FROM DISCRETE TO CONTINUOUS TIME FINANCE: WEAK CONVERGENCE OF THE FINANCIAL GAIN PROCESS

by

Darrell Duffie
Stanford University

and

Philip Protter
Purdue University

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Department of Statistics
Purdue University

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Darrell Duffie and Philip Protter

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Abstract: Conditions are provided for the weak convergence of stochastic integrals defining financial gains from trading securities. Examples include convergence from discrete to continuous time settings, and in particular, for generalizations of the convergence of binomial option pricing models to the Black–Scholes model.

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1. Introduction

Although a large part of financial economic theory is based on models with continuous-time security trading, it is widely felt that these models are relevant insofar as they characterize the behavior of models in which trades occur discretely in time. It seems natural to check that the limit of discrete-time trading as the periods between trades shrink to zero produces the effect of continuous-time trading. That is one of the principal aims of this paper.

An easier way to describe the purpose of the paper is to recall Cox, Ross, and Rubinstein's (1979) proof that the Black–Scholes (1973) Option Pricing Formula is the limit of a discrete–time binomial option pricing formula (due to William Sharpe) as the number of time periods per unit of real time goes to infinity. Aside from providing a simple interpretation of the Black–Scholes formula, this connection between discrete and continuous time financial models led to a standard technique for estimating continuous–time derivative asset prices by using numerical methods based on discrete–time reasoning.

In general terms, this approach to asset pricing works because of the following sort of reasoning. Suppose, in an environment with n trading periods per unit of time, that S^n is the stochastic security price process and θ^n is the trading strategy; that is, S^n_t is the price of the asset at time t and θ^n_t is the number of units of the asset held by the investor at time t. The corresponding financial gain process is $G^n = \int \theta^n_t \, dS^n_t$. At time T, that is, the cumulative financial gain is the stochastic integral $G^n_T = \int_0^T \theta^n_t \, dS^n_t$. Suppose that (θ^n, S^n) converges weakly to (θ, S). (A precise definition of weak convergence of processes is given in Section 2.) Under recently developed regularity conditions, the gain G^n converges in distribution to the continuous–time gain $G = \int \theta_t \, dS_t$. This paper is designed to give simple statements of these regularity conditions, showing how they can be applied in various financial models to answer a number of different types of questions.

The results are general: S^n and θ^n can be continuous–time processes (defined, for example, by an economic environment indexed by n); both S^n and θ^n can be path–dependent. An example of the sort of result one can obtain is the following extension of the Cox–Ross–Rubinstein result for general price processes (in discrete or continuous time): if the risky asset price process S^n converges in distribution to exponential Brownian Motion, then there is a self-financing trading strategy (in the risky asset and a riskless bond) such that the distribution of the strategy's payoff at the exercise date of the option converges to the corresponding Black–Scholes option payoff, and such that the required initial investment converges to that of the Black–Scholes formula.

We also present special tools to handle weak convergence of stochastic differential equations and to deal directly with return processes, rather than price processes. For example, suppose X^n is the cumulative return process for environment n; the corresponding price process is $S^n = \mathcal{E}(X^n)$, the stochastic exponential of X^n (defined in Section 3). We show that if X^n converges weakly to X in a natural way, then S^n converges weakly to $S = \mathcal{E}(X)$. Under regularity conditions directly on X^n and θ^n, we can then show weak convergence of $\int \theta_t^n \, dS_t^n$ to $\int \theta_t \, dS_t$.

Our general goal is to provide a useful set of tools for exploring the boundaries between discrete and continuous time financial models, as well as the stability of the financial gain process $\int \theta \, dS$ with respect to perturbations of the price process S and trading strategy θ.

2. Preliminaries

This section sets out some of the basic definitions and notation. We let $I\!D^d$ denote the space of $I\!R^d$–valued càdlàg[1] sample paths on a fixed time interval $\mathcal{T}$, either $[0,T]$ or $[0,\infty)$. The Skorohod topology (as explained by Billingsley (1968), p. 111) on $I\!D^d$ is used throughout, unless otherwise noted. A càdlàg process is a random variable S on some probability space into $I\!D^d$. A sequence $\{S^n\}$ of càdlàg processes (which may be defined on different probability spaces) converges in distribution to a càdlàg process S, denoted $S^n \Longrightarrow S$, if $E[h(S^n)] \to E[h(S)]$ for any bounded continuous real–valued function h on $I\!D^d$.

A famous example is Donsker's Theorem, whereby a normalized "coin toss" random walk converges in distribution to Brownian Motion. That is, let $\{Y_k\}$ be a sequence of independent random variables with equally likely outcomes $+1$ and -1, and let $X_t^n = (Y_1 + \cdots + Y_{[nt]})/\sqrt{n}$ for any time t, where $[t]$ denotes the smallest integer less than or equal to t. Then $X^n \Longrightarrow B$, where B is Standard Brownian Motion. Donsker's Theorem applies to more general forms of random walk and to certain classes of martingales; Billingsley

[1] That is, $f \in I\!D$ means that $f : \mathcal{T} \to I\!R$ has a limit $f(t-) = \lim_{s \uparrow t} f(s)$ from the left for all t, and that the limit from the right $f(t+)$ exists and is equal to $f(t)$ for all t. By convention, $f(0-) = f(0)$. The expression "RCLL" (right continuous with left limits) is a also used in place of càdlàg (continue à droit, limites à gauche).

(1968), Ethier–Kurtz (1986), or Jacod–Shiryayev (1988) are good general references.

In financial models, we are more likely to think of $\{Y_k\}$ as a discrete–time return process, so that X^n is the normalized cumulative return process. The corresponding price process S^n is defined by $S_t^n = S_0\mathcal{E}(X^n)_t$, for some initial price $S_0 > 0$, where the stochastic exponential $\mathcal{E}(X)$ of X^n is given in this case by

$$\mathcal{E}(X^n)_t = \prod_{k=1}^{[nt]} \left(1 + \frac{Y_k}{\sqrt{n}}\right).$$

The definition of the stochastic exponential $\mathcal{E}(X)$ for general X is given in Section 3. It is well known that $S^n \Longrightarrow S$, where $S_t = S_0 e^{B_t - t/2}$. That is, with returns generated by a coin toss random walk, the asset price process converges in distribution to the solution of the stochastic differential equation $dS_t = S_t\, dB_t$. This is the classical Black–Scholes example (leaving out, for simplicity, constants for the interest rate and the mean and variance of stock returns). We return later in the paper to extend this example, showing that the Black–Scholes formula can be found as the limit of discrete–time models with a general class of cumulative return processes X^n converging in distribution to Brownian Motion.

A process X is a *semimartingale* if there exists a decomposition $X = M+A$ where M is a local martingale and A is an adapted càdlàg process with paths of finite variation on compact time intervals. Semimartingales are the most general stochastic "differentials." Protter (1989) is an introductory treatment of stochastic integration and differential equations; Dellacherie–Meyer (1982) is a comprehensive treatment of semimartingales and stochastic integration.

3. Weak Convergence Results for Stochastic Integrals

This section presents recently demonstrated conditions for weak convergence of stochastic integrals, as well as simplified versions of these conditions designed for applications in financial economic models. Readers interested only in our applications could skip to Section 4.

The following setup is fixed for this section. The time set $\mathcal{T}$ is $[0,T]$ or $[0,\infty)$. For each n, there is a probability space $(\Omega^n, \mathcal{F}^n, P^n)$ and a filtration $\{\mathcal{F}_t^n : t \in \mathcal{T}\}$ of sub–σ–fields of $\mathcal{F}^n$ (satisfying the usual conditions) on which X^n is a semimartingale and H^n is a càdlàg adapted process. (We can, and do, always fix a càdlàg version of any semimartingale.)

There is also a probability space and filtration on which the corresponding properties hold for X and H, respectively. Moreover, $(H^n, X^n) \Longrightarrow (H, X)$, where $\Longrightarrow$ in this case refers to convergence in distribution on the Skorohod space $\mathbb{D}^2$ of càdlàg sample paths on $\mathcal{T}$, valued in $\mathbb{R}^2$. It is important to realize that convergence in $\mathbb{D}^2$ is not convergence in $\mathbb{D}^1 \times \mathbb{D}^1$. By $(H^n, X^n) \Longrightarrow (H, X)$ we mean that there exists one (and not two) sequence of changes of time λ_n such that $\lambda_n(s)$ converges to s uniformly, and $(H^n_{\lambda^n(s)}, X^n_{\lambda^n(s)})$ converges in law uniformly in s to (H, X).

First of all, we need conditions under which $\int H^n_{s-}\, dX^n_s \Longrightarrow \int H_{s-}\, dX_s$.

ASSUMPTION JC (JACOD'S CONDITION). *For a sequence (A^n) of càdlàg, adapted processes with paths of finite variation on compact intervals, for each time t,*

$$\lim_{b \to \infty} \sup_n P_n\big(\mid A \mid_t > b\big) = 0,$$

where $\mid A \mid_t = \int_0^t \mid dA \mid_s$ denotes the total variation of A at t. That is, the finite variation processes (A^n) are tight on $\mathbb{R}$, in total variation.

Comment: By Chebyshev's Inequality, for (A^n) to satisfy Jacod's Condition, it suffices that, for any t, $\sup_n E(\mid A^n \mid_t)$ is finite. In particular, if $A^n_t = \int_0^t h^n_s\, ds$, for (JC) it is enough that, for any t, $\sup_n E\left(\int_0^t \mid h^n_s \mid ds\right)$ is finite.

The following condition was introduced by Jakubowski, Mémin, and Pages (1988). For simple notation, let $\sim X$ denote the distribution of a random variable X.

DEFINITION (UNIFORM TENSION). *Let $\mathcal{H}^n_t$ be the set of processes of the form:*

$$H^n_s = Y^n_0 + \sum_{i=0}^{k} Y^n_{t_i} 1_{(t_i, t_{i+1}]}(s),$$

where $Y^n_{t_i}$ is $\mathcal{F}^n_{t_i}$-measurable, $\mid Y^n_{t_i} \mid \leq 1$, and $\{t_0, t_1, \ldots, t_k\}$ is a finite partition of $[0, t]$ with $t_0 = 0$ and $t_k = t$. A sequence of semimartingales $(X^n)_{n \geq 1}$ has uniform tension *if $\{\sim \int H^n_s\, dX^n_s : n \in \mathbb{N},\ H^n \in \mathcal{H}^n_t\}$ is tight for each $t > 0$.*

The condition of *uniform tension* was inspired by the theorem of Bichteler–Dellacherie, which states that an adapated, càdlàg process X has uniform tension (where the "sequence" consists of $X^n = X$ for all n) if and only if X is a semimartingale. See Protter (1989).

THEOREM 1 (JAKUBOWSKI–MÉMIN–PAGES). *Let (X^n) be a sequence of semimartingales and (K^n) a sequence of adapted càdlàg processes, where, for each n, (X^n, K^n) is defined on $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\geq 0}, P^n)$. Suppose the sequence (X^n) has uniform tension. If $(K^n, X^n) \Longrightarrow (K, X)$, then X is a semimartingale and $\int K_{s-}^n\, dX_s^n \Longrightarrow \int K_{s-}\, dX_s$.*

A precursor of Theorem 1 is due to Strasser (1986). The following is a useful consequence of Theorem 1 which is not given in Jakubowski, Mémin, and Pages (1989).

THEOREM 2. *Let (K^n, X^n) be as in Theorem 1, with (X^n) having uniform tension, and with $(K^n, X^n) \Longrightarrow (K, X)$. Then the semimartingales $Y^n = \int K_{s-}^n\, dX_s^n$ also have uniform tension.*

PROOF: Since K^n converges weakly, we know that for every $t > 0$, $\epsilon > 0$, there exists n_0 and $p > 0$ such that $n \geq n_0$ implies $P_n\left((K^n)_t^* > p\right) < \epsilon$, where $(K^n)_t^* = \sup_{s\leq t} \mid K_s^n \mid$. Fix $t > 0$ and set $T^{n,p} = \inf\{t > 0 : \mid K_s^n \mid > p\}$. Then $(K_{s-}^n)^{T^{n,p}}$ is bounded by p, and for large enough n we have $P(T^{n,p} < t) < \epsilon$. Let $H^n \in \mathcal{H}_t^n$, where $\mathcal{H}_t^n$ is as defined in the definition of uniform tension. Let $Y^n = \int K_{s-}^n\, dX_s^n$, or in the usual shorthand notation that we shall henceforth adopt, $Y^n = K_-^n \cdot X^n$. To show the uniform tension of Y^n it suffices to show that $\{\sim(H^n \cdot Y^n)_t^* : n \in \mathbb{N}, H^n \in \mathcal{H}_t^n\}$ is tight (see Lemma 1.1 of Jakubowski, Mémin, and Pages (1989)). That is, we need to show that, given $\epsilon > 0$, for some constant q and some n_0, we have $P_n\left[(H^n \cdot Y^n)_t^* > q\right] < 2\epsilon$, all $H \in \mathcal{H}_t^n$, all $n \geq n_0$. Since

$$\begin{aligned} P_n\left[(H^n \cdot Y^n)_t^* > q\right] &\leq P_n\left[\left((H^n \cdot Y^n)^{T^{n,p}}\right)_t^* > q\right] + P_n(T^{n,p} < t). \\ &\leq P_n\left[\left((H^n \cdot Y^n)^{T^{n,p}}\right)_t^* > q\right] + \epsilon, \end{aligned}$$

Without loss of generality we can assume Y^n is stopped at $T^{n,p}$, hence $\mid K_-^n \mid \leq p$. But also

$$P_n\left(\frac{1}{p}(H^n \cdot Y^n)_t^* > q\right) = P_n\Big((H^n \cdot Y^n)_t^* > pq\Big),$$

and therefore we can assume without loss of generality that $\mid K_-^n \mid \leq 1$, since q was arbitrary.

It is elementary to show that $\mathcal{H}_t^n$ is dense in the set of left continuous, adapted processes with right limits that are bounded by one (see Theorem 10 of Chapter II of Protter (1989), where this is done in detail), with the topology of uniform convergence on compacts in probability (ucp). Let $K^{n,m} \in \mathcal{H}_t^n$ converging ucp to K^n. Then

$$\begin{aligned} &P_n\left[\left((H^n K^n) \cdot X^n\right)_t^* > 2pq\right] \\ \leq &P_n\left[\left((H^n K^n) \cdot X^n - (H^n K^{n,m}) \cdot X^n\right)_t^* > pq\right] + P_n\left[\left((H^n K^{n,m}) \cdot X^n\right)_t^* > pq\right]. \end{aligned}$$

Since $H^n K^{n,m} \in \mathcal{H}_t^n$, the second term on the right is less than $\epsilon/2$ for large enough n (and all m) by the uniform tension of X^n. Then, for each n, the first term on the right is less than $\epsilon/2$ for $m > m_0(n)$. This gives the uniform tension of Y^n. ∎

The hypothesis of uniform tension is difficult to verify in practice. We have the following simpler version of Theorem 1. For a process $Z \in I\!D$, let $\Delta Z_t = Z_t - Z_{t-}$, the jump of Z at time t.

THEOREM 3. *Let (X^n) be a sequence of semimartingales with decompositions $X^n = M^n + A^n$, and let K^n be an adapted càdlàg processes on $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\geq 0}, P_n)$, each $n \in I\!N$. Suppose that*

(a) (A^n) satisfies (JC).

(b) For each t, $\sup_n E_n \left(\sup_{s\leq t} \mid \Delta M_s^n \mid\right) < \infty$.

If $(K^n, X^n) \Longrightarrow (K, X)$, then X is a semimartingale, $(X^n)_{n\geq 1}$ has uniform tension, and $\int K_{s-}^n \, dX_s^n \Longrightarrow \int K_{s-} \, dX_s$.

PROOF: By Theorem 1 it suffices to show that the sequence (X^n) has uniform tension. To this end, let $H^n \in \mathcal{H}_t^n$.

For a process Z, let $Z_t^* = \sup_{s\leq t} \mid Z_s \mid$. Since $(\int_0^t H_s^n \, dA_s^n)^* \leq \mid A^n \mid_t$, condition (a) implies the (uniform) tension of $\{\sim \int H_s^n \, dA_s^n\}$, which further implies the tension of $\{\sim (A^n)_t^*\}$. [See Stricker (1985).] Since $(M^n)_t^* \leq (X^n)_t^* + (A^n)_t^*$, and since $X^n \Longrightarrow X$ implies the tension of $\{\sim (X^n)_t^* : n \in I\!N\}$, we deduce the tension of $\{\sim (M^n)_t^*\}$.

We next establish the tension of $\{\sim \int H_s^n \, dM_s^n\}$. We will use Davis' inequality (Dellacherie and Meyer (1982), VII. 90): For any martingale M and any stopping time T, there exist universal constants α and β such that

$$E\left([M,M]_T^{1/2}\right) \leq \alpha E\left(M_T^*\right) \leq \beta E\left([M,M]_T^{1/2}\right),$$

where $[M, M]$ denotes the quadratic variation process of the martingale M.

Let $\epsilon > 0$. By the tightness of $\{\sim (M^n)^*\}$, there exists a constant K such that, if

$$T^{n,K} = \inf\{s \leq t : \mid M_s^n \mid > K\} \wedge t,$$

then $P_n(T^{n,K} < t) < \epsilon$. Since

$$(M^n)_{T^{n,k}}^* \leq K + \sup_{s\leq t} \mid \Delta M_s^n \mid,$$

we have

$$\sup_n E_n[(M^n)^*_{T^{n,K}}] \leq K + \sup_n E_n\left(\sup_{s\leq t} \mid \Delta M^n_s \mid\right) \leq C.$$

(We let the constant C vary from place to place in what follows.) Then by Davis' inequality,

$$E_n\left([M^n, M^n]^{1/2}_{T^{n,K}}\right) \leq \alpha E_n\left[(M^n)^*_{T^{n,K}}\right] \leq C.$$

Moreover,

$$E_n\left[\left(\int H^n\, dM^n\right)^*_{T^{n,K}}\right] \leq E_n\left[\left(\int_0^{T^{n,K}} (H^n_s)^2\, d[M^n, M^n]_s\right)^{1/2}\right]$$
$$\leq E_n\left([M^n, M^n]^{1/2}_{T^{n,K}}\right) \leq C.$$

Therefore

$$P_n\left[\left(\int H^n\, dM^n\right)^*_{T^{n,K}} > \frac{C}{\epsilon}\right] \leq \epsilon.$$

However $\sup_n P_n(T^{n,K} < t) < \epsilon$, and thus

$$\sup_n P_n\left[\left(\int H^n\, dM^n\right)^*_t > \frac{C}{\epsilon}\right] \leq 2\epsilon,$$

which implies the tension of $\{\sim \int H^n\, dM^n\}$. Combined with the tension of $\{\sim \int H^n\, dA^n\}$, this yields the (uniform) tension of $\{\sim \int H^n\, dX^n\}$. ∎

Comment: If the semimartingales X^n have uniformly bounded jumps, then they are *special*: that is, there exists a unique decomposition $X^n = M^n + A^n$, where the finite variation process A^n is taken to be predictably measurable. Such a decomposition is called *canonical*. For the canonical decomposition, it can be shown that the jumps of M^n (and hence of A^n) are also bounded, and therefore for the canonical decomposition in the case of bounded jumps, condition (b) is always satisfied, and one need only check that (JC) holds in order to conclude the convergence of the stochastic integrals. Of course, if $X^n = N^n + C^n$ is any sequence of decompositions with the jumps of $(N^n)_{n\geq 1}$ uniformly bounded, then again clearly condition (a) [(C^n) satisfies (JC)] is sufficient for the conclusions of Theorem 2.

LEMMA 1. *Suppose (Z^n), $Z^n = M^n + A^n$, is a sequence of special martingales with (A^n) satisfying (JC). Then, for the canonical decomposition $Z^n = N^n + \widetilde{A}^n$, the sequence $(\widetilde{A}^n)$ satisfies (JC).*

PROOF: Since Z^n is special, A^n is locally of integrable variation [Dellacherie and Meyer (1982), page 214]. Since $\widetilde{A}^n$ is the predictable compensator of A^n, the result follows from the Corollary of Appendix Lemma A1. ∎

3.1. Stochastic Differential Equations

We now address the case of stochastic differential equations of the form

$$Z_t^n = H_t^n + \int_0^t f_n(s, Z_{s-}^n)\, dX_s^n,$$
$$Z_t = H_t + \int_0^t f(s, Z_{s-})\, dX_s,$$

where f_n and f are real–valued functions on $\mathbb{R}_+ \times \mathbb{R}$ such that:

(A) $x \mapsto f_n(t, x)$ is Lipschitz, each n,

(B) $t \mapsto f_n(t, x)$ is LCRL (left continuous with right limits, or "càglàd") for each x, each n, and

(C) for any sequence (x_n) of càdlàg functions with $x_n \to x$ in the Skorohod topology, (y_n, x_n) converges to (y, x) (Skorohod), where $y_n(s) = f_n(s+, x_n(s))$, $y(s) = f(s+, x(s))$.

(If $f_n(t, x) = f(t, x)$, all x, then condition (C) is automatically true.)

THEOREM 4 (SŁOMINSKI). *Let (H^n, X^n) be as in Theorem 1, and let $(f_n)_{n\geq 1}$ and f satisfy (A), (B), and (C) above. Let Z^n, Z be solutions*[2] *of*

$$Z_t^n = H_t^n + \int_0^t f_n(s, Z_{s-}^n)\, dX_s^n$$
$$Z_t = H_t + \int_0^t f(s, Z_{s-})\, dX_s.$$

Then $(Z^n, H^n, X^n) \Longrightarrow (Z, H, X)$.

The following theorem in not in Słominski, but it is essential for the applications we have in mind.

THEOREM 5. *Let X^n have uniform tension, let Z_0^n be random variables on Ω^n which are $\mathcal{F}_0^n$ measurable, and let $(f_n)_{n\geq 1}$, f satisfy (A), (B), and (C) above. Let Z^n, Z be solutions of*

$$Z_t^n = Z_0^n + \int_0^t f_n(s, Z_{s-}^n)\, dX_s^n$$
$$Z_t = Z_0 + \int_0^t f(s, Z_{s-})\, dX_s\,.$$

[2] Unique solutions exist. See Protter (1989) or Emery (1979).

If $(Z_0^n, X^n) \Longrightarrow (Z_0, X)$*, then* $(Z^n, X^n) \Longrightarrow (Z, X)$ and $(Z^n)_{n\geq 1}$ *have uniform tension.*

PROOF: By Theorem 4 we have $(Z^n, X^n) \Longrightarrow (Z, X)$, and therefore $(f_n(s, Z_s^n), X^n) \Longrightarrow (f(s, Z_s), X)$. The theorem now follows from Hypothesis (C) and Theorem 2. ∎

THEOREM 6 (SŁOMINSKI, SIMPLIFIED). *Let* (H^n, X^n) *be as in Theorem 3, and let* $(f_n)_{n\geq 1}$ *and* f *satisfy (A), (B), and (C). Let* Z^n *and* Z *be as in Theorem 4. Then* $(Z^n, H^n, X^n) \Longrightarrow (Z, H, X)$*. Moreover if* $H^n = Z_0^n$*, with* $H = Z_0$ *as in Theorem 5, then* $(Z^n)_{n\geq 1}$ *has uniform tension.*

An important special case is the stochastic differential equation

$$Z_t = 1 + \int_0^t Z_{s-}\, dX_s,$$

which defines the stochastic exponential[3] $Z = \mathcal{E}(X)$ of X. The solution, extending the special case of the previous section, is

$$Z_t = \exp\left(X_t - \frac{1}{2}[X,X]_t^c\right) \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s},$$

where $[X,X]^c$ denotes the continuous part of the quadratic variation $[X,X]$ of X. With a Standard Brownian Motion B, for example, $[B,B]_t^c = [B,B]_t = t$ and $\mathcal{E}(B)_t = e^{B_t - t/2}$.

Note that if $\Delta X_T = -1$ for some finite stopping time T, then $Z_t = \mathcal{E}(X)_t$ is identically zero after T. Also if $\inf_s \Delta X_s \geq -1$, then $\mathcal{E}(X)$ is always nonnegative. Finally, one can show that the process $V_t = \prod_{s\leq t}(1 + \Delta X_s)e^{-\Delta X_s}$ is a well-defined, adapted, càdlàg process with paths of finite variation on compacts. (As is clear from the equation it satisfies, the stochastic exponential plays a role analogous to the ordinary exponential of elementary calculus.)

For a semimartingale X, let $\mathcal{E}(\mathcal{E}(X)) = \mathcal{E}^{(2)}(X)$, the exponential of the exponential of X. In general let $\mathcal{E}^{(n)}(X) = \mathcal{E}(\mathcal{E}(\cdots \mathcal{E}(X))\cdots)$, where $\mathcal{E}^{(n)}(X)$ denotes the n-fold iterated stochastic exponential. We have the following corollary; the case $n = 1$ is originally due to Avram (1988).

COROLLARY 1. *Let* (X^n) *be semimartingales having uniform tension with* $X^n \Longrightarrow X$*. Then* $\left(X^n, \mathcal{E}^{(k)}(X^n)\right) \Longrightarrow \left(X, \mathcal{E}^{(k)}(X)\right)$*, each* $k \geq 1$*.*

[3] This is also known as the Doléans–Dade exponential.

COROLLARY 2. *Let X^n be semimartingales satisfying the hypotheses of Theorem 3. If $X^n \Longrightarrow X$ then $(X^n, \mathcal{E}^{(k)}(X^n)) \Longrightarrow (X, \mathcal{E}^{(k)}(X))$, each $k \geq 1$.*

PROOF: Since the hypotheses of Theorem 3 imply that $(X^n)_{n\geq 1}$ have uniform tension, both corollaries will be proved if $\mathcal{E}^{(k-1)}(X)$ has uniform tension, each $k-1$. However, this follows by Theorem 5, taking $f_n(s,x) = x$, each n, and induction on k. ▮

4. Convergence of Discrete-Time Strategies

In order to apply our results to "discrete-time" trading strategies θ^n and corresponding price processes S^n, we need conditions under which $(\theta^n, S^n) \Longrightarrow (\theta, S)$. We will consider strategies that are discrete-time with respect to a *grid*, defined by times $\{t_0, \ldots, t_k\}$ with $0 = t_0 < t_1 < \cdots < t_k = T$. The *mesh size* of the grid is $\sup_k \mid t_k - t_{k-1} \mid$.

The following convergence result is sufficient for many purposes. This result is trivial if f is uniformly continuous. The content of the lemma is to reduce it to that case.

LEMMA 2. *Let (S^n) and S be $\mathbb{D}^d$-valued on the same probability space, S be continuous, and $S^n \Longrightarrow S$ in the uniform metric topology. For each n, let the random times $\{T_k^n\}$ define a grid on $[0,T]$ with mesh size converging with n to 0 almost surely. For some continuous $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, let $H_t^n = f[S(T_k^n), T_k^n]$, $t \in [T_k^n, T_{k+1}^n)$, and $H_t = f(S_t, t)$. Then $(H^n, S^n) \Longrightarrow (H, S)$ in the uniform metric topology on $\mathbb{D}^{d+1}$.*

PROOF: Since S is continuous, for each ω, there is some bound $r(\omega)$ for $S(\omega)$. Since $S^n \Longrightarrow S$ (uniform) for each ω, there is some $N(\omega)$ such that, for $n > N(\omega)$,

$$\sup_s |S_s^n(\omega) - S_s(\omega)| < 1\,,$$

which implies that

$$\sup_{n>N(\omega)} \sup_s |S_s^n(\omega)| < r(\omega) + 1\,,$$

and therefore that

$$\sup_n \sup_s |S_s^n(\omega)| \leq \max\left\{ r(\omega) + 1,\ \sup_{k\leq N(\omega)} \sup_s |S_s^k(\omega)| \right\}.$$

Thus S^n is uniformly bounded, ω by ω. For each ω, we can therefore assume without loss of generality that f is defined on a compact set and therefore for each $\delta > 0$, there is an $\epsilon(\omega)$ such that $|f(y,t) - f(x,s)| < \delta$ whenever $|(y,t) - (x,s)| < \epsilon(\omega)$.

On $\{T_k^n \leq t < T_{k+1}^n\}$, for each ω there is some $N(\omega)$ such that, for $n \geq N(\omega)$,

$$|(S^n(T_k^n), T_k^n) - (S(t), t)| < \epsilon(\omega),$$

implying that

$$|H^n(t) - H(t)| = |f[S^n(T_k^n), T_k^n] - f[S(t), t]| \leq \delta.$$

It follows that

$$\sup_t |(S_t^n, H_t^n) - (S_t, H_t)| \to_n 0 \quad \text{a.s.},$$

which implies that $(S^n, H^n) \Longrightarrow (S, H)$ (uniform metric topology). ∎

COROLLARY 1. *The result follows if the assumption $S^n \Longrightarrow S$ (uniform) is replaced by $S^n \Longrightarrow S$ (Skorohod), provided S^n is continuous for all n or S^n is "discrete," that is, $S^n(t) = S^n(T_k^n)$ for $t \in [T_k^n, T_{k+1}^n)$.*

PROOF: It is well known that if α is a continuous function, a sequence (α_n) converges to α in the Skorohod topology if and only if it converges to α locally uniformly [e.g., Jacod–Shiryaev (1987), p. 292]. Thus, for S^n continuous, we have $S^n \Longrightarrow S$ in the uniform topology. Alternatively, if S^n is "discrete" as above, and $S^n \Longrightarrow S$ Skorohod, then $S^n \Longrightarrow S$ uniform. ∎

The next corollary is immediate.

COROLLARY 2. *Suppose (for $d = 1$) that $S^n = M^n + A^n$ also satisfies assumptions (a) and (b) of Theorem 3. Then $\int H_{t-}^n \, dS_t^n \Longrightarrow \int H_{t-} \, dS_t$.*

The following corollary allows the function f defining the trading strategies to depend on n. The proof involves only a slight adjustment.

COROLLARY 3. *Suppose $f_n : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ is continuous for each n such that: For any $\epsilon > 0$, there is some N large enough that, for any (x,t) and $n \geq N$, $|f_n(x,t) - f(x,t)| < \epsilon$. Then, with $H_t^n = f_n[S(T_k^n), T_k^n]$, $t \in [T_k^n, T_{k+1}^n)$, the same conclusion follows.*

5. Example: Convergence to the Black–Scholes Model

The objective of this section is to show that the weak convergence methods presented in this paper are easy to apply to a standard situation: the Black–Scholes (1973) option pricing formula. Under standard regularity conditions, the unique arbitrage-free price of a call option with time τ to expiration and exercise price K, when the current stock price is x, and the continuously compounding interest rate is $r \geq 0$, is

$$C(x,\tau) = \Phi(h)x - Ke^{-r\tau}\Phi(h - \sigma\sqrt{\tau}),$$

where Φ is the standard normal cumulative distribution function and $h = \frac{\log(x/K)+r\tau+\sigma^2\tau/2}{\sigma\sqrt{\tau}}$, provided the stock price process S satisfies the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t; \quad S_0 = x > 0, \tag{1}$$

for constants μ, r and $\sigma \neq 0$.[4] We will show convergence to the Black–Scholes formula in two cases:

(*a*) A fixed stock-price process S satisfying (1) and a sequence of stock trading strategies $\{\theta^n\}$ corresponding to discrete–time trading with trading frequency increasing in n, with limit equal to the Black–Scholes stock trading strategy $\theta_t = C_x(S_t, T-t)$, where T is the expiration date of the option and $C_x(x,\tau) = \frac{\partial}{\partial x}C(x,\tau)$.

(*b*) A sequence of stock price processes $\{S^n\}$ constructed as the stochastic exponentials of cumulative return processes $\{X^n\}$ converging in distribution to a Brownian Motion X, and trading strategies $\{\theta^n\}$ defined by $\theta^n(t) = C_x(S_t^n, T-t)$ for discretely chosen t.

Case (a) handles applications such as those of Leland (1986); Case (b) handles extensions of the Cox–Ross–Rubinstein (1979) results.

CASE (*a*) INCREASING TRADING FREQUENCY.

Let $T > 0$ be fixed, and let the set of stopping times $\mathcal{T}_n = \{T_k^n\}$ define a sequence of grids (as in Lemma 2) with mesh size shrinking to zero almost surely. In the n–th environment, the investor is able to trade only at stopping times in $\mathcal{T}_n$. That is, the trading strategy θ^n must be chosen from the set Θ^n of square–integrable predictable processes with $\theta^n(t) = \theta^n(T_k^n)$ for $t \in (T_{k-1}^n, T_k^n]$. For a simple case, let $t_k^n = k/n$, or n trades per unit of time, deterministically.

We take the case $r = 0$ for simplicity, since this allows us to consider stock gains alone, bond trading gains being zero. For $r > 0$, a standard trick of Harrison and Kreps (1979)

[4] Note that S is the stochastic exponential of the semimartingale $X_t = \mu t + \sigma B_t$.

allows one to normalize to this case without loss of generality. We consider the stock trading strategy $\theta^n \in \Theta^n$ defined by $\theta^n(0)$ arbitrary and

$$\theta^n(t) = C_x[S(T_k^n),\, T - T_k^n], \qquad t \in (T_k^n, T_{k+1}^n]. \tag{2}$$

For riskless discount bonds maturing after T, with a face value of one dollar (the unit of account) and bearing zero interest, we define the bond trading strategy $\alpha^n \in \Theta^n$ by the self–financing restriction

$$\alpha^n(t) = \alpha^n(0) + \int_0^{T_k^n} \theta_t^n \, dS_t - \theta^n(T_k^n)S(T_k^n) + \theta^n(0)S(0), \qquad t \in (T_k^n, T_{k+1}^n], \tag{3}$$

where

$$\alpha^n(0) = C(S_0, T) - \theta^n(0)S_0.$$

The total initial investment $\alpha_0^n + \theta_0^n S_0$ is the Black–Scholes option price $C(S_0, T)$. (Note that, $\alpha^n \in \Theta^n$.) The total payoff of this self–financing strategy (α^n, θ^n) at time T is $C(S_0, 0) + \int_0^T \theta_t^n \, dS_t$. For our purposes, it is therefore enough to show that

$$C(S_0, 0) + \int_0^T \theta_t^n \, dS_t \Longrightarrow (S_T - K)^+,$$

the payoff of the option. This can be done by direct (tedious) calculation (as in, say, Leland), but our general weak convergence results are quite simple to apply here.

PROPOSITION 1. *In the limit, the discrete-time self-financing strategy θ^n pays off the option. That is, $C(S_0, 0) + \int_0^T \theta_t^n \, dS_t \Longrightarrow (S_T - K)^+$.*

PROOF: For $X^n = X = S$ and $H_t^n = C_x(S_t, t)$, $t \in [T_k^n, T_{k-1}^n)$, the conditions of Theorem 2 are satisfied: Assumption (a) is satisfied because, since $S^n = S$, all n, we have $A^n = A$, all n (the finite variation terms) and obviously $\lim_{b\to\infty} P(|\, A\,|_t > b) = 0$. Assumption (b) is also satisfied because $S^n = S$ is continuous and therefore [as is well known—see Protter (1989) or Dellacherie–Meyer (1982)] all decompositions of $S = M + A$ are such that M and A are continuous.

Since $\theta_t^n = H_{t-}^n$ and C_x is continuous, Corollary 2 of Lemma 2 implies that $C(S_0, 0) + \int_0^T \theta_t^n \, dS_t \Longrightarrow C(S_0, 0) + \int_0^T \theta_t \, dS_t$. By Black and Scholes (1973), $C(S_0, 0) + \int_0^T \theta_t \, dS_t = (S_T - K)^+$ a.s. [For the details, see, for example, Duffie (1988), Section 22.] Thus $C(S_0, 0) + \int_0^T \theta_t^n dS_t \Longrightarrow (S_T - K)^+$. ∎

We can generalize the result as follows. We can allow S to be any diffusion process of the form $dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dB_t$. Then, subject to technical restrictions, for any terminal payoff $g(S_T)$, there is a sequence of discrete-time trading strategies whose terminal payoff converges in distribution to $g(S_T)$. The following technical regularity conditions are far in excess of the minimum known sufficient conditions. For weaker conditions, see, for example, the references cited in Section 21 of Duffie (1988).

CONDITION A. *The functions* $\mu : I\!R \times [0,T] \to I\!R$, $\sigma : I\!R \times [0,T] \to I\!R$, *and* $g : I\!R \to I\!R$ *together satisfy Condition A if they are Lipschitz and have Lipschitz first and second derivatives.*

PROPOSITION 2. *Let* (μ, σ, g) *satisfy Condition A. Suppose* $dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dB_t$, *and* $S^n \Longrightarrow S$ *(uniform topology), where* $S^n = M^n + A^n$ *satisfies assumption (a) and (b) of Theorem 3. Then there exist (discrete-time self-financing) strategies* (θ^n) *in* Θ^n *such that*

$$E\left[g(X_T)\right] + \int_0^T \theta_t^n \, dS_t^n \Longrightarrow g(S_T),$$

where $X_t = S_0 + \int_0^t \sigma(X_s, s)dB_s$, $t \in [0,T]$.

PROOF: Let $F(x,t) = E\left[g(X_T^{x,t})\right]$, where $X_\tau^{x,t} = x + \int_t^\tau \sigma(X_s^{x,t}, s)\, dB_s$, $\tau \geq t$. Then, as in Duffie (1988) Section 22, the partial F_x is a well-defined continuous function and $\theta_t = F_x(S_t, t)$ satisfies $E\left[g(X_T)\right] + \int_0^T \theta_t \, dS_t = g(S_T)$ a.s. For the trading strategies $\theta_t^n = f[S(T_k^n), T_k^n]$, $t \in (T_k^n, T_{k+1}^n]$, the result then follows as in the proof of Proposition 1. ▮

Of course, one can extend the result much further.

CASE (*b*) (CUMULATIVE RETURNS THAT ARE APPROXIMATELY BROWNIAN MOTION).

The cumulative return process X corresponding to the price process S of (1) is the Brownian Motion X defined by

$$X_t = \mu t + \sigma B_t. \tag{4}$$

That is, $S = S_0 \mathcal{E}(X)$, where $\mathcal{E}(X)$ is the stochastic exponential of X as defined in Section 2. We now consider a sequence of cumulative return processes $\{X^n\}$ with $X^n \Longrightarrow X$.

Example 1. (Binomial Returns)

A classical example is the coin–toss walk "with drift" used by Cox, Ross, and Rubinstein (1979). That is, let

$$X_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} Y_k^n, \tag{5}$$

where, for each n, $\{Y_k^n\}$ is a sequence of independent and identically distributed binomial trials with $\sqrt{n}\,E[Y_1^n] \to \mu$ and $\sqrt{n}\,\mathrm{var}(Y_1^n) \to \sigma^2$. It is easy to show that $X^n \Longrightarrow X$. (See, for example, Duffie (1988), Section 22.)

Let us show that the assumptions of Theorem 3 (for example) are satisfied in this case. For any number t, recall that $[t]$ denotes the largest integer less than or equal to t. Since the $(Y_k^n)_{k \geq 1}$ are independent and have finite means, we know that

$$M_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} [Y_k^n - E\,(Y_k^n)]$$

is a martingale, and thus a decomposition of X^n is:

$$X_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} [Y_k^n - E(Y_1^n)] + \frac{1}{\sqrt{n}} [nt] E\,(Y_1^n) = M^n + A^n.$$

Clearly the jumps of M^n are bounded. As for (JC), we have

$$\lim_{b \to \infty} \sup_n P\,(\mid A^n \mid_t > b) \cong \lim_{b \to \infty} \sup_n P\left(\frac{1}{\sqrt{n}} n \mid E(Y_1^n) \mid > b\right),$$

and since $\lim_{n \to \infty} \sqrt{n} \mid E(Y_1^n) \mid = \mid \mu \mid < \infty$, (JC) is satisfied.

Therefore, if $S^n = S_0^n \mathcal{E}(X^n)$ and $S_0^n \to S_0$, then $S^n \Longrightarrow S$ by the Corollaries of Theorems 5 and 6, or by direct calculation (and an application of Billingsley (1968), Theorem 5.1, if $S^n \neq S_0$). This ends Example 1.

We consider the discrete–time stock–trading strategy $\theta^n \in \Theta^n$ defined by

$$\theta_t^n = C_x\,[S^n(T_k^n), T_k^n], \qquad t \in (T_k^n, T_{k+1}^n],$$

where $S^n = S_0^n \mathcal{E}(X^n)$. To show that Black–Scholes applies in the limit, we must show that $C(S_0^n, 0) + \int_0^T \theta_t^n \, dS_t^n \Longrightarrow (S_T - K)^+$. [The self–financing bond trading strategy α^n is defined by the obvious analogue to (3), and the initial investment is the Black-Scholes value of the option, $C(S_0^n, T)$.] It is implicit in the following statement that all processes are defined on the same probability space unless the stopping times $\{T_k^n\}$ are deterministic.

PROPOSITION 3. *Suppose $S_0^n \to S_0 > 0$, $(X^n)_{n \geq 1}$ satisfies the hypotheses of Theorem 5, and $X^n \Longrightarrow X$, where X is the Black–Scholes cumulative return process (4). Then $S^n = \mathcal{E}(X^n)S_0^n \Longrightarrow \mathcal{E}(X)S_0 = S$. If $\{X^n\}$ satisfies the conditions of Theorem 1, then $C(S_0^n, 0) + \int_0^T \theta_t^n \, dS_t^n \Longrightarrow (S_T - K)^+$.*

PROOF: To apply Corollary 2 of Lemma 2, we need only show that $S^n \Longrightarrow S$ and that S^n satisfies the conditions of Theorem 1. This is true by the Corollary to Theorem 6. Since $C(S_0, 0) + \int_0^T \theta_t \, dS_t = (S_T - K)^+$ a.s., we are done. ∎

What examples, in addition to the coin–toss random walks $\{X^n\}$ satisfy the hypotheses of Proposition 3?

Example 2. (iid Returns). Suppose $\{X^n\}$ is a sequence of stock return processes defined by (5), where:

(i) $\{Y_k^n\}$ are uniformly bounded,

(ii) for each n, $\{Y_k^n\}$ is i.i.d.,

(iii) $\sqrt{n}\, E[Y_k^n] \to_n \mu$, and

(iv) $\sqrt{n}\, \mathrm{var}(Y_k^n) \to_n \sigma^2$.

Then, using Lindeberg's Central Limit in the proof of Donsker's Theorem, we have $X^n \Longrightarrow X$, where X is given by (4). Furthermore, $\{X^n\}$ satisfies the hypotheses of Theorem 6. Thus, the hypotheses of Proposition 3 are satisfied. This ends Example 2.

Example 3. (Mixing returns). Let the sequence $\{X^n\}$ of cumulative return processes be defined by (5), where the following conditions apply:

(i) $\{Y_k^n\}$ are uniformly bounded, $\mathbb{R}$–valued, and stationary in k (for each n).

(ii) For $\mathcal{F}_m^n = \sigma\{Y_k^n; k < m\}$, $\mathcal{G}_n^m = \sigma\{Y_k^n; k \geq m\}$, and $\varphi_p^n(m) = \varphi_p^n(\mathcal{G}_n^{m+\ell} \mid \mathcal{F}_\ell^n)$, where

$$\varphi_p^n(\mathcal{A} \mid \mathcal{B}) = \sup_{A \in \mathcal{A}} \|P_n(A \mid \mathcal{B}) - P_n(A)\|_{L^p},$$

$C_n = \sum_{m=1}^\infty [\varphi_p^n(m)]^\alpha < \infty$, each n, where $p = \frac{2+\delta}{1+\delta}$, $\alpha = \frac{\delta}{1+\delta}$, for some $\delta > 0$.

(iii) $\sqrt{n} E_n(Y_k^n) \to \mu$ and, for $U_k^n = Y_k^n - E_n[Y_k^n]$, $\sup_n \sqrt{n} C_n \|U_1^n\|_{L^{2+\delta}} < \infty$.

(iv) $\sigma_n^2 = E_n[(U_1^n)^2] + 2\sum_{k=2}^\infty E_n(U_1^n U_k^n)$ is well-defined and $\sigma_n^2 \to_n \sigma^2$.

Under (i)–(iv), for X^n defined by (5), we have $X^n \Longrightarrow X$. [See Ethier–Kurtz (1986), pp. 350–353, for calculations not given here.]

We wish to invoke the Corollaries to Theorems 5 and 6 and thus Proposition 3. To verify the jump condition and (JC) we need to find suitable semimartingale decompositions of X^n.

To this end, following Ethier–Kurtz (1986) (p. 350 ff), define:

$$M_\ell^n = \sum_{k=1}^{\ell} U_k^n + \sum_{m=1}^{\infty} E_n\left(U_{\ell+m}^n \mid \mathcal{F}_\ell^n\right).$$

The series on the right is convergent as a consequence of the mixing hypotheses (see Ethier-Kurtz (1986), p. 351), and M_ℓ^n is a martingale with bounded jumps with respect to the filtration $(\mathcal{F}_\ell^n)_{\ell\geq 1}$. We have

$$X_t^n = \frac{1}{\sqrt{n}} M_{[nt]}^n + A_t^n ,$$

where

$$A_t^n = \frac{1}{\sqrt{n}} V_{[nt]}^n + \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} E_n(Y_k^n),$$

with

$$V_\ell^n = -\sum_{m=1}^{\infty} E_n(U_{\ell+m}^n \mid \mathcal{F}_\ell^n).$$

Note that the total variation of the paths of the process V^n are majorized in that

$$E_n(\mid V^n \mid_\ell) \leq E_n\left(\sum_{k=1}^{\ell} \left| \sum_{m=1}^{\infty} E_n(U_{k+m}^n \mid \mathcal{F}_k^n) \right|\right),$$

and using a standard estimate [Ethier–Kurtz (1986), p. 351], this expression is

$$\leq \sum_{k=1}^{\ell} \left(\sum_{m=1}^{\infty} \delta \varphi_p^{\delta/1+\delta}(m)\right) \|U_1^n\|_{L^{2+\delta}} \leq \ell \delta C_n \|U_1^n\|_{L^{2+\delta}}.$$

Then

$$\begin{aligned}\lim_{b\to\infty} \sup_n P_n\left(\frac{1}{\sqrt{n}} \mid V^n \mid_{[nt]} > b\right) &\leq \lim_{b\to\infty} \sup_n \frac{1}{b\sqrt{n}} E_n\left(\mid V^n \mid_{[nt]}\right) \\ &\leq \lim_{b\to\infty} \sup_n \frac{16}{b} \sqrt{n}\, t\, C_n \|U_1^n\|_{L^{2+\delta}} = 0,\end{aligned}$$

by hypothesis (iii). Thus $\left(V_{[nt]}^n/\sqrt{n}\right)$ satisfies (JC). Since $\sqrt{n}E_n(Y_k^n) \to_n \mu$, (A^n) satisfies (JC), and the hypotheses of Proposition 3 are met. This ends Example 3.

Appendix

The following Lemma allows one to work with the canonical decompositions of special semimartingales for purposes of checking (JC).

LEMMA A1. *Suppose (A^n) satisfies (JC) and A^n is of integrable variation for all n. If $\widetilde{A}^n$ is the predictable compensator of A^n, then $(\widetilde{A}^n)$ satisfies (JC).*

PROOF: Let

$$H_t^n = \frac{d\widetilde{A}_t^n}{d \mid \widetilde{A}^n \mid_t},$$

where $\mid \widetilde{A}^n \mid_t = \int_0^n \mid d\widetilde{A}_s^n \mid$ denotes the total variation of the process of the paths of $\widetilde{A}^n$. Then H^n is predictable and, for any stopping time T,

$$E\left[\int_0^T H_t^n \, dA_t^n\right] = E\left[\int_0^T H_t^n \, d\widetilde{A}_t^n\right] = E\left[\mid \widetilde{A}^n \mid_T\right].$$

Since $\mid H_t^n \mid = 1$,

$$E\left[\mid A_T^n \mid\right] \geq E\left[\int_0^T H_t^n \, dA_t^n\right] = E\left[\mid \widetilde{A}^n \mid_T\right].$$

By the Lenglart Domination Theorem [Jacod and Shiryaev (1987), Lemma 3.30 (b), page 35, with $\epsilon = b$ and $\eta = \sqrt{b}$],

$$\lim_{b\to\infty} \sup_n P_n\left(\mid \widetilde{A}^n \mid_T \geq b\right) \leq \lim_{b\to\infty} \sup_n \left\{\frac{1}{b}\left[\sqrt{b} + E_n\left(\sup_{t\leq T} \mid \Delta A_t^n \mid\right)\right] + P_n\left(\mid A_T^n \mid \geq \sqrt{b}\right)\right\} = 0.$$

Since (A^n) satisfies (JC), it follows that $\widetilde{A}^n$ satisfies (JC). ▮

COROLLARY. *Suppose (A^n) satisfies (JC) and A^n is locally of integrable variation for all n. If $\widetilde{A}^n$ is the predictable compensator for A^n, then $(\widetilde{A}^n)$ satisfies (JC).*

PROOF: For given b, let $T^n = \inf\{t \geq 0 : |A^n|_t \geq b\}$. The stopped process $(A^n)^{T^n_-}$ is of bounded total variation. For given $t_0 > 0$ and $\epsilon > 0$, there exists b large enough that

$$\sup_n P_n(T^n \leq t_0) \leq \sup_n P_n\left(\mid A^n \mid_{t_0} \geq b\right) < \epsilon, \tag{6}$$

since (A^n) satisfies (JC). It follows that

$$\begin{aligned}\lim_{b\to\infty} \sup_n P_n\left(\mid \widetilde{A}^n \mid_{t_0} \geq b\right) &\leq \lim_{b\to\infty} \sup_n \left\{P_n\left(\left\{\mid \widetilde{A}^n \mid_{t_0} \geq b\right\} \bigcap \{T^n > t_0\}\right) + P_n(T^n \leq t_0)\right\} \\ &\leq \lim_{b\to\infty} \sup_n \left\{P_n\left(\left|(\widetilde{A}^n)^{T^n_-}\right|_{t_0} \geq b\right) + P_n(T^n \leq t_0)\right\} = 0,\end{aligned}$$

by the lemma and (6). Thus $(\widetilde{A}^n)$ satisfies (JC). ▮

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