A LARGE DEVIATION RESULT FOR PROCESSES

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Abstract

A large deviation result is proved for processes. The result is applicable when regeneration points exist for the sequence. An example of such an application is given.

Key words: Large deviations, break points.

1. Introduction

In many stochastic processes one can find regeneration points, so that the changes in state and intervals of time between these regeneration points comprise an i.i.d. sequence of random vectors $\{(X_i, Y_i)\}$. If this is the case, then one can use standard techniques from the theory of collective risks to study the process of interest. The purpose of this paper is to apply the results of Lalley (1984) to prove a large deviation result which is applicable to processes where such regeneration points exist.

In particular, we show the following. If we let

$$\tau_n = \inf\{m : \sum_{i=1}^{n} Y_i \geq n\}$$

denote the time of the first regeneration after time n, and

$$S_n = (S_n^{(1)}, S_n^{(2)}) = \sum_{i=1}^{n} (X_i, Y_i),$$

then we show that, under certain conditions,

$$P[S_{\tau_n}^{(1)} \geq an] \sim \frac{c_1}{\sqrt{n}} \exp\{-c_2 n\}$$

and

$$P[S_{\tau_n}^{(1)} \leq an] \sim \frac{c_3}{\sqrt{n}} \exp\{-c_4 n\}$$

where the c_i are positive constants. An example of such a process with regeneration points may be found in Kuczek (1989), where it was shown that the right edge process of oriented percolation has such regeneration points. This provides at least one setting where the results in this paper apply.

2. Notation and Preliminary Results

The purpose of this section is to introduce notation and terminology and to present the machinery needed to obtain the large deviation results. The theory in this section parallels special cases of results in Lalley (1984). Throughout the paper $\{(X_i, Y_i)\}$ will denote a sequence of i.i.d. random vectors in Z^2 with finite first and second moments. We denote $\mu_1 = EX_1, \mu_2 = EY_1$, and let Σ be the covariance matrix. We assume that $\mu_2 > 0$. Let τ_n and S_n be as defined in section 1.

The major result of this section is the following theorem. The proof is somewhat technical and is not important for understanding the rest of the paper.

Theorem 1 *On the set where $|k_1 - n\mu_2^{-1}|/\sqrt{n}$ remains bounded,*

$$P[\tau_n = k_1, S_{\tau_n}^{(1)} = a_n^{(1)}, S_{\tau_n}^{(2)} - n = k_2]$$

$$\sim P[S_{\tau_n}^{(2)} - n = k_2]P[\tau_n = k_1, S_{\tau_n}^{(1)} = a_n^{(1)}].$$

as $n \to \infty$, uniformly in k_1 for k_2 on compact sets and for $\{a_n^{(1)}\}$ such that $|a_n^{(1)} - n\mu_1\mu_2^{-1}|/\sqrt{n}$ remains bounded. In particular, (on this same set), the uniform convergence implies

$$P[S_{\tau_n}^{(1)} = a_n^{(1)}, S_{\tau_n}^{(2)} - n = k_2] \sim P[S_{\tau_n}^{(1)} = a_n^{(1)}]P[S_{\tau_n}^{(2)} - n = k_2].$$

Before proving Theorem 1, we need some technical results.

Proposition 1 *Let $\{(a_n^{(1)}, a_n^{(2)})\}$ be a sequence such that*

$$\left| \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix} - n \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right| \Big/ \sqrt{n}$$

remains bounded. Then for any fixed integer $k \geq 1$,

$$P[Y_n = j_0, \ldots Y_{n-k} = j_k | S_n^{(1)} = a_n^{(1)}, S_n^{(2)} = a_n^{(2)}] \longrightarrow \prod_{i=0}^{k} P^{(2)}(j_i),$$

as $n \to \infty$, where $P^{(2)}(\cdot)$ is the marginal probability distribution of Y_1.

Proof:

$$P[X_n = m_0, \ldots, X_{n-k} = m_k; Y_n = j_0, \ldots, Y_{n-k} = j_k | S_n^{(1)} = a_n^{(1)}, S_n^{(2)} = a_n^{(2)}]$$
$$= \left(\prod_{i=0}^{k} P[X_i = m_i, Y_i = j_i] \right) \left(\frac{P[S_{n-k-1}^{(1)} = a_n^{(1)} - \Sigma m_i, S_{n-k-1}^{(2)} = a_n^{(2)} - \Sigma j_i]}{P[S_n^{(1)} = a_n^{(1)}, S_n^{(2)} = a_n^{(2)}]} \right).$$

By a result of Stone (1967, Corollary 1), the ratio on the right converges to 1, uniformly for $(\overline{m}, \overline{j}) = (m_0, \ldots, m_k, j_0, \ldots, j_k)$ in compact sets. Since the joint distribution converges to what we want, the marginal distributions must also converge to the desired result.

Lemma 2 *For each integer k_2 and for each $\epsilon > 0$ there is an integer $k \geq 1$ for which*

$$(1) \qquad P[S_{k_1}^{(2)} - S_{k_1-l}^{(2)} \leq k_2, \text{ some } l, k \leq l \leq k_1 | S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2] < \epsilon,$$

for n sufficiently large where $k_1 = k_1(u) = [n\mu_2^{-1} + u(\sigma_{22}\mu_1^{-3}n)^{\frac{1}{2}}]$. Furthermore, k can be chosen so that (1) holds for all k_2 in a compact set and all $k_1(u)$ for u in a compact set, provided n is sufficiently large.

Proof:

$$P[S_{k_1}^{(2)} - S_{k_1-l}^{(2)} \leq k_2, \text{ some } l, k \leq l \leq k_1 | S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2]$$
$$\leq P[S_l^{(2)} \leq k_2, \text{ some } l, k \leq l \leq k_1]$$
$$\times \frac{P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2 | S_l^{(2)} \leq k_2, \text{ some } l, k \leq l \leq k_1]}{P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2]}$$

Now

$$P[S_l^{(2)} \leq k_2, \text{ some } l, k \leq l \leq k_1] \leq P[S_l^{(2)} \leq k_2, \text{ some } l \geq k]$$

which $\to 0$ as $k \to \infty$ by the strong law and the convergence is uniform in k_2 since the probability is an increasing function of k_2. By Stone (1967 Corollary 1)

$$\frac{1}{P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2]} = \mathcal{O}(k_1) = \mathcal{O}(n).$$

Thus it suffices to show that

$$\max_{0\leq l\leq k_1} P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2 | S_l^{(2)} \leq k_2] = \mathcal{O}(n^{-1}).$$

For $0 \leq l \leq k_1/2$, the result is true by Stone (1967, Corollary 1).

For $k_1/2 \leq l \leq k_1$, we have

$$\begin{aligned}
&\max_{\frac{k_1}{2}\leq l\leq k_1} P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2 | S_l^{(2)} \leq k_2] \\
&\leq \max_{\frac{k_1}{2}\leq l\leq k_1} P[S_{k_1}^{(2)} - n = k_2 | S_l^{(2)} \leq k_2] \\
&\leq \mathcal{O}(k_1^{-1}) = \mathcal{O}(n^{-1})
\end{aligned}$$

by Chebyshev's inequality.

That the convergence is uniform on compact sets follows from the fact that compact sets in Z^2 are finite. This completes the proof.

We are now ready to prove Theorem 1.

Proof of Theorem 1: For $k_2 \geq 0$, consider the event

$$\begin{aligned}
A &= \{\tau_n = k_1, S_{\tau_n}^{(1)} = a_n^{(1)}, S_{\tau_n}^{(2)} - n = k_2\} \\
&= \{S_{k_1}^{(2)} - S_{k_1-l}^{(2)} > k_2, \text{ for all } l, 1\ \leq l \leq k_1, S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2\}.
\end{aligned}$$

Then

$$\begin{aligned}
P[A] &= P[S_{k_1}^{(2)} - S_{k_1-l}^{(2)} > k_2, \text{ for all } l, 1 \leq l \leq k_1, | S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2] \\
&\quad \times P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2] \\
&= \left(1 - P[S_{k_1}^{(2)} - S_{k_1-l}^{(2)} \leq k_2, \text{ some } l,\ 1 \leq l \leq k_1, | S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2]\right) \\
&\quad \times P[S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2].
\end{aligned}$$

By Lemma 2, there is a k such that

$$P[S_{k_1}^{(2)} - S_{k_1-l}^{(2)} \leq k_2, \text{ some } l, k \leq l \leq k_1, | S_{k_1}^{(1)} = a_n^{(1)}, S_{k_1}^{(2)} - n = k_2] < \epsilon.$$

Thus

$$\left(1-P[S^{(2)}_{k_1}-S^{(2)}_{k_1-l}\le k_2,\ \text{some } l, 1\le l\le k, |S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2]-\epsilon\right)$$

$$\times P[S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2]$$

$$\le P[A]$$

$$\le \left(1-P[S^{(2)}_{k_1}-S^{(2)}_{k_1-l}\le k_2,\ \text{some } l, 1\le l\le k, |S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2]+\epsilon\right)$$

$$\times P[S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2].$$

By Proposition 1, for large n

$$P[S^{(2)}_{k_1}-S^{(2)}_{k_1-l}\le k_2,\ \text{some } l, 1\le l\le k, |S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2]$$

$$\sim P[S^{(2)}_l\le k_2,\ \text{some } l, 1\le l\le k],$$

and by the strong law

$$\lim_{k\to\infty} P[S^{(2)}_l\le k_2,\ \text{for some } l\ge k]=0.$$

Thus for large enough k, n

$$\left(1-P[S^{(2)}_l\le k_2,\ \text{some } l\ge 1]-2\epsilon\right) P[S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2].$$

$$\le P[A]$$

$$\le \left(1-P[S^{(2)}_l\le k_2,\ \text{some } l\ge 1]+2\epsilon\right) P[S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2].$$

By Spitzer (1976, Proposition P7.10)

$$P[S^{(1)}_{k_1}=a^{(1)}_n, S^{(2)}_{k_1}-n=k_2]$$

$$\sim \frac{1}{2\pi k_1|\Sigma|^{\frac{1}{2}}}\exp\left\{\frac{1}{2k_1}\begin{pmatrix} a^{(1)}_n & -k_1\mu_1 \\ n & -k_1\mu_2\end{pmatrix}'\Sigma^{-1}\begin{pmatrix} a^{(1)}_n & -k_1\mu_1 \\ n & -k_1\mu_2\end{pmatrix}\right\}$$

as $k_1\to\infty$.

Therefore,

$$P[A]\sim\left(1-P[S^{(2)}_l\le k_2,\ \text{some } l\ge 1\right)$$

$$\times\frac{1}{2\pi k_1|\Sigma|^{\frac{1}{2}}}\exp\left\{\frac{1}{2k_1}\begin{pmatrix} a^{(1)}_n & -k_1\mu_1 \\ n & -k_1\mu_2\end{pmatrix}'\Sigma^{-1}\begin{pmatrix} a^{(1)}_n & -k_1\mu_1 \\ n & -k_1\mu_2\end{pmatrix}\right\}$$

as $n \to \infty$. Since the first factor does not depend on $a_n^{(1)}$ or k_1 and the second factor does not depend on k_2, the independence is proved. By Corollary 1 of Stone (1967), the convergence is uniform for k_1 and $a_n^{(1)}$ satisfying the conditions of the theorem. □

3. Large Deviation Results

In this section we will prove a large deviation result which, by using the i.i.d. sequence of Kuczek (1989), is applicable to the right edge of oriented percolation. We will use the same notation as in section one. (Our Y_i is the same as Kuczek's τ_i.) We will assume that $\phi(\vec{\theta}) \equiv E(\exp\{\theta_1 X_1 + \theta_2 Y_1\})$ exists in a neighborhood R of $(0,0)$. We define $\psi(\vec{\theta}) \equiv \log \phi(\vec{\theta})$, and let $m_1 = \mu_1/\mu_2$ (remember that $\mu_2 > 0$).

Our approach will be to imbed the distribution of (X_1, Y_1) in an exponential family (as in section 3 of Lalley (1984)) and then choose an appropriate member to obtain our results. With this in mind we define (for $\vec{\theta} \in R$)

$$P_{\vec{\theta}}(k_1, k_2) = \exp\{\theta_1 k_1 + \theta_2 k_2 - \psi(\vec{\theta})\} P[X_1 = k_1, Y_1 = k_2].$$

It is easy to see that this is a probability measure on Z^2.

The following lemma shows that one of these $P_{\vec{\theta}}$ has certain properties that we desire.

Lemma 4 *There exists a neighborhood $N(m_1)$ of m_1 such that for any $a \in N(m_1)$, there is a $\vec{\theta}_0 = \vec{\theta}_0(a) \in R$ satisfying*

1. $\psi(\vec{\theta}_0) = 0$ *and*
2. $\left.\frac{\partial \psi(\vec{\theta})}{\partial \theta_1}\right|_{\vec{\theta}=\vec{\theta}_0} = a \left.\frac{\partial \psi(\vec{\theta})}{\partial \theta_2}\right|_{\vec{\theta}=\vec{\theta}_0}.$

Proof: Let

$$\psi_i(\vec{\theta}) = \frac{\partial \psi(\vec{\theta})}{\partial \theta_i}.$$

Then

$$\psi_1(\vec{\theta}1) = \frac{E(X_1 \exp\{\theta_1 X_1 + \theta_2 Y_1\})}{E(\exp\{\theta_1 X_1 + \theta_2 Y_1\})}$$

$$= E(X_1 \exp\{\theta_1 X_1 + \theta_2 Y_1 - \psi(\vec{\theta})\})$$

$$= E_{\vec{\theta}}(X_1),$$

and similarly $\psi_2(\vec{\theta}) = E_{\vec{\theta}}(Y_1)$.

Now $\psi(0,0) = 0$; so by the implicit function theorem, there exists a function $f(\theta_1)$, which has a continuous derivative, such that $f(0) = 0$ and $\psi(\theta_1, f(\theta_1)) = 0$ for θ_1 in a neighborhood of 0. Along the curve $(\theta_1, f(\theta_1)), \psi$ defines a function of θ_1 which satisfies

$$\psi_1(\theta_1, f(\theta_1)) + f'(\theta_1)\psi_2(\theta_1, f(\theta_1)) = 0,$$

or

$$-f'(\theta_1) = \frac{\psi_1(\theta_1, f(\theta_1))}{\psi_2(\theta_1, f(\theta_1))},$$

for θ_1 in a neighborhood of 0.

In particular

$$-f'(0) = \frac{E(X_1)}{E(Y_1)} = m_1.$$

Since ψ is strictly convex, we have

$$0 = \alpha\psi(\theta_1, f(\theta_1)) + (1-\alpha)\psi(\tilde{\theta}_1, f(\tilde{\theta}_1))$$

$$> \psi(\alpha\theta_1 + (1-\alpha)\tilde{\theta}_1, \alpha f(\theta_1) + (1-\alpha) f(\tilde{\theta}_1)),$$

which implies

(2) $\psi(\alpha\theta_1 + (1-\alpha)\tilde{\theta}_1, f(\alpha\theta_1 + (1-\alpha)\tilde{\theta}_1)) = 0 > \psi(\alpha\theta_1 + (1-\alpha)\tilde{\theta}_1, \alpha f(\theta_1) + (1-\alpha) f(\tilde{\theta}_1)).$

Because $E(Y_1)$ is positive, $\psi(\theta_1, \theta_2)$ attains its minimum when θ_2 is negative for all values of θ_1 in a neighborhood of 0. Therefore, $\psi(\vec{\theta})$ is increasing in its second argument for all

$\vec{\theta}$ in a neighborhood of $\vec{0}$. Hence, inequality (2) implies

$$f(\alpha\theta_1 + (1-\alpha)\tilde{\theta}_1) > \alpha f(\theta_1) + (1-\alpha)f(\tilde{\theta}_1),$$

i.e., f is strictly concave in a neighborhood of 0. The strict concavity of f implies $-f'(\theta)$ is increasing in a neighborhood of 0, and the image of this neighborhood under $-f'$ is a neighborhood of m_1. Thus the lemma is proved. □

We are now ready to prove the large deviation result for the i.i.d. process. We note here that the α of Kuczek (1989) is the same as our m_1. This can be seen by first noting that there is a subsequence of $\{S^{(1)}_{\tau_n}/n\}$ which is the same as a subsequence of$\{r_n/n\}$ and then comparing their limits. In what follows we will use α instead of m_1.

Theorem 5 *There exists a neighborhood $N(\alpha)$ of α such that for $a \in N(\alpha)$*

1. *if $a > \alpha$, then*

$$P[S^{(1)}_{\tau_n} \geq an] \sim \frac{c_1}{\sqrt{n}} \exp\{-c_2 n\}$$

as $n \to \infty$, and

2. *if $a < \alpha$, then*

$$P[S^{(1)}_{\tau_n} \leq an] \sim \frac{c_3}{\sqrt{n}} \exp\{-c_4 n\}$$

as $n \to \infty$, where c_1, c_2, c_3, and c_4 are positive constants.

Proof: Since we are working with the right edge, we have $\alpha > 0$. Since τ_n is a stopping time having a proper distribution, we have (using Theorem 1.1, on page 4 of Woodroofe (1982))

$$\begin{aligned} &P[S^{(1)}_{\tau_n} \geq an] \\ &= E_{\vec{\theta}}(\exp\{-\theta_1 S^{(1)}_{\tau_n} - \theta_2 S^{(2)}_{\tau_n} + \tau_n \psi(\vec{\theta})\}\chi_{[an,\infty)}(S^{(1)}_{\tau_n})) \\ &= \exp\{-n(\theta_1 a - \theta_2)\} E_{\vec{\theta}}(\exp\{-\theta_1(S^{(1)}_{\tau_n} - an) - \theta_2(S^{(2)}_{\tau_n} - n) + \tau_n \psi(\vec{\theta})\}\chi_{[an,\infty)}(S^{(1)}_{\tau_n})). \end{aligned}$$

Using the $\vec{\theta}$ guaranteed by Lemma 4, we can write the second factor as (remember $\psi(\vec{\theta}) = 0$)

$$
\begin{aligned}
&E_{\vec{\theta}}(\exp\{-\theta_1(S^{(1)}_{\tau_n} - an) - \theta_2(S^{(2)}_{\tau_n} - n) + \tau_n\psi(\vec{\theta})\}\chi_{[an,\infty)}(S^{(1)}_{\tau_n})) \\
&\quad = \int_{y=n}^{\infty}\int_{x=an}^{\infty} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy) \\
&\qquad = \int_{y=n}^{n+n^{\frac{1}{2}}}\int_{x=an}^{an+n^{\frac{1}{2}}} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy) \\
&\qquad + \int_{y=n+n^{\frac{1}{2}}}^{\infty}\int_{x=an}^{\infty} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy) \\
&\qquad + \int_{y=n}^{n+n^{\frac{1}{2}}}\int_{x=an+n^{\frac{1}{2}}}^{\infty} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy) \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}
$$

In the same manner we have

$$P[S^{(1)}_{\tau_n} \le an] = \exp\{-n(a\theta_1 + \theta_2)\}(J_1 + J_2 + J_3),$$

where

$$
\begin{aligned}
J_1 &= \int_{y=n}^{n+n^{\frac{1}{2}}}\int_{x=an-n^{\frac{1}{2}}}^{an} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy), \\
J_2 &= \int_{y=n+n^{\frac{1}{2}}}^{\infty}\int_{x=-\infty}^{an} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy),
\end{aligned}
$$

and

$$J_3 = \int_{y=n}^{n+n^{\frac{1}{2}}}\int_{x=-\infty}^{an-n^{\frac{1}{2}}} \exp\{-\theta_1(x - an) - \theta_2(y - n)\}P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx, S^{(2)}_{\tau_n} \in dy).$$

The proof of the theorem will be complete if we can show (for a on the correct side of α)

1. $a\theta_1 + \theta_2 > 0$;
2. I_1 and J_1 converge to $c/\sqrt{n}$ for some constant c; and
3. I_2, I_3, J_2 and J_3 are $o(n^{-\frac{1}{2}})$.

The first statement follows easily from the choice of $\vec{\theta}$ and the concavity of f in lemma 1.

Before proving the other two statements, we need to make a few comments. First, $S^{(2)}_{\tau_n} - n$ converges asymptotically to a proper distribution on the nonnegative integers. In addition, $S^{(2)}_{\tau_n} - n$ has a moment generating function in the same region that Y_1 has a moment generating function.

Second $S^{(1)}_{\tau_n}$ is asymptotically normal. Under $P_{\vec{\theta}}$, the mean is

$$nE_{\vec{\theta}}(X_1)[E_{\vec{\theta}}(Y_1)]^{-1} = n\frac{\psi_1(\vec{\theta})}{\psi_2(\vec{\theta})} = na,$$

and the variance is a finite multiple of n. (We will use the generic $\sigma^2 n$, even though σ^2 depends on a.)

Finally, θ_1 and θ_2 have opposite signs. When $a > \alpha$, we have $\theta_2 < 0 < \theta_1$ and when $a < \alpha$, we have $\theta_1 < 0 < \theta_2$.

In statement two, the proofs for I_1 and J_1 are similar; so we will only do the one for I_1.

Since the range of integration is finite and $E_{\vec{\theta}}(S^{(1)}_{\tau_n}) = an$, Theorem 3 says we can factor the probability distribution. Thus

$$\begin{aligned} I_1 &\sim \int_{y=n}^{n+n^{\frac{1}{2}}} \exp\{-\theta_2(y-n)\} P_{\vec{\theta}}(S^{(2)}_{\tau_n} \in dy) \int_{x=an}^{an+n^{\frac{1}{2}}} \exp\{-\theta_1(x-an)\} P_{\vec{\theta}}(S^{(1)}_{\tau_n} \in dx) \\ &= \int_{y=0}^{n^{\frac{1}{2}}} \exp\{-\theta_2(y)\} P_{\vec{\theta}}(S^{(2)}_{\tau_n} - n \in dy) \int_{x=0}^{n^{\frac{1}{2}}} \exp\{-\theta_1(x)\} P_{\vec{\theta}}(S^{(1)}_{\tau_n} - an \in dx). \end{aligned}$$

Since $\theta_2 > 0, \exp\{-\theta_2(y)\} > 1$ when $y > 0$. Since the moment generating function for $S_{\tau_n} - n$ exists in a neighborhood of 0, we can always choose our neighborhood $N(\alpha)$ so that $-\theta_2$ is in the region where the moment generating function exists. Thus, in this region, the first integral converges to a constant. For the second integral, we replace $dP_{\bar{\theta}}$ with the appropriate normal density:

$$\int_{x=0}^{n^{\frac{1}{2}}} \exp\{-\theta_1 x\} P_{\bar{\theta}}(S_{\tau_n}^{(1)} - an \in dx) = \int_0^{n^{\frac{1}{2}}} \exp\{-\theta_1 x\} \frac{1}{\sqrt{2\pi n\sigma^2}} \exp\{-\frac{x^2}{2n\sigma^2}\} dx.$$

Now make the substitution $y = x/\sqrt{n}$ to get

$$\int_0^1 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\theta_1\sqrt{n}y - \frac{y^2}{2\sigma^2}\} dy.$$

Then

$$\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-(\theta_1\sqrt{n} - \frac{1}{2\sigma^2})y\} dy \\
&\leq \int_0^1 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\theta_1\sqrt{n}y - \frac{y^2}{2\sigma^2}\} dy \\
&\leq \int_0^1 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\theta_1\sqrt{n}y\} dy.
\end{aligned}$$

Since both the upper and lower bounds are asymptotically $1/\sqrt{2\pi\sigma^2 n}\theta_1$ plus terms of order smaller than $1/\sqrt{n}$, we have the desired result for I_1.

In statement three for I_3 and J_3, we note that over the range of integration $-\theta_1(S_{\tau_n}^{(1)} - an) < 0$ and is a maximum at the finite endpoint. Thus

$$I_3, J_3 \leq (\rho)^{n^{\frac{1}{2}}} \int_n^{n+n^{\frac{1}{2}}} \exp\{-\theta_2(y-n)\} dP_{\bar{\theta}}(S_{\tau_n}^{(2)} \in dy) = o(n^{-\frac{1}{2}}),$$

where

$$1 > \rho = \begin{cases} e^{-\theta_1} & \text{for} \quad I_3 \\ e^{\theta_1} & \text{for} \quad J_3 \end{cases}.$$

For J_2, θ_2 and $S_{\tau_n}^{(2)} - n$ are both positive; so

$$J_2 \leq (e^{-\theta_2})^{n^{\frac{1}{2}}} \int_{-\infty}^{an} \exp\{-\theta_1(y-an)\} dP_{\bar{\theta}}(S_{\tau_n}^{(1)} \in dy) = o(n^{-\frac{1}{2}}).$$

Finally we come to I_2.

$$I_2 = \int_{y=n+n^{\frac{1}{2}}}^{\infty} \int_{x=an}^{\infty} \exp\{-\theta_1(x-an) - \theta_2(y-n)\} P_{\vec{\theta}}(S_{\tau_n}^{(1)} \in dx, S_{\tau_n}^{(2)} \in dy)$$
$$\leq \int_{y=n+n^{\frac{1}{2}}}^{\infty} \exp\{-\theta_2(y-n)\} P_{\vec{\theta}}(S_{\tau_n}^{(2)} \in dy)$$
$$= \sum_{y=\sqrt{n}}^{\infty} \sum_{l=0}^{n-1} \sum_{k=0}^{\infty} \exp\{-\theta_2 y\} P_{\vec{\theta}}[S_k^{(2)} = l, Y_{k+1} = y+n-l]$$

(where the first sum runs over all integers $\geq \sqrt{n}$)

$$= \sum_{y=\sqrt{n}}^{\infty} \sum_{l=0}^{n-1} \sum_{k=0}^{\infty} \exp\{-\theta_2 y\} P_{\vec{\theta}}[S_k^{(2)} = l] P_{\vec{\theta}}[Y_{k+1} = y+n-l]$$
$$= \sum_{y=\sqrt{n}}^{\infty} \exp\{-\theta_2 y\} \sum_{l=0}^{n-1} P_{\vec{\theta}}[Y_1 = y+n-l] \sum_{k=0}^{\infty} P_{\vec{\theta}}[S_k^{(2)} = l]$$
$$\leq C \sum_{y=\sqrt{n}}^{\infty} \exp\{-\theta_2 y\} \sum_{l=0}^{n-1} P_{\vec{\theta}}[Y_1 = y+n-l]$$
$$\leq C \sum_{y=\sqrt{n}}^{\infty} \exp\{-\theta_2 y\} \sum_{x=y}^{\infty} P_{\vec{\theta}}[Y_1 = x]$$

We conclude the proof at this point by noting that

$$P_{\theta}(Y_1 \geq y) \leq Ce^{-\epsilon y}.$$

4. Conclusion

In this paper we have obtained a large deviation result which can be applied to processes with regeneration type points. Our methods paralleled those of Lalley (1984, sections 2 and 3).

As obtained, the result does not depend on any special properties of a process other than the existence of regeneration points. A non-trivial example of an application is to apply the result to the i.i.d. sequence of Kuczek (1989).

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