving $n$ factors, say $a_1, \ldots, a_n$, with $n_i \geq 1$ of these factors at $s_i$ levels, for $i = 1, \ldots, q$, where each $s_i$ is a *distinct prime number.* Then $n = n_1 + \cdots + n_q$. Let the levels of any factor with $s_i$ levels be the set of integers $\{0, 1, \ldots, s_{i-1}\}$, with all arithmetic on these levels being performed modulo $s_i$. A complete factorial design in all $n$ factors would require $\prod_{i=1}^{q} s_i^{n_i}$ runs or treatment combinations.

We use the notation

$$G = s_1^{(n_1, p_1, r_1)} \cdots s_q^{(n_q, p_q, r_q)} \tag{2.1}$$

to denote a (possibly) blocked and/or fractional factorial design in the $n$ factors. For each $i = 1, \ldots, q$ and the $n_i$ factors with $s_i$ levels, let $G_i = s_i^{(n_i, p_i, r_i)}$ denote an $s_i^{-p_i}$ fraction of the complete factorial design found by selecting a set of $p_i$ independent defining effects, $n_i > p_i \geq 0$, involving only these $n_i$ factors. We assume that the fraction chosen always contains the treatment combination **1** in which all the $n_i$ factors are at level 0. Another $r_i$ independent effects are used to block the $s_i^{n_i - p_i}$ runs into $s_i^{r_i}$ blocks each of size $s_i^{n_i - p_i - r_i}$. Then the notation of (2.1) implies that $G = G_1 \times G_2 \times \cdots \times G_q$, the product of the sub-designs $G_i$, so $G$ contains all the treatment combinations formed from products of the runs in each sub-design $G_i$. We assume that the principal blocks of each sub-design generate the principal block of $G$, the block containing the run **1**. Thus, $G$ contains $N = \prod_{i=1}^{q} s_i^{n_i - p_i}$ runs blocked into $B = \prod_{i=1}^{q} s_i^{r_i}$ blocks each of size $R = N / B$. While other methods may be used to define mixed-level fractional factorial designs, this product structure for $G$ has, for our purposes, the two-fold advantage that (i) it is an easily applied method that is in common use and (ii) it proves compatible with the GFS used in Section 3 to achieve the design objective of trend orthogonality.

Define $r = \sum_{i=1}^{q} r_i$ and $p = \sum_{i=1}^{q} p_i$. Let $B_1$ denote the principal block of $G$. Then the treatment combinations of $G$ may be found by choosing $h = n - p - r$ independent runs in $B_1$ and forming all possible powers and products among them to generate $B_1$. There are $h_i = n_i - p_i - r_i$ *within-block* generators of $B_1$ contributed by each sub-design $G_i$, say $\mathbf{g}_{ij}$, $j = 1, \ldots, h_i$, $i = 1, \ldots, q$. Then every run in $B_1$ has the general form

$$\mathbf{g} = \prod_{i=1}^{q} [\prod_{j=1}^{h_i} \mathbf{g}_{ij}^{\xi_{ij}}],$$

where $\xi_{ij}$ ranges over the set $\{0, , \ldots, s_i - 1\}$ for each $i = 1, \ldots, q$. The remaining $B_1 - 1$ blocks of $G$ may be generated in a similar fashion using independent *between-block* runs $\mathbf{g}_1, \ldots, \mathbf{g}_r$ from distinct blocks of $G$.

In Coster and Cheng (1988), trend orthogonality via the GFS was developed for main effects (only) plans with ALL $n$ factors having the *same* prime power number of levels, $s^m$, $m \geq 1$, $s$ prime. The results of Section 3 allow us to cover not only this prime-powered case but more general mixed-level fractional factorial designs by modeling non-prime leveled factors by products of prime leveled *pseudofactors* and requiring all main effects *and* interactions among these pseudofactors to achieve the trend orthogonality condition. Theorem 2 provides the primary construction results for this purpose.

Following the development in Coster and Cheng (1988), we define below the form of the polynomial trend present in each block of $G$ and the main effects components of each of the $n$ factors. In these definitions, $s$ is any one of the prime numbers $s_1, \ldots, s_q$.

DEFINITION 1. The system of orthogonal polynomials on m equally spaced points $l = 0, \ldots, m-1$ is the set $\{P_{km}, k = 0, 1, 2, \ldots, m-1\}$ of polynomials satisfying

$$\sum_{l=0}^{m-1} P_{km}(l) = 0 \quad \text{for } all \ k \geq 1 \tag{2.2}$$

$$\sum_{l=0}^{m-1} P_{km}(l) P_{k'm}(l) = 0 \quad \text{for } all \ k \neq k', \tag{2.3}$$

where $P_{0m}(l) = 1$ and $P_{km}(l)$ is a polynomial of degree $k$. We assume that each polynomial in the system is scaled so that its values are always integers.

DEFINITION 2. (Factor effects). The $s$ coefficients of the $j$th main effects component of a factor, $1 \leq j \leq s-1$, are $P_{js}(l)$, $0 \leq l \leq s-1$, the values of the orthogonal polynomial of degree $j$ on $s$ equally spaced points.

DEFINITION 3. (Trend effects). The $R$ values of a polynomial trend of degree $j$, $1 \leq j \leq R-1$, in a block of size $R$ are $P_{jR}(l)$, $0 \leq l \leq R-1$, the values of the orthogonal polynomial of degree $j$ on $R$ equally spaced points.

The linear model for the $N$ observations is

$$\mathbf{Y} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{T})\,(\beta'_1, \beta'_2, \beta'_3)' \ + \ \varepsilon, \tag{2.4}$$

where $\varepsilon$ is an $N$-vector of zero mean, uncorrelated random errors, $\mathbf{X}_1$ is an $N \times \tau$ matrix of factor effect coefficients, $\mathbf{X}_2$ is an $N \times B$ matrix of block effect coefficients, and $\mathbf{T}$ is an $N \times k$ matrix of polynomial trend coefficients, the same in every block, of degrees $1, \ldots, k$. The first $R$ rows of $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{T})$ are the $R$ treatment combinations in the principal block $B_1$, the next $R$ rows the treatment combinations in the second block, and so on. The terms $\beta_1$, $\beta_2$ and $\beta_3$ are the corresponding factor, block and trend parameter effects, respectively. The product design definition of $G$, the assumption of the same degree trend in each block, and the requirements that only one effect from each alias set and no effects confounded with blocks are included in $\mathbf{X}_1$ imply that:

$$\mathbf{X}'_1 \mathbf{X}_1 = \mathbf{0}, \qquad \mathbf{X}'_1 \mathbf{X}_2 = \mathbf{0}, \qquad \mathbf{X}'_2 \mathbf{T} = \mathbf{0}.$$

We may now define the design criterion for trend elimination.

DEFINITION 4. (Design optimality). A run order of design $G$ is optimal for the estimation of the factor effects of interest, $\beta_1$, in the presence of a nuisance $k$-degree polynomial trend in each block, if

$$\mathbf{X}'_1 \mathbf{T} = \mathbf{0}. \tag{2.5}$$

If condition (2.5) is satisfied, we say that the run order of $G$ is $k$-trend free.

If $\mathbf{x}$ is any column of $\mathbf{X}_1$ and $\mathbf{t}$ any column of $\mathbf{T}$, then we call the usual inner product $\mathbf{x}'\mathbf{t}$ the time count between $\mathbf{x}$ and $\mathbf{t}$. Criterion (2.5) states that all the time counts are zero for an optimal run order. As stated in the Introduction, our primary objective is to satisfy optimality condition (2.5) in a setting where $\mathbf{X}_1$ contains columns representing two- and higher-factor interactions among factors not constrained to have the same prime power number of levels.

**3. Construction of Optimal Run Orders by the GFS.** We begin by modifying the Generalized Foldover Scheme, GFS, of Coster and Cheng (1988) for the mixed-level fractional factorial designs defined in the previous section. We then present conditions under which both main effect and interaction components of the $n$ factors become othogonal to the polynomial trend. This leads to a stepwise construction method for optimal run orders of $G$. In what follows, we may assume that $G$ is run in a single block of size $N$. The usual block structure is replaced after Theorem 2 and the advantage of blocking is demonstrated by Theorem 3.

DEFINITION 5. (GFS for $G$). Suppose that $\{\mathbf{g}_1, \ldots, \mathbf{g}_{n-p}\}$ are $n-p$ independent generators of $G$. Assume that $\mathbf{g}_j$, $j=1,\ldots,n-p$, contains at a *non-zero* level at least one factor with $f_j$ levels, $f_j \in \{s_1, \ldots, s_q\}$. Let $U_0 = \mathbf{1}$. Then the run order of $G$ produced by the GFS with respect to generator sequence $\{\mathbf{g}_1, \ldots, \mathbf{g}_{n-p}\}$ and *foldover sequence* $\{f_1, \ldots, f_{n-p}\}$ is given by $U_{n-p}$ where

$$U_j = U^*_{j-1}(\mathbf{g}_j) = (U_{j-1}, U_{j-1}\mathbf{g}_j, \ldots, U_{j-1}\mathbf{g}_j^{f_j-1}), \quad j=1,\ldots,n-p. \tag{3.1}$$

DEFINITION 6. ($k$-trend free factor effects over $U_j$). Let factors $a_1, \ldots, a_m$, $m \geq 1$, have $t_1, \ldots, t_m$ levels, respectively. Then the $m$-factor interaction in factors $a_1, \ldots, a_m$ is $k$-trend free for some $k \geq 0$ over the run order of $U_j$ if

(a) at least one of the $m$ factors, say $a_1$, occurs equally often at each of its levels over $U_j$ and

(b) *all* $t = \prod_{l=1}^{m} (t_l - 1)$ interaction components are orthogonal to the trend polynomials $P_{0N}, \ldots, P_{kN}$.

Condition (a) of Definition 6 ensures that our definition of $k$-trend free interactions is in keeping with the definition for $k$-trend free main effects only, that is, $m=1$, in Coster and Cheng (1988). When constructing run orders with the GFS (3.1) of Definition 5, if at least one of the $m$ factors, say $a_1$ with $t_1$ levels, is at a non-zero level in a generator $g_v$ of $U_j$ for which $f_v = t_1$, then $a_1$ meets condition (a) and is 0-trend free.

Among the foldover levels $\{f_j, j=1, \ldots, n-p\}$ there are exactly $n_i - p_i$ appearances of the level $s_i$, $i=1, \ldots, q$. If this last condition were not met, $G$ would not be correctly generated. Note that generator $g_j$ may contain, at non-zero levels, other factors with numbers of levels not equal to the foldover level $f_j$. This has the advantage that it easily produces run orders of $G$ that are not simply the product of the separately ordered sub-designs $G_i$. This latter run order, while it might have the trend orthogonality properties we seek, would be considered too systematic for many practical applications.

EXAMPLE 1. Let $G = 2^{(2,1,0)} 3^{(2,1,0)}$ be defined by $I = AB$ for the two factors with 2 levels and by $I = CD$ for those with 3 levels. If we choose $g_1 = abc^2d$ and $f_1 = 2$ followed by $g_2 = cd^2$ and $f_2 = 3$, by Definition 5, the GFS (3.1) generates the run order

$$G = U_2 = (1, abc^2d, cd^2, ab, c^2d, abcd^2),$$

which is not the simple product of $G_1 = (1, ab)$ and $G_2 = (1, cd^2, c^2d)$.

We now state our primary construction results. Theorem 1 is a generalization, for our mixed-level factorial design structure, of the results in Coster and Cheng (1988) that guarantee $k$-trend free main effects components. Essentially the same conditions must be met. Our primary result, Theorem 2, provides sufficient conditions on the generator and foldover sequences that ensure trend orthogonality for two- and (possibly) higher-factor interactions. We then recover the usual block structure and show how this is useful in Theorem 3. Before presenting these theorems, we state in Lemma 1 the essential requirement that any $m$-factor interaction that is $k$-trend free over $U_{v-1}$ remains $k$-trend free over $U_v$ (and hence over $G$). In what follows, let the generator and foldover sequences be as given in Definition 5.

LEMMA 1. Suppose that the $m$-factor interaction in factors $a_1, \ldots, a_m$, $m \geq 1$, is $k$-trend free, $k \geq 0$, over $U_{v-1}$ according to Definition 6. Then this same interaction is also $k$-trend free over each piece $U_{v-1}\mathbf{g}_v^j$, $j = 1, \ldots, f_v - 1$, of $U_v$ and hence is $k$-trend free over $U_v$.

Note that some of the components of the $m$-factor interaction of interest may not be part of the columns of the effects matrix $\mathbf{X}_1$ of (2.4) because of aliasing or confounding in the blocking and fractionating schemes. However, our definition applies to all the components of the interaction even though some of these components are not part of the final estimation problem.

THEOREM 1. Let $a$ be any one of the $n$ factors and let $a$ have $s$ levels, $s \in \{ s_1, \ldots, s_q \}$. Suppose that factor $a$ is at a non-zero level in $(k+1)$, $k \geq 0$, of the generators for which the corresponding foldover level is $s$. Then all $s-1$ main effects components, as given by Definition 2, are $k$-trend free over $G$.

Each factor must be non-zero at its foldover level in at least one generator, that is, all factors are necessarily 0-trend free over $G$. The conditions of Theorem 1 must be met for all $n$ factors if the resulting run order is to be an optimal $k$-trend free main effects plan. Note one limitation of the GFS for constructing $k$-trend free main effects run orders of $G$: if $n_i - p_i = 1$ for one (or more) $i$, so that there is only one use of foldover level $s_i$, then the factor(s) with $s_i$ levels cannot be made $k$-trend free for any $k \geq 1$. Therefore, we exclude this possibility in all practical examples. Theorem 1 is, in fact, a special case of Theorem 2 (iii) below and no proof of Theorem 1 is given in the Appendix. However, we have stated the sufficient conditions for main effects to be $k$-trend free in a separate theorem because of the assumed importance to the experimenter of achieving trend orthogonality for main effects before being concerned about interaction components.

THEOREM 2. Suppose that all the components of an $(m-1)$-factor interaction, $(m \geq 2)$, involving factors $a_1, \ldots, a_{m-1}$ are $k$-trend free over $U_{v-1}$, $2 \leq v \leq n-p$. Suppose that a factor $a_m$ with $s$ levels is at a non-zero level $\xi_{mv}$ in $\mathbf{g}_v$ and the foldover level is $f_v$. Then one of the following cases may apply:

(i) if $a_m \notin \{ a_1, \ldots, a_{m-1} \}$, then the $m$-factor interaction in factors $a_1, \ldots, a_m$ is $k$-trend free over $U_v$ if factor $a_m$ is at level zero in generators $g_1, \ldots, g_{v-1}$ of $U_{v-1}$;

(ii) if $a_m \notin \{ a_1, \ldots, a_{m-1} \}$, then the $m$-factor interaction in factors $a_1, \ldots, a_m$ is $(k+1)$-trend free over $U_v$ if all of the factors $a_1, \ldots, a_{m-1}$ are at level zero in $g_v$ and $f_v = s$;

(iii) if $a_m \in \{a_1, \ldots, a_{m-1}\}$, say $a_m = a_1$, then the $(m-1)$-factor interaction in factors $a_1, \ldots, a_{m-1}$ is $(k+1)$-trend free over $U_v$ if $a_2, \ldots, a_{m-1}$ are at level zero in $g_v$ and $f_v = s$.

Case (i) of Theorem 2 indicates that the $m$-factor interaction inherits the $k$-trend free property of the $(m-1)$-factor interaction, provided factor $a_m$ has yet to appear in any previous generator. Of greater import are the sufficient conditions, expressed in the second and third cases of Theorem 2, that produce an increase in the degree of trend orthogonality. Case (ii) shows how the degree of trend orthogonality may be increased from $k$ to $(k+1)$ provided that the new factor, $a_m$, appears in $g_v$ in isolation to the other $(m-1)$ factors. A similar requirement is needed in case (iii).

Our construction objectives are now apparent. To achieve main effects orthogonality to a $k$th degree trend, we seek a generator (and foldover) sequence that has each factor appearing at a non-zero level and at its foldover level in $(k+1)$ generators. Interactions involving two factors are $k$-trend free if the two factors are non-zero in isolation to one another in $(k+1)$ generators with the appropriate foldover levels, or if case (i) applies to the two factors of interest. We may similarly procede to conditions for higher factor interactions to be trend free. When attempting to meet the trend orthogonality conditions for main effects, if the foldover level for a generator is $f_v$, we would like to have as many as is possible of the factors with $f_v$ levels appearing at a non-zero level in $g_v$. Conversely, trend orthogonality for two-factor interactions requires these factors to be isolated from one another in different generators. Clearly, these conditions compete with each other and in some cases run orders with, say, all main effects and two-factor interaction components $k$-trend free cannot be generated by the GFS.

EXAMPLE 2. Let $G$ be as in Example 1, but generated by $g_1 = ab$ and $g_2 = cd^2$, with $f_1 = 2$ and $f_2 = 3$ as before. Then the two $ac$ interaction components are 1-trend free over $G$. (Note: the same is true of the $ad$, $bc$ and $bd$ components, but these are not part of the effects matrix $X_1$ of (2.4) because of the aliasing of A with B and C with D.) Both main effects are 0-trend free but neither is 1-trend free in this (impractically) small example.

Before moving to examples and applications in the next section, we replace the usual block structure of $G$ defined by (2.1) and state a theorem that exploits the block structure and, in particular, the assumption that the same degree trend is present in every block. Recall that the principal block uses $h=n-p-r$ generators. Let $\mathbf{g}_v$, $h+1 \le v \le n-p$, be a between block generator with foldover level $f_v$. Then, with the GFS construction method, the following conditions hold.

THEOREM 3. (Blocked designs). Suppose that factor $a_1$ with $t_1=f_v$ levels is at a non-zero level in $\mathbf{g}_v$. Then,

(i) all $t_1-1$ main effects components of $a_1$ are orthogonal to *all* the trend columns of matrix **T** of model (2.4).

(ii) if factors $a_2, \ldots, a_m$ are at level zero in $\mathbf{g}_v$, then the $m$-factor interaction in $a_1, \ldots, a_m$ is orthogonal to *all* the trend columns of matrix **T** of model (2.4).

**4. Examples of k-Trend Free Runs Orders.** Coster and Cheng (1988) gave an example of a run order of $G=2^{(4,0,0)}$, a complete factorial in four factors each at two levels, for which all four main effects and six two-factor interactions are 1-trend free. We generalize this individual example as follows.

EXAMPLE 3. Suppose that the design is $G=s^{(n,0,0)}$, a complete factorial design in $n$ factors each with $s$ levels, where $s$ is a prime number and $n \ge 4$. Then, for the following two cases:

(i) $n$ is *even*, and we choose generators

$$\mathbf{g}_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n\,,\quad i=1, \ldots, n-1\,,\qquad \mathbf{g}_n = a_1 \cdots a_{n-2} a_{n-1}^{[q(s-1)(mod\; s)]};$$

(ii) $n$ is *odd*, and we choose generators

$$\mathbf{g}_1 = a_2 \cdots a_n\,,\qquad \mathbf{g}_{n-1} = a_1 \cdots a_{n-2} a_{n-1}^{[q(s-1)(mod\; s)]}\,,\qquad \mathbf{g}_n = a_n$$

$$\mathbf{g}_i = a_1 \cdots a_{i-1} a_{i+1} a_{n-1}\,,\quad i=2, \ldots, n-2;$$

the resulting run order of $G$ has all $n(s-1)$ main effects components and all $1/2\,n(n-1)(s-1)^2$ two-factor interaction components 1-trend free. Note that the integer $q$ in the generator expressions above is any choice of $1 \le q \le s-1$ such that we obtain a complete set of $n$ independent generators. It is sufficient to choose $q \ne (n-2)\,(mod\; s)$. The case for $n$ odd simply uses the sequence for an even number $(n-1)$ of factors, modifies $\mathbf{g}_1$ by including $a_n$ and adds the last generator $\mathbf{g}_n = a_n$ to meet the requirement for the main effects to be 1-trend free that each factor be at a non-zero level at least twice in the generator sequence. Each pair of distinct factors $a_i$, $a_j$, $i \neq j$ are isolated from one another in two different generators. Hence, by Theorem 2 (ii), all two-factor interactions are 1-trend free.

For example, the design $G = 2^{(4,0,0)}$ presented in Coster and Cheng (1988) would now use generator sequence $\{ bcd, acd, abd, abc \}$, while design $G = 2^{(5,0,0)}$ would have generator sequence $\{ bcde, acd, abd, abc, e \}$. In a similar fashion, design $G = 3^{(4,0,0)}$ is generated by the sequence $\{ bcd, acd, abd, abc^2 \}$, except that the foldover level is now 3, not 2, at every stage. For this last case, the value $q = 1$ was used for $g_4$ since $(4-2) = 2 \ (mod\ 3)$ cannot be used.

The examples shown here are generally large, complete factorial designs. In practice, such large designs are often blocked, especially in our trend elimination setting where the experimenter may believe the trend to be linear in each block when the blocks are relatively small but may doubt this assumption for one very large block of size $s^n$. As Theorem 3 demonstrates, blocking typically makes the trend elimination problem easier, although success does depend on the choice of block confounding effects. For example, if $G = 2^{(4,0,1)}$ with the ABC interaction confounded with blocks, the same generator sequence shown above for the unblocked $2^4$ is sufficient. However, if the highest factor interaction ABCD is confounded with blocks, the GFS fails to find a run order for which all main effects and two-factor interactions are 1-trend free. As stated in the Introduction, Cheng and Jacroux (1988) provide an alternative approach to the trend elimination problem for the $2^n$ series.

EXAMPLE 4. Consider the design $G = 2^{(8,3,2)}$ defined by $I = ABEGH = ACFG = ABCD$ with blocking effects $ABEF$ and $ACE$, a design in 4 blocks of size 8 for 8 factors each at 2 levels. This is plan 8.8.8 in Applied Mathematics Series 48 (1957) from the National Bureau of Standards. There are 12 estimable two-factor interactions (out of a total of 28 two-factor interactions), and these are listed in AMS 48 (1957). The following sequence of five generators, the first three generating the principal block and the last two the remaining three blocks, make all 8 main effects and 12 estimable two-factor interactions 1-trend free:

$$abcd, abefh, bcegh, abcdefg, eh.$$

EXAMPLE 5. Consider now designs for $n$ factors each with a proper prime-power number of levels. For our purposes, we will restrict attention to the prime powers 4, 8 and 9. (The prime powers 16, 25, 27 and so on seem unreasonably large for most practical factorial designs in at least $n \geq 2$ factors, especially complete factorial designs.)

To use the GFS construction method of Section 3 and the results developed there, we first define pseudofactors, each with a prime number of levels, to represent each real factor. If each factor $a_i$ has $s^m$ levels, $s$ prime, $m > 1$, let $m$ pseudofactors each with $s$ levels be $a_{ij}, j = 1, \ldots, m$. Then *all* main effects and interactions among the $m$ pseudofactors are equivalent to the $s^m - 1$ main effects components of the real factor. Thus, a factor with 4 levels requires two pseudofactors each at 2 levels, a factor with 9 levels is represented by two pseudofactors with 3 levels, and so on. Then, we have the following construction results:

(a) For $n \geq 2$ factors each at $t = 4$ or $t = 9$ levels, there is a run order of $G = t^n$ with all main effects components 1-trend free. This follows from the fact that for the $2n \geq 4$ pseudofactors with 2 or 3 levels, respectively, Example 3 above produces a run order with all main effects and two-factor interactions 1-trend free, which is more than sufficient to make the main effects of the real factors 1-trend free.

(b) For $n \geq 2$ factors each at $t = 8$ levels, there is a run order of $G = t^n$ with all main effects components 1-trend free. In this case, we have the additional requirement that the three factor interaction between each set of pseudofactors $(a_{i1}, a_{i2}, a_{i3})$ also be 1-trend free. A sequence of generators having the required properties is

$$\mathbf{g}_1 = \prod_{i=1}^{n} a_{i1}, \quad \mathbf{g}_2 = \prod_{i=1}^{n} a_{i2}, \quad \mathbf{g}_3 = \prod_{i=1}^{n} a_{i3}, \quad \mathbf{g}_4 = \left[ \prod_{i=1}^{n-1} \prod_{j=1}^{3} a_{ij} \right] a_{n1} a_{n2}, \quad \mathbf{g}_5 = a_{11} a_{n3}, \cdots$$

where the remaining $(n-5)$ generators may be anything to complete the design. Note that the two- and three-factor interactions among each set of three pseudofactors are 1-trend free over $U_3$, by Theorem 2 (i) and (ii), while $\mathbf{g}_4$ and $\mathbf{g}_5$ simply meet the main effects requirement that each pseudofactor appear at least twice in the sequence.

Similar results are possible for the other prime powers but we omit the details since these other designs would rarely be used in practice.

Our final two examples illustrate the application of the GFS to designs with a mixture of levels among the $n$ factors. Before presenting the example, note that the GFS cannot produce a run order of a $6^2$ design with both 6-leveled factors 1-trend free. This is a reflection of the fact that no run order of a $2^2$ has both main effects 1-trend free.

EXAMPLE 6. Consider a design $G = 6^n$, $n \geq 3$. Each factor $a_i$, $i = 1, \ldots, n$ may be represented by a pseudofactor with 2 levels, say $b_i$, and another with 3 levels, $c_i$. For the main effects of the factors in $G$ to be 1-trend free, we require that the main effects of the $b_i$ and $c_i$ be 1-trend free, as also must be each interaction between $b_i$ and $c_i$. By Theorem 1, each $b_i$ and $c_i$ must appear in at least two generators using its foldover level, while Theorem 2 tells us that each pair $(b_i, c_i)$ must be isolated from each other in different generators using the appropriate foldover level. It is sufficient to use as the first two generators

$$g_1 = \prod_{i=1}^{n} b_i, \ f_1 = 2, \qquad g_2 = \prod_{i=1}^{n} c_i, \ f_2 = 3,$$

and continue with $(n-3)$ other generators that complete the main effects requirements among the $\{b_i\}$ separately and the $\{c_i\}$ separately. For example, when $n = 3$, the generator and foldover sequence

$$b_1b_2, \ f_3 = 2, \qquad b_2b_3, \ f_4 = 2, \qquad c_1c_2, \ f_5 = 3, \qquad c_2c_3, \ f_6 = 3$$

completes the required run order of $G$. Similar results apply for cases having factors with 10, 12, 14 and so on levels.

EAMPLE 7. Consider now the specific example of a 1/6 fraction of a $6^4$ design involving four factors, $a_1, \ldots, a_4$, each with six levels. If $a, b, c, d$ are four pseudofactors each at two levels and $e, f, g, h$ are four pseudofactors each with three levels, let the pairs $(a, e)$, $(b, f)$, $(c, g)$, $(d, h)$ represent the four real factors $a_1, \ldots, a_4$, respectively. To choose the fraction, use the defining effects $ABC$ and $EFGH$ for the two and three level pseudofactors, respectively. Then the main effects of the four real factors are 1-trend free if the main effects of the eight pseudofactors plus the interactions $AE$, $BF$, $CG$ and $DH$ are 1-trend free. In terms of the pseudofactors, if six generators are given by

$$bcd, \ acd, \ abd, \ efg, \ efh, \ fgh,$$

then all eight main effects are 1-trend free by Theorem 1, while any interaction between the pseudofactors with two levels and those with three levels are 1-trend free by Theorem 2 (i), since none of the factors with three levels appears in the generator sequence until each factor with two levels has become 1-trend free. Hence, the four main effects each with 5 degrees of freedom are estimated with full efficiency.

## APPENDIX

The proofs of Lemma 1 and Theorems 2 and 3 are presented here.

PROOF OF LEMMA 1. Without loss of generality, let the $m$ factors be $a_1, \ldots, a_m$, with $a_l$ having $t_l$ levels, $l = 1, \ldots, m$. Let factor $a_1$ be present equally often at each of its levels $\eta_1 \varepsilon \{0, \ldots, t_1-1\}$ over $U_{v-1}$. Let $\eta_l \varepsilon \{0, \ldots, t_l-1\}$ be the possible levels of each factor $a_l$, $l = 1, \ldots, m$.

Let the treatment combination $(\eta_1, \ldots, \eta_m)$ occur $c(\eta_1,..,\eta_m)$ times over $U_{v-1}$ in run positions $i_{(\eta_1, \ldots, \eta_m)j}$, $j = 1, \ldots, c(\eta_1, \ldots, \eta_m)$. Then the assumptions of the Lemma imply that, for each $0 \le u \le k$ and every $q_l = 1, \ldots, t_l-1$, $l = 1, \ldots, m$,

$$0 = \sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} \left\{ \left[ \sum_{j=1}^{c(\eta_1, \ldots, \eta_m)} P_{uN}(i_{(\eta_1, \ldots, \eta_m)j}) \right] \prod_{l=1}^{m} P_{q_l t_l}(\eta_l) \right\}. \tag{A.1}$$

Let the term in square brackets "[...]" in expression (A.1) be denoted by $W(\eta_1, \ldots, \eta_m)$ if $c(\eta_1, \ldots, \eta_m) > 0$, with the dependence on $u$ and $N$ suppressed. Set the function $W(\eta_1, \ldots, \eta_m)$ to be a constant, say $C$, for those level combinations for which $c(\eta_1, \ldots, \eta_m) = 0$. Using the constant $C$ in this fashion in (A.1) does not affect the sum since summing over those combinations of levels that do not occur involves summing the product at the end of the expression over *all* the levels $\eta_1$ of $a_1$ and by Defintion 1 this sum is always 0.

The function $W(\eta_1, \ldots, \eta_m)$ represents the contribution from the trend of degree $u$ to the time count with each component $(q_1, \ldots, q_m)$ of the $m$-factor interaction. We now show that $W(\eta_1, \ldots, \eta_m) = C$ for all the level combinations, not just those that do not appear in $U_{v-1}$. From this, the statement of the Lemma easily follows.

For fixed $\eta_1, \ldots, \eta_{m-1}$, the $t_m$ points $(\eta_m, W(\eta_1, \ldots, \eta_m))$, $\eta_m = 0, \ldots, t_m-1$, may be fitted by a polynomial of degree at most $t_m-1$, say $Q_{t_m-1}(\eta_m)$, with coefficients that depend on $\eta_1, \ldots, \eta_{m-1}$ but not on $\eta_m$. Then,

$$Q_{t_m-1}(\eta_m) = \sum_{j_1=0}^{t_m-1} \omega(\eta_1, \ldots, \eta_{m-1}; j_1) P_{j_1 t_m}(\eta_m). \tag{A.2}$$

Substitute (A.2) into (A.1) and sum over $\eta_m$ and $j_1$. By expression (2.3) of Definition 1, only the term for which $j_1 = q_m$ makes a non-zero contribution to the time count. Then, expression (A.1) reduces to

$$0 = \sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_{m-1}=0}^{t_{m-1}-1} \left\{ \left[ d(q_m)\,\omega(\eta_1, \ldots, \eta_{m-1}; q_m) \right] \prod_{l=1}^{m-1} P_{q_l t_l}(\eta_l) \right\}, \tag{A.3}$$

where $d(q_m) = \sum_{\eta_m=0}^{t_m-1} (P_{q_m t_m}(\eta_m))^2 > 0$. We have shown that $W(\eta_1, \ldots, \eta_m)$ contributes to the time count only through a positive constant $d(q_m)$ and a coefficient $\omega(\eta_1, \ldots, \eta_{m-1}; q_m)$ that does not involve $\eta_m$. Continuing this induction for $\eta_{m-1}, \ldots, \eta_1$, we may eventually reduce (A.3) to the form

$$0 = d(q_1, \ldots, q_m)\,\omega(q_1, \ldots, q_m),$$

where $d(q_1, \ldots, q_m) > 0$. So each of the coefficients that contribute to the original expression for $W(\eta_1, \ldots, \eta_m)$, when expressed as a polynomial in $\eta_l$ in successive iterations, must satisfy the condition

$$\omega(q_1, \ldots, q_m) = 0$$

and this is true for all $1 \le q_l \le t_l - 1$, $l = 1, \ldots, m$. Thus, $W(\eta_1, \ldots, \eta_m)$ is constant as a function of its arguments.

We now complete the proof of the Lemma by showing that the $m$-factor interaction remains $k$-trend free over $U_v$. For each $l = 1, \ldots, m$, let factor $a_l$ be at level $\xi_l$ in $g_v$. Let $f$ be the number of runs in $U_{v-1}$. Then the run positions in $U_{v-1} g_v^w$, $w = 1, \ldots, f_v - 1$, are $wf, \ldots, (w+1)f - 1$. For the $(q_1, \ldots, q_m)$ component of the interaction over these run positions, the time count with the trend of degree $k$ is given by

$$\sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} \left\{ \left[ \sum_{j=1}^{c(\eta_1, \ldots, \eta_m)} P_{kN}(wf + i_{(\eta_1, \ldots, \eta_m)j}) \right] \prod_{l=1}^{m} P_{q_l t_l}(\eta_l + w\xi_l) \right\}, \tag{A.4}$$

where it is understood that the sum $\eta_l + w\xi_l$ is reduced modulo $t_l$ in expression (A.4). Now expressing the trend polynomial in (A.4) in the form

$$P_{kN}(wf + i_{(\eta_1, \ldots, \eta_m)}) = \sum_{u=0}^{k} \alpha_u(wf)\,P_{uN}(i_{(\eta_1, \ldots, \eta_m)}),$$

where $\alpha_u(wf)$ does not depend on $(\eta_1, \ldots, \eta_m)$, and substituting this expression into (A.4) gives

$$\sum_{u=0}^{k} \alpha_u(wf) \sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} \left\{ \left[ W(\eta_1, \ldots, \eta_m) \right] \prod_{l=0}^{m} P_{q_l t_l}(\eta_l + w\xi_l) \right\}$$

$$= \sum_{u=0}^{k} \left\{ \alpha_u(wf) W(u,N) \sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} \left[ \prod_{l=0}^{m} P_{q_l t_l}(\eta_l + w\xi_l) \right] \right\}, \tag{A.5}$$

by the result for $W(\eta_1, \ldots, \eta_m)$ developed earlier. Now, since $\eta_1$ ranges over all its values in $U_{v-1}$, so too does $\eta_1 + w\xi_1$ for any $w$ and $\xi_1$ by the cyclic group properties of addition and multiplication modulo $t_1$, so the inner sum over $\eta_1$ is 0, establishing the claim that the $m$-factor interaction remains $k$-trend free over $U_{v-1}\mathbf{g}_v^w$ and so over all the runs of $U_v$.

PROOF OF THEOREM 2. Before establishing the results, we introduce a slightly more convenient notation. Let the level of factor $a_l$ be $\xi_{lj}$ in generator $\mathbf{g}_j$, $l=1, \ldots, m$ and $j=1, \ldots, v$, and the foldover levels be $f_j$, as usual. For index $i_j \in \{0, \ldots, f_j-1\}$, $j=1, \ldots, v$, the run position in $U_v$ given by a fixed choice of $(i_1, \ldots, i_v)$ may be expressed as

$$\sum_{j=1}^{v} \left(\prod_{l_1=1}^{j-1} f_{l_1}\right) i_j = f i_v + d\,, \quad f = \prod_{l_1=1}^{v-1} f_{l_1}, \quad d = \sum_{j=1}^{v-1} \left(\prod_{l_1=1}^{j-1} f_{l_1}\right) i_j,$$

where $d$ depends on $i_1, \ldots, i_{v-1}$ but not on $i_v$. The level of factor $a_l$ in this run position is of the form

$$\sum_{j=1}^{v} \xi_{lj} i_j = \xi_l^* + i_v \xi_{lv},$$

where $\xi_l^*$ is also independent of $i_v$. Arithmetic in the above expression is understood to be carried out modulo $t_l$. Then the assumptions of the theorem together with Lemma 1 imply that, for any $0 \le u \le k$ and all $1 \le q_l \le t_l - 1$, $l=1, \ldots, m$,

$$0 = \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_{v-1}=0}^{f_{v-1}-1} \left[ P_{uN}(f i_v + d) \prod_{l=1}^{m-1} P_{q_l t_l}(\xi_l^* + i_v \xi_{lv}) \right], \tag{A.6}$$

for any $i_v = 0, \ldots, f_v - 1$.

PROOF OR PART (i). The time count over $U_v$ between the trend of degree $k$ and any component of the $m$-factor interaction is expression (A.6) summed also over $i_v$, except that the product at the end now runs from $l=1$ to $l=m$ and includes a term of the form $P_{q_m t_m}(\xi_m^* + i_v \xi_{mv})$. By the assumptions of part (i), $\xi_m^* = 0$, independent of $i_1, \ldots, i_{v-1}$, so this additional term in the product may be taken outside the summations over $i_1, \ldots, i_{v-1}$ and by expression (A.6) the result is 0 for each value of $i_v$.

Note that part (i) allows a factor that has not yet appeared in any previous generator to inherit, in its interactions with those other factors already used in the earlier generators, any trend free properties existing among this first set of factors.

PROOF OF PART (ii). The time count over $U_v$ between the trend of degree $(k+1)$ and any component of the $m$-factor interaction is of the form

$$T(k+1,v) = \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_v=0}^{f_v-1} P_{k+1,N}(fi_v+d) \prod_{l=0}^{m} P_{q_l t_l}(\xi_l^* + i_v \xi_{lv}). \tag{A.7}$$

By the assumptions of part (ii), $\xi_{mv} \neq 0$, $\xi_{lv} = 0$, $l=1,\ldots,m-1$, and $f_v = s$. If we express the trend polynomial in (A.7) in the form

$$P_{k+1,N}(fi_v+d) = \sum_{u=0}^{k+1} \alpha_u(fi_v) P_{uN}(d), \tag{A.8}$$

where $\alpha_{k+1}(fi_v)=1$ independent of $i_v$, we may write (A.7) as

$$T(k+1,v) = \sum_{u=0}^{k+1} \sum_{i_v=0}^{s-1} \alpha_u(fi_v) \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_{v-1}=0}^{f_{v-1}-1} \left\{ \left[ P_{uN}(d) \prod_{l=1}^{m-1} P_{q_l t_l}(\xi_l^*) \right] P_{q_m t_m}(\xi_m^* + i_v \xi_{mv}) \right\}. \tag{A.9}$$

For $u=0,\ldots,k$, the inner summations over $i_1,\ldots,i_{v-1}$ yield 0 for any value of $i_v$ by the assumptions of the theorem and Lemma 1. Then, since $\alpha_{k+1}(fi_v)=1$, expression (A.9) reduces to

$$T(k+1,v) = \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_{v-1}=0}^{f_{v-1}-1} \left\{ P_{k+1,N}(d) \prod_{l=1}^{m-1} P_{q_l t_l}(\xi_l^*) \left[ \sum_{i_v=0}^{s-1} P_{q_m t_m}(\xi_m^* + i_v \xi_{mv}) \right] \right\}.$$

Finally, the inner summation over $i_v$ always yields 0 by (2.2) of Definition 1. Hence, the $m$-factor interaction is $(k+1)$-trend free over $U_v$.

PROOF OF PART (iii). Since $a_m = a_1$, the expression for $T(k+1,v)$, the time count over $U_v$ between the trend polynomial of degree $k+1$ and any component of the $(m-1)$-factor interaction, is the same as expression (A.9) except that the product inside the square brackets "[...]" now runs over $l=2$ to $l=m-1$. Then, the same arguments that follow (A.9) now apply to this time count and the proof is complete.

PROOF OF THEOREM 3. Let factor $a_1$ be at level $\xi \neq 0$ in $g_v$.

PROOF OF PART (i). When $U_{v-1}$ is folded over with respect to generator $\mathbf{g}_v$ at foldover level $f_v = t_1$, each block of size $R$ of $U_{v-1}$ generates $f_{v-1} - 1$ new blocks of the same size. If factor $a_1$ is at level $\xi_j$ in run position $j$, $j = 1, \ldots, R$, in any block of $U_{v-1}$, then $a_1$ is at level $\xi_j + i_v\xi$ in some block of $U_v$ generated from this starting block. Since $\xi_j + i_v\xi$ takes each possible level of $a_1$ exactly once as $i_v$ runs from 0 to $f_v-1$, and since the trend polynomial of any degree $k$ is the same in every such block in run position $j$, the contribution to the time count from starting position $j$ for any main effects component $q_1$ of $a_1$ is given by

$$\sum_{i_v=0}^{f_v-1} P_{q_1t_1}(\xi_j + i_v\xi)P_{kR}(j) \tag{A.10}$$

which is 0 for every $j$ by (2.2) of Definition 1. Hence, $a_1$ is $k$-trend free over $G$ for any $0 \le k \le R-1$.

PROOF OF PART (ii). For starting position $1 \le j \le R$ of any fixed block of $U_{v-1}$ and trend polynomial of degree $0 \le k \le R-1$, since $a_2, \ldots, a_m$ are at level zero in $\mathbf{g}_v$, the product of $P_{kR}(j)$ and any component of the $(m-1)$-factor interaction in factors $a_2, \ldots, a_m$ is constant for this same position $j$ in all $f_v-1$ blocks generated from the current starting block of interest. So the time count contributed by position $j$ is the same as (A.10) except for the addition of a constant product, that does not depend on $i_v$, involving any component of the $(m-1)$-factor interaction. The resulting sum is again 0 for every $j$ and hence the $m$-factor interaction of $a_1, \ldots, a_m$ is $k$-trend free over $G$.

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