ON THE OPTIMAL PARKING PROBLEM*

by

Shengwu He
Department of Mathematical Statistics
East China Normal University
Shanghai, CHINA
and
Department of Mathematics
Purdue University
West Lafayette, IN USA

Jiangang Wang
Institute of Mathematics
Fudan University
Shanghai, CHINA

Technical Report # 88-53

Department of Statistics
Purdue University

October, 1988

* Research supported by National Natural Science Foundation of China.

△

ON THE OPTIMAL PARKING PROBLEM*

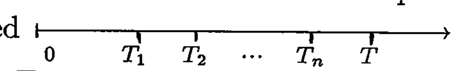
SHENGWU HE
Department of Mathematical Statistics
East China Normal University
Shanghai, China
and
Department of Mathematics
Purdue University
West Lafayette, Indiana, USA
and
JIAGANG WANG
Institute of Mathematics
Fudan University
Shanghai, China

1. Introduction and the main result.

The optimal parking problem, presented by Sakaguchi and Tamaki [4], is described as follows. A motorist is driving his car along a street toward his destination, and is looking for a parking place. If he finds an unoccupied parking place, he must decide either to park there and to walk the distance to his destination or to continue driving, expecting to find another parking place nearer to his destination. It is assumed that

1) The location of the destination T is a random variable, $0 < T < \infty$, with a known distribution function $F(x)$ and finite expectation;

2) The unoccupied parking places appear randomly in accordance with a Poisson process with parameter $\lambda > 0$, i.e. if denote the unoccupied



parking places by $0 < T_1 < T_2 < \cdots < T_n < \cdots$, then $T_1, T_2 - T_1, \cdots, T_n - T_{n-1}, \cdots$ are i.i.d. and the common distribution is exponential with rate λ;

3) T and $\{T_1, T_2, \cdots, T_n, \cdots\}$ are independent;

4) The speed of walking is 1, and the speed of driving is $\frac{1}{r}$, $0 < r < 1$.

If the car stops at S, then the whole time duration spent to reach the destination is

$$rS + |T - S|.$$

What we observe is the two stochastic processes,

$$X_t = \sum_{n=1}^{\infty} 1_{T_n \le t}, t \ge 0, \tag{1}$$

$$Y_t = 1_{T \le t}, t \ge 0, \tag{2}$$

*Research supported by National Natural Science Foundation of China

i.e. at time t we know if T or $T_1, T_2, \cdots$ do appear. Therefore, S should be a stopping time with respect to $(X_t, Y_t)_{t\geq 0}$, and $S = T_N$, where N is random. Obviously, S is a finite stopping time, so–called a stopping rule in [1].

Now the problem is to choose S to minimize the expected time

$$E\{rS + |S - T|\}.$$

Let $C = \inf\{x : F(x) = 1\}$, and

$$\varphi(x) = \frac{1}{1 - F(x)} \int_x^\infty e^{-\lambda(y-x)} F(dy), \ x \in (0, C).$$

It is easy to see that the function φ is well defined on the interval $(0, C)$, and

$$0 \leq \varphi(x) \leq 1, \ x \in (0, C). \tag{3}$$

Now our main result can be formulated as follows.

THEOREM. *Suppose that $\varphi(x)$ is increasing on $(0, C)$, and*

$$a = \inf\{x : \varphi(x) \geq \frac{1-r}{2}\}.$$

Then the optimal stopping rule for the parking problem is

$$S^* = \inf\{T_n : n \geq 1, \ T_n \geq T \text{ or } T \geq a\}. \tag{4}$$

We give some simple examples to illustrate the condition assumed in the Theorem.

Example 1. When $F(x)$ is the degenerated distribution function at point $C > 0$, we have

$$\varphi(x) = e^{-\lambda(C-x)}, \ x \in (0, C),$$

and φ is increasing on $(0, C)$.

Example 2. When $F(x)$ is the uniform distribution function on the interval $(0, C)$, we have

$$\varphi(x) = \frac{1}{\lambda(C-x)}[1 - e^{-\lambda(C-x)}], \ x \in (0, C).$$

It is easy to verify that φ is increasing on $(0, C)$.

Example 3. When $F(x)$ is the exponential distribution function with rate μ, we have $C = \infty$, and

$$\varphi(x) = \frac{\lambda}{\lambda + \mu}, \ x \geq 0.$$

By the way, if $C < \infty$, it is not difficult to show

$$\lim_{x \to C} \varphi(x) = 1. \tag{5}$$

But if $C = \infty$, (5) is not true in general, as indicated by Example 3.

Since the stopping rules we considered have the form $S = T_N$, where N takes integer values, our optimal parking problem essentially is one of discrete time case. And under our assumption about the function φ, it will be reduced to a special monotone case, the optimal stopping rule will be found out directly. We need not appeal to the general results of the optimal stopping theory, presented in [1]. Our idea to deal with the problem is somehow different from that in [4]. Even the forms of the results are slightly different. In fact, [4] didn't give explicitly the conditions, which guarantee the existence of the optimal stopping rule. In our opinion, our treatment is strictly rigorous, simpler and easier to understand. In the view of the general theory of stochastic processes, what we use is just the concept of dual predictable projection. Essentially, the idea is the same as in [2].

2. A special monotone case.

Let $(\mathfrak{F}_n)_{n\geq 1}$ be a discrete time filtration, i.e. an increasing sequence of σ–fields: $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots \subset \mathfrak{F}_n \subset \cdots$, and $(X_n)_{n\geq 1}$ be an adapted sequence of integrable random variables, i.e. for each n, X_n is measurable with respect to $\mathfrak{F}_n$. Set

$$Y_n = E[X_n|\mathfrak{F}_{n-1}] - X_{n-1}, \ n \geq 2.$$

PROPOSITION. *Suppose*

$$Y_2 \leq Y_3 \leq \cdots \leq Y_n \leq \cdots \tag{6}$$

and

$$S^* = \inf\{n \geq 1 : Y_{n+1} \geq 0\} \tag{7}$$

is finite. Then

$$EX_{S^*} = \inf\{EX_S : S \text{ is a stopping rule}\},$$

i.e. S^* *is the optimal stopping rule for* $(X_n, \mathfrak{F}_n)_{n\geq 1}$.

Proof. Let S be a stopping rule. Then

$$\begin{aligned} EX_S &= E\left\{X_1 + \sum_{k=2}^{S}(X_k - X_{k-1})\right\} \\ &= E\left\{X_1 + \sum_{n=2}^{\infty} 1_{S=n} \sum_{k=2}^{n}(X_k - X_{k-1})\right\} \\ &= E\left\{X_1 + \sum_{k=2}^{\infty}(X_k - X_{k-1})1_{S\geq k}\right\} \\ &= E\left\{X_1 + \sum_{k=2}^{\infty} E[X_k - X_{k-1}|\mathfrak{F}_{k-1}]1_{S\geq k}\right\} \\ &= E\left\{X_1 + \sum_{k=2}^{\infty} Y_k 1_{S\geq k}\right\} \qquad (8) \end{aligned}$$

where we commutate the summations to get the third equality and use the fact that $\{S \geq k\} \in \mathfrak{F}_{k-1}, k \geq 2$, for the fourth one.

Now for (6) and (8) it is easy to see that in order to minimize EX_S for $k \geq 2$ we should make

$$\begin{cases} 1_{S\geq k} = 1, \text{ if } Y_k < 0, \\ 1_{S\geq k} = 0, \text{ if } Y_k \geq 0. \end{cases}$$

Hence, S^* defined by (7) is the optimal stopping rule. □

According to [1], the monotone case means that

$$A_n = \{E[X_{n+1}|\mathfrak{F}_n] \geq X_n\}, \ n \geq 1$$

is an increasing sequence of events. Obviously, if the condition (6) is satisfied, i.e. $(Y_n)_{n\geq 2}$ is increasing, so is $A_n = \{Y_{n+1} \geq 0\}$. Hence, the case we consider is just a special monotone one, in which the stopping rule defined by (7) is always optimal.

On the other hand, in the view of the general theory of stochastic processes established by French school, $(X_1 + \sum_{k=2}^{\infty} Y_k)_{n\geq 1}$ is just the dual predictable projection or compensator of $(X_n)_{n\geq 1}$. Compared with [2], it can be clearly realized that the idea by using the concept of dual predictable projection to deal with the optimal stopping problem is the same as in [2]. The Proposition here is also a certain predictable criterion, which is practically useful as we'll see in the following.

3. The proof of the main result.

Set

$$\mathfrak{F}_t = \sigma\{X_s, Y_s, s \leq t\}, \ t \geq 0,$$

i.e. $(\mathfrak{F}_t)_{t\geq 0}$ is the filtration of observation. Since $(X_t, Y_t)_{t\geq 0}$ is a multivariate point process, from the results in [3] we know that $(\mathfrak{F}_t)_{t\geq 0}$ is right continuous and the following Lemma is true.

LEMMA 1. *For each $(\mathcal{F}_t)$–stopping time S, we have*

$$\mathcal{F}_S = \sigma\left\{X_{S\wedge t}, Y_{S\wedge t}, t \geq 0\right\}.$$

We should make decisions at T_n, $n = 1, 2, \cdots$. What we concern with is the filtration $(\mathcal{F}_{T_n})_{n\geq 1}$ indeed. From Lemma 1 and definitions (1) and (2), the following Lemma can be derived immediately.

LEMMA 2. *For each $n \geq 1$, we have*

$$\mathcal{F}_{T_n} \cap \{T_n < T\} = \sigma\{T_1, \ldots, T_n\} \cap \{T_n < T\},$$
$$\mathcal{F}_{T_n} \cap \{T_n \geq T\} = \sigma\{T_1, \cdots, T_n, T\} \cap \{T_n \geq T\}.$$

LEMMA 3. *For every integrable random variable ξ, we have*

$$E[\xi|\mathcal{F}_{T_n}] = \frac{E[\xi 1_{T_n<T}|T_1, \cdots, T_n]}{E[1_{T_n<T}|T_1, \cdots, T_n]} 1_{T_n<T} + E[\xi|T_1, \cdots, T_n, T] 1_{T_n\geq T}. \tag{9}$$

Lemma 3 is a direct consequence of Lemma 2. Notice that the event $\{T_n < T\}$ is not measurable with respect to $\{T_1, \cdots, T_n\}$, which incurs some complexity in the formula (9). The details are referred to the Appendix.

Now we are ready for the proof of the Theorem. At first, we observe that the objective function has the form:

$$\begin{aligned}
&E\left\{rS + |S - T|\right\}\\
=&E\left\{\int_0^S r dt + \int_0^S Y_t dt + \int_S^\infty (1 - Y_t) dt\right\}\\
=&E\left\{\int_0^S (r - 1 + 2Y_t) dt + \int_0^\infty (1 - Y_t) dt\right\}\\
=&E\left\{\int_0^S (r - 1 + 2Y_t) dt\right\} + ET
\end{aligned} \tag{10}$$

Secondly, we calculate

$$Z_n = E\left[\int_{T_n}^{T_{n+1}} (r - 1 + 2 1_{T\leq t}) dt \,|\, \mathcal{F}_{T_n}\right], n \geq 1$$

by the formula (9). Noting that T, $T_{n+1} - T_n$ and $\{T_1, \cdots, T_n\}$ are independent, we have

$$\begin{aligned}
Z_{T_n} 1_{T_n\geq T} &= E\left[\int_{T_n}^{T_{n+1}} (r - 1 + 2 1_{T\leq t}) dt 1_{T_n\geq T} | T_1, \cdots, T_n, T\right]\\
&= E\left[(r + 1)(T_{n+1} - T_n) \,|\, T_1, \cdots, T_n, T\right] 1_{T_n\geq T}\\
&= \frac{r + 1}{\lambda} 1_{T_n\geq T},
\end{aligned} \tag{11}$$

$$Z_{T_n} 1_{T_n<T} = \frac{E\left[\{(r - 1)(T_{n+1} - T_n) + 2(T_{n+1} - T)^+\} 1_{T_n<T} \,|\, T_1, \cdots, T_n\right]}{E\left[1_{T_n<T} \,|\, T_1, \cdots, T_n\right]} 1_{T_n<T}. \tag{12}$$

On the other hand,

$$\begin{aligned}
&E\left[(T_{n+1}-T_n)1_{T_n<T}\,|\,T_1,\cdots,T_n\right]\\
=&E\left[1_{T_n<T}E[T_{n+1}-T_n\,|\,T_1,\cdots,T_n,T]\,|\,T_1,\cdots,T_n\right]\\
=&\frac{1}{\lambda}E\left[1_{T_n<T}\,|\,T_1,\cdots,T_n\right] \qquad (13)\\
&E\left[1_{T_n<T}\,|\,T_1,\cdots,T_n\right]\\
=&E\left[1_{x<T}\,|\,T_1,\cdots,T_n\right]|_{x=T_n}\\
=&1-F(T_n) \qquad (14)\\
&E\left[(T_{n+1}-T)^+1_{T_n<T}\,|\,T_1,\cdots,T_n\right]\\
=&E\left[(T_{n+1}-T_n+z-T)^+1_{z<T}\,|\,T_1,\cdots,T_n\right]|_{z=T_n}\\
=&\int_z^\infty F(dy)\int_0^\infty (x+z-y)^+\lambda e^{-\lambda x}dx\,|_{z=T_n}\\
=&\frac{1}{\lambda}\int_{T_n}^\infty e^{-\lambda(y-T_n)}F(dy) \qquad (15)
\end{aligned}$$

Substituting (13), (14), (15) into (12) yields

$$Z_n 1_{T_n<T} = \frac{1}{\lambda}[r-1+2\varphi(T_n)]1_{T_n<T}. \qquad (16)$$

Combining (16) with (11) gives

$$Z_n = \frac{r+1}{\lambda}1_{T\le T_n} + \frac{1}{\lambda}[r-1+2\varphi(T_n)]1_{T_n<T}. \qquad (17)$$

Observing that from (3) and our assumption on φ we have

$$\begin{gathered}
\{T\le T_n\}\subset\{T\le T_{n+1}\},\\
r-1+2\varphi(T_n)\le r+1,\ \text{on } \{T_n<T\le T_{n+1}\},\\
T_n\le T_{n+1}<T\le C,\ \varphi(T_n)\le\varphi(T_{n+1}),\ \text{on } \{T_{n+1}<T\},
\end{gathered}$$

we conclude that

$$Z_n\le Z_{n+1},\ n\ge 1.$$

Since $S=T_N$, by using the Proposition in section 2, we obtain the optimal stopping rule

$$S^* = \inf\{T_n:\ n\ge 1,\ Z_n\ge 0\}. \qquad (18)$$

From (17) and (18) we know immediately that S^* has the form (4). □

Appendix.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub–σ–field of $\mathcal{F}$, and ξ an integrable random variable. Then for every $A \in \mathcal{F}$ and $G \in \mathcal{G}$, we have

$$\int_{AG} \xi dP = \int_{AG} \frac{E[\xi 1_A|\mathcal{G}]}{E[1_A|\mathcal{G}]} dP. \tag{19}$$

Proof. Set $H = \{E[1_A \,|\, \mathcal{G}] \neq 0\}$, then $H \in \mathcal{G}$ and

$$P(AH^c) = \int_{H^c} E[1_A \,|\, \mathcal{G}] dP = 0$$

where $H^c = \{E[1_A \,|\, \mathcal{G}] = 0\}$ is the complement of H. Hence, the right side of (9) makes sense. Define

$$P_A(F) = P(AF), \;\; F \in \mathcal{F}.$$

Then for every $G \in \mathcal{G}$ we have

$$P_A(G) = \int_G E[1_A \,|\, \mathcal{G}] dP$$

$$\begin{aligned} \int_{AG} \frac{E[\xi 1_A \,|\, \mathcal{G}]}{E[1_A \,|\, \mathcal{G}]} dP &= \int_{AGH} \frac{E[\xi 1_A \,|\, \mathcal{G}]}{E[1_A \,|\, \mathcal{G}]} dP \\ &= \int_{GH} \frac{E[\xi 1_A \,|\, \mathcal{G}]}{E[1_A \,|\, \mathcal{G}]} dP_A \\ &= \int_{GH} E[\xi 1_A \,|\, \mathcal{G}] dP \\ &= \int_{GH} \xi 1_A dP \\ &= \int_{AG} \xi dP \end{aligned}$$

(19) follows. □

REFERENCES

[1] Y. S. Chow, H. Robbins and D. Siegmund, *Great Expectation: the Theory of Optimal Stopping*, Houghton Mifflin Company, Boston, 1971.

[2] S. W. He, *Optimization applications of compensators of Poisson random measures*, To appear in Probability in Engineering and Information Science, 3 (1989).

[3] J. Jacod, *Multivariate point processes: predictable projection, Radon–Nikodym derivative, representation of martingales*, Z. Wahrsch. Verw. Geb., 31 (1975), pp. 235–253.

[4] M. Sakaguchi and M. Tamaki, *On the optimal parking problem in which spaces appear randomly*, Bulletin of Informatics and Cybernetics, 20 (1982), pp. 1–10.