Summary

Tables of Minimum Cost, Linear Trend-Free Run Sequences of Two- and Three-Level Fractional Factorial Designs

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Run orders of two- and three-level fractional factorial designs, tabled in National Bureau of Standards Applied Mathematics Series publications 48 and 54, which minimize (or nearly minimize) a cost function equal to the number of times the factors change levels during the time sequence in which the runs are performed and which simultaneously have all factor main effects components orthogonal to a polynomial time trend (usually linear) may be found by applying the Generalized Foldover Scheme of Coster and Cheng (1988) to the run sequences tabled in this report.

1. Introduction. Suppose an experiment is to be performed according to a given fractional factorial plan. In some cases, the time order in which the runs or treatment combinations are performed need not be randomized. Instead, certain systematic run orders may be preferred. For example, if the runs are made in some time or space sequence, each observation may be affected by a trend which is a function of time or position. In the presence of a time trend, a non-randomized run order may improve the efficiency with which factor effects are estimated. A design objective of full efficiency is attained when the factor effects are orthogonal to the time trend effects.

The cost of conducting an experiment is often of practical importance. A second design criterion of interest is a cost function based on the number of times each factor changes levels. The practical interpretation is that it costs a certain amount to change the levels of each factor, for example, to reset a measurement instrument, change the fertilizer on a field trial, restart an industrial plant and so on. If all level changes are equally expensive, run orders that minimize the total number of factor level changes are optimal with respect to this second criterion.

Cox (1951) began the study of systematic designs, for replicated variety trials, with the single criterion of efficient estimation of treatment effects in the presence of a smooth polynomial trend. Certain 2^n factorial designs robust to both linear and quadratic trends were found by Daniel and Wilcoxon (1966). The cost criterion was introduced by Draper and Stoneman (1968) in their exhaustive searches of some eight-run factorial plans. Dickinson (1974) extended the work of Draper and Stoneman to 2^4 and 2^5 complete factorial plans with the search restricted to minimum cost run orders. In an unpublished report, P.W.M. John extended the method of Daniel and Wilcoxon to certain designs for factors at two and three levels and discussed the foldover properties of such systematic run orders. Cheng (1985) gave a theoretical description of the cost structure in two-level factorial designs and provided some examples of run orders optimal with respect to both our design criteria. The method of Daniel and

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Wilcoxon was further extended by Cheng and Jacroux (1987) for constructing trend free run orders of two-level fractional factorial designs.

Coster and Cheng (1988) developed a Generalized Foldover Scheme (GFS) for generating k-trend free, minimum cost run orders of fractional factorial designs with all n factors at the same prime power number of levels. The tables of Section 3 list optimal (or nearly optimal) minimum cost run sequences, which produce generator sequences for the GFS of Coster and Cheng, for the two- and three-level factorial plans tabled in the National Bureau of Standards Applied Mathematics Series publications 48 and 54, (1957, 1959). It is found that a majority of these designs can be optimally ordered with respect to both design criteria. Before giving the tables of run sequences, in Section 2 we briefly summarize the Generalized Foldover Scheme and cost-structured decomposition of factorial run orders detailed in Coster and Cheng.

2. Generalized Foldover Scheme and Run-Order Cost. Attention is restricted to designs in which all factors are at the same number of levels. Consider n factors, each at s levels where s is a prime power. Let the s levels of each factor be the s elements of the Galois field of order s, GF(s). We denote the s factor levels by $0, 1, \ldots, s-1$, with 0 the additive identity and 1 the multiplicative identity in GF(s).

A complete factorial design in all n factors requires s^n runs. Let $G = (s_r^{n-p})$ denote a s^{-p} fraction of the complete factorial, blocked in s^r blocks each of size s^{n-p-r}. Let $N = s^{n-p}$ be the number of runs in the design G. Let $R = s^{n-p-r}$ be the size of each block.

Design G is a group with elements $\{\mathbf{g}_1, \ldots, \mathbf{g}_N\}$. Without loss of generality, G is generated by some set of $(n-p)$ linearly independent generators, say $\{\mathbf{g}_1, \ldots, \mathbf{g}_{n-p}\}$, the first $h = n-p-r$ of which generate the principal block. We call $\{\mathbf{g}_1, \ldots, \mathbf{g}_h\}$ the within block generators. The between block generators are $\{\mathbf{g}_{h+1}, \ldots, \mathbf{g}_{n-p}\}$. Then, any treatment combination in G is of the form

$$\mathbf{g} = \mathbf{g}_1^{b_1} \mathbf{g}_2^{b_2} \cdots \mathbf{g}_{n-p}^{b_{n-p}}, \quad b_j \in \mathrm{GF}(s), \quad j = 1, \ldots, n-p. \tag{2.1}$$

In (2.1), if $b_{h+1} = \cdots = b_{n-p} = 0$, the corresponding treatment combination lies in the principal block; if $b_1 = \cdots = b_{n-p} = 0$, we write $\mathbf{g} = \mathbf{1}$ to denote the treatment combination corresponding to all factors at level 0. Note that we assume that any design G is a main effects plan, that is, no main effect is aliased with another main effect nor confounded with any block effect.

We now give the definition of the Generalized Foldover Scheme (GFS) of Coster and Cheng (1988) for generating a run order of a fractional factorial plan with n factors at s levels. Following this definition is a restatement of Theorem 3 of Coster and Cheng giving a sufficient condition for the resulting run order to be k-trend free.

DEFINITION 1. (GFS for G). Suppose that $\{ \mathbf{g}_1, \ldots, \mathbf{g}_{n-p} \}$ are $n-p$ independent generators of G. Let $U_0 = \mathbf{1}$. Then the run order of G produced by the GFS with respect to this generator sequence is given by U_{n-p} where

$$U_j = U_{j-1}^* = (U_{j-1}, U_{j-1}\mathbf{g}_j, \ldots, U_{j-1}\mathbf{g}_j^{s-1}), \quad j = 1, \ldots, n-p. \tag{2.2}$$

Thus, the s^j runs of U_j are formed in proper order by first repeating the runs of U_{j-1} a total of s times and then multiplying the runs in the ith repetition by $\mathbf{g}_j^i$, for $i = 0, \ldots, s-1$. Recall that all arithmetic on the levels of each factor in the generated runs is performed according to the group operations of addition and multiplication in GF(s).

The resulting run order of G is said to be k-trend free, for $k \geq 1$, if *all* $n(s-1)$ main effects components are orthogonal to the same smooth polynomial trend of degree k over the run positions in every block. A sufficient condition for this trend free property to be achieved, when the run order is generated by the GFS as shown in Definition 1, is:

THEOREM 1. For G generated according to (2.2), the run order is k-trend free if each factor appears at least $(k+1)$ times at a *non-zero* level in the generator sequence or, if a factor does not appear at least $(k+1)$ times, that factor is at a *non-zero* level at least once in some between block generator.

Note that the second condition for trend orthogonality simply exploits the assumption that the same polynomial trend is present in every block.

The remainder of this section discusses the cost of a run order when the GFS is used to generate the runs of the design. The details, which are in Coster and Cheng (1988), are themselves an extension of the two-level results in Cheng (1985).

Recall the assumption that all factor level changes are equally expensive. Begin by defining a cost or distance function between any two subsets A and B of G by

$$d(A, B) = \min_{\omega \varepsilon A, \nu \varepsilon B} d(\omega, \nu),$$

where $d(\omega, \nu)$ is the number of factor level changes between runs ω and ν. That is, for the n factors named $a_1, \ldots, a_n$, if $\omega = a_1^{\omega_1} \cdots a_n^{\omega_n}$ and $\nu = a_1^{\nu_1} \cdots a_n^{\nu_n}$, $\omega_i, \nu_i \ \varepsilon \ GF(s)$, then $d(\omega, \nu) = \sum I(\omega_i \neq \nu_i)$ where $I(\omega_i \neq \nu_i)$ equals 1 if $\omega_i \neq \nu_1$ and is 0 otherwise. In particular, $d(\mathbf{1}, \omega)$ is the number of factors at a non-zero level in run ω. In what follows, assume that the first block of G is the principal block, denoted by B_1, a subgroup of G. Blocks $B_2, \ldots, B_{s^r}$ are cosets of B_1 in G.

LEMMA 1. Let $\{ \mathbf{g}_1, \ldots, \mathbf{g}_{n-p} \}$ generate G by the generalized foldover scheme of Definition 1. Let

$$d_i = d(\mathbf{g}_i, \prod_{j=0}^{i-1} \mathbf{g}_j^{s-1}), \quad i = 1, \ldots, n-p. \tag{2.3}$$

Then the cost of the run order so generated is

$$C = \sum_{i=1}^{n-p} (s-1) s^{n-p-i} d_i \, . \tag{2.4}$$

Consider the following group structured decomposition of the principal block, B_1. Beginning with $H_1^{(0)} = \{\mathbf{1}\}$, we iteratively define a sequence of quotient groups $G_i = B_1 / H_1^{(i)}$ and subgroups $H_1^{(i)}$ of B_1 along with a set of minimum within block costs $\{c_i\}$, $i = 0, 1, \ldots, t-1$, by

$$H_1^{(i+1)} = \bigcup_{H \varepsilon S_1^{(i)}} H \quad \text{and} \quad c_{i+1} = \min_{H, K \varepsilon G_i, H \neq K} d(H, K) \, ,$$

where $S_1^{(i)}$ = subgroup of G_i generated by $\{H : d(H_1^{(i)}, H) = c_{i+1}\}$. Let $m_i = |S_1^{(i-1)}| = s^{r_i}$, $N_i = N_{i-1}/m_i$ and $N_0 = s^{n-p}$. Note that $G_0 = B_1$.

Each N_i equals s^r multiplied by the number of cosets of $H_1^{(i)}$ in B_1, where for convenience we count $H_1^{(i)}$ as a coset of itself, each coset being of size $m_1 m_2 \cdots m_i$, while r_{i+1} is the number of independent generators of $S_1^{(i)}$, the subgroup of the quotient group G_i generated by those elements of G_i distance c_{i+1} from the current subgroup $H_1^{(i)}$ of B_1. The elements of $S_1^{(i)}$ are cosets of $H_1^{(i)}$. The $H_1^{(i)}$'s form a nested sequence of subgroups, of strictly increasing size, of B_1. The sequence of costs $\{c_i, i = 1, \ldots, t\}$ is strictly increasing. The iterations terminate when $N_t = s^r$ for some t at which time $H_1^{(t)} = B_1$. Note that $r_1 + \cdots + r_t = n-p-r$. At each stage $i = 0, \ldots, t-1$, there are arrangements of the $s^{r_{i+1}}$ elements of $S_1^{(i)}$ that have cost c_{i+1} between any two adjacent elements in the arrangement. This produces a minimum cost ordering of the elements of $S_1^{(i)}$. Theorem 2 below shows how the generalized foldover scheme may be used to find such arrangements. When the principal block has been minimally ordered, we repeat the above induction, starting with $H_1^{(t)} = B_1$ and G replacing B_1, until some $N_{t+t'} = 1$ and $H_1^{(t+t')} = G$. The between block minimum costs $\{c_{t+1}, \ldots, c_{t+t'}\}$ found from this second iterative procedure, although strictly increasing, may be less than the within block costs found when ordering B_1.

This cost structured decomposition of G may be combined with the generalized foldover scheme to produce minimum cost run orders as follows. At each stage $i = 1, \ldots, t+t'$, suppose $S_1^{(i-1)}$ is generated by $\{K_{i1}, \ldots, K_{ir_i}\} \in G_{i-1}$. By definition of $S_1^{(i-1)}$, there must exist independent runs $\mathbf{z}_{ij} \in K_{ij}$, $j = 1, \ldots, r_i$ each distance c_i from run $\mathbf{1}$. Thus, at each stage, $\mathbf{z}_{ij}$ has the minimum possible number of factors at a non-zero level. Setting $r_0 = 1$ and $\mathbf{z}_{01} = \mathbf{1}$, define a set of n-p independent generators of G by

$$\mathbf{g}_{ij} = \left(\prod_{q=1}^{i-1} \prod_{j_1=1}^{r_q} \mathbf{g}_{qj_1}^{s-1}\right)\left(\prod_{j_1=1}^{j-1} \mathbf{g}_{ij_1}^{s-1}\right)\mathbf{z}_{ij}, \quad j = 1, \ldots, r_i, \quad i = 1, \ldots, t+t'. \tag{2.5}$$

Thus, $\mathbf{g}_{ij}$ is $\mathbf{z}_{ij}$ multiplied by the product of all previous generators raised to the power (s-1).

Since the $\mathbf{z}_{ij}$ are independent in $H_1^{(i)}$, the collection

$$\{\, \mathbf{g}_{ij}, \quad j=1,\ldots,r_i, \quad i=1,\ldots,t+t' \,\} \tag{2.6}$$

are n-p independent generators of G. With the help of Lemma 1, the following theorem is true.

THEOREM 2. If a run order of G is constructed by the generalized foldover scheme (2.2) applied to the sequence of generators (2.6), the resulting run order has minimum cost given by

$$C_{\min} = \sum_{i=1}^{t+t'} (N_{i-1} - N_i)\, c_i \,. \tag{2.7}$$

EXAMPLE 1. Consider the design $G = 2_1^{8-4}$, a design for 8 factors in 2 blocks of size 8, defined by $I = ABEGH = ACFG = ABCD = ABEF$ with blocking effect ACE. (Note that this design is too small to be of much practical use and serves only as an example here.) The principal block contains three runs, **abcd, acfg** and **bdfg**, each with four factors at a non-zero level. Any two of these three runs are independent. Thus $c_1=4$, $r_1=2$, $m_1=4$ and $N_1=4$. Choosing $\mathbf{z}_{11}=\mathbf{abcd}$ and $\mathbf{z}_{12}=\mathbf{bdfg}$, by (2.5) $\mathbf{g}_{11}=\mathbf{abcd}$ and $\mathbf{g}_{12}=\mathbf{acfg}$. With these generators, the subgroup $H_1^{(1)}$ and its coset $H_2^{(1)}$ are

$$H_1^{(1)} = \{\mathbf{1, abcd, acfg, bdfg}\}$$

$$H_2^{(1)} = \{\mathbf{cdefh, abefh, adegh, bcegh}\}.$$

Now $G_1 = B_1 / H_1^{(1)}$ consists of $H_1^{(1)}$ and its coset $H_2^{(1)}$. Also, $S_1^{(1)} = G_1$ in this example. Since each run in $H_2^{(1)}$ has five factors at a non-zero level, $c_2=5$, $r_2=1$ and $N_2=2=s^r$. If we choose $\mathbf{g}_{21}=\mathbf{cdefh}$, then the final minimum cost ordering of B_1 by the foldover method is $H_1^{(1)}$ followed by $H_2^{(1)}$ with the runs in the order shown. The second block of the design has three runs with three factors at a non-zero level. Any one of these may be used as the required between block minimum cost run. Thus $c_3=3$. If we set $\mathbf{z}_{31}=\mathbf{bde}$, then, by (2.5), $\mathbf{g}_{31}=\mathbf{cdgh}$ and the resulting minimum cost ordering of B_2 is

$$B_2 = \{\, \mathbf{cdgh, abgh, adfh, bcfh, efg, abcdefg, ace, bde} \,\}.$$

By (2.7), the overall minimum cost is 61 level changes.

Including the between block costs $\{c_{t+1}, \ldots, c_{t+t'}\}$ in the cost decomposition described above implies that the the observations for treatment combinations in each block are made before the next block's observations are begun. In reality, observations for runs in each block may be made concurrently and there will be no between block costs. If this is the case, a run order will have minimum cost of level changes if each block is minimally ordered according to the within block costs found above and *any* r independent between block generators may be used in the GFS (2.2). With this added freedom, minimum cost run orders that satisfy the orthogonality design criterion above are more readily found. Expression (2.7) becomes

$$C_{\min} = s^r \sum_{i=1}^{t} (N_{i-1} - N_i)\, c_i \; . \qquad (2.8)$$

The results above provide a sufficient condition under which a run order of G is optimal with respect to both design criteria: trend elimination and minimum cost of level changes. Assume that the trend is of degree k. Let the cost structure of G be given by

$$\{ (c_1, r_1), (c_2, r_2), \ldots, (c_{t+t'}, r_{t+t'}) \} \qquad \text{where} \qquad \sum_{j=1}^{t+t'} r_j = n - p \; .$$

Let $\{\mathbf{z}_{ij},\ j=1, \ldots, r_i,\ i=1, \ldots, t+t'\}$ be some choice of n-p independent minimum distance runs with respect to this cost structure. Let $\{\mathbf{g}_{ij}\}$ be formed from these as in expression (2.5). All preceding results may be combined to give:

THEOREM 3. If each factor appears at some non-zero level at least $(k+1)$ times in the sequence of runs $\{\mathbf{g}_{ij}\}$ which generate G by the generalized foldover scheme (2.2), or at least once in a between block generator, the resulting run order, having minimum cost (2.7), or (2.8) if the between block costs are zero, and being k-trend free by Theorem 2, is optimal with respect to both design criteria.

The proofs of all results stated in this section are detailed in Coster and Cheng (1988).

3. NBS AMS 48 and AMS 54 Examples. The National Bureau of Standards Applied Mathematics Series publication number 48 lists 125 fractional factorial designs with all factors at two levels. The designs range in size from 16 run plans for 5 factors to 256 run plans for 16 factors. All plans are blocked in two or more blocks. The defining relations are chosen so that all main effects and some two factor interactions are estimable if higher order interactions are assumed to be negligible.

In all, 96 of the 125 plans in AMS 48 may be optimally ordered by applying the generalized foldover construction technique of Coster and Cheng, (1988). In Table II below, we present one possible sequence of minimum cost runs $\{\mathbf{z}_i\}$ with which an optimal sequence of generators may be produced for the GFS by expression (2.5). For the 29 plans that do not have an optimal ordering by the construction method of Theorem 5 of Coster and Cheng (1988), Table II gives a sequence of runs that produces a nearly optimal run order in the sense that all main effects are linear trend free while the cost of the run order is slightly more than minimum. In such cases, column 4 of Table II lists the number of level changes needed; otherwise, an * indicates a minimum cost order is obtained. The plan description in column 1 of Table II is similar to the notation used in AMS 48. Thus, plan *p.n.r* of Table II represents a design $G = (2_r^{n-p})$ with 2^{n-p} runs for n factors in 2^r blocks of size 2^{n-p-r}. The specific defining relation and blocking effects used to find the runs of each design are available in AMS 48. Column 3 of Table II lists the minimum cost of an optimal run order. The format of Table II is summarized in Table I.

TABLE I

Column	Table II Column Description
1	Plan *p.n.r*
2	Minimum cost sequence $\{\mathbf{z}_i,\ i=1,\ldots,n-p\}$
3	Minimum cost given by expression (2.7)
4	Optimal run order (*) or cost of non-optimal order
5	Number of quadratic trend free factors (* means all)

For small plans, the search for an optimal sequence of runs is readily done by hand. For larger plans, the search was made by computer. This was achieved as follows: the program would read the variables n, p and r (number of factors, defining effects and blocking effects respectively) followed by the defining and blocking effects. The runs of the design would be generated from this information. The minimum cost structure would be found by following the iterative decomposition scheme described in Section 2 and all possible candidates for the r_i independent runs with cost c_i, $i = 1, \ldots, t+t'$, would be found and saved by the program. All sequences of $n-p$ independent runs, r_i of cost c_i for each i, would then be formed systematically from these candidate sets until one was found that met the linear orthogonality condition or all such minimum cost run sequences had been tried without finding an optimal run sequence. Note that only the variables n, p and r and $p+r$ independent defining and blocking effects were required as input. If the assumption of zero between block costs was added, all non-principal block runs were candidates for between block generators. This made the search for optimal run sequences more successful at the expense of increased search time over larger candidate sets. If an optimal generator sequence was found for a particular plan, the search was continued in an attempt to maximize the number of factors that were also orthogonal to a quadratic trend. In 63 cases, a minimum cost generator sequence that produces a 2-trend free run order was found. However, the search time greatly increased when 2-trend free run orders were sought since all minimum cost linear trend free run sequences had to be checked for quadratic orthogonality.

Note that the letters of the alphabet used to name the factors in AMS 48 do not include "i", "q" or "r" while we use these letters in their usual places. Furthermore, plans 8.10.32 and 8.10.64 in AMS 48 are incorrectly blocked. For ease of presentation, Table II begins on the next page.

TABLE II

Optimal or Near Optimal Minimum Cost Run Sequences				
Plan	**Run Sequence**	**C_{min}**	**Cost**	**Quad**
1. 5. 2	de abce be cd	38	*	*
1. 5. 1	de ab ce bd	30	*	4
1. 6. 3	bcef adef cf cd bd	110	*	*
1. 6. 2	de af bcef cf ad	70	*	*
1. 6. 1	de bf ce af bd	62	*	5
1. 7. 4	defg abcd be cd af ag	222	*	*
1. 7. 3	cdfg bdeg adef dg ce bg	238	*	*
1. 7. 2	de bg af cefg dg ab	134	*	*
1. 7. 1	fg de bg ce af bd	126	*	6
1. 8. 5	acgh bdefgh ae eg bc ab dg	510	*	*
1. 8. 4	cdeh bcfg adeg dg eg bc fh	478	*	*
1. 8. 3	efgh cdgh bdfg adfh cg eg ab	494	*	*
1. 8. 2	gh ef cd ab bdfh dg ac	262	*	7
1. 8. 1	gh ef cd fh bd ac bg	254	*	7
1. 9. 5	cefi aegh abcd hi fi fg bi be	958	*	*
1. 9. 4	degh cefi befg adfi eh fg bc gh	990	*	*
1. 9. 3	cd eghi dfgi bfhi afgh di bc eh	750	*	*
1. 9. 2	gi eh df cf ab bfhi dg ac	518	*	8
1. 9. 1	ab cd gh ei af cf hi fg	510	*	6
2. 6. 3	abcdef ab df bcd	63	*	*
2. 6. 2	cdef abef cd bde	55	*	*
2. 6. 1	bdf bce ade cd	44	*	4
2. 7. 3	abce bcdfg dg fg ce	118	*	6
2. 7. 2	beg acg cdef dg ab	94	*	*
2. 7. 1	eg df ab cg bef	63	*	5
2. 8. 4	bcdeg adefh cg ab dh eg	270	*	*
2. 8. 3	adg efgh abce dh fh cg	206	*	*
2. 8. 2	bfg adg beh ach eg df	186	*	*
2. 8. 1	fh eg bgh acf adg ce	140	*	7
2. 9. 4	bdhi abefi ceghi fh ab dh eg	526	*	*

TABLE II cont.

Optimal or Near Optimal Minimum Cost Run Sequences				
Plan	Run Sequence	C_{min}	Cost	Quad
2. 9. 3	adg cfi bdhi efgh bi cg ai	398	*	*
2. 9. 2	bfg adg beh ach dei cg fh	378	*	*
2. 9. 1	ai fh eg bgh cfi adg ce	268	*	8
2. 10. 5	bcdej abdfh abcgi cd eg ab ef hj	1182	*	*
2. 10. 4	bej eghi bcfg adef cd fg hi ij	862	*	*
2. 10. 3	fgij eghj cdij bdhi adhj eg cd bfh	1007	*	*
2. 10. 2	dgi dfj deh cgj bfh aei bc cd	762	*	*
2. 10. 1	ij fg cd ab dgj bfh deh ac	524	*	9
3. 7. 2	abcd efg adg abf	49	53	6
3. 7. 1	abcd efg abf ace	45	53	5
3. 8. 3	eh abcdfgh bdh cdf efg	109	*	*
3. 8. 2	abcd acfg abefh cdf ace	123	125	7
3. 8. 1	eh abcd efg ach abf	77	85	6
3. 9. 4	eghi abcdfhi gi bde abf fhi	277	*	*
3. 9. 3	fgh efi abcdgi eh bde abf	209	*	*
3. 9. 2	efi abhi bdfg abcd gi eh	214	*	*
3. 9. 1	gi eh bdh ace fhi adg	141	*	7
3. 10. 4	abcd aegij cfhij cj gi eh abf	527	*	9
3. 10. 3	bdfi afgj cdgh abefh gi cj eh	502	*	*
3. 10. 2	gi aej cdf abf dhij eh cj	318	*	7
3. 10. 1	cj gi aej cdf fhi abf eh	284	*	8
3. 11. 5	cfhij adfgjk bdefgh cj dk gi eh agk	1279	*	*
3. 11. 4	abcd bghik cfhij defik cj gi eh abf	1103	*	*
3. 11. 3	afgj bdfi hijk abck cefhk gi dk eh	1014	*	*
3. 11. 2	cj bde efi bfgk hijk acfg dk eh	662	*	*
3. 11. 1	cj dk gi ace bde efg abgh bij	542	543	8
4. 8. 2	abcd efgh adeg abef	60	none	6
4. 8. 1	abcd abef cdgh aceh	60	none	5
4. 9. 3	hi abcdefgi acei cdgi efgh	124	*	*
4. 9. 2	hi abcd efgi acei cdgh	92	*	8

TABLE II cont.

Optimal or Near Optimal Minimum Cost Run Sequences				
Plan	Run Sequence	C_{min}	Cost	Quad
4. 9. 1	hi cdef abgi efgi adfh	92	*	7
4. 10. 3	adeg bcij bcfh dgj agi chj	245	*	9
4. 10. 2	bfj bhi abdefgj chj aej dei	193	221	8
4. 10. 1	bfj bhi abdefgj cfj adfh agi	195	223	7
4. 11. 4	befhjk acdgjk defgik aej ack bdk ehk	717	*	*
4. 11. 3	bcdej afghj cdfhk ceghi aej ack bfj	621	*	*
4. 11. 2	fgk dei ahjk cdgh abef cfi chj	409	*	*
4. 11. 1	fgk chj dgj ack bhi dei bfj	381	*	10
4. 12. 5	bdghij acefij ceghjkl aej dgj dei bfj efl	1469	*	*
4. 12. 4	bcfgl adehl befhjk defgik efl ghl abl chj	1293	*	*
4. 12. 3	ijkl cdfhk abegk bcfgl bcdej aej bhi fgk	1133	*	*
4. 12. 2	fgk dei cdgh abef bckl ahjk bhi cdl	825	*	*
4. 12. 1	efl abl fgk chj ack dgj bhi cdl	765	*	11
5. 10. 3	ghij abcdefij efij abgh bcehj	157	*	*
5. 10. 2	ghij cdef abij efgh bdfhj	125	*	9
5. 10. 1	ghij cdef abij efgh bdfhj	125	*	8
5. 11. 3	bgjk bhik abcdefgh cfk dek efij	278	*	*
5. 11. 2	cdef ahjk bgjk abij cfk efgh	250	*	10
5. 11. 1	cdef ahjk efij abcd ghij cfk	251	*	10
5. 12. 4	abcd acefkl abefghij bfl ael ahjk cdij	624	*	*
5. 12. 3	bfl dek abcd aeghijl cfk ahjk cdij	432	*	*
5. 12. 2	ael bfl cfk ghij dek bgjk abgh	388	392	8
5. 12. 1	ael bfl cfk ghij dek abgh agik	388	392	7
6. 11. 2	efgk abcdk hijk adgi cdej	128	132	7
6. 11. 1	efgk abcdk hijk cdej acfh	126	132	6
6. 12. 3	efgh abcdegikl efij adkl abcd cdefl	293	333	10
6. 12. 2	abcd efgh abcdegikl efij adkl abefl	257	293	10
6. 12. 1	abcd efgh abcdegikl efij abefl adkl	258	284	7
6. 13. 4	ehjkl efgikm abcdgjkm flm cdem bckl hikm	692	*	*

TABLE II cont.

Optimal or Near Optimal Minimum Cost Run Sequences				
Plan	Run Sequence	C_{min}	Cost	Quad
6. 13. 3	flm ghij egikl abcdefij cdem bckl efgh	492	*	*
6. 13. 2	flm gjkm abcd efij ghij cdem adkl	444	*	12
6. 13. 1	flm abcd efij gjkm cdem ghij adkl	444	*	12
6. 14. 5	cdhjklm acefgjm bfgijkln flm bin ghij abem abcd	1700	*	*
6. 14. 4	abfhjk aghiln defikn cdhjklm flm bin adkl cegn	1504	*	*
6. 14. 3	hikm bcegi cdijl agklmn abfhjk flm bin cdem	1158	*	*
6. 14. 2	flm hikm abcd dehln afgkn bghjn bin cdem	918	*	*
6. 14. 1	bin flm gjkm cegn ajln hikm abcd efgh	828	*	*
7. 12. 2	defghi acdghj abcjkl hijk cegil	181	*	10
7. 12. 1	abcjkl defghi hijk acefij bcfgk	153	177	8
7. 13. 3	efgh abcdegiklm efij adkl abcd abehik	302	350	*
7. 13. 2	abcd egiklm efij efgh adkl cdijlm	262	286	12
7. 13. 1	abcd efgh abegjk abeflm efij adkl	264	286	9
7. 14. 4	efij ghijmn abcdehjklm mn bckl abcd bdfgil	654	*	*
7. 14. 3	mn efgh abcdehjkln efij adkl abcd abegjk	430	478	13
7. 14. 2	mn abcd egiklm efij efgh adkl abeflm	390	414	10
7. 14. 1	mn abcd efgh abegjk abeflm efij adkl	392	404	10
7. 15. 5	abfgijmo acdjlmno bdeghjkmn abjo dehm cklo cfjn hijk	1948	*	*
7. 15. 4	acdghj ghlmno abcekmn abfgijmo abjo dehm cklo aein	1564	*	*
7. 15. 3	hijk ceflm abfkln acdgik ghlmno abjo cfjn dehm	1196	*	*
7. 15. 2	cfjn hijk egiko ejlmn acdeo abcjkl dehm fgim	1084	*	*
7. 15. 1	abjo cfjn cklo fgim acdeo abegh hijk aein	1026	1044	12
8. 13. 2	cdfghm bdejkm adilm abghk bcgjlm	168	180	10

TABLE II cont.

Optimal or Near Optimal Minimum Cost Run Sequences				
Plan	**Run Sequence**	**C_{min}**	**Cost**	**Quad**
8. 13. 1	abcdehl cdhijk bdefgi bcehim abghk	169	201	7
8. 14. 3	mn bcefghjk adilm abghk bdfhjl bcfikl	246	270	12
8. 14. 2	mn cdfghm bdejkm adilm abghk bcgjlm	232	244	11
8. 14. 1	mn abcdehl cdhijk bdefgi bcehim abghk	233	265	8
8. 15. 4	dhklo iklmn abcdefghjn adkn bcfmn defmo cijlo	707	*	*
8. 15. 3	dhklo bgimo iklmn acefj adkn cegho bcfmn	631	*	14
8. 15. 2	bgimo dhklo bcfmn iklmn acefj adkn cegho	633	*	*
8. 15. 1	dhklo bgimo acefj iklmn cegho bcfmn adkn	634	*	14
8. 16. 5	abklmnp adghinp acdefhjklo abip adkn defmo bgimo cegho	1795	*	*
8. 16. 4	abdglo ahikmo dgkmnop bcefghjk abip adkn defmo ejklp	1567	*	*
8. 16. 3	abip iklmn bdghn ahlno cdefjkn adkn cijlo fhjmp	1159	*	15
8. 16. 2	abip adkn agmop bdlmp acefj ghikp bcfmn cijlo	1083	*	15
8. 16. 1	abip adkn bgimo acefj bdlmp fgjln cegho ejklp	1083	*	*

As stated earlier, 63 of the 125 plans listed in Table II have at least one 2-trend free minimum cost run sequence. Another search was made for 1- and 2-trend free generator sequences under the relaxed assumption of zero between block costs. Table III lists minimum cost run sequences for 33 plans for which some improvement was obtained. For 12 designs for which an optimal generator sequence did not exist under the conditions of Table II, namely plans: 3.8.2, 3.11.1, 6.12.3, 6.12.2, 6.12.1, 7.13.3, 7.13.2, 7.13.1, 7.14.3, 7.14.2, 7.14.1 and 7.15.1, optimal run sequences are given in Table III. For each of the remaining 21 plans, the number of 2-trend free factors was increased, in most cases to a 2-trend free order. The minimum costs listed in Table III are given by (2.8).

TABLE III

Improved Run Sequences with Zero Between Block Costs				
Plan	**Run Sequence**	**C_{min}**	**Cost**	**Quad**
1. 6. 1	de ce bf af bcde	60	*	*
1. 7. 1	fg de ce bg ag bcdefg	124	*	*
1. 8. 2	gh ef cd ab bdfh dh acefgh	256	*	*
1. 8. 1	gh fh eg cd bd ac bcdefh	252	*	*
1. 9. 2	gi eh df cf ab bfhi fi acdefghi	512	*	*
1. 9. 1	hi gi eh df cf bf af bcdegi	508	*	*
2. 6. 1	bdf bce ade bcf	42	*	*
2. 7. 3	abce bcdfg dg abcefg cdef	104	*	*
2. 8. 1	fh eg bgh acf adg bcefg	138	*	*
2. 9. 1	ai fh eg bgh dgi bcd abefgh	266	*	*
2. 10. 1	ij fg cd ab dgj deh bgh abcdefij	522	*	*
3. 8. 2	bdfg acfg adegh bdh abcdefg	116	*	*
3. 10. 4	abcd aegij cfhij eghi beghj bcegh acdefj	496	*	*
3. 10. 2	gi aej cdf abf dhij eh bcefgi	312	*	*
3. 10. 1	cj gi aej dfj fhi bde acdfghij	282	*	*
3. 11. 1	dk cj gi bek efi aej bfhj abcdegjk	540	*	*
4. 10. 3	bcfh adeg bcij dgj abcdfhij abdeghi	224	*	*
4. 11. 1	fgk ack chj dgj bhi dei abcfjk	378	*	*
4. 12. 1	efl abl fgk ack chj dgj bhi bcdefhjkl	762	*	*
5. 11. 1	cdef ahjk bgjk efij abcd cfghijk	248	*	*
6. 12. 3	ghij efij abcdehjkl acijk abghl fhikl	264	*	*
6. 12. 2	abcd ghij efij ehjkl acghk cdefghijl	244	*	11
6. 12. 1	abcd ghij efij ehjkl cdghl acefghijk	254	*	11
7. 12. 2	defghi acdghj befikl adehijl acdgik	168	*	*
7. 13. 3	ghij efij abcdehjklm bdehil cdehik fgjklm	272	*	*
7. 13. 2	abcd ghij efij ehjklm acehil cdfgik	248	*	12
7. 13. 1	abcd ghij efij cdijlm abehik adfgjm	260	*	12
7. 14. 3	mn ghij efij abcdehjkln acijkn abghln fhiklm	400	*	*
7. 14. 2	mn abcd ghij efij ehjkln acghkn cdefghijlm	376	*	13
7. 14. 1	mn abcd ghij efij cdghln abehik acfhjlmn	388	*	*
7. 15. 1	cklo abjo cfjn fgim hijk efkmo bdefn acghijkln	1022	*	*
8. 15. 3	dhklo bgimo iklmn acefja bdejkm abdfjno cdfghikln	600	*	*
8. 16. 3	abip iklmn bdghn ahlno cdefjkn ejklp adfhjkmnp bgimo	1128	*	*

A source of fractional factorial designs for factors at three levels is National Bureau of Standards Applied Mathematics Series publication number 54. For the forty one designs presented in AMS 54, Table IV lists one possible minimum cost run sequence with which the generalized foldover scheme (2.2) produces an optimal run order. As before, unless otherwise stated, a run order is optimal if it is linear trend free (1-trend free) and has minimum cost of level changes. All 41 designs may be optimally ordered: in general, as the number of levels s increases, optimal generator sequences are more readily found.

As before, for design $G = (3_r^{n-p})$ having independent minimum cost run sequence $\{\mathbf{z}_1, \ldots, \mathbf{z}_{n-p}\}$, the sequence of generators to be used in the GFS is found from (2.5). This expression simplifies to:

$$\mathbf{g}_i = \mathbf{z}_{i-1}^2 \mathbf{z}_i \,. \tag{3.1}$$

So the generator sequence to be used in (2.2) is easily constructed from the minimum cost run sequence listed in Table IV by applying expression (3.1).

A factor is linear trend free if it appears at a non-zero level in two or more generators $\{\mathbf{g}_i\}$. In terms of the run sequences listed in Table IV, the linear orthogonality condition becomes: each factor must change levels at least once after its first appearance in some run $\mathbf{z}_j$. Because of the size of the search required for the larger designs, it is not guaranteed that the run sequences shown maximize the number of quadratic trend free factors.

TABLE IV

Optimal Minimum Cost Run Sequences for AMS 54 Designs

Plan	Run Sequence	C_{min}	Cost	Quad
1. 4. 2	ab^2cd^2 ab^2 a^2c	88	*	*
1. 4. 1	ab^2 cd^2 ac^2	52	*	3
1. 5. 3	ab^2cde a^2b ae^2 c^2d	322	*	*
1. 5. 2	ace a^2b^2cd bc^2 b^2e	250	*	*
1. 5. 1	bc^2 ad^2 $a^2b^2e^2$ b^2e	166	*	3

TABLE IV cont.

Optimal Minimum Cost Run Sequences for AMS 54 Designs				
Plan	**Run Sequence**	**C_{min}**	**Cost**	**Quad**
1. 6. 3	abc^2f $c^2de^2f^2$ ae cd^2 b^2c^2	916	*	*
1. 6. 2	ae b^2df cd^2ef e^2f^2 ad^2	574	*	5
1. 6. 1	ae a^2f cd^2 b^2de^2 bc	490	*	5
1. 7. 3	adg bce^2g^2 $c^2de^2f^2$ ef a^2e^2 bc	2374	*	*
1. 7. 2	ae cfg bd^2f^2 adg bg dg^2	1690	*	5
1. 7. 1	ae bg cd^2 a^2f ab^2c dg^2	1462	*	5
2. 6. 3	$abc^2de^2f^2$ b^2f ce^2 ad^2f	378	*	*
2. 6. 2	bce^2f^2 $ab^2d^2f^2$ b^2f bde^2	306	*	5
2. 6. 1	bf^2 de^2f a^2cf ade	186	*	5
2. 7. 3	$ab^2cd^2f^2$ $b^2c^2deg^2$ ce^2 fg^2 ab^2	1132	*	*
2. 7. 2	de^2f ab^2ce^2 $abdg^2$ f^2g ce^2	790	*	*
2. 7. 1	fg^2 bcd ac^2f^2 d^2ef^2 ab^2	562	*	6
2. 8. 3	adg^2h^2 $a^2b^2c^2eh^2$ a^2bc^2df f^2g ce^2 ab^2	3076	*	*
2. 8. 2	bh d^2ef^2 adg^2h^2 $a^2c^2eh^2$ fg^2 ce^2	1762	*	7
2. 8. 1	bh fg^2 a^2cg de^2g c^2d^2h ab^2	1534	*	6
3. 7. 3	$abcdef^2g$ $c^2d^2g^2$ bde^2 bc^2f^2	456	*	*
3. 7. 2	ab^2df^2 $bc^2e^2g^2$ cdg ef^2g	312	*	6
3. 7. 1	cdg b^2d^2e a^2bd^2f abg^2	246	*	5
3. 8. 3	$bcdefg$ abc^2d^2gh a^2c^2f abd^2 bce	1374	*	*
3. 8. 2	$abdf^2g^2$ $cd^2f^2gh^2$ a^2degh a^2c^2f abd^2	1194	*	*
3. 8. 1	$acdg$ a^2bef $b^2c^2d^2h^2$ c^2def^2 bce	966	*	6
3. 9. 3	$bcdefg$ $a^2d^2f^2g^2hi$ $c^2ef^2ghi^2$ a^2e^2g bce cg^2h^2	4290	*	*
3. 9. 2	bcd^2g^2i ac^2e^2fg a^2degh aef^2h^2i a^2e^2g bce	3624	*	*
3. 9. 1	$bdfi^2$ $cegi$ ad^2fh be^2h^2i $efgh^2$ a^2e^2g	2910	*	8

TABLE IV cont.

Optimal Minimum Cost Run Sequences for AMS 54 Designs				
Plan	**Run Sequence**	**C_{min}**	**Cost**	**Quad**
4. 8. 3	$abc^2d^2ef^2g^2h$ $a^2b^2gh^2$ ef^2gh^2 $bc^2f^2h^2$	536	*	*
4. 8. 2	abg^2h $c^2d^2ef^2$ e^2fg^2h $bc^2f^2h^2$	320	*	7
4. 8. 1	abg^2h $c^2d^2ef^2$ e^2fg^2h $bc^2f^2h^2$	320	*	6
4. 9. 3	ab^2cf^2gi bcd^2efh^2i ab^2gh^2 $bcdh$ $c^2e^2g^2i^2$	1454	*	*
4. 9. 2	abc^2d^2gh $ce^2fg^2h^2i$ $b^2cd^2fhi^2$ $ab^2e^2f^2$ bd^2e^2g	1436	*	*
4. 9. 1	ab^2cd^2e $bfghi$ $bc^2eg^2h^2$ $a^2c^2gh^2i$ $efgh^2$	1208	*	7
5. 9. 3	$abc^2e^2f^2g^2hi$ $d^2efg^2i^2$ b^2c^2efi $ad^2e^2g^2h^2$	562	*	*
5. 9. 2	de^2f^2gi abc^2d^2gh b^2c^2efi $ad^2e^2g^2h^2$	418	*	*
5. 9. 1	de^2f^2gi b^2c^2efi acg^2hi^2 $cd^2f^2gh^2$	400	*	8
5. 10. 3	ab^2cf^2gi $abde^2fghj$ $b^2d^2f^2ij$ b^2ce^2gh a^2befj	1534	*	*
5. 10. 2	$d^2efg^2i^2$ $a^2bc^2de^2$ abf^2hi^2j $b^2cfg^2j^2$ be^2h^2ij	1228	*	*
5. 10. 1	$d^2efg^2i^2$ $a^2bc^2de^2$ ad^2fhj^2 $a^2bd^2h^2i^2$ bc^2f^2gj	1210	*	*

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