THE LINEAR BIRTH AND DEATH PROCESS UNDER THE INFLUENCE OF INDEPENDENTLY OCCURRING DISASTERS

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Abstract

A population developing according to a time homogeneous linear birth and death process is subjected to an independently occurring random sequence of disasters. Using an embedded Galton-Watson process with random environments explicit results about the probability of extinction and the asymptotic behavior of the process are obtained.

Key words: Birth and death process, disasters, random environments, integrals of branching processes.

1. Introduction

This paper may be viewed as a sequel to that by Bartoszyński et al (1989) considering the following model. A population whose rule of development is that of a linear time homogeneous birth and death process with intensities λ and μ is subjected to a sequence of disasters occurring at random times $\tau_1, \tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3, \ldots$. Each disaster affects all members of the population independently, killing an individual with probability $\varepsilon = 1 - \delta$. Thus our model constitutes a special case of a branching process with disasters as originally introduced by Kaplan et al (1975) and further studied by Athreya and Kaplan (1976); - see also Sankanarayanan and Krishnamoorthy (1978). Recently Altenburg (1986, 1987) has studied a discrete time version. For more recent work concentrating on the case where the risk of catastrophes depends on the present population size see e.g. Brockwell (1985), Pakes (1987, 1988) and the references given there.

By making use of our detailed knowledge of the birth and death process it is possible to obtain more explicit results than appears possible for the general case. Information about the general structure of the results can already be extracted from the above mentioned articles. We derive our results by a systematic use of an embedded Galton-Watson process with random environments (GWRE process), a technique also implicitely used by Kaplan et al (1975), Athreya and Kaplan (1976); - see also Neuts (1968).

The probability generating function of this embedded GWRE process describing the state of the population immediately after the n^{th} disaster $(n = 0, 1, 2, \ldots)$ can be obtained explicitly (conditionally on the sequence of disaster times). Its limit as $n \to \infty$ is almost surely a random variable Q (sometimes degenerate at 1) independent of the dummy variable

s. This shows that our process shares with other branching and related processes the property that almost surely only extinction or explosion is possible. Also the explicit representation of Q as a function of the sequence $\underset{\sim}{\tau} = (\tau_1, \tau_2, \ldots)$ gives some insight into the distribution of Q. In this connection we show that the additional risk of extinction introduced into the process by the disasters is for small ε proportional to the severeness ε of the disasters.

Furthermore we investigate the analogues of the classical limit theorems for branching processes and the behaviour of the integral of processes with disasters which may be relevant for the study of interacting populations subjected to disasters (see Bartoszyński et al 1987b).

2. The embedded GWRE process

One aspect complicating the study of processes with disasters is the fact that, as the progeny of different individuals are subjected to the same disasters, the independence property of the underlying branching process is lost. Consider, however, the process with disasters $\{Z(t), t \geq 0\}$ at the times $\tau_1, \tau_1 + \tau_2, \ldots$, that is let $\tilde{Z}_n = Z(\tau_1 + \tau_2 + \ldots + \tau_n)$ be the population size immediately after the n^{th} disaster. Then the independence structure for the progeny of different individuals is restored. Instead the law of reproduction changes randomly from one generation to the next. Thus, in fact we have a GWRE process whose random environments are provided by the random variables $\tau_n, n = 1, 2, \ldots$. This embedded GWRE process has at least implicitly already been used in the first papers on the subject. Its analogue also plays an important role in the Galton-Watson process with disasters as studied by Altenburg (1986, 1987). It has however not been fully exploited e.g. in Bartoszyński et al (1989) in their study of the extinction probability.

Consider the sequence $\underset{\sim}{\tau}$ of environments as given and let $H_n(s, \underset{\sim}{\tau})$ be the p.g.f. of $\tilde{Z}_n$. Then, given $\tilde{Z}_1$ the process $Z_k^* = \tilde{Z}_{k+1}$ develops just like the process $\{Z_n, n \geq 0\}$ except that its environments are given by $T\underset{\sim}{\tau} = (\tau_2, \tau_3, \ldots)$ and that it starts with $Z_0^* = \tilde{Z}_1$. Thus $E\left(s^{\tilde{Z}_{n+1}} | \tilde{Z}_1\right) = \left[H_n\left(s, T\underset{\sim}{\tau}\right)\right]^{\tilde{Z}_1}$ which leads to the recursion

$$H_{n+1}(s, \underset{\sim}{\tau}) = H_1[H_n(s, T\underset{\sim}{\tau}), \underset{\sim}{\tau}]. \tag{2.1}$$

Here $H_1(s, \underset{\sim}{\tau}) = Es^{Z(\tau_1)} = F_0(\delta s + \varepsilon, \tau_1)$ where F_0 is the p.g.f. of the process undisturbed by disasters. For the linear birth and death process with time independent intensities λ and μ this p.g.f. is explicitly known (see e.g. Goel & Richter-Dyn, 1974) and has a simple form, namely

$$F_0(s, t) = \frac{\left(\mu - \mu e^{(\lambda-\mu)t}\right) - \left(\lambda - \mu e^{(\lambda-\mu)t}\right) s}{\left(\mu - \lambda e^{(\lambda-\mu)t}\right) - \left(\lambda - \lambda e^{(\lambda-\mu)t}\right) s} \, . \tag{2.2}$$

Replacing s by $\delta s + \varepsilon$ in (2.2) after some manipulation brings H into the form

$$\begin{aligned} H(s,t) &= \frac{(\mu - \delta\lambda y(t))s + ((\mu - \lambda\varepsilon)y(t) - \mu)}{(\lambda - \delta\lambda y(t))s + ((\mu - \lambda\varepsilon)y(t) - \lambda)} \\ &=: \frac{A(t)s + B(t)}{C(t)s + D(t)} \, , \end{aligned} \tag{2.3}$$

where $y(t) = \delta^{-1} \exp(-(\lambda - \mu)t)$. Composition of fractional linear functions is most easily done by multiplying the corresponding matrices of coefficients. From the recursion (2.1) we thus see that $H_n(s, \underset{\sim}{\tau})$ corresponds to the product of matrices

$$\begin{pmatrix} A(\tau_1) & B(\tau_1) \\ C(\tau_1) & D(\tau_1) \end{pmatrix} \begin{pmatrix} A(\tau_2) & B(\tau_2) \\ C(\tau_2) & D(\tau_2) \end{pmatrix} \cdots \begin{pmatrix} A(\tau_n) & B(\tau_n) \\ C(\tau_n) & D(\tau_n) \end{pmatrix} . \tag{2.4}$$

It is an elementary but somewhat laborious exercise to carry out these matrix multiplications. If we introduce the notations

$$Y_j = \delta^{-1} \exp(-(\lambda - \mu)\tau_j), U_n = \prod_{j=1}^{n} Y_j \text{ and } X_n = \sum_{i=1}^{n} U_i,$$

then the resulting p.g.f. is given as

$$H_n(s, \underset{\sim}{\tau}) = \frac{(\mu + \lambda\varepsilon X_n)(1-s) + (\lambda s - \mu)U_n}{(\lambda + \lambda\varepsilon X_n)(1-s) + (\lambda s - \mu)U_n} \,. \tag{2.5}$$

This explicit representation of the (conditional) p.g.f. of $\tilde{Z}_n$ (given the disaster times) is basic for most of the sequel. We may note that it does not require any assumptions about the joint distribution of the interarrival times τ_j of the disasters.

Versions of the following lemma have been used in the theory of GWRE processes.

Lemma 2.1: *If the sequence $\{\tau_n, n = 1, 2, \ldots\}$ is stationary and ergodic with $1/\beta := E\tau$ then as $n \to \infty$*

a) $-\log\delta - 1/\alpha < 0 \Rightarrow U_n \to 0,$

$-\log\delta - 1/\alpha > 0 \Rightarrow U_n \to \infty,$

$-\log\delta - 1/\alpha = 0 \Rightarrow -\infty = \liminf_{n\to\infty} U_n < \limsup_{n\to\infty} U_n = +\infty,$

where as usual we may call these three cases supercritical, subcritical and critical, respectively.

b) $-\log\delta - 1/\alpha \geq 0 \Rightarrow \sum_{j=1}^{\infty} \prod_{i=1}^{j} Y_i = \lim_{n\to\infty} X_n = \infty,$

$-\log\delta - 1/\alpha < 0 \Rightarrow \lim_{n\to\infty} X_n = X_\delta,$

where X_δ is almost surely positive and finite.

The idea of the proof is to study the limit behaviour of $\log U_n = \sum_{j=1}^{n} \log Y_j$ instead of that of U_n (see also Puri 1987). Here $E\log Y = -\log\delta - 1/\alpha$ with $\alpha = \beta/(\lambda - \mu)$ such that the first two statements in a) follow immediately from the strong law of large numbers. For the last statement one has to consider the fluctuations of sums of variables around their expectation. Actually in the cases $E\log Y < 0$ the strong law shows even that the product of the first n variables Y_j tends to zero at a geometric rate which makes the sum of these products convergent. We close this section by applying Lemma 2.1 to obtain the asymptotic behavior of H_n.

Theorem 2.1: *If the sequence* $\underset{\sim}{\tau}$ *is stationary and ergodic, then almost surely*

$$\lim_{n\to\infty} H_n(s,\underset{\sim}{\tau}) = \begin{cases} 1 & \text{if } -\log\delta - 1/\alpha \geq 0 \\ \frac{\mu+\lambda\varepsilon X_\delta}{\lambda+\lambda\varepsilon X_\delta} & \text{if } -\log\delta - 1/\alpha < 0 \end{cases} . \tag{2.6}$$

Corollary 2.1: *The process* $\{\tilde{Z}_n, n = 0,1,2,\ldots\}$, *and therefore the process* $\{Z(t), t \geq 0\}$, *shares the "extinction or explosion" property common to branching process models, i.e.*

$$P(\tilde{Z}_n \longrightarrow 0 \text{ or } \tilde{Z}_n \longrightarrow \infty) = P(Z_t \longrightarrow 0 \text{ or } Z_t \longrightarrow \infty) = 1 .$$

This is seen by noting that the limit of H_n does not depend on s. Since Corollary 2.1 holds for almost all realizations of the "environmental" sequence $\underset{\sim}{\tau}$ it is also true as an unconditional statement and we have formulated it in that way.

3. The probability of extinction

Bartoszyński et al (1989) have used the renewal property expressed by (2.1) for $s = 0$ to obtain information about the (random) probability Q of extinction of the process. By further exploiting the embedded GWRE process we can display additional structure. Actually Theorem 2.1 gives Q in terms of the sequence of disaster times as

$$Q = \frac{\mu + \lambda\varepsilon X_\delta}{\lambda + \lambda\varepsilon X_\delta} \text{ with } X_\delta = \sum_{n=1}^{\infty} \prod_{j=1}^{n} Y_j \ , \tag{3.1}$$

where $Y_j = \delta^{-1}\exp(-(\lambda - \mu)\tau_j)$, and where $X_\delta < \infty$ if and only if $-\log\delta - 1/\alpha < 0$.

The study of the random variable Q and that of X_δ are obviously equivalent activities. We chose to first investigate the behaviour of X_δ. A similar series arising in a quite different context has been studied recently by Todorovic and Gani (1987) by a similar approach.

The key step is to assume the sequence $\{Y_1, Y_2, \ldots\}$ to be i.i.d. and to see that then

$$X_\delta = \sum_{n=1}^{\infty} \prod_{i=1}^{n} Y_i = Y_1 \left(1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} Y_{i+1}\right) = Y_1 \left(1 + \tilde{X}_\delta\right) ,$$

where X_δ and $\tilde{X}_\delta$ have the same distribution and the factors in the last term are independent.

If we assume not only that the interarrival times of the disasters are i.i.d. but that they even are given by a Poisson process with parameter β, then we get a result that is explicit enough at least for numerical purposes.

Theorem 3.1: *If $\underset{\sim}{\tau}$ is the sequence of interarrival times of a time homogeneous Poisson process with intensity β and X_δ is given by (3.1), then the density h and the distribution function H of X_δ satisfy for $x \neq 0$*

$$h(x) = \frac{\alpha}{x}\{H(x) - H(\delta x - 1)\} , \tag{3.2}$$

with $\alpha = \beta/(\lambda - \mu)$. Note that $H(0) = 0$ as per the definition of X_δ.

Proof: This can be shown by direct calculation from $H(x) = P(Y \cdot (1 + X_\delta) \leq x) = EP(\tilde{X}_\delta \leq (x/Y) - 1|Y)$. We prefer to use the Laplace transform $L(\theta) = Ee^{-\theta X_\delta}$. Note first that in our case Y has the density $f(y) = \alpha\delta(\delta y)^{\alpha-1} 1_{[0,1/\delta]}(y)$ such that

$$L(\theta) = \int_0^\infty E(\exp(-\theta(X+1)Y)|Y = y)f(y)dy = \int_0^{1/\delta} \exp(-\theta y)L(\theta y)f(y)dy$$

$$= (1/\theta) \int_0^{\theta/\delta} \exp(-u)L(u)f(u/\theta)du = \alpha(\delta/\theta)^\alpha \int_0^{\theta/\delta} e^{-u}u^{\alpha-1}L(u)du,$$

and therefore

$$L'(\theta) = -(\alpha/\theta)L(\theta) + \alpha(\delta/\theta)\exp(-\theta/\delta)L(\theta/\delta) = (\alpha/\theta)\{\delta \exp(-\theta/\delta)L(\theta/\delta) - L(\theta)\}.$$

This equation for the Laplace transform is inverted into $-xh(x) = \alpha P((X_\delta + 1)/\delta \leq x) - \alpha H(x) = \alpha\{H(x\delta - 1) - H(x)\}$, which is the desired relation (3.2).

Theorem 3.1 gives us two possibilities to find the distribution function $G(z)$ of $Q = (\mu + \lambda\varepsilon X_\delta)/(\lambda + \lambda\varepsilon X_\delta)$. The first is to solve (3.2) for H and then determine $G(z) = H[(\lambda z - \mu)/\lambda\varepsilon(1-z)]$. The second way is to substitute the inverse of this relation and the corresponding transformation of densities into (3.2) to obtain an equation connecting G with its density g. The results of these two approaches are stated as corollaries; the details of the calculations are left to the reader.

Corollary 3.1: *Under the assumptions of Theorem 3.1, $G(z) = H[(\lambda z - \mu)/\lambda\varepsilon(1 - z)]$ where H is obtained from (3.2) as*

$$H(x) = cx^\alpha, \quad \textit{for} \quad 0 \leq x \leq 1/\delta\ , \tag{3.3a}$$

$$H(x) = x^\alpha \left\{ H(x_n)x_n^{-\alpha} - \alpha \int_{x_n}^{x} u^{-(1+\delta)} H(\delta u - 1)du \right\}, \quad \text{for} \quad x_n \leq x \leq x_{n+1}. \tag{3.3b}$$

Here the sequence $x_1 < x_2 < \ldots$ has to be chosen in such a manner that $x_n \to \infty$ and that for each n, if $x_n \leq u \leq x_{n+1}$, the value $H(\delta\mu - 1)$ must already have been determined in a previous step. It seems natural to use the maximal values that are possible, i.e. to determine the sequence by $x_1 = 1/\delta, x_n = \delta x_{n+1} - 1$, leading to $x_n = 1/\delta + 1/\delta^2 + \ldots + 1/\delta^n$. Unfortuantely Corollary 3.1 contains the constant c in (3.3a) which can only be determined as a normalizing constant after the integration has been carried out all the way to infinity.

Corollary 3.2: *Under the assumptions of Theorem 3.1 the density g and the distribution function G of the random probability of extinction satisfy*

$$g(z) = \frac{\beta}{(1-z)(\lambda z - \mu)}\{G(z) - G((z-\varepsilon)/(1-\varepsilon))\}\ , \tag{3.4}$$

with $G(0) = 0$.

Remarks:

1) The relation (3.4) was obtained by Bartoszyński et al (1989) without the use of the explicit almost sure structure of Q as given by (3.1). They noted that (3.4) can be integrated in a manner quite analogous as that described in connection with (3.3) above.

2) Remembering the definition of X_δ in (3.1) we see that as a function of δ it is nonincreasing. Thus for $\delta \to 1$, or equivalently $\varepsilon \to 0$, it converges to $X^* = \sum\limits_{n=1}^{\infty} \prod\limits_{j=1}^{n} \exp(-(\lambda - \mu)\tau_j)$. Further we observe that therefore $(Q - \mu/\lambda)/\varepsilon = (\lambda - \mu)X_\delta/(\lambda + \lambda\varepsilon X_\delta) \to (1 - \mu/\lambda)X^*$. Thus for small ε we may consider εX^* as the additional risk of extinction introduced into the process by the disasters. Note that (3.3) remains valid for $\delta = 1$ and even simplifies. Furthermore in this case the equation for L' in the proof of Theorem 3.1 becomes an ordinary differential equation, namely

$$L'(\theta) = (\alpha/\theta)\{\exp(-\theta) - 1\}L(\theta) \ , \tag{3.5}$$

which we can solve to yield

$$L(\theta) = \exp\left\{-\alpha \int_0^\theta \left[\left(1 - e^{-u}\right)/u\right]\right\} du \ . \tag{3.6}$$

This relation can be used to evaluate the moments of X^*. Expectation and variance may be of interest. They are

$$EX^* = \alpha \quad , \quad V(X^*) = \alpha/2 \ . \tag{3.7}$$

Going back to the relation $X_\delta = Y_1(1 + \tilde{X}_\delta)$ we can calculate the expectation of X_δ for $\delta > 0$ from $EX_\delta = EY_1(1 + EX_\delta)$. It is easily seen that $EY_1 = \alpha/[(1 + \alpha)\delta]$.

As a consequence $EX_\delta = \infty$ if $\alpha/(1+\alpha) \leq \delta$ and $EX_\delta = (\alpha/\delta)/(1-\alpha+\alpha/\delta)$ if $\alpha/(1+\alpha) > \delta$. For $\delta \to 1$ the first of the relations (3.7) is reproduced. In a similar manner higher moments of X_δ can be obtained. The conditions on the parameters α and δ for these moments to be finite become successively more restrictive.

3) Surprisingly there is no "rate of approach" of Q to 1 when the process approaches criticality, i.e. when $\delta \to \exp(-1/\alpha)$ from above. In fact for any bounded B_δ

$$B_\delta/(1-Q) = (\lambda + \lambda\varepsilon X_\delta)B_\delta/(\lambda-\mu) \sim X_\delta B_\delta \lambda(1-\exp(-1/\alpha))/(\lambda-\mu)$$

as $\delta \to \exp(-1/\alpha)$. Here, again using the monotonicity of X_δ as a function of δ, we know that $X_\delta \to \infty$, a.s.. Assume this could be counter-balanced by $B_\delta \to 0$, such that $B_\delta X_\delta \to U$. Then

$$U \longleftarrow B_\delta X_\delta = B_\delta Y_1(1+\tilde{X}_\delta) = B_\delta Y_1 + B_\delta Y_1 \tilde{X}_\delta \longrightarrow 0 + Y_1 U,$$

where the latter convergence holds in distribution. Equality of the distributions of U and of $Y_1 U$ is only possible when U is concentrated in (at most) the two points 0 and ∞.

4. Limit theorems

In this section we study the analogues of the classical branching process theorems in the present model. In the setup of GWRE processes these limit theorems have been given by Athreya and Karlin (1972b), see also Kaplan (1972) for limitations to their validity in full generality. At least for the noncritical cases our special assumptions about the environmental process lead to much simpler and more explicit formulations of these results so that it seems justified to present them here. Let us first turn to the supercritical case.

Theorem 4.1: *Let $\{Z(t), t \geq 0\}$ be a time homogeneous linear birth and death process with intensities λ and μ subjected to disasters of strength $\varepsilon = 1 - \delta$ arriving according to a Poisson process $N(t)$ with intensity β. Let $\tilde{Z}_n = Z(\tau_1 + \tau_2 + \ldots + \tau_n)$. Then, if $-\log \delta - 1/\alpha < 0$,*

a) *$W_t := \delta^{-N(t)} e^{-(\lambda-\mu)t} Z(t)$ converges almost surely to a nondegenerate limit W.*

b) *$\tilde{W}_n := \delta^{-n} e^{-(\lambda-\mu)(\tau_1+\ldots+\tau_n)} \tilde{Z}_n$ converges almost surely to a nondegenerate limit $\tilde{W}$.*

c) *$W = \tilde{W}$ almost surely.*

d) *The conditional distribution of $\tilde{W}$, given the sequence $\underset{\sim}{\tau}$, has point mass Q at the origin and the remaining mass distributed exponentially with parameter $(1 - \mu/\lambda)/(\mu/\lambda + \varepsilon X_\delta)$.*

Proof: The proof of a) is first given for the general case of branching processes with disasters by Kaplan et al (1975). It uses the fact that $\{W_t, t \geq 0\}$ is a martingale. We may note in this connection that the "usual" martingale $Z(t)/EZ(t)$ cannot converge to a nondegenerate limit because even after long time disasters would force it to jump again away from such a limit. For b) we first work under the condition of $\underset{\sim}{\tau}$ being given. Then, evaluating the appropriate conditional expectations we see that $\tilde{W}_n$ is a martingale conditionally. Being nonnegative it has to converge a.s.. The nondegeneracy of $\tilde{W}$ can then be taken from either c) with a) or from d). To prove c) note that for all n by definition $N(\tau_1 + \tau_2 + \ldots + \tau_n) = n$ and $Z(\tau_1 + \ldots + \tau_n) = \tilde{Z}_n$. This embeds $\tilde{W}_n = W_{\tau_1+\tau_2+\ldots+\tau_n}$ into the process $\{W_t, t \geq 0\}$. As $\tau_1 + \tau_2 + \ldots + \tau_n \to \infty$ the limits of W_t and $\tilde{W}_n$ must be the same. The limit distribution is obtained from rewriting (2.5) as

$$H_n(s, \underset{\sim}{\tau}) = \frac{(\lambda s - \mu) + (1-s)U_n^{-1}(\mu + \lambda \varepsilon X_n)}{(\lambda s - \mu) + (1-s)U_n^{-1}(\lambda + \lambda \varepsilon X_n)} \,. \tag{4.1}$$

Replacing s by $\exp(-U_n v)$ in (4.1) shows after an elementary calculation that

$$Ee^{-v\tilde{W}_n} = H_n(\exp(-U_n v), \underset{\sim}{\tau}) \longrightarrow \frac{(\lambda - \mu) - v(\mu + \lambda \varepsilon X_\delta)}{(\lambda - \mu) - v(\lambda + \lambda \varepsilon X_\delta)} \,, \tag{4.2}$$

from which the assertions of part d) of the theorem can be easily extracted.

Theorem 4.2: *Consider the same situation as in the previous Theorem except that now* $-\log \delta - 1/\alpha > 0$ *(subcritical case). Then the conditional distribution of* $\tilde{Z}_n$ *given* $\tilde{Z}_n \neq 0$ *converges to the geometric distribution with p.g.f.* $(\lambda - \mu)s/(\lambda s - \mu)$ *for almost all realizations of* $\underset{\sim}{\tau}$.

The proof consists in explicitly calculating the conditional p.g.f. $(H_n(s, \underset{\sim}{\tau}) - H_n(0, \underset{\sim}{\tau}))/(1 - H_n(0, \underset{\sim}{\tau}))$ and letting $n \to \infty$.

Surprisingly the result does not depend on intensity and severity of the disasters except for the fact that disasters must be sufficiently frequent and strong to turn the process subcritical.

Turning to the critical case we try to use the information given in Athreya and Karlin (1971) for the critical GWRE process. "To avoid unimportant technical details" they work under boundedness conditions of the first three moments of the (random) progeny distribution. In our case these conditions are satisfied if and only if the interarrival times $\tau_1, \tau_2, \ldots$ are bounded which excludes the Poisson process case. Even under this possibly unnecessarily restrictive condition we have so far not succeeded to verify the condition of Athreya and Karlin (1971, p. 1857). Thus we formulate the expected result only as a conjecture.

Conjecture 4.3: *In the critical case $E \log Y = 0$, the conditional distribution of population size $\tilde{Z}_n$ after the n^{th} disaster given nonextinction up to this time properly normalized converges to an exponential distribution.*

With some additional effort it should be possible to also show that the corresponding result holds for the (corresponding) conditional distribution of Z_t itself.

5. The integral of the process

In this section we study the process

$$I(t) = \int_0^t Z(s)ds, \quad t \geq 0. \tag{5.1}$$

Almost surely the realization of $\{Z(s), s \geq 0\}$ is a pure step function and thus its integral is well defined. For the case without disasters its properties have been studied by e.g. Puri (1966). We restrict our attention to the asymptotic behaviour.

In the critical and subcritical cases almost surely $Z(s)$ tends to zero and thus $I(t)$ converges almost surely to a finite random variable I_∞.

For the supercritical case we use the following version of de l'Hôpital's rule.

Lemma 5.1: *Let f and g be two nonnegative functions such that $\int_0^t f(s)ds$ and $\int_0^t g(s)ds$ exist and that*

$$0 < \lim_{s \to \infty} (f(s)/g(s)) = c < \infty.$$

Then $\int_0^\infty f(s)ds = \infty \Leftrightarrow \int_0^\infty g(s)ds = \infty$ and in this case

$$\lim_{t \to \infty} \frac{\int_0^t f(s)ds}{\int_0^t g(s)ds} = c\,.$$

There are two ways of combining this lemma with Theorem 4.1 both of which may be of some interest.

Theorem 5.1: *Under the conditions of Theorem 4.1 we have on* $[W > 0]$

a) $(\lambda - \mu)\int_0^t \delta^{-N(s)} Z(s)ds \sim We^{(\lambda-\mu)t} \qquad (t \to \infty),$

b) $I(t) = \int_0^t Z(s)ds \sim W\int_0^t \delta^{N(s)} e^{(\lambda-\mu)s} ds \qquad (t \to \infty).$

Proof: The two parts correspond to two different ways of writing

$$W_s = \delta^{-N(s)} e^{-(\lambda-\mu)s} Z(s)$$

in the form $f(s)/g(s)$ and applying Lemma 5.1.

Remark: The third decomposition, namely $W_s = (Z(s)e^{-(\lambda-\mu)s})/(\delta^{N(s)})$, cannot be used in a similar way since here numerator and denominator individually converge to zero fast enough to have finite integrals.

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