JACKKNIFE VARIANCE ESTIMATOR FOR TWO SAMPLE LINEAR RANK STATISTICS[1]

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ABSTRACT

The jackknife estimator of the asymptotic variance of a two sample linear rank statistic is shown to be strongly consistent. Statistical applications of the result are discussed. The technique used in proving the consistency of the jackknife variance estimator can be applied to general situations.

Keywords: strong consistency; linear rank test; influence function.

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1. Introduction and the main result

Consider the following test problem concerning two (not necessarily continuous) population distributions F and G:

$$H_0: F=G \quad \text{vs.} \quad H_1: F \neq G. \tag{1.1}$$

Let $\{ X_1,...,X_n \}$ and $\{ Y_1,...,Y_m \}$ be independent samples from F and G, respectively. For simplicity, we assume that $m=n$. The results obtained in the following can be extended to the case $n/m \to \lambda$, $0<\lambda<1$, with some modifications. The statistic for the test problem (1.1) is the following two-sample simple linear rank statistic (see, e.g., Hájek and Sidák, 1967; Huber, 1981):

$$S(F_n,G_n) = \int J[{}^1\!/\!{}_2 F_n(x) + {}^1\!/\!{}_2 G_n(x)] \mathrm{d}F_n(x), \tag{1.2}$$

where F_n and G_n are empirical distribution functions corresponding to the samples $\{ X_1,...,X_n \}$ and $\{ Y_1,...,Y_n \}$, respectively, and J is a score function satisfying $J(1-t) = -J(t)$, $t \in [0,1]$. Let $H = {}^1\!/\!{}_2 F + {}^1\!/\!{}_2 G$ and $H_n = {}^1\!/\!{}_2 F_n + {}^1\!/\!{}_2 G_n$. $S(F_n,G_n)$ can be used as a point estimator of the quantity

$$S(F,G) = \int J[H(x)] \mathrm{d}F(x).$$

We assume that $S(F,G) = 0$ under the null hypothesis H_0 (which is satisfied if F is symmetric or F is continuous). Thus, we reject H_0 if $|S(F_n,G_n)|$ is large.

An asymptotic analysis of the sampling distribution of $S(F_n,G_n)$ is needed for obtaining the critical value for the test problem (1.1) and for calculating the power of the test procedure. Chernoff and Savage (1958) showed that under certain conditions (see also Hájek and Sidák, 1967, pp.233-237), $(2n)^{1/2}[S(F_n,G_n) - S(F,G)]$ converges in distribution to $N(0, \sigma^2)$ with

$$\sigma^2 = Var_F \phi(X_1) + Var_G \psi(Y_1), \tag{1.3}$$

where

$$\begin{aligned} \phi(x) &= {}^1\!/\!{}_2 \int J'[H(y)][I_{(x \le y)} - F(y)] \mathrm{d}F(y) + J[H(x)] - \int J[H(y)] \mathrm{d}F(y) \\ \psi(x) &= {}^1\!/\!{}_2 \int J'[H(y)][I_{(x \le y)} - G(y)] \mathrm{d}F(y), \end{aligned} \tag{1.4}$$

I_A is the indicator function of the set A and J' is the derivative of J. Note that $\phi(x)$ and $\psi(x)$ in (1.4) are influence functions of $S(F,G)$ by using a statistical functional approach (see

Hampel, 1974; Huber, 1981).

Suppose that we have a consistent estimator s^2 of σ^2 given in (1.3), i.e., $s^2 \to \sigma^2$ *a.s.* Then

$$(2n)^{1/2}[S(F_n,G_n) - S(F,G)]/s \to N(0,1) \quad \text{in distribution.}$$

Hence a test procedure with approximate level α ($0<\alpha<1/2$) concludes H_1 if

$$(2n)^{1/2}|S(F_n,G_n)|/s > \Phi^{-1}(1-\alpha/2), \tag{1.5}$$

where Φ is the distribution function of $N(0,1)$. (1.5) gives the critical region of the test for (1.1).

In this note we prove that an estimator of σ^2 obtained by using the jackknife method (Tukey, 1958) is strongly consistent and therefore can be used in the above test procedure.

For $i=1,\ldots,n$, let F_{ni} and G_{ni} be the empirical distribution functions corresponding to the samples $\{X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n\}$ and $\{Y_1,\ldots,Y_{i-1},Y_{i+1},\ldots,Y_n\}$, respectively, and $H_{ni} = \frac{1}{2}F_{ni} + \frac{1}{2}G_{ni}$. Let $S(F_{ni},G_{ni})$ be defined as in (1.2) with F_n and G_n replaced by F_{ni} and G_{ni}. The jackknife estimator of σ^2 is defined to be

$$s_J^2 = (n-1)\sum_{i=1}^n [S(F_{ni},G_{ni}) - S(F_n,G_n)]^2.$$

We shall assume the following condition.

Condition A. J' is continuous on [0,1] and $\|J'\|_V$ is finite, where $\|\ \|_V$ is the total variation norm (see Natanson, 1961).

Note that J' satisfies $J'(1-t) = J'(t)$ for $t \in [0,1]$. Hence the condition $\|J'\|_V < \infty$ is satisfied if J' is monotone on $[0,1/2]$. If J'' exists, then $\|J'\|_V = \int_0^1 |J''(t)|\,dt$ and therefore $\|J'\|_V < \infty$ if J'' is integrable. An example of J satisfying condition A is $J(t) = t - 1/2$ (corresponding to Wilcoxon statistic).

The following is our main result.

Theorem. Assume condition A. Then the jackknife estimator is strongly consistent, i.e.,

$$s_J^2 \to \sigma^2 \ \ a.s.$$

2. Proof of the theorem

Let $\phi_n(x)$ and $\psi_n(x)$ be defined as in (1.4) with F, G and H replaced by F_n, G_n and H_n, respectively. We prove the following result first.

Lemma. Assume condition A. Then

$$\|\phi_n - \phi\|_\infty \to 0 \ \ a.s. \quad \text{and} \quad \|\psi_n - \psi\|_\infty \to 0 \ \ a.s., \tag{3.1}$$

where $\| \ \|_\infty$ is the sup norm.

Proof. Under condition A, $\|J'\|_\infty < \infty$. From $\|F_n - F\|_\infty \to 0$ and $\|G_n - G\|_\infty \to 0$ $a.s.$,

$$|J[H_n(x)] - J[H(x)]| \le \|J'\|_\infty \ \|H_n - H\|_\infty \to 0 \ \ a.s.$$

and

$$\int |J[H_n(x)] - J[H(x)]| \mathrm{d}F_n(x) \le \|J'\|_\infty \ \|H_n - H\|_\infty \to 0 \ \ a.s.$$

From the strong law of large numbers (SLLN),

$$\int J[H(x)]\mathrm{d}[F_n(x) - F(x)] \to 0 \ \ a.s.$$

Hence

$$\|J[H_n] - \int J[H_n(x)]\mathrm{d}F_n(x) - J[H] - \int J[H(x)]\mathrm{d}F(x)\|_\infty \to 0 \ \ a.s.$$

For the first assertion in (3.1), it remains to show that

$$\sup_x \left| \int J'[H_n(y)][I_{(x\le y)} - F_n(y)]\mathrm{d}F_n(y) - \int J'[H(y)][I_{(x\le y)} - F(y)]\mathrm{d}F(y) \right| \to 0 \ \ a.s. \tag{3.2}$$

The quantity in (3.2) is bounded by

$$\begin{aligned} &\left| \int J'[H_n(y)][F(y) - F_n(y)]\mathrm{d}F_n(y) \right| + \sup_x \left| \int J'[H(y)][I_{(x\le y)} - F(y)]\mathrm{d}[F_n(y) - F(y)] \right| \\ &+ \sup_x \left| \int \{J'[H_n(y)] - J'[H(y)]\}[I_{(x\le y)} - F(y)]\mathrm{d}F_n(y) \right|. \end{aligned} \tag{3.3}$$

The first term in (3.3) is bounded by $\|J'\|_\infty \|F_n - F\|_\infty \to 0$ $a.s.$ The third term in (3.3) is bounded by $\|J'[H_n] - J'[H]\|_\infty$, which $\to 0$ $a.s.$ since J' is continuous on [0,1]. From the SLLN, $\int J'[H(y)]F(y)\mathrm{d}[F_n(y) - F(y)] \to 0$ $a.s.$ Hence (3.2) follows from

$$\sup_x \left| \int J'[H(y)]I_{(x\le y)}\mathrm{d}[F_n(y) - F(y)] \right| \to 0 \ \ a.s. \tag{3.4}$$

Let $I_x(y) = I_{(x\le y)}$ and $g_x(y) = J'[H(y)]I_x(y)$. From Natanson (1961, p.232),

$$\left| \int J'[H(y)]I_{(x\le y)}\mathrm{d}[F_n(y) - F(y)] \right| \le \|g_x\|_V \|F_n - F\|_\infty.$$

Note that $\|g_x\|_V \le \|J'\|_V \|I_x\|_\infty + \|J'\|_\infty \|I_x\|_V \le \|J'\|_V + \|J'\|_\infty$. Hence (3.4) holds and the first assertion follows. The proof of the second assertion is similar. □

Proof of Theorem. Let

$$V_{ni} = \int \phi(x) \mathrm{d}[F_{ni}(x) - F_n(x)] + \int \psi(x) \mathrm{d}[G_{ni}(x) - G_n(x)],$$

$$U_{ni} = \int [\phi_n(x) - \phi(x)] \mathrm{d}[F_{ni}(x) - F_n(x)] + \int [\psi_n(x) - \psi(x)] \mathrm{d}[G_{ni}(x) - G_n(x)],$$

$$W_{ni} = \int \{J[H_{ni}(x)] - J[H_n(x)]\} \mathrm{d}F_{ni}(x) - \int J'[H_n(x)][H_{ni}(x) - H_n(x)] \mathrm{d}F_n(x)$$

and $R_{ni} = U_{ni} + W_{ni}$. Then

$$S(F_{ni}, G_{ni}) - S(F_n, G_n) = V_{ni} + R_{ni}$$

and therefore

$$s_J^2 = (n-1) \sum_{i=1}^n (V_{ni}^2 + R_{ni}^2 + 2V_{ni} R_{ni}).$$

Let $\xi_i = \phi(X_i)$, $\zeta_i = \psi(Y_i)$, $\bar{\xi} = n^{-1} \sum_{i=1}^n \xi_i$ and $\bar{\zeta} = n^{-1} \sum_{i=1}^n \zeta_i$. Then

$$\begin{aligned}(n-1) \sum_{i=1}^n V_{ni}^2 &= (n-1) \{ \sum_{i=1}^n [(n-1)^{-1} \sum_{j \ne i} \xi_j - \bar{\xi}]^2 + \sum_{i=1}^n [(n-1)^{-1} \sum_{j \ne i} \zeta_j - \bar{\zeta}]^2 \\ &\quad + 2 \sum_{i=1}^n [(n-1)^{-1} \sum_{j \ne i} \xi_j - \bar{\xi}][(n-1)^{-1} \sum_{j \ne i} \zeta_j - \bar{\zeta}] \} \\ &= (n-1)^{-1} \sum_{i=1}^n (\xi_i - \bar{\xi})^2 + (n-1)^{-1} \sum_{i=1}^n (\zeta_i - \bar{\zeta})^2 + 2(n-1)^{-1} \sum_{i=1}^n (\xi_i - \bar{\xi})(\zeta_i - \bar{\zeta}),\end{aligned}$$

which converges *a.s.* to σ^2 according to the SLLN. From Cauchy-Schwarz inequality, it remains to show that

$$(n-1) \sum_{i=1}^n R_{ni}^2 \to 0 \quad a.s.,$$

which is implied by

$$\max_{i \le n} |U_{ni}| = o(n^{-1}) \quad a.s. \tag{3.5}$$

and

$$\max_{i \le n} |W_{ni}| = o(n^{-1}) \quad a.s. \tag{3.6}$$

Since

$$\begin{aligned}\left| \int [\phi_n(x) - \phi(x)] \mathrm{d}[F_{ni}(x) - F_n(x)] \right| &= (n-1)^{-1} \left| \phi(X_i) - \phi_n(X_i) - n^{-1} \sum_{i=1}^n \phi(X_i) \right| \\ &\le (n-1)^{-1} \|\phi_n - \phi\|_\infty + [n(n-1)]^{-1} \left| \sum_{i=1}^n \xi_i \right|,\end{aligned}$$

$\max_{i \le n} \left| \int [\phi_n(x) - \phi(x)] \mathrm{d}[F_{ni}(x) - F_n(x)] \right| = o(n^{-1})$ *a.s.* follows from $\|\phi_n - \phi\|_\infty \to 0$ *a.s.* (Lemma) and $n^{-1} \sum_{i=1}^n \xi_i \to 0$ *a.s.* (SLLN). Similarly, we can prove that

$$\max_{i\le n} \left|\int [\psi_n(x)-\psi(x)] \mathrm{d}[G_{ni}(x)-G_n(x)]\right| = o(n^{-1}) \ \ a.s.$$

Hence (3.5) holds. From the continuity of J' and $\|H_{ni} - H_n\|_\infty \le n^{-1}$,

$$\max_{i\le n} \left|\int \{J[H_{ni}(x)]-J[H_n(x)] - J'[H_n(x)][H_{ni}(x)-H_n(x)]\} \mathrm{d}F_n(x)\right| = o(n^{-1}) \ \ a.s.$$

Then (3.6) follows from

$$\max_{i\le n} \left|\int \{J[H_{ni}(x)]-J[H_n(x)]\} \mathrm{d}[F_{ni}(x)-F_n(x)]\right| = o(n^{-1}) \ \ a.s. \tag{3.7}$$

Again from the continuity of J', (3.7) follows from

$$\max_{i\le n} \left|\int J'[H_n(x)][H_{ni}(x)-H_n(x)] \mathrm{d}[F_{ni}(x)-F_n(x)]\right| = o(n^{-1}) \ \ a.s. \tag{3.8}$$

Note that

$$\left|\int J'[H_n(x)][H_{ni}(x)-H_n(x)] \mathrm{d}[F_{ni}(x)-F_n(x)]\right| \le \|F_{ni} - F_n\|_\infty \|J'[H_n][H_{ni}-H_n]\|_V$$
$$\le n^{-1} \|J'[H_n][H_{ni}-H_n]\|_V \le n^{-1}(\|J'\|_V \|H_{ni}-H_n\|_\infty + \|J'\|_\infty \|H_{ni}-H_n\|_V).$$

Since $F_{ni}(y)-F_n(y) = (n-1)^{-1}[F_n(y)-I_{X_i}(y)]$, where $I_{X_i}(y) = I_{(X_i\le y)}$,

$$\|F_{ni} - F_n\|_V = (n-1)^{-1} \|F_n - I_{X_i}\|_V \le (n-1)^{-1}(\|F_n\|_V + \|I_{X_i}\|_V) = 2(n-1)^{-1}.$$

Similarly, $\|G_{ni} - G_n\|_V \le 2(n-1)^{-1}$ and therefore $\|H_{ni} - H_n\|_V \le 2(n-1)^{-1}$. Hence (3.8) holds. This proves the theorem. □

3. Comments

In some situations (e.g., F is continuous), the variance of $S(F_n,G_n)$ under the null hypothesis H_0 can be calculated using theory of rank statistics (see Hájek and Sidák, 1967). Hence the critical region of the test procedure for (1.1) can be constructed by using $s^2 = (2n)Var[S(F_n,G_n)|\mathrm{H}_0]$, provided $(2n)Var[S(F_n,G_n)|\mathrm{H}_0] \to \sigma^2$. In these situations, the jackknife just gives an alternative method. Computing the jackknife estimator is routine and simple and does not require a theoretical derivation of $Var[S(F_n,G_n)|\mathrm{H}_0]$. Furthermore, the consistency of the jackknife estimator holds under both null and alternative hypotheses and therefore the jackknife may provide other statistical analysis procedures in some situations. For example, suppose that under the alternative H_1, $G(x) = qF + pF^2$ (see Serfling, 1980, p.293), where p may or may not be known, $0<p\le 1$ and $q=1-p$. Suppose also that F is continuous. Then

$$S(F,G) = \int_0^1 J[t-p(t-t^2)]\mathrm{d}t.$$

Denote this quantity by $g(p)$. Then the power of the test at $p = p_1$ is approximately

$$1 - \Phi[\Phi^{-1}(1-\alpha/2) - n^{1/2} g(p_1)/s_J] + \Phi[\Phi^{-1}(\alpha/2) - n^{1/2} g(p_1)/s_J].$$

Assume that J is strictly increasing. Then $g(p)$ is strictly decreasing in p. If p is unknown, an approximate $100(1-\alpha)\%$ confidence interval for p has limits

$$g^{-1}[S(F_n,G_n) \pm \Phi^{-1}(1-\alpha/2) n^{-1/2} s_J].$$

Finally, the technique used in the proof of the consistency of jackknife estimator can be applied to general situations where $S(F,G)$ is a functional with inference functions satisfying (3.1).

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