EXACT CONVOLUTION OF t DISTRIBUTIONS, WITH APPLICATION TO BAYESIAN INFERENCE FOR A NORMAL MEAN WITH t PRIOR DISTRIBUTIONS[1]

by

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ABSTRACT

An exact formula for the convolution of two t densities with odd degrees of freedom is derived. Such convolutions are of great importance to statistics. For instance, for basic normal inference problems concerning the mean when the variance is unknown, Bayesian analyses tend to end up dealing with marginal likelihood functions for the mean that are t-densities. For such marginal likelihoods, it is natural and desirable to perform Bayesian analysis with t priors. Bayesian testing and posterior normalization then require calculation of convolutions of t densities. This, as well as the posterior mean and variance, are evaluated in closed form when the sample size is even and the prior has odd degrees of freedom. Convolutions of t densities also arise in the Behrens-Fisher problem and in Bayesian inference concerning the common mean of two samples. From the viewpoint of Bayesian robustness in the basic normal inference problem, a Cauchy prior is of special interest. For this case, in addition to the exact formulas for even sample size, interpolation formulas for the marginal, posterior mean and posterior variance are derived when the sample size is odd. In comparison with the IMSL numerical integration subroutine, our algorithms turn out to be substantially more efficient and accurate.

KEY WORDS: Convolution; marginal likelihood function; posterior mean; posterior variance; Behrens-Fisher problem.

1 Introduction

The basic quantities that will be considered are integrals of the form

$$I^0_{n,m}(w,z) = \int \frac{1}{\sqrt{w}(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi, \tag{1}$$

and

$$I^1_{n,m}(w,z) = \int \frac{\xi}{\sqrt{w}(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi. \tag{2}$$

These integrals are typically evaluated by numerical integration, but this can be quite expensive when it must be done simultaneously for many w and z (cf. section 1.2). Thus we consider closed form evaluation of (1) and (2) using residues.

1.1 Statistical Motivation

This section mentions several important statistical problems in which evaluation of (1) and/or (2) is necessary.

1.1.1 Bayesian analysis : normal sample, t-prior

The data consists of i.i.d. random variables $X_1, X_2, \ldots, X_n$ from a $N(\theta, \sigma^2)$ distribution, both θ and σ^2 unknown. Letting $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ denote the observed data, the joint likelihood function for θ and σ^2 is

$$l_{\mathbf{x}}(\theta, \sigma^2) = \frac{1}{(2\pi)^{n/2}\,\sigma^n} e^{-n(\bar{x}-\theta)^2/(2\sigma^2)} e^{-(n-1)s^2/(2\sigma^2)},$$

where $\bar{x} = \sum x_i/n$ and $s^2 = \sum(x_i - \bar{x})^2/(n-1)$ are the usual sample mean and variance respectively.

In a Bayesian analysis, it will typically be the case that prior opinions about θ and σ^2 are independent. Assuming this and letting $\pi^{**}(\sigma^2)$ denote the prior for σ^2 and $\pi(\theta)$ denote the prior for θ, the marginal posterior density for θ can be written as

$$\pi(\theta|\mathbf{x}) = \frac{\int l_{\mathbf{x}}(\theta,\sigma^2)\pi(\theta)\pi^{**}(\sigma^2)\,d\sigma^2}{\int l_{\mathbf{x}}(\theta,\sigma^2)\pi(\theta)\pi^{**}(\sigma^2)\,d\sigma^2\,d\theta} = \frac{l_{\mathbf{x}}(\theta)\pi(\theta)}{m(\mathbf{x})},$$

where

$$l_{\mathbf{x}}(\theta) = \int l_{\mathbf{x}}(\theta,\sigma^2)\pi^{**}(\sigma^2)d\sigma^2, \tag{3}$$

and

$$m(\mathbf{x}) = \int l_{\mathbf{x}}(\theta)\pi(\theta)d\theta.$$

It is common to give σ^2 the noninformative prior $\sigma^{-2}d\sigma^2$. When this is used in (3), calculation yields (upon renormalization to obtain a density in θ)

$$l_{\mathbf{x}}(\theta) = \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\bar{x}-\theta)^2}{(n-1)s^2})^{n/2}}, \tag{4}$$

where

$$K_i = \Gamma(\frac{i+1}{2})/[\Gamma(\frac{i}{2})\sqrt{i\pi}]. \tag{5}$$

The common informative choice for $\pi^{**}(\sigma^2)$ is the inverse gamma distribution with subjectively specified parameters (α, β),

$$\pi^{**}(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha(\sigma^2)^{\alpha+1}}e^{-1/(\sigma^2\beta)}. \tag{6}$$

For this π^{**}, $l_{\mathbf{x}}(\theta)$ is of the same form as (4), with n and s^2 replaced, respectively, by

$$n^* = n + 2\alpha \tag{7}$$

and

$$(s^*)^2 = \frac{(n+\alpha)((n-1)s^2\beta+2)}{(n+2\alpha-1)n\beta}. \tag{8}$$

In what follows we will explicitely consider only the noninformative prior scenario, but all results can be applied to the general case, using (7) and (8) in place of n and s^2 in (4), providing n^* is an integer.

In the above situations, the marginal likelihood function for the mean (i.e. $l_{\mathbf{x}}(\theta)$) is thus a t-density. If a unimodal, symmetric prior distribution for θ is desired, it is then natural to choose $\pi(\theta)$ to also be a t-density. (Note that it may well be desirable to use a subjective prior distribution for θ even when a noninformative prior distribution for σ^2 is used.) Indeed, robustness considerations suggest that the tail of the prior should typically be at least as flat as the tail of the likelihood (cf. Berger (1985)). Thus we consider

$$\pi(\theta) = \frac{K_{m-1}}{\tau(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}}, \tag{9}$$

so that the posterior density for θ is

$$\pi(\theta|\mathbf{x}) = \frac{1}{m(\mathbf{x})} \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\bar{x}-\theta)^2}{(n-1)s^2})^{n/2}} \cdot \frac{K_{m-1}}{\tau(1+\frac{(\theta-\mu)^2}{\tau^2(m-1)})^{m/2}}. \tag{10}$$

Note first that

$$m(\mathbf{x}) = \int \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\bar{x}-\theta)^2}{(n-1)s^2})^{n/2}} \cdot \frac{K_{m-1}}{\tau(1+\frac{(\theta-\mu)^2}{\tau^2(m-1)})^{m/2}} d\theta.$$

A linear transformation provides a more convenient form for $m(\mathbf{x})$, namely

$$m(\mathbf{x}) = \frac{K_{n-1}\sqrt{n}K_{m-1}\sqrt{m-1}}{s} \; I^0_{n,m}(w,z) \tag{11}$$

(see (1)), where

$$w = n(m-1)\tau^2/[(n-1)s^2], \tag{12}$$

$$z = \sqrt{n}(\bar{x}-\mu)/(\sqrt{n-1}s). \tag{13}$$

The posterior mean, $\delta^\pi(\mathbf{x})$, is then given by

$$\begin{aligned} \delta^\pi(\mathbf{x}) &= \int \theta\pi(\theta|\mathbf{x})d\theta \\ &= \bar{x} + \frac{\rho^\pi(\mathbf{x})}{m(\mathbf{x})}, \end{aligned} \tag{14}$$

where

$$\rho^\pi(\mathbf{x}) = K_{n-1}K_{m-1}\sqrt{(n-1)(m-1)} \; I^1_{n,m}(w,z). \tag{15}$$

A third quantity of general interest is the posterior variance of θ,

$$V^\pi(\mathbf{x}) = \frac{1}{m(\mathbf{x})} \int \frac{K_{n-1}K_{m-1}\theta^2}{(s/\sqrt{n})(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}\tau(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}} d\theta - (\delta^\pi(\mathbf{x}))^2. \tag{16}$$

It turns out, however (see section 4), that the posterior variance can be written in terms of $I^0_{n,m}, I^0_{n,m-2}$ and $\delta^\pi(\mathbf{x})$ as

$$V^\pi(\mathbf{x}) = \frac{\sqrt{m-1}\tau^2 I^0_{n,m-2}(w,z)}{I^0_{n,m}(w,z)} - (m-1)\tau^2 - (\mu - \delta^\pi(\mathbf{x}))^2, \tag{17}$$

so that evaluation of (11) and (14) suffices to also determine this quantity.

1.1.2 Bayesian analysis : two samples, noninformative prior for θ

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from the $N(\theta, \sigma^2_{\mathbf{x}})$ distribution where $\sigma^2_{\mathbf{x}}$ is unknown, and $Y_1, Y_2, \ldots, Y_m$ be another random sample from the $N(\theta, \sigma^2_{\mathbf{y}})$ distribution with $\sigma^2_{\mathbf{y}}$ unknown. If both samples are independent and the noninformative prior $\pi^{**}(\sigma^2) = \sigma^{-2}$ is used (independently) for both $\sigma^2_{\mathbf{x}}$ and $\sigma^2_{\mathbf{y}}$, the marginal likelihood for θ is (see (4))

$$l(\theta) = \frac{\sqrt{n}K_{n-1}}{s_{\mathbf{x}}(1+\frac{n(\bar{x}-\theta)^2}{(n-1)s^2_{\mathbf{x}}})^{n/2}} \frac{\sqrt{m}K_{m-1}}{s_{\mathbf{y}}(1+\frac{m(\bar{y}-\theta)^2}{(m-1)s^2_{\mathbf{y}}})^{m/2}},$$

where $\bar{x}, \bar{y}, s^2_{\mathbf{x}}$ and $s^2_{\mathbf{y}}$ are the usual sample means and variances for the corresponding samples. Informative gamma priors for the variances could also be used here, as in the previous section, with versions of (7) and (8) replacing $(n, s^2_{\mathbf{x}})$ and $(m, s^2_{\mathbf{y}})$ in the above expression.

If the noninformative prior $\pi(\theta) \equiv 1$ is chosen for θ, the posterior density is thus given by (10), letting $\mathbf{x} = (x_1, \ldots, x_n; y_1, \ldots, y_m)$ and replacing $\bar{x}, s^2$, and μ, τ^2 by $\sum x_i/n, s^2_{\mathbf{x}}$ and $\sum y_i/m, s^2_{\mathbf{y}}/m$ respectively. Therefore, the equations in (11) through (17) also apply to the problem of making inferences about a common mean θ under a noninformative prior.

1.1.3 The Behrens-Fisher density

For a classical example in which the integrals in (1) and (2) are encountered, suppose that two random samples are drawn from two normal populations, $N(\mu_i, \sigma^2_i)$, with unknown σ^2_i, for $i = 1, 2$, i indexing the two populations. Let n_i denote the sample size, $\bar{x}_i$ denote the sample mean and s_i^2 denote the sample variance. It is well known that

$$t_i = \frac{\sqrt{n_i}(\bar{x}_i - \mu_i)}{s_i}$$

is distributed according to a central t-distribution with $n_i - 1$ degrees of freedom. Moreover, t_1 and t_2 are independent. Then

$$\mu_1 - \mu_2 = (\bar{x}_1 - \bar{x}_2) - (\frac{s_1}{\sqrt{n_1}}t_1 - \frac{s_2}{\sqrt{n_2}}t_2)$$

yields the fiducial distribution of the difference between the population means as a linear combination of two independent t distributions by treating $\bar{x}_i$ and s_i as constants. The Behrens-Fisher density is therefore the density of

$$Y_{n_1,n_2} = U_{n_1} - V_{n_2},$$

where $U_{n_1} \sim T(n_1 - 1; \bar{x}_1, s_1^2/n_1)$ and $V_{n_2} \sim T(n_2 - 1; \bar{x}_2, s_2^2/n_2)$. Calculation yields the density of Y_{n_1,n_2} to be

$$f_{n_1,n_2}(y) = \frac{K_{n_1-1}K_{n_2-1}\sqrt{n_2-1}}{s_1/\sqrt{n_1}} I^0_{n,m}(w', z'), \tag{18}$$

where

$$w' = \frac{\sqrt{n_2-1}s_2/\sqrt{n_2}}{\sqrt{n_1-1}s_1/\sqrt{n_1}}, \quad z' = \frac{y - (\bar{x}_1 - \bar{x}_2)}{\sqrt{n_1-1}s_1/\sqrt{n_1}}.$$

Fisher(1935) proposed a solution to the problems of estimating and testing the difference between the means of two normal populations with unknown variances based on the Behrens-Fisher distribution. A test equivalent to Fisher's was given earlier for a special case by Behrens (1929). Chapman (1950) reported a two-stage procedure for testing the difference of the means. Fisher (1939), Fisher and Healy (1956), and Fisher and Cornish (1960) also discussed certain features of the Behrens-Fisher distribution. Scheffé (1943,1944) and Welch (1947) also offered solutions to the Behrens-Fisher problem. For the problem of estimating the difference of the means, Jeffreys(1940) arrived at the same conclusion as Fisher, but Jeffreys' argument was based on a Bayesian argument with the usual noninformative prior for the means and variances.

Fisher (1935) also described a fiducial interval for $\mu_1 - \mu_2$ which is based on the so-called d-statistic (or Behrens-Fisher statistic)

$$d_{n_1,n_2,\alpha} = (\cos\alpha)t_1 - (\sin\alpha)t_2,$$

for a specified $\alpha \in [0, \pi/2]$; the interval requires determination of percentage points of the distribution of $d_{n_1,n_2,\alpha}$. Calculation gives the density of $d_{n_1,n_2,\alpha}$ to be

$$f_{n_1,n_2,\alpha}(d) = \frac{K_{n_1-1}K_{n_2-1}\sqrt{n_2-1}}{\cos\alpha}\, I^0_{n,m}(w'', z''),$$

where

$$w'' = \frac{\sqrt{n_2-1}}{\sqrt{n_1-1}}\tan\alpha, \quad z'' = \frac{d}{(\cos\alpha)\sqrt{n_1-1}}.$$

In frequentist statistics, $d_{n_1,n_2,\alpha}$ is used for testing hypothses about $\mu_1 - \mu_2$ or for constructing a confidence interval for $\mu_1 - \mu_2$ (Fisher (1935), Chapman (1950), Ruben (1960)).

1.2 A Statistical Scenario Involving Many Integrals

A statistical scenario involving many integrals of form of (1) is given in Berger and Delampady (1987). Suppose one observes $X_1, \ldots, X_n$, distributed as i.i.d. $N(\theta, \sigma^2)$ random variables, θ and σ^2 unknown. It is desired to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. With the noninformative prior for σ^2 and a Cauchy prior with parameters (μ, τ^2) for θ under H_1, the Bayes factor for testing H_0 versus H_1 is given by

$$B(\mu, \tau^2) = \frac{l_{\mathbf{x}}(\theta_0)}{m(\mathbf{x})},$$

where $m(\mathbf{x})$ and $l_{\mathbf{x}}(\theta)$ are given by (11) (for $m = 2$) and (4) respectively.

To perform a robustness study with respect to a particular choice of μ and τ^2, or to present the Bayesian inference to a range of consumers having different μ and τ^2, it is useful to present a contour graph of $B(\mu, \tau^2)$ as a function of the prior parameters. A method to efficiently compute $m(\mathbf{x})$ over a grid of (μ, τ) is thus needed. The exact formula for $m(\mathbf{x})$ that we develop is easily adapted to inexpensively calculate $m(\mathbf{x})$ over a grid. Figure 1 is the resulting contour graph of $B(\mu, \tau^2)$ when $\theta_0 = 0, n = 15, \bar{x} = 20.93$ and $s = 37.79$. The graph required 2501 integrations of the form of (1), but took only 17.8 cpu seconds on a Vax11/780.

Other Bayesian analyses that require many integrations are discussed in Fan and Berger (1989b). Note also that utilization of the Behrens-Fisher density often requires integration over y in (18) and hence evaluation of (18) at many y.

1.3 History and Overview

Fisher (1941) proposed calculating (1) by using infinite series expansions of the two t-distributions and integrating term by term in the ensuing double series. Such an approach was also taken by Tiao and Zellner (1964) in the multivariate case. An alternate series expansion was advocated by Dickey (1968), who also developed methods for dealing with higher dimensions or products of more than two t-densities. Ruben (1960) obtained an integral form for the density of the $d-$statistic such that d can be interpreted as the ratio of two independent random variables, namely a t-variable and a function of a Beta variable. Some related work about the cumulative distribution function of the Behrens-Fisher density can be seen in James (1959), Patil (1965), Ghosh (1975) and Franck (1981).

Spiegelhalter (1985) gives exact expressions for the posterior distributions of the location and scale parameters of a Cauchy distribution assuming vague prior information. For two observations, his marginal likelihood for fixed scale parameter is essentially equivalent to our result with $n = m = 2$.

In section 4, exact closed form expressions will be obtained for (1) and (2), when n and m are even integers. (Unfortunately, calculation by residues can only be done when n and m

are even.) Sections 2 and 3 concentrate on the case $m = 2$, corresponding to the important Bayesian statistical application in section 1.1.1 when the often recommended Cauchy prior is assumed. In section 2, closed form expressions for $m(\mathbf{x}), \delta^{\pi}(\mathbf{x})$ and the posterior variance $V^{\pi}(\mathbf{x})$ are developed when n is an even integer, and accurate interpolation formulas are given when n is an odd integer. In section 3, a comparison of the calculation time needed for the exact formulas and the IMSL numerical integration subroutine will be given.

For the Bayesian applications discussed in sections 1.1.1 and 1.1.2, observe that choice of a slightly different noninformative prior for the normal variance(s) can ensure that n (and m) are even, so that the interpolation formulas can be avoided. Indeed, if n (or m in section 1.1.2) are odd, choose the noninformative prior to be of the form $\pi^{**}(\sigma^2) = \sigma^{-1}$ rather than $\pi^{**}(\sigma^2) = \sigma^{-2}$, and (see (7)) replace n (or m) by $n^* = n - 1$ (or $m^* = m - 1$). There is some support for use of $\pi(\sigma^2)^{**} = \sigma^{-1}$ as a noninformative prior for σ^2 (essentially, an argument for uniformity in $\log \sigma$ rather than uniformity in $\log \sigma^2$), and except for very small n (or m) there will be little difference. In the same vein, when choosing an informative prior for σ^2 and/or θ, it is calculationally convenient if n^* and m (or m^*) are made to be even.

2 Calculation When $m = 2$

The $m = 2$ situation is of particular interest in the scenario of section 1.1.1, since it corresponds to choice of the Cauchy prior

$$\pi(\theta) = \frac{1}{\pi\tau(1 + (\theta - \mu)^2/\tau^2)}. \tag{19}$$

Reasons for choosing a Cauchy prior include the following. We assume that a unimodal and symmetric prior is desired.

1. Specification of a prior median and interquartile range for θ is relatively simple, even for nonstatisticians. The median of (19) is μ and the interquartile range is 2τ, so the prior parameters will have direct meaning for practitioners.

2. Cauchy priors have desirable robustness properties. By robustness here is roughly

meant that features of the prior which are not carefully considered will not have a great effect on the analysis. The location and scale features of a prior, such as μ and τ above, often are carefully considered, and in any case one can present conclusions for a range of μ and τ as discussed in section 1.2. The functional form of $\pi(\theta)$, however, is often very difficult to specify, beyond general shape features such as unimodality and/or symmetry. Evidence has gradually accumulated to the effect that the functional form is nonobtrusive providing the tails of the prior are less sharp than those of the likelihood function. Use of Cauchy priors guarantees, in the problems we consider here, that this will be so.

3. Constructing graphs such as Figure 1 in section 1.2 is computationally onerous for t-type likelihoods. Cauchy priors are among the easiest priors to handle computationally.

For a Cauchy prior for θ and the noninformative prior $\pi^{**}(\sigma^2) = \sigma^{-2}$ for σ^2, (11) and (14) reduce to

$$m(\mathbf{x}) = \frac{K_{n-1}\sqrt{n}}{\pi s} I^0_{n,2}(w,z), \tag{20}$$

$$\delta^\pi(\mathbf{x}) = \bar{x} + \frac{s\sqrt{n-1}}{\sqrt{n}} \frac{I^1_{n,2}(w,z)}{I^0_{n,2}(w,z)}, \tag{21}$$

where w and z are given by (12) and (13), respectively, with $m = 2$. The calculation of $m(\mathbf{x})$, $\delta^\pi(\mathbf{x})$ and $V^\pi(\mathbf{x})$ will be considered in sections 2.1, 2.2, and 2.3, respectively. Note that, for the general $\pi^{**}(\sigma^2)$ in (6), all formulas below are valid if n and s^2 are replaced by (7) and (8), respectively.

2.1 Calculation of the Marginal Distribution

The exact formula for $m(\mathbf{x})$, for even n, will be given in section 2.1.1, and an interpolation result for odd n will be given in section 2.1.2.

2.1.1 Exact Formula for Even n

When n is even, the integral in (20) can be evaluated in closed form. The formula is a special case of the more general result in Theorem 4.1, but is recorded here for later development of an interpolation formula and numerical study. Let

$$r = \sqrt{(w+z^2-1)^2+4z^2},\ v = cos^{-1}((w+z^2-1)/r) \ \text{ and } \ v' = cos^{-1}((z^2-w+1)/r). \quad (22)$$

Let $\langle j \rangle$ denote the largest integer less than or equal to j, and define

$$\underline{n} = 2\langle \frac{n-4}{4} \rangle,\ \bar{n} = 2\langle \frac{n-2}{4} \rangle,\ \tilde{k} = \langle k/2 \rangle. \quad (23)$$

Theorem 2.1 *If n is even and $r \neq 0$, then $m(\mathbf{x})$ in (20) can be calculated as follows:*

$$\begin{aligned} m(\mathbf{x}) &= (\sqrt{n}/s)\, K_{n-1} \\ &\quad \times \{\sum_{k=0}^{\underline{n}} r^{-(k+2)}[\cos(k+2)v - |z| \sin(k+2)v] \sum_{j=k_1(n)}^{\tilde{k}} w^{j+\frac{1}{2}} A_{1,n/2}(k,j) \\ &\quad + \sum_{k=0}^{\bar{n}} r^{-(k+1)} \cos[(k+1)v] \sum_{j=k_2(n)}^{\tilde{k}} w^{j+\frac{1}{2}} B_{1,n/2}(k,j) + r^{-n/2}\cos(\frac{n}{2}v')\}, \quad (24) \end{aligned}$$

where

$$k_1(n) = \max\{0, k-\underline{n}/2\},\ k_2(n) = \max\{0, k-\bar{n}/2\}, \quad (25)$$

and $A_{1,n/2}$ and $B_{1,n/2}$ are defined in Theorem 4.1. (In any of the summations above, $\sum_s^t$ with $s > t$ is to be defined as zero.)

When $r = 0$ (which happens when $w = 1$ and $z = 0$),

$$m(\mathbf{x}) = \sqrt{\frac{n-1}{n}}\frac{1}{\pi s}. \quad (26)$$

Proof: Follows directly from Theorem 4.1. □

For example, if $n = 6$, then $\bar{n} = 2$, $\underline{n} = 0$,

$$A_{1,3}(0,0) = -(3/4), \qquad B_{1,3}(0,0) = 3/8,$$

$$B_{1,3}(1,0) = -(3/4), \qquad B_{1,3}(2,1) = 1.$$

Therefore,

$$
\begin{aligned}
m(\mathbf{x}) &= \frac{8\sqrt{6/5}}{3\pi s}\left\{\frac{3}{8}\frac{\cos v}{r}\sqrt{w} - \frac{3}{2}\frac{\cos(2v)}{r^2}\sqrt{w} + \frac{\cos(3v)}{r^3}w\sqrt{w}\right. \\
&\quad \left. + \frac{3}{4}|z|\frac{\sin(2v)}{r^2}\sqrt{w} + \frac{\cos(3v')}{r^3}\right\}. \qquad (27)
\end{aligned}
$$

Everything in (24) is easily calculable. Unfortunately, as $r \to 0$, (24) becomes numerically unstable. We have observed that this instability occurs in a region of the form

$$
R_1 = \{(w,z) : r^2 = (w-1+z^2)^2 + 4z^2 < C^{-4/n}\}, \qquad (28)
$$

where C is computer dependent. For example, if (24) is implemented in a double precision FORTRAN program, C can be 10^{23} on a CDC6000, but only 10^{12} on a VAX11/780. Note that R_1 is a very small region, and would probably never occur for specific prior and data. However, in graphical displays which present the Bayesian answers for a wide range of priors, such as that discussed in section 1.2, consideration of R_1 is needed.

To deal with this instability, for moderate and large n, one can use the following approximation to $m(\mathbf{x})$ if r is small:

$$
\begin{aligned}
\hat{m}(\mathbf{x}) &= \frac{\sqrt{n}}{\pi s\sqrt{(n-1)w(1+z^2/w)}} \\
&\quad \times\left\{1 + \frac{(3z^2/w-1)}{(n-3)w(1+z^2/w)^2} + \frac{3(5z^4/w^2 - 10z^2/w + 1)}{(n-3)(n-5)w^2(1+z^2/w)^4}\right\}. \qquad (29)
\end{aligned}
$$

(See Fan and Berger (1989a) for a derivation.) For small n, (26) suffices as an approximation. Thus, for $(w,z) \in R_1$, we suggest approximating $m(\mathbf{x})$ by $\tilde{m}(\mathbf{x})$ defined as

$$
\tilde{m}(\mathbf{x}) = \begin{cases} \hat{m}(\mathbf{x}) & \text{if } n > 14 \\ (\sqrt{n-1}/\sqrt{n})(\pi s)^{-1} & \text{if } n \le 14. \end{cases} \qquad (30)
$$

The relative error of $\tilde{m}(\mathbf{x})$ is less than 0.8% on a Vax11/780 machine. Note that the distance between the modes of $l_{\mathbf{x}}(\theta)$ and $\pi(\theta)$ in R_1 is rather small, making numerical integration simple and accurate in this region. If one requires accuracy higher than 0.8%, using numerical integration in R_1 will not cost much because R_1 is such a small set.

2.1.2 Interpolation of $m(\mathbf{x})$ for Odd n

Unfortunately, exact calculation of $m(\mathbf{x})$ is not possible if n is odd. In this subsubsection, a suitable linear interpolation of $m(\mathbf{x})$ for odd n will be considered.

It is helpful to consider (20) as a function of $\sqrt{w}$ and z, namely

$$g_n(\sqrt{w}, z) = \frac{\sqrt{n}K_{n-1}}{\pi s}\, I^0_{n,2}(w, z). \tag{31}$$

Theorem 2.1 gives an exact formula for $m(\mathbf{x})$ (i.e. $g_n(\sqrt{w}, z)$) when n is even. The rescaled linear interpolation,

$$\begin{aligned} \tilde{g}_n(\sqrt{w}, z) &= \alpha_n g_{n-1}\left(\frac{\sqrt{n-1}\sqrt{w}}{\sqrt{n-2}}, \frac{\sqrt{n-1}}{\sqrt{n-2}} z\right) \\ &+ (1-\alpha_n) g_{n+1}\left(\frac{\sqrt{n-1}\sqrt{w}}{\sqrt{n}}, \frac{\sqrt{n-1}}{\sqrt{n}} z\right), \end{aligned} \tag{32}$$

will be considered as an approximation to $g_n(\sqrt{w}, z)$ for odd n, where α_n is given by

$$\alpha_3 = \begin{cases} 0.28 + 0.18/(1+z) & \text{if } \sqrt{w} \le 1 \text{ and } z \ge 1.5 \\ 0.43 & \text{otherwise,} \end{cases}$$

and for $n \ge 5$,

$$\alpha_n = \begin{cases} 0.38 & \text{if } \sqrt{w} \le 10^{-(n/4)} \text{ and } z \ge 1.5 \\ \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n-1}} & \text{otherwise.} \end{cases}$$

(The formula for α_n was derived using the approximate formula (29).) If n is odd, calculate $\sqrt{w}$ and z using (12) and (13) and then calculate $g_{n-1}(\frac{\sqrt{n-1}}{\sqrt{n-2}}\sqrt{w}, \frac{\sqrt{n-1}}{\sqrt{n-2}}z)$ and $g_{n+1}(\frac{\sqrt{n-1}}{\sqrt{n}}\sqrt{w}, \frac{\sqrt{n-1}}{\sqrt{n}}z)$ using Theorem 2.1. When $(w, z) \in R_1$ (see (28)), g_{n-1} and g_{n+1} should be replaced by $\tilde{m}(\mathbf{x})$ of (30).

For $(\sqrt{w}, z) \in \tilde{R}_1$, where

$$\tilde{R}_1 = \{(\sqrt{w}, z) | \sqrt{w} < 3 \times 10^{-\langle n/3 \rangle}, \text{ and } z > 1.9 + \langle \frac{n}{5} \rangle + \sqrt{w}\}, \tag{33}$$

the interpolation can have a relative error of 2.5 % or more, and numerical integration is recommended. On $\tilde{R}_1^c$, $\tilde{g}_n$ differs from g_n by at most 2.5 %, and is typically extremely accurate as discussed below. Note that $(\sqrt{w}, z) \in \tilde{R}_1$ rarely occurs in practice.

Table 1 shows, for various odd n, the maximum relative errors of $\tilde{g}_n$ at $\sqrt{w} = 0.1$, 0.5, 1, 2, and 3, respectively. The first number in each of the last 5 columns is the maximum relative error and the second is where the maximum relative error occurs over $0 < z \leq 6$ (to the nearest 0.1). It was separately observed that, when $n \geq 7$, the relative error of $\tilde{g}_n$ increases and then decreases as z varies from 0 to infinity for small $\sqrt{w}$. Thus, in Table 1, the entries for $n \geq 7$ appear to be global maxima. (For $n = 25$ and 27, the relative errors in $\tilde{g}_n$ were so small – less than 10^{-5} – that it was very difficult to determine the maximizing z.) It was also observed that for larger $\sqrt{w}$ the interpolation appears to be very accurate.

2.2 Calculation of the Posterior Mean

In this subsection, an exact formula for $\delta^\pi(\mathbf{x})$ will be given for even n, and an interpolation will be given for odd n.

2.2.1 Exact Calculation of δ^π for even n

Because of (14), we need only consider calculation of $\rho^\pi(\mathbf{x})$ in the following theorem.

Theorem 2.2 *If $r \neq 0$ and $n \geq 2$ is an even integer, an exact formula for $\rho^\pi(\mathbf{x})$ (cf. Eq. (15)) is*

$$\rho^\pi(\mathbf{x}) = K_{n-1}\sqrt{n-1}\, sgn(z)\, (I_1 + I_2 + I_3), \tag{34}$$

where

$$I_1 = -r^{-n/2}[\sin(\frac{n}{2}v')\sqrt{w} + |z| cos(\frac{n}{2}v')],$$

$$I_2 = \sum_{k=0}^{\frac{n}{2}} r^{-(k+2)}[\sin(k+2)v + |z|\cos(k+2)v] \sum_{j=k_1(n)}^{\bar{k}} w^{j+\frac{1}{2}} \frac{(2(k-j)+1)\, A_{1,n/2}(k,j)}{(n-2(k-j)-3)},$$

$$I_3 = \begin{cases} \sum_{k=0}^{\bar{n}} r^{-(k+1)}[\sin(k+1)v] \sum_{j=k_2(n)}^{\bar{k}} w^{j+\frac{1}{2}} \frac{(k-j)\, B_{1,n/2}(k,j)}{(n/2-(k-j)-1)} & \text{if } n > 2; \\ r^{-1}(\sin v)\sqrt{w} B_{1,1}(0,0) & \text{if } n = 2, \end{cases}$$

with all quantities being defined in section 2.1.1. (Again, $\sum_s^t$ is defined to be zero if $s > t$.)

If $r = 0$, $\rho^\pi(\mathbf{x}) = 0$.

Proof: Apply Theorem 4.1 for $m = 2$. □

For example, if $n = 6$, then

$$\rho^\pi(\mathbf{x}) = \frac{8}{3\pi s}\left\{\frac{\sqrt{w}\sin v}{8r} - \frac{9}{4}\frac{\sqrt{w}\sin(2v)}{r^2} + \frac{3w\sqrt{w}\sin(3v)}{r^3} - \frac{\sqrt{w}\sin(3v')}{r^3} - \frac{|z|\cos(3v')}{r^3}\right\}.$$

Note that $\delta^\pi(\mathbf{x})$ can be obtained by combining this with (14) and (27).

Analogously to the situation with $m(\mathbf{x})$ in section 2.1.1, the calculation of $\rho^\pi(\mathbf{x})$ is unstable if (w,z) is inside the region

$$R_2 = \{(w,z) : ((w-1)^2 + z^2)^2 + 4z^2 < D^{-4/n}\}. \tag{35}$$

For instance, on a Vax11/780 computer, instability can occur when D equals $10^{10.5}$. If $(w,z) \in R_2$, we suggest approximating δ^π by

$$\tilde{\delta}^\pi(\mathbf{x}) = \begin{cases} \bar{x} & \text{if } n \le 12 \\ \hat{\delta}^\pi(\mathbf{x}) & \text{if } n \ge 14, \end{cases} \tag{36}$$

where

$$\begin{aligned}\hat{\delta}^{\pi}(\mathbf{x}) &= \bar{x} - \frac{2sz\sqrt{n-1}}{\sqrt{n}(n-3)w(1+z^2/w)}\{1 + \frac{3nz^2/w - 3z^2/w - 5n + 13}{(n-3)(n-5)w(1+z^2/w)^2} \\ &+ \frac{1}{w^2(1+z^2/w)^4}[\frac{15(3z^2/w-1)(z^2/w-3)}{(n-5)(n-7)} + \frac{(3z^2/w-1)^2}{(n-3)^2} \\ &- \frac{33z^4/w^2 - 54z^2/w + 9}{(n-3)(n-5)}]\}. \end{aligned} \tag{37}$$

(See Fan and Berger (1989a) for development of this approximation.) On a Vax 11/780, the difference between δ^{π} and $\tilde{\delta}^{\pi}$ was observed to be less than 1/200 of the posterior standard deviation (see section 2.3) so that, in the context of statistical estimation, $\tilde{\delta}$ is generally an accurate enough approximation. Again, R_2 is where the modes of the likelihood and the prior are close to each other, in which case numerical integration is likely to be accurate and fast, should greater accuracy be required.

An Alternative Expression

Express $\rho^{\pi}(\mathbf{x})$ in (15) as a function of $\sqrt{w}$ and z, namely

$$k_n(\sqrt{w}, z) = \frac{K_{n-1}\sqrt{n-1}}{\pi} I^1_{n,2}(w, z). \tag{38}$$

It is of interest that $k_n(\sqrt{w}, z)$ can be given in terms of $g_n(\sqrt{w}, z)$ (see (31)) and $g_{n-2}(\sqrt{w}, z)$.

Lemma 2.1 *For $n \geq 2$ and $z \neq 0$,*

$$k_n(\sqrt{w}, z) = \frac{\sqrt{w}}{2\pi z} - \frac{\sqrt{n-2}}{\sqrt{n-3}}\frac{s}{2z} g_{n-2}(\sqrt{w}, z) + (1 - w - z^2)\frac{\sqrt{n-1}}{\sqrt{n}}\frac{s}{2z} g_n(\sqrt{w}, z), \tag{39}$$

and $k_n(\sqrt{w}, 0) = 0$.

Proof: Follows directly from Theorem 4.3 with $m = 2$. □

Equation (39) looks much neater than the complicated formula of Theorem 2.2, and is equally efficient computationally. However, it is less stable for z close to zero, and is hence not preferred.

2.2.2 Interpolation of $\delta^\pi(\mathbf{x})$ for Odd n

It is not possible to calculate $\delta^\pi(\mathbf{x})$ exactly for odd n; thus we again consider an appropriate linear interpolation of values for surrounding even n. Actually, we will find a linear interpolation for

$$\begin{aligned} h_n(\sqrt{w}, z) &= \delta^\pi(\mathbf{x}) - \bar{x} \\ &= \frac{k_n(\sqrt{w}, z)}{g_n(\sqrt{w}, z)}, \end{aligned} \tag{40}$$

where $g_n(\sqrt{w}, z)$ and $k_n(\sqrt{w}, z)$ are defined by (31) and (38), respectively. When n is odd, g_{n-1}, g_{n+1} and k_{n-1}, k_{n+1} can all be calculated by the formulas for $m(\mathbf{x})$ and $\delta^\pi(\mathbf{x})$ in Theorems 2.1 and 2.2. The interpolation is then given by

$$\begin{aligned} \tilde{h}_n(\sqrt{w}, z) &= \beta_n h_{n-1}\left(\frac{\sqrt{n-1}}{\sqrt{n-2}}\sqrt{w}, \frac{\sqrt{n-1}}{\sqrt{n-2}}z\right) \\ &\quad + (1-\beta_n) h_{n+1}\left(\frac{\sqrt{n-1}}{\sqrt{n}}\sqrt{w}, \frac{\sqrt{n-1}}{\sqrt{n}}z\right). \end{aligned} \tag{41}$$

An analysis based on the approximation (37) suggests choosing β_n to be

$$\beta'_n = \left(\frac{n-1}{\sqrt{n}(n-3)} - \frac{n}{\sqrt{n+1}(n-2)}\right)\left(\frac{n-2}{\sqrt{n-1}(n-4)} - \frac{n}{\sqrt{n+1}(n-2)}\right)^{-1}. \tag{42}$$

Based on numerical studies, we actually suggest choosing

$$\beta_3 = 0.225 + \frac{0.15}{1+\sqrt{w}}$$

and, for $n \geq 5$,

$$\beta_n = \begin{cases} 0.68 + 0.01\langle \frac{n}{7} \rangle - 0.1z & \text{if } (\sqrt{w}, z) \in R_2^* \\ \beta'_n & \text{otherwise,} \end{cases}$$

where

$$R_2^* = \{(\sqrt{w}, z) | \sqrt{w} \leq 10^{-\langle\frac{n-1}{2}\rangle+1} \text{ and } z \leq 6.8 + 0.1\langle\frac{n}{7}\rangle - 10\beta_n'\}.$$

When (w, z) falls in R_2 defined by (35), h_{n-1} and h_{n+1} should be replaced by the $\tilde{\delta}^\pi(\mathbf{x}) - \bar{x}$ given by (36). When $(\sqrt{w}, z)$ falls in

$$\tilde{R}_2 = \{(\sqrt{w}, z) | \sqrt{w} \leq 1.53e^{-n/2.82} - 0.027 \text{ and } z \geq (2.25n - 5.25)\sqrt{w} + 3 - 0.1n\}, \quad (43)$$

the standardized error of $\tilde{h}_n$, defined as $|\tilde{h}_n - h_n|/\sqrt{V^\pi}$ where V^π is the posterior variance, can be 2.5 % or more. (On $\tilde{R}_2^c$ the standardized error is no more than 2.5 %.) Thus numerical integration should be used if $(\sqrt{w}, z) \in \tilde{R}_2$. Again, however, $\tilde{R}_2$ will not frequently occur.

Table 2 shows, for various odd n, the maximum standardized error of $\tilde{h}_n$ for $\sqrt{w}$ = 0.1, 0.5, 1, 2, and 3, respectively. As in Table 1, the first number in each of the last 5 columns is the maximum error and the second is where the maximum error occurs over $0 \leq z \leq 6$ (to the nearest 0.1). Clearly the interpolations are very accurate, especially when $\sqrt{w}$ is large.

2.3 Calculation of the Posterior Variance

2.3.1 Representation in terms of $m(\mathbf{x})$ and $\delta^\pi(\mathbf{x})$

The posterior variance, $V^\pi(\mathbf{x})$, can be easily obtained from $m(\mathbf{x})$ and $\delta^\pi(\mathbf{x})$ due to the following lemma.

Lemma 2.2 *The posterior variance is given by*

$$V^\pi(\mathbf{x}) = \frac{\tau}{\pi m(\mathbf{x})} - \tau^2 - (\mu - \delta^\pi(\mathbf{x}))^2. \quad (44)$$

Proof: Apply Theorem 4.2 with $m = 2$. □

The exact formulas for $m(\mathbf{x})$ and $\delta^\pi(\mathbf{x})$, when n is even, of course yield an exact expression for $V^\pi(\mathbf{x})$, through (44). However, when r is small, $V^\pi(\mathbf{x})$ may be inaccurate due to the numerical instability of $m(\mathbf{x})$ and $\delta^\pi(\mathbf{x})$. (See sections 2.1.1 and 2.2.1.) We found that the unstable region for $V^\pi(\mathbf{x})$ is the same as R_2 defined in (35), where $\delta^\pi(\mathbf{x})$ is inaccurate. If $(w, z) \in R_2$, we recommend that $V^\pi(\mathbf{x})$ be approximated by

$$\tilde{V}^\pi(\mathbf{x}) = \begin{cases} s^2/n & \text{if } n \le 44 \\ \hat{V}^\pi(\mathbf{x}) & \text{if } n \ge 46, \end{cases} \tag{45}$$

where $\hat{V}^\pi(\mathbf{x})$ is defined by (44) with $m(\mathbf{x}), \delta^\pi(\mathbf{x})$ replaced by $\hat{m}(\mathbf{x})$ and $\hat{\delta}^\pi(\mathbf{x})$ of (29) and (37), respectively. The resulting relative error of $\sqrt{\tilde{V}^\pi(\mathbf{x})}$ compared with $\sqrt{V^\pi(\mathbf{x})}$ is then bounded by .9 %.

2.3.2 Interpolation of V^π for odd n

The posterior variance, V^π, can also be viewed as a function of $\sqrt{w}$ and z for fixed $\bar{x}$ and s^2. Indeed, (44) can be written as

$$d_n(\sqrt{w}, z) = \frac{\sqrt{(n-1)ws}}{\pi\sqrt{n}g_n(\sqrt{w}, z)} - \frac{(n-1)w\,s^2}{n} - \left(\frac{\sqrt{n-1}sz}{\sqrt{n}} + h_n(\sqrt{w}, z)\right)^2, \tag{46}$$

where g_n and h_n are given, respectively, in (31) and (40). When n is even, this can be calculated explicitely using the exact expressions for g_n and h_n. For odd n, the rescaled interpolation,

$$\begin{aligned} \tilde{d}_n(\sqrt{w}, z) &= \gamma_n\, d_{n-1}\left(\frac{\sqrt{n-1}}{\sqrt{n-2}}\sqrt{w}, \frac{\sqrt{n-1}}{\sqrt{n-2}}z\right) \\ &+ (1-\gamma_n) d_{n+1}\left(\frac{\sqrt{n-1}}{\sqrt{n}}\sqrt{w}, \frac{\sqrt{n-1}}{\sqrt{n}}z\right), \end{aligned} \tag{47}$$

will be considered. We recommend choosing

$$\gamma_3 = \max(0.15, 0.28 - 0.032\sqrt{w});$$

$$\gamma_5 = \begin{cases} (2.4 + 0.5\sqrt{w})^{-1} & \text{if } \sqrt{w} \le 0.2 \\ 0.33(1 + 0.2(z-4)^+)^{-1} + 0.001\sqrt{w} & \text{otherwise,} \end{cases}$$

where $(z-4)^+ = \max(0, z-4)$; and, for $n \geq 7$,

$$\gamma_n = \begin{cases} 0.31 + 0.01(n-7) & \text{if } (\sqrt{w}, z) \in R_3^* \\ \gamma_n' & \text{otherwise,} \end{cases}$$

with

$$\gamma_n' = \frac{(n-1)(n-4)(n^2-n+2)}{2n(n-3)(n^2-2n+2)} \tag{48}$$

and

$$R_3^* = \{(\sqrt{w}, z) | \sqrt{w} < 0.7 \times 4^{-(n+1)/2+4} \text{ and } z > \frac{3}{n-5} + 1.8\}.$$

If $(w, z) \in R_2$, then use $\tilde{V}^\pi(\mathbf{x})$ of (45) for d_{n-1} and d_{n+1}. We recommend using numerical integration if $(\sqrt{w}, z)$ is in

$$\tilde{R}_3 = \begin{cases} \{(\sqrt{w}, z) | \sqrt{w} \leq 1 \text{ and } z \geq 3.3; \text{ or for any } \sqrt{w} \geq 6\} & \text{if } n = 3 \\ \{(\sqrt{w}, z) | \sqrt{w} < (2 + \frac{|n-8|}{3})10^{-(\frac{n-2}{3})} \text{ and} & \\ z > 1.3 + 0.1n + 10\sqrt{w}\} & \text{if } n \geq 5, \end{cases} \tag{49}$$

since the relative error of $\sqrt{\tilde{d}_n}$ can then be 2.5 % or more. Note that, for $n = 3$, $\tilde{R}_3$ includes a region with larger $\sqrt{w}$; this is because $V^\pi(\mathbf{x})$, for $n = 2$, becomes large as $\sqrt{w}$ becomes large (cf. Fan and Berger (1989a)), resulting in inaccurate interpolation when $n = 3$.

Table 3 shows, for various odd n, the maximum relative error of $\sqrt{\tilde{d}_n}$ compared with the posterior standard deviation, $\sqrt{d_n}$, for $\sqrt{w} = 0.1$, 0.5, 1, 2 and 3, respectively. The first number in each of the last 5 columns is the maximum error and the second is where the maximum occurs over the range $0 < z \leq 6$ (to the nearest 0.1) for large n. The error increases for $z > 6$, since $V^\pi(\mathbf{x})$ tends to infinity as z goes to infinity (cf. Fan and Berger (1989a)).

Another possible way to approximate d_n is to plug in $\tilde{g}_n$ and $\tilde{h}_n$, given by (32) and (41), respectively, into (46). However, this approximation is not as accurate as $\tilde{d}_n$, especially for small n.

3 Comparison with Numerical Integration

Table 4 shows the cpu seconds used in computing $m(\mathbf{x})$, $\delta^{\pi}(\mathbf{x}) - \bar{x}$ and $V^{\pi}(\mathbf{x})$ by the exact (and interpolation) formulas and by the IMSL numerical integration subroutine with nominally specified accuracy of 2 % for $m(\mathbf{x})$ and $\delta^{\pi}(\mathbf{x}) - \bar{x}$, using VAX11/780 double precision programs. Columns $\sqrt{w}$ and z indicate the range of $\sqrt{w}$ and z for which $m(\mathbf{x}), \delta^{\pi}(\mathbf{x}) - \bar{x}$ and $V^{\pi}(\mathbf{x})$ were calculated. For each n, the calculations were performed first for $\sqrt{w} = 0.1$ and $z = 4$; then for $1 \leq \sqrt{w} \leq 2$ (by steps of 0.1) and $z = 1.5$; then for $0 < z \leq 2$ (by steps of 0.1) and $\sqrt{w} = 1$; and finally over the grid $0.1 \leq \sqrt{w} \leq 4, 0 \leq z \leq 4$ (to the nearest 0.1), respectively. Note that the interpolation formulas may cause as much as 2.5% relative error over some small set of $(\sqrt{w}, z)$. However, they are typically much more accurate than the 2% nominal accuracy for the numerical integration, especially since the nominal accuracy for IMSL is sometimes not even actually attained. (For instance, when $n = 14$ and the nominal accuracy was specified as .1%, the actual accuracy for $m(\mathbf{x})$ using the IMSL routine is only 10% for $(\sqrt{w}, z) = (0.1, 0.9)$.) Indeed, one must specify quite high nominal accuracy for the IMSL numerical integration subroutine to have a reasonable degree of assurance as to accuracy; hence, if anything, the comparison here is biased in favor of numerical integration based on such consideration. (By numerical study, indeed, we found that 10^{-4} and 10^{-5} are appropriate nominal accuracies for the IMSL routine to compute $m(\mathbf{x})$ and $\delta^{\pi}(\mathbf{x})$, respectively, over the grid $0.1 \leq \sqrt{w} \leq 4, 0 \leq z \leq 4$ for most n. However, for 0.001 % nominal accuracy, it took about twice the cpu time compared with that using 2 % accuracy in calculating $\delta^{\pi}(\mathbf{x}) - \bar{x}$ and $V^{\pi}(\mathbf{x})$ over the region $0.1 \leq \sqrt{w} \leq 4, 0 \leq z \leq 4$.) The following observations can be made.

1. For a single value of $(\sqrt{w}, z)$, the exact and interpolation formulas are only one to five times faster than the IMSL subroutine. However, when calculating over an interval of $\sqrt{w}$ or z (or a lattice of points $(\sqrt{w}, z)$), the difference in cpu time between using the exact formulas and using the IMSL subroutine becomes quite large. The reason is that the exact formulas for $m(\mathbf{x})$ and $\rho^{\pi}(\mathbf{x})$ have many common quantities depending on n or on w, so that

the calculations over an interval or grid of values can be organized efficiently.

2. Recall that $\delta^\pi(\mathbf{x}) - \bar{x} = \rho^\pi(\mathbf{x})/m(\mathbf{x})$, and the essential difference between the exact formulas for $m(\mathbf{x})$ and $\rho^\pi(\mathbf{x})$ is that between the $\psi_i^{2,n}$ and $\psi_i^{*2,n}$ (see Theorem 4.1). Even these expressions share many common terms, however, so that there is almost no additional cost in calculating $\rho^\pi(\mathbf{x})$ past that in calculating $m(\mathbf{x})$. On the other hand, the numerical integration routine must be separately called to compute $m(\mathbf{x})$ and $\rho^\pi(\mathbf{x})$ if δ^π is needed.

3. As expected, the interpolation formulas for odd n needed twice the cpu time needed by the exact formulas for surrounding even n. Of some surprise was that the IMSL subroutine also was often slower for odd n.

A change of variables was made to transform the integral limits from $(-\infty, \infty)$ to (0,1) when using the IMSL subroutine. The IMSL subroutine, *dcadre*, is a competitive quadrature subroutine for performing one-dimensional numerical integration, since it breaks down the integral into integrals over suitably chosen subintervals according to the shape of the function (de Boor (1971)). Therefore, it is a reasonably sophisticated target for comparison with our exact and interpolation results.

4 Calculation of $I^0_{n,m}$ and $I^1_{n,m}$

In this section, exact formulas for $I^0_{n,m}(w,z)$ and $I^1_{n,m}(w,z)$ are given for even n and m. These results can be applied to all three situations mentioned in subsection 1.1.

4.1 Exact Formulas for Even n and m.

Using the notation defined in section 2 and Appendix A, we have

Theorem 4.1 *If $r \neq 0$ and n and m are positive even integers, then*

$$I^0_{n,m}(w,z) = \pi\{\sum_{k=0}^{\bar{n}} r^{-(k+m/2)} \cos(\frac{m}{2} + k)v \; \psi_2^{m,n}(w,k)$$
$$+ \sum_{k=0}^{\underline{n}} r^{-(k+m/2+1)} [\cos(\frac{m}{2} + k + 1)v - |z| \sin(\frac{m}{2} + k + 1)v] \psi_1^{m,n}(w,k)$$

$$
\begin{aligned}
&+ \sum_{k=0}^{\bar{m}} r^{-(k+n/2)} \cos(\frac{n}{2}+k)v' \, \varphi_2^{n,m}(w,k) \\
&+ \sum_{k=0}^{\underline{m}} r^{-(k+n/2+1)} [w\cos(\frac{n}{2}+k+1)v' + |z|\sqrt{w}\sin(\frac{n}{2}+k+1)v'] \varphi_1^{n,m}(w,k)\},
\end{aligned}
$$

and

$$
I^1_{n,m}(w,z) = \pi \; sgn(z) \; \{J_1 + J_2 + J_3 + J_4 - |z|(J_5 + J_6)\},
$$

where

$$
\begin{aligned}
J_1 &= \sum_{k=0}^{\bar{n}} r^{-(m/2+k)} \sin(\frac{m}{2}+k)v \; \psi_2^{\star m,n}(w,k), \\
J_2 &= \sum_{k=0}^{\underline{n}} r^{-(m/2+k+1)} [\sin(\frac{m}{2}+k+1)v + |z|\cos(\frac{m}{2}+k+1)v] \; \psi_1^{\star m,n}(w,k), \\
J_3 &= -\sum_{k=0}^{\bar{m}} r^{-(n/2+k)} \sqrt{w} \sin(\frac{n}{2}+k)v' \; \varphi_2^{\star n,m}(w,k), \\
J_4 &= \sum_{k=0}^{\underline{m}} r^{-(n/2+k+1)} \sqrt{w} [w\sin(\frac{n}{2}+k+1)v' - |z|\sqrt{w}\cos(\frac{n}{2}+k+1)v'] \; \varphi_1^{\star n,m}(w,k), \\
J_5 &= \sum_{k=0}^{\bar{m}} r^{-(n/2+k)} \; \cos(\frac{n}{2}+k)v' \; \varphi_2^{n,m}(w,k), \\
J_6 &= \sum_{k=0}^{\underline{m}} r^{-(n/2+k+1)} [w\cos(\frac{n}{2}+k+1)v' + |z|\sqrt{w} \; \sin(\frac{n}{2}+k+1)v'] \; \varphi_1^{n,m}(w,k),
\end{aligned}
$$

with

$$
\begin{aligned}
\psi_1^{m,n}(w,k) &= w^{(m-1)/2} \sum_{j=k_1(n)}^{\tilde{k}} w^j A_{m/2,n/2}(k,j), \\
\psi_2^{m,n}(w,k) &= w^{(m-1)/2} \sum_{j=k_2(n)}^{\tilde{k}} w^j B_{m/2,n/2}(k,j), \\
\varphi_1^{n,m}(w,k) &= w^k \sum_{j=k_1(n)}^{\tilde{k}} w^{-j} A_{m/2,n/2}(k,j), \\
\varphi_2^{n,m}(w,k) &= w^k \sum_{j=k_2(n)}^{\tilde{k}} w^{-j} B_{m/2,n/2}(k,j), \\
\psi_1^{\star m,n}(w,k) &= w^{(m-1)/2} \sum_{j=k_1(n)}^{\tilde{k}} w^j A_{m/2,n/2}(k,j) \frac{(2(k-j)+1)}{(n-2(k-j)-3)},
\end{aligned}
$$

$$
\begin{aligned}
\psi_2^{*m,n}(w,k) &= w^{(m-1)/2}\sum_{j=k_2(n)}^{\tilde{k}} w^j B_{m/2,n/2}(k,j)\frac{(k-j)}{(n/2-(k-j)-1)} \quad \text{if } n>2;\\
\psi_2^{*m,2}(w,0) &= \sqrt{w}B_{m/2,1}(0,0),\\
\varphi_1^{*n,m}(w,k) &= w^k\sum_{j=k_1(m)}^{\tilde{k}} w^{-j}A_{n/2,m/2}(k,j)\frac{(2(k-j)+1)}{(m-2(k-j)-3)},\\
\varphi_2^{*n,m}(w,k) &= w^k\sum_{j=k_2(m)}^{\tilde{k}} w^{-j}B_{n/2,m/2}(k,j)\frac{(k-j)}{(m/2-(k-j)-1)} \quad \text{if } m>2;\\
\varphi_2^{*n,2}(w,0) &= B_{n/2,1}(0,0),
\end{aligned}
$$

where $k_i(\cdot),\tilde{k}$ and $\underline{n},\bar{n}$ are defined by (25) and (23), and $A_{p,q}$ and $B_{p,q}$ are given in the Appendix A. If $r=0$, $I^0_{n,m}(w,z)=\Gamma(\frac{n+m-1}{2})\sqrt{\pi}/\Gamma(\frac{n+m}{2})$ and $I^1_{n,m}(w,z)=0$.

Proof: See Appendix B. □

Another quantity of interest is $I^2_{n,m}(w,z)$ defined as

$$
I^2_{n,m}(w,z)=\int\frac{\xi^2}{\sqrt{w}(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}}d\xi.
$$

Lemma 4.1 $I^2_{n,m}(w,z)=wI^0_{n,m-2}(w,z)-(w+z^2)I^0_{n,m}(w,z)-2zI^1_{n,m}(w,z)$.

Proof:

$$
\begin{aligned}
I^2_{n,m}(w,z) &= \int\frac{w+(\xi+z)^2-2z\xi-z^2-w}{\sqrt{w}(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}}d\xi\\
&= wI^0_{n,m-2}(w,z)-(w+z^2)I^0_{n,m}(w,z)-2zI^1_{n,m}(w,z).
\end{aligned}
$$
□

Thus Theorem 4.1 can also be used to calculate $I^2_{n,m}(w,z)$, and hence for instance the posterior variance, $V^\pi(\mathbf{x})$, defined by (16) of subsection 1.1.1.

Theorem 4.2 *The posterior variance defined by (16) can be expressed as*

$$
V^\pi(\mathbf{x})=\frac{\sqrt{m-1}\tau^2 I^0_{n,m-2}(w,z)}{I^0_{n,m}(w,z)}-(m-1)\tau^2-(\mu-\delta^\pi(\mathbf{x}))^2,
$$

where w, z and $\delta^\pi(\mathbf{x})$ are given by (12), (13) and (14).

Note that $I^0_{n,0}(w,z) = (\sqrt{n-1}K_{n-1})^{-1}$, where K_i is defined by (5).

Proof: Changing variables in (16) yields

$$
\begin{aligned}
V^\pi(\mathbf{x}) &= \frac{1}{I^0_{n,m}(w,z)} \int \frac{(\bar{x} + \sqrt{n-1}s/\sqrt{n}\xi)^2}{\sqrt{w}(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi - (\delta^\pi(\mathbf{x}))^2 \\
&= \bar{x}^2 + 2\bar{x}\frac{\sqrt{n-1}s}{\sqrt{n}} \frac{I^1_{n,m}(w,z)}{I^0_{n,m}(w,z)} + \frac{(n-1)s^2}{n}\frac{I^2_{n,m}(w,z)}{I^0_{n,m}(w,z)} - (\delta^\pi(\mathbf{x}))^2.
\end{aligned}
$$

Applying Lemma 2.3 together with (12), (13) and (14) yields

$$
\begin{aligned}
V^\pi(\mathbf{x}) &= \bar{x}^2 + 2(\bar{x} - z\frac{\sqrt{n-1}s}{n})\frac{\sqrt{n-1}s}{n}\frac{I^1_{n,m}(w,z)}{I^0_{n,m}(w,z)} - \frac{(n-1)s^2}{n}z^2 \\
&\quad -(\delta^\pi(\mathbf{x}))^2 + \frac{(n-1)s^2 w}{n}\left(\frac{I^0_{n,m-2}(w,z)}{I^0_{n,m}(w,z)} - 1\right) \\
&= \bar{x}^2 + 2\mu(\delta^\pi(\mathbf{x}) - \bar{x}) - (\bar{x}-\mu)^2 - (\delta^\pi(\mathbf{x}))^2 + (m-1)\tau^2\left(\frac{I^0_{n,m-2}(w,z)}{I^0_{n,m}(w,z)} - 1\right),
\end{aligned}
$$

from which the result is immediate. □

4.2 Recursive Expressions for $I^0_{n,m}$ and $I^1_{n,m}$

The following Theorem gives recursive formulas for $I^1_{n,m}$ and $I^0_{n,m}$.

Theorem 4.3 *For $z \neq 0$,*

$$
\begin{aligned}
i)\ I^1_{n,m}(w,z) &= \frac{(1-z^2-w)}{2z}I^0_{n,m}(w,z) + \frac{w}{2z}I^0_{n,m-2}(w,z) - \frac{1}{2z}I^0_{n-2,m}(w,z), \\
ii)\ I^0_{n,m}(w,z) &= \frac{1}{(1+z^2-w)^{n/2-1}}I^0_{2,m}(w,z) \\
&\quad + \frac{w}{(m-2)(1+z^2-w)}\sum_{k=2}^{n/2}\frac{(2k-m+2)}{(1+z^2-w)^{n/2-k}}I^0_{2k,m-2}(w,z) \\
&\quad + \frac{(1-z^2-w)w}{(m-2)(1+z^2-w)}\sum_{k=2}^{n/2}\frac{2k}{(1+z^2-w)^{n/2-k}}I^0_{2k+2,m-2}(w,z) \\
&\quad + \frac{w^2}{(m-2)(1+z^2-w)}\sum_{k=2}^{n/2}\frac{2k}{(1+z^2-w)^{n/2-k}}I^0_{2k+2,m-4}(w,z).
\end{aligned}
$$

Proof: i) $I^0_{n,m}(w,z) = \int \frac{1}{(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi$

$$= \int \frac{1+(\xi+z)^2 - 2\xi z - z^2}{(1+\xi^2)^{n/2+1}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi$$

$$= (1-z^2) \int \frac{1}{(1+\xi^2)^{(n+2)/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi$$

$$+ w \int \frac{1+w^{-1}(\xi+z)^2 - 1}{(1+\xi^2)^{(n+2)/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi - 2z I^1_{n+2,m}(w,z)$$

$$= (1-z^2-w) I^0_{n+2,m}(w,z) + w I^0_{n+2,m-2}(w,z) - 2z I^1_{n+2,m}(w,z).$$

This proves i).

ii) $I^1_{n,m}(w,z) = \int \frac{\xi + z - z}{(1+\xi^2)^{n/2}(1+w^{-1}(\xi+z)^2)^{m/2}} d\xi$

$$= -\frac{wn}{2(m-2)} \int \frac{2\xi}{(1+\xi^2)^{n/2+1}(1+w^{-1}(\xi+z)^2)^{m/2-1}} d\xi - z I^0_{n,m}(w,z).$$

Thus,

$$I^1_{n,m}(w,z) = -\frac{nw}{m-2} I^1_{n+2,m-2}(w,z) - z I^0_{n,m}(w,z). \tag{50}$$

Plugging the result of i) for $I^1_{n,m}$ and $I^1_{n+2,m-2}$ into (50) yields

$$\begin{aligned} I^0_{n,m}(w,z) &= \frac{1}{1+z^2-w} I^0_{n-2,m}(w,z) + \frac{w(n-m+2)}{(m-2)(1+z^2-w)} I^0_{n,m-2}(w,z) \\ &\quad - \frac{nw(1-z^2-w)}{(m-2)(1+z^2-w)} I^0_{n+2,m-2}(w,z) \\ &\quad - \frac{nw^2}{(m-2)(1+z^2-w)} I^0_{n+2,m-4}(w,z). \end{aligned}$$

The result then follows by induction. □

Although the above formulas for $I^0_{n,m}$ and $I^1_{n,m}$ are considerably shorter than those in Theorem 4.1, recursive calculation turned out to be much more expensive. It is also less accurate for large n, due to round off error.

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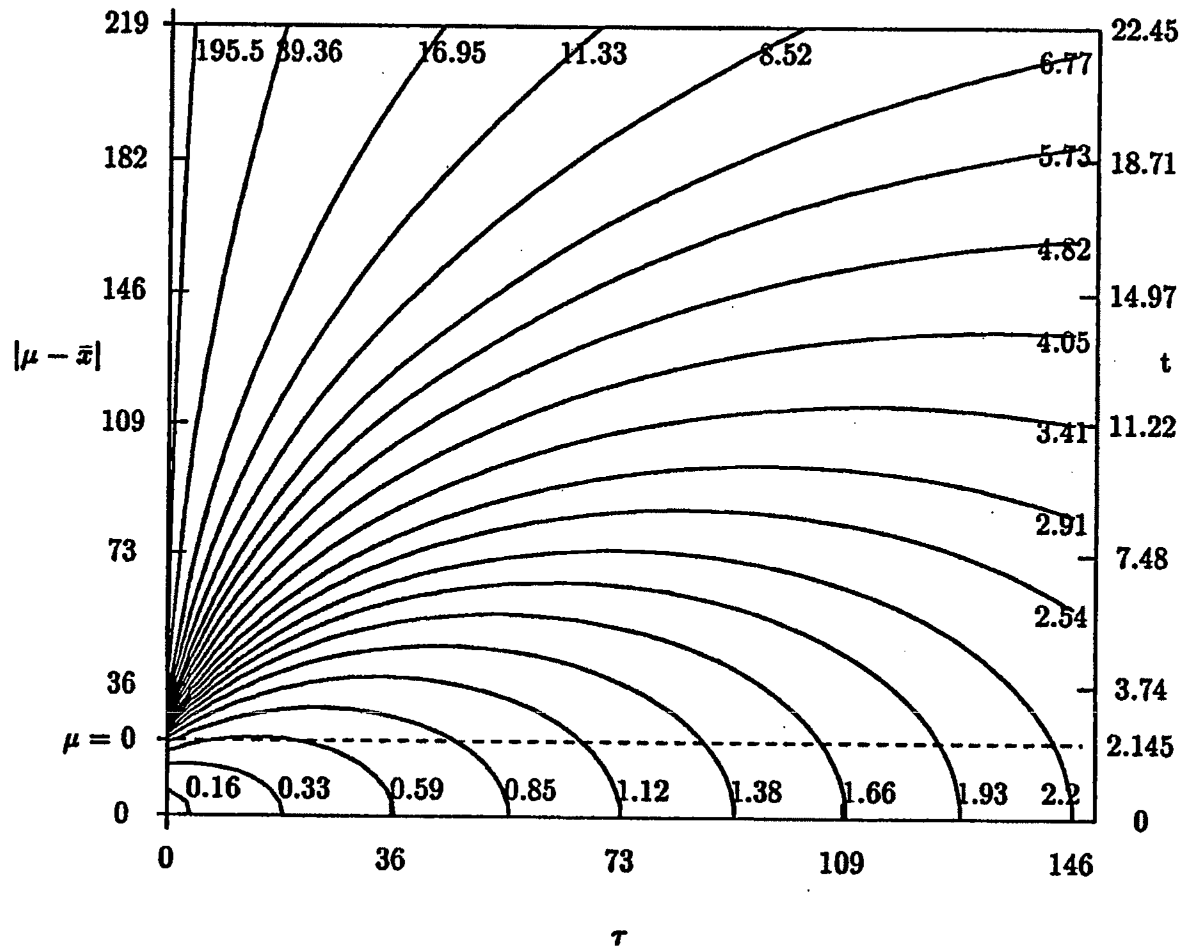

Figure 1: Contours of B as a function of $|\mu - \bar{x}|$ and τ when $\bar{x} = 20.93, s = 37.79, \theta_0 = 0$ and $n = 15$.

Table 1: The Relative Accuracy of $\tilde{g}_n$.

n	$\sqrt{w}$				
	0.1	0.5	1	2	3
3	(.0216, 2.0)*	(.0221, 6.0)	(.0250, 5.6)	(.0092, 6.0)	(.0052, 2.0)
5	(.0244, 4.3)*	(.0124, 4.9)	(.0057, 5.6)	(.0174, 3.0)	(.0012, 6.0)
7	(.0168, 4.1)	(.0021, 2.9)	(.0008, 3.4)	(.0003, 4.4)	(.0002, 0.1)
9	(.0036, 1.9)	(.0006, 2.2)	(.0002, 2.7)	(.0001, 0.1)	(.0000, 0.1)
11	(.0015, 1.6)	(.0002, 1.8)	(.0001, 2.3)	(.0001, 0.1)	(.0000, 0.1)
13	(.0007, 1.4)	(.0001, 1.6)	(.0000, 2.0)	(.0000, 0.1)	(.0000, 0.1)
15	(.0004, 1.2)	(.0000, 1.5)	(.0000, 1.9)	(.0000, 0.1)	(.0000, 0.1)
17	(.0002, 1.0)	(.0000, 1.3)	(.0000, 0.1)	(.0000, 0.1)	(.0000, 0.1)
19	(.0001, 1.0)	(.0000, 1.3)	(.0000, 0.1)	(.0000, 0.1)	(.0000, 0.1)
21	(.0001, 0.9)	(.0000, 1.2)	(.0000, 0.1)	(.0000, 0.1)	(.0000, 0.1)
23	(.0001, 0.9)	(.0000, 1.1)	(.0000, 0.1)	(.0000, 0.1)	(.0000, 0.1)
25	(.0000, 0.8)	(.0000, 1.1)	(.0000, 1.6)	(.0000, 2.7)	(.0000, 4.0)
27	(.0000, 0.8)	(.0000, 1.0)	(.0000, 1.5)	(.0000, 2.7)	(.0000, 4.0)
29	(.0000, 0.7)	(.0000, 0.1)	(.0001, 0.1)†	(.0000, 0.1)	(.0000, 0.1)
31	(.0000, 0.1)	(.0000, 0.1)	(.0001, 0.1)†	(.0000, 0.1)	(.0000, 0.1)

* for $(\sqrt{w}, z) \notin \tilde{R}_1$, † for $(\sqrt{w}, z) \in R_1$

Table 2: Standardized Relative Accuracy of $\tilde{h}_n$.

n	$\sqrt{w}$ 0.1	0.5	1	2	3
3	(.0248, 2.7)*	(.0212, 3.6)	(.0191, 6.0)	(.0171, 6.0)	(.0134, 6.0)
5	(.0217, 3.2)*	(.0240, 2.4)	(.0178, 2.4)	(.0153, 2.8)	(.0102, 3.3)
7	(.0246, 4.4)	(.0062, 1.1)	(.0036, 1.2)	(.0014, 1.6)	(.0008, 2.2)
9	(.0083, 2.7)	(.0018, 0.8)	(.0009, 0.9)	(.0003, 1.3)	(.0001, 1.6)
11	(.0035, 2.0)	(.0007, 0.6)	(.0003, 0.8)	(.0001, 1.1)	(.0000, 1.4)
13	(.0017, 1.7)	(.0004, 0.6)	(.0001, 0.7)	(.0000, 1.0)	(.0000, 1.3)
15	(.0010, 1.5)	(.0002, 0.5)	(.0001, 0.6)	(.0000, 1.0)	(.0000, 1.3)
17	(.0006, 1.3)	(.0001, 0.5)	(.0000, 0.6)	(.0000, 0.9)	(.0000, 1.2)
19	(.0004, 1.2)	(.0001, 0.4)	(.0000, 0.6)	(.0000, 0.9)	(.0000, 1.2)
21	(.0002, 1.1)	(.0000, 0.4)	(.0000, 0.6)	(.0000, 0.9)	(.0000, 1.2)
23	(.0002, 1.0)	(.0000, 0.4)	(.0000, 0.5)	(.0000, 0.8)	(.0000, 1.2)
25	(.0001, 0.3)	(.0000, 0.4)	(.0000, 0.5)	(.0000, 0.8)	(.0000, 1.2)
27	(.0001, 0.3)	(.0000, 0.4)	(.0000, 0.1)†	(.0000, 0.8)	(.0000, 1.2)
29	(.0001, 0.3)	(.0000, 0.4)	(.0000, 0.1)†	(.0000, 0.8)	(.0000, 1.2)
31	(.0001, 0.3)	(.0000, 0.3)	(.0000, 0.1)†	(.0000, 0.8)	(.0000, 1.2)

* for $(\sqrt{w}, z) \notin \tilde{R}_2$, † for $(\sqrt{w}, z) \in R_2$

Table 3: Relative Accuracy of $\sqrt{\tilde{d}_n}$.

n	$\sqrt{w}$				
	0.1	0.5	1	2	3
3	(.0236, 4.6)*	(.0240, 3.2)*	(.0208, 6.0)	(.0243, 6.0)	(.0188, 6.0)
5	(.0196, 4.5)*	(.0213, 4.0)	(.0230, 4.0)	(.0181, 4.1)	(.0218, 6.0)
7	(.0232, 4.4)*	(.0140, 3.3)	(.0136, 6.0)	(.0057, 6.0)	(.0029, 6.0)
9	(.0164, 2.5)	(.0058, 3.3)	(.0026, 3.7)	(.0010, 4.8)	(.0006, 6.0)
11	(.0090, 2.4)	(.0019, 2.5)	(.0008, 2.9)	(.0003, 4.2)	(.0002, 6.0)
13	(.0040, 1.9)	(.0008, 2.1)	(.0003, 2.5)	(.0002, 6.0)	(.0002, 6.0)
15	(.0021, 1.6)	(.0004, 1.8)	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)
17	(.0012, 1.4)	(.0002, 1.6)	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)
19	(.0007, 1.3)	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)
21	(.0005, 1.2)	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)	(.0003, 6.0)
23	(.0003, 1.1)	(.0002, 6.0)	(.0002, 6.0)	(.0003, 6.0)	(.0003, 6.0)
25	(.0018, 6.0)	(.0018, 6.0)	(.0018, 6.0)	(.0020, 6.0)	(.0023, 6.0)
27	(.0016, 6.0)	(.0016, 6.0)	(.0016, 6.0)	(.0018, 6.0)	(.0020, 6.0)
29	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)	(.0002, 6.0)	(.0003, 6.0)
31	(.0004, 6.0)	(.0004, 6.0)	(.0004, 6.0)	(.0005, 6.0)	(.0005, 6.0)

* for $(\sqrt{w}, z)$ not in $\tilde{R}_3$ and $z \leq 6$

Table 4: Comparison of the IMSL subroutine and the exact (and interpolation) formulas.

n	$\sqrt{w}$	z	$m(\mathrm{x})$		$\delta^*(\mathrm{x}) - \bar{x}$		$V^*(\mathrm{x})$	
			exact	IMSL	exact	IMSL	exact	IMSL
5	(0.1, 0.1)	(4.0, 4.0)	0.1	0.3	0.1	0.5	0.1	0.5
	(1.0, 2.0)	(1.5, 1.5)	0.2	0.4	0.2	1.2	0.2	1.3
	(1.0, 1.0)	(0.1, 2.0)	0.2	1.2	0.2	2.6	0.2	2.6
	(0.1, 4.0)	(0.0, 4.0)	12.0	96.5	12.8	214.9	13.5	217.0
10	(0.1, 0.1)	(4.0, 4.0)	0.1	0.1	0.1	0.3	0.1	0.3
	(1.0, 2.0)	(1.5, 1.5)	0.1	0.5	0.2	1.3	0.2	1.3
	(1.0, 1.0)	(0.1, 2.0)	0.1	1.0	0.2	2.1	0.2	2.2
	(0.1, 4.0)	(0.0, 4.0)	11.2	67.4	12.0	179.5	12.1	190.4
15	(0.1, 0.1)	(4.0, 4.0)	0.1	0.2	0.1	0.2	0.1	0.2
	(1.0, 2.0)	(1.5, 1.5)	0.2	1.0	0.3	1.6	0.2	1.7
	(1.0, 1.0)	(0.1, 2.0)	0.3	1.9	0.3	3.1	0.3	3.3
	(0.1, 4.0)	(0.0, 4.0)	14.5	144.2	16.0	232.2	16.9	244.8
20	(0.1, 0.1)	(4.0, 4.0)	0.1	0.1	0.1	0.3	0.1	0.3
	(1.0, 2.0)	(1.5, 1.5)	0.2	0.6	0.2	1.4	0.2	1.5
	(1.0, 1.0)	(0.1, 2.0)	0.2	1.1	0.2	2.7	0.3	2.8
	(0.1, 4.0)	(0.0, 4.0)	12.5	75.9	14.0	202.3	14.4	207.3
25	(0.1, 0.1)	(4.0, 4.0)	0.1	0.1	0.1	0.2	0.1	0.2
	(1.0, 2.0)	(1.5, 1.5)	0.3	0.5	0.4	1.2	0.3	1.3
	(1.0, 1.0)	(0.1, 2.0)	0.3	0.9	0.4	2.1	0.3	2.2
	(0.1, 4.0)	(0.0, 4.0)	16.8	68.7	19.3	174.1	20.1	179.0

Appendices

Appendix A. Notation for Theorem 4.1.

Letting $p = m/2$ and $q = n/2$, define

$$\alpha_p(k) = \begin{cases} -2 & \text{if } k = 0 \\ (-2)(p+k)\alpha_p(k-1) & \text{if } k > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{51}$$

$$\beta_q(i) = \begin{cases} 4^{(\bar{n}/2+1)}/(\bar{n}+1)! & \text{if } i = \bar{n}/2 \\ \dfrac{(\bar{n}+1)\,\bar{n}\,n\,(n+2)\,\beta_q(i+1)}{16\,(q-\bar{n})!} & \text{if } i = \frac{\bar{n}}{2}-1,\ \bar{n} = \underline{n} \\ \dfrac{(\bar{n}+1)\,\bar{n}\,n\,\beta_q(i+1)}{8\,(q-\bar{n})!} & \text{if } i = \frac{\bar{n}}{2}-1,\ \bar{n} \neq \underline{n} \\ \frac{(2i+3)(i+1)(q-i-2)(n-2i-3)\,\beta_q(i+1)}{(q-2i-2)(q-2i-3)} & \text{if } 0 \leq i \leq \bar{n}-2 \\ 0 & \text{otherwise,} \end{cases} \tag{52}$$

$$\gamma_p(k,j) = \begin{cases} 1 & \text{if } k = j = 0, \text{ or,} \\ & \text{if } k \text{ is even and } j = \tilde{k} \\ \prod_{l=1}^{k}(2p+2l-1) & \text{if } k \geq 1, j = 0 \\ \frac{(k-2j)(k-2j-1)\gamma_p(k,j-1)}{(k-j)(j+1)(2p+2j+1)(2p+2(k-j)-1)} & \text{if } k \geq 1, 0 < j \leq \tilde{k}-2 \\ (j+1)(2p+2j+1) & \text{if } k \text{ is odd, } j = \tilde{k} \\ 0 & \text{otherwise,} \end{cases} \tag{53}$$

$$\lambda_q(i) = \begin{cases} 2\,(q-i-1)(2i+1)/(q-2i-1) & \text{if } 0 \leq i \leq \underline{n}/2 \\ (2i+1) & \text{if } i = \bar{n}/2 \neq \underline{n}/2 \\ 0 & \text{otherwise,} \end{cases} \tag{54}$$

$$A_{p,q}(k,j) = 2^{(1-2q)}\alpha_p(k)\beta_q(k-j)\gamma_p(k,j), \tag{55}$$

$$B_{p,q}(k,j) = -A_{p,q}(k,j)\lambda_q(k-j)/[4\,(k+p)]. \tag{56}$$

The formulas above have been written in a fashion amenable to easy iterative calculation.

Appendix B. Proof of Theorem 4.1.

Let $\xi_1(\zeta) = (w + (\zeta + z)^2)^{-m/2}$, and $\xi_2(\zeta) = (1 + \zeta^2)^{-n/2}$, where w and z are defined by (12) and (13) respectively. Then

$$I^0_{n,m}(w,z) = w^{(m-1)/2} \int \xi_1(\zeta)\xi_2(\zeta)d\zeta.$$

Also let

$$\phi_1(\zeta) = (\zeta + i)^{-n/2}, \quad \text{and} \quad \phi_2(\zeta) = (\zeta + z + i\sqrt{w})^{-m/2}.$$

Note that $\xi_1(\zeta) \cdot \xi_2(\zeta)$ has poles at $\zeta_1 = i$ of order $n/2$ and $\zeta_2 = -z + i\sqrt{w}$ of order $m/2$ on the upper plane. The residue theorem gives

$$I^0_{n,m}(w,z) = w^{(m-1)/2} \times \text{ the real part of } \{2\pi i \cdot (\text{Res}(\zeta_1) + \text{Res}(\zeta_2))\}, \tag{57}$$

where

$$\begin{aligned} \text{Res}(\zeta_1) &= \frac{1}{(\frac{n}{2}-1)!} \frac{d^{n/2-1}}{d\zeta^{n/2-1}} (\zeta - \zeta_1)^{n/2} f(\zeta)|_{\zeta=\zeta_1} \\ &= \frac{1}{(\frac{n}{2}-1)!} \sum_{j=0}^{\frac{n}{2}-1} \begin{pmatrix} \frac{n}{2}-1 \\ j \end{pmatrix} \frac{d^{n/2-j-1}}{d\zeta^{n/2-j-1}} \phi_1(\zeta) \frac{d^j}{d\zeta^j} \xi_1(\zeta)|_{\zeta=\zeta_1}, \end{aligned} \tag{58}$$

and

$$\text{Res}(\zeta_2) = \frac{1}{(\frac{m}{2}-1)!} \sum_{j=0}^{\frac{m}{2}-1} \begin{pmatrix} \frac{m}{2}-1 \\ j \end{pmatrix} \frac{d^{m/2-j-1}}{d\zeta^{m/2-j-1}} \phi_2(\zeta) \frac{d^j}{d\zeta^j} \xi_2(\zeta)|_{\zeta=\zeta_2}. \tag{59}$$

Note next that

$$\begin{aligned} \frac{d^k}{d\zeta^k}\phi_1(\zeta)|_{\zeta=\zeta_1} &= (-1)^k \frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)(\zeta+i)^{-(\frac{n}{2}+k)}|_{\zeta=\zeta_1} \\ &= (-1)^k \frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)(2i)^{-(\frac{n}{2}+k)}, \end{aligned} \tag{60}$$

$$\frac{d^k}{d\zeta^k}\phi_2(\zeta)|_{\zeta=\zeta_2} = (-1)^k \frac{m}{2}(\frac{m}{2}+1)\cdots(\frac{m}{2}+k-1)(2i\sqrt{w})^{-(\frac{m}{2}+k)}. \tag{61}$$

Define

$$a_{j,s,r} = \frac{1}{2}(-1)^{s+1}\alpha_{r-1}(s+j)\gamma_r(s+j,j),$$

where $\alpha_p(\cdot), \gamma_p(\cdot,\cdot)$ are given by (51) and (53) respectively; then

i) if $l = 2s$,

$$\begin{aligned}
\frac{d^l}{d\zeta^l}\xi_1(\zeta)|_{\zeta=\zeta_1} &= \sum_{j=0}^{s} \frac{a_{j,s,\frac{m}{2}}\ \ w^j}{(w+(\zeta+z)^2)^{m/2+s+j}}|_{\zeta=\zeta_1} \\
&= \sum_{j=0}^{s} a_{j,s,\frac{m}{2}}\ \ w^j r^{-(m/2+s+j)}(\cos v - i\sin v)^{m/2+s+j}, \qquad (62) \\
\frac{d^l}{d\zeta^l}\xi_2(\zeta)|_{\zeta=\zeta_2} &= \sum_{j=0}^{s} a_{j,s,\frac{n}{2}}\ \ r^{-(n/2+s+j)}(\cos v' - i\sin v')^{n/2+s+j}, \qquad (63)
\end{aligned}$$

ii) if $l = 2s+1$,

$$\begin{aligned}
\frac{d^l}{d\zeta^l}\xi_1(\zeta)|_{\zeta=\zeta_1} &= \sum_{j=0}^{s} \frac{a_{j,s,\frac{m}{2}}\ \ w^j(-2)(\frac{m}{2}+s+j)(\zeta+z)}{(w+(\zeta+z)^2)^{m/2+s+j+1}}|_{\zeta=\zeta_1} \\
&= \sum_{j=0}^{s} a_{j,s,\frac{m}{2}}\ \ w^j(-2)(\frac{m}{2}+s+j)(i+z) \\
&\quad \times r^{-(m/2+s+j+1)}(\cos v - i\sin v)^{m/2+s+j+1}, \qquad (64) \\
\frac{d^l}{d\zeta^l}\xi_2(\zeta)|_{\zeta=\zeta_2} &= \sum_{j=0}^{s} a_{j,s,\frac{n}{2}}\ \ (-2)(\frac{n}{2}+s+j)(\sqrt{w}i - z) \\
&\quad \times r^{-(n/2+s+j+1)}(\cos v' - i\sin v')^{n/2+s+j+1}, \qquad (65)
\end{aligned}$$

where r, v, and v' are defined in (22). Combining (58), (60), and (62) yields

$$\begin{aligned}
\mathrm{Res}(\zeta_1) &= \sum_{j=0}^{q-1} \begin{pmatrix} n-j-2 \\ q-1 \end{pmatrix} \frac{(-1)^{j+1}}{j!} 2^{-(n-j-1)} i^{j+1} \frac{d^j}{d\zeta^j}\xi_1(\zeta)|_{\zeta=\zeta_1} \\
&= \sum_{s=0}^{\langle\frac{q-1}{2}\rangle} \begin{pmatrix} n-2s-2 \\ q-1 \end{pmatrix} \frac{(-1)^{s+1}}{(2s)!} 2^{-(n-2s-1)} i \sum_{j=0}^{s} a_{j,s,p}\ \ w^j \\
&\quad \cdot r^{-(m/2+s+j)}(\cos(\frac{m}{2}+s+j)v - i\sin(\frac{m}{2}+s+j)v)
\end{aligned}$$

$$+ \sum_{s=0}^{\langle\frac{q-2}{2}\rangle} \begin{pmatrix} n-2s-3 \\ q-1 \end{pmatrix} \frac{(-1)^s}{(2s+1)!} 2^{-(n-2s-3)} \sum_{j=0}^{s} a_{j,s,p}\ w^j (\frac{m}{2}+s+j)$$
$$\cdot r^{-(m/2+s+j+1)}(i+z)(\cos(\frac{m}{2}+s+j+1)v - i\sin(\frac{m}{2}+s+j+1)v).$$

The real part of $\{2\pi i \mathrm{Res}(\zeta_1)\}$ becomes

$$\begin{aligned}
\mathrm{Re}\ \{2\pi i \mathrm{Res}(\zeta_1)\} &= 2\pi \left\{ \sum_{s=0}^{\langle\frac{q-1}{2}\rangle} \begin{pmatrix} n-2s-2 \\ q-1 \end{pmatrix} \frac{(-1)^s}{(2s)!} 2^{-(n-2s-1)} \right. \\
&\quad \cdot \sum_{j=0}^{s} a_{j,s,p}\ w^j r^{-(m/2+s+j)} \cos(\frac{m}{2}+s+j)v \\
&\quad + \sum_{s=0}^{\langle\frac{q-2}{2}\rangle} \begin{pmatrix} n-2s-3 \\ q-1 \end{pmatrix} \frac{(-1)^s}{(2s+1)!} 2^{-(n-2s-2)} \\
&\quad \cdot \sum_{j=0}^{s} a_{j,s,p}\ w^j (\frac{m}{2}+s+j) \cdot r^{-(m/2+s+j+1)} \\
&\quad \left. (\cos(\frac{m}{2}+s+j+1)v - |z|\sin(\frac{m}{2}+s+j+1)v) \right\}.
\end{aligned}$$

Rewriting it in terms of $\sum_{k=s+j}$ and $\sum_{j}$ yields

$$\begin{aligned}
\mathrm{Re}\ \{2\pi i \mathrm{Res}(\zeta_1)\} &= \pi \left\{ \sum_{k=0}^{\bar{n}} r^{-(m/2+k)} \cos(\frac{m}{2}+k)v \right. \\
&\quad \cdot \sum_{j=k_2(n)}^{\bar{k}} \begin{pmatrix} n-2(k-j)-2 \\ q-1 \end{pmatrix} \frac{(-1)^{k-j}}{(2(k-j))!} 2^{-(n-2(k-j)-2)} a_{j,s,p}\ w^j \\
&\quad + \sum_{k=0}^{\underline{n}} r^{-(m/2+k+1)} (\cos(\frac{m}{2}+k+1)v - |z|\sin(\frac{m}{2}+k+1)v) \\
&\quad \cdot \sum_{j=k_1(n)}^{\bar{k}} \begin{pmatrix} n-2(k-j)-3 \\ q-1 \end{pmatrix} \frac{(-1)^{k-j+1}}{(2(k-j)+1)!} \\
&\quad \left. \cdot 2^{-(n-2(k-j)-4)} (\frac{m}{2}+k) a_{j,s,p}\ w^j \right\} \\
&= \pi \left\{ \sum_{k=0}^{\bar{n}} r^{-(m/2+k)} \cos(\frac{m}{2}+k)v \sum_{j=k_2(n)}^{\bar{k}} w^j B_{p,q}(k,j) \right.
\end{aligned}$$

$$+\sum_{k=0}^{n} r^{-(m/2+k+1)}(\cos(\frac{m}{2}+k+1)v - |z|\sin(\frac{m}{2}+k+1)v)$$
$$\cdot \sum_{j=k_1(n)}^{\tilde{k}} w^j A_{p.q}(k,j)\Bigg\}. \quad (66)$$

A similar argument yields

$$\begin{aligned} \text{Re }\{2\pi i\text{Res}(\zeta_2)\} &= \pi w^{-(m-1)/2}\Bigg\{\sum_{k=0}^{m} r^{-(n/2+k)}\cos(\frac{n}{2}+k)v' \sum_{j=k_2(m)}^{\tilde{k}} w^{k-j}B_{p,q}(k,j) \\ &+ \sum_{k=0}^{m} r^{-(n/2+k+1)}(w\cos(\frac{n}{2}+k+1)v' + |z|\sqrt{w}\sin(\frac{n}{2}+k+1)v') \\ &\quad \cdot \sum_{j=k_1(m)}^{\tilde{k}} w^{k-j}A_{p,q}(k,j)\Bigg\}. \end{aligned} \quad (67)$$

Using (57), together with (66) and (67), completes the proof for $I^0_{n,m}(w,z)$.

Now, consider

$$I^1_{n,m}(w,z) = w^{(m-1)/2}\int \zeta\xi_1(\zeta)\xi_2(\zeta)d\zeta.$$

Let

$$\phi_3(\zeta) = \zeta(\zeta+i)^{-n/2}, \text{ and } \phi_2(\zeta) = \zeta(\zeta+z+i\sqrt{w})^{-m/2};$$

then

$$\begin{aligned} \phi_3(\zeta) &= (\zeta+i)^{-(n/2-1)} - i(\zeta+i)^{-n/2}, \\ \phi_4(\zeta) &= (\zeta+z+i\sqrt{w})^{-(m/2-1)} - (\zeta+z+i\sqrt{w})^{-m/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d^k}{d\zeta^k}\phi_3(\zeta)|_{\zeta=\zeta_1} &= (-1)^k(\frac{n}{2}-1)(\frac{n}{2})\cdots(\frac{n}{2}+k-2)(2i)^{-(\frac{n}{2}+k-1)} \\ &\quad -i(-1)^k(\frac{n}{2})(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)(2i)^{-(\frac{n}{2}+k)} \\ &= (-1)^k\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)2^{-(\frac{n}{2}+k)}i^{-(\frac{n}{2}+k-1)}, \end{aligned}$$

$$\begin{aligned}
\frac{d^k}{d\zeta^k}\phi_4(\zeta)|_{\zeta=\zeta_2} &= (-1)^k(\frac{m}{2}-1)(\frac{m}{2})\cdots(\frac{m}{2}+k-2)(2i\sqrt{w})^{-(\frac{m}{2}+k-1)} \\
&\quad -(z+i\sqrt{w})(-1)^k(\frac{m}{2})(\frac{m}{2}+1)\cdots(\frac{m}{2}+k-1)(2i\sqrt{w})^{-(\frac{m}{2}+k)} \\
&= (-1)^k\frac{m}{2}(\frac{m}{2}+1)\cdots(\frac{m}{2}+k-2)2^{-(\frac{m}{2}+k)}(i\sqrt{w})^{-(\frac{m}{2}+k-1)} \\
&\quad -z\frac{d^k}{d\zeta^k}\phi_2(\zeta)|_{\zeta=\zeta_2}.
\end{aligned}$$

Following a similar argument to the proof of $I^0_{n,m}(w,z)$ yields

$$\begin{aligned}
\mathrm{Res}(\zeta_1) &= \frac{1}{(\frac{n}{2}-1)!}\sum_{j=0}^{\frac{n}{2}-1}\begin{pmatrix}\frac{n}{2}-1\\ j\end{pmatrix}\frac{d^{n/2-j-1}}{d\zeta^{n/2-j-1}}\phi_3(\zeta)\frac{d^j}{d\zeta^j}\xi_1(\zeta)|_{\zeta=\zeta_1}, \\
&= \sum_{j=0}^{q-1}\begin{pmatrix}n-j-2\\ q-1\end{pmatrix}\frac{(-1)^j i^j}{(j-1)!(n-j-2)}2^{-(n-j-1)}\frac{d^j}{d\zeta^j}\xi_1(\zeta)|_{\zeta=\zeta_1} \\
&= \sum_{s=0}^{\langle\frac{q-1}{2}\rangle}\begin{pmatrix}n-2s-2\\ q-1\end{pmatrix}\frac{(-1)^s}{(2s-1)!}\frac{2^{-(n-2s-1)}}{(n-2s-2)}\sum_{j=0}^{s}a_{j,s,p}\ w^j \\
&\quad \cdot r^{-(m/2+s+j)}(\cos(\frac{m}{2}+s+j)v - i\sin(\frac{m}{2}+s+j)v) \\
&+ \sum_{s=0}^{\langle\frac{q-2}{2}\rangle}\begin{pmatrix}n-2s-3\\ q-1\end{pmatrix}\frac{(-1)^{s+1}i}{(2s)!}\frac{2^{-(n-2s-3)}}{(n-2s-3)} \\
&\quad \cdot\sum_{j=0}^{s}a_{j,s,p}\ w^j(\frac{m}{2}+s+j)(i+z) \\
&\quad \cdot r^{-(m/2+s+j+1)}(\cos(\frac{m}{2}+s+j+1)v - i\sin(\frac{m}{2}+s+j+1)v).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Re}\ \{2\pi i\mathrm{Res}(\zeta_1)\} &= 2\pi\left\{\sum_{s=0}^{\langle\frac{q-1}{2}\rangle}\begin{pmatrix}n-2s-2\\ q-1\end{pmatrix}\frac{(-1)^s}{(2s-1)!}\frac{2^{-(n-2s)}}{(\frac{n}{2}-s-1)}\right. \\
&\quad \cdot\sum_{j=0}^{s}a_{j,s,p}\ w^j r^{-(m/2+s+j)}\sin(\frac{m}{2}+s+j)v \\
&\quad +\sum_{s=0}^{\langle\frac{q-2}{2}\rangle}\begin{pmatrix}n-2s-3\\ q-1\end{pmatrix}\frac{(-1)^s}{(2s)!}\frac{2^{-(n-2s-3)}}{(n-2s-3)}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\sum_{j=0}^{s} a_{j,s,p}\ w^j(\frac{m}{2}+s+j)r^{-(m/2+s+j+1)} \\
&\cdot(\sin(\frac{m}{2}+s+j+1)v+|z|\cos(\frac{m}{2}+s+j+1)v)\Big\} \\
= \ & \pi\left\{\sum_{k=0}^{\bar{n}} r^{-(m/2+k)}\sin(\frac{m}{2}+k)v\sum_{j=k_2(n)}^{\bar{k}}\begin{pmatrix} n-2(k-j)-2 \\ q-1\end{pmatrix}\right. \\
&\cdot\frac{(-1)^{k-j}}{(2(k-j))!}\frac{2(k-j)}{(\frac{n}{2}-(k-j)-1)}2^{-(n-2(k-j)-1)}a_{j,s,p}\ w^j \\
&+\sum_{k=0}^{\underline{n}} r^{-(m/2+k+1)}(\sin(\frac{m}{2}+k+1)v+|z|\cos(\frac{m}{2}+k+1)v) \\
&\cdot\sum_{j=k_1(n)}^{\tilde{k}}\begin{pmatrix} n-2(k-j)-3 \\ q-1\end{pmatrix}\frac{(-1)^{k-j}}{(2(k-j)+1)!}\frac{(2(k-j)+1)}{(n-2(k-j)-3)} \\
&\left.\cdot 2^{-(n-2(k-j)-4)}(\frac{m}{2}+k)a_{j,s,p}\ w^j\right\} \\
= \ & \pi\left\{\sum_{k=0}^{\bar{n}} r^{-(m/2+k)}\sin(\frac{m}{2}+k)v\right. \\
&\cdot\sum_{j=k_2(n)}^{\bar{k}} w^j B_{p,q}\frac{k-j}{(\frac{n}{2}-(k-j)-1)} \\
&+\sum_{k=0}^{\underline{n}} r^{-(m/2+k+1)}(\sin(\frac{m}{2}+k+1)v+|z|\cos(\frac{m}{2}+k+1)v) \\
&\left.\cdot\sum_{j=k_1(n)}^{\tilde{k}} w^j A_{p,q}(k,j)\frac{(2(k-j)+1)}{(n-2(k-j)-3)}\right\} \\
= \ & \pi w^{-(m-1)/2}(J_1+J_2). \qquad (68)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathrm{Res}(\zeta_2) \ = \ & \frac{1}{(\frac{m}{2}-1)!}\sum_{j=0}^{\frac{m}{2}-1}\begin{pmatrix}\frac{m}{2}-1 \\ j\end{pmatrix}\frac{d^{m/2-j-1}}{d\zeta^{m/2-j-1}}\phi_4(\zeta)\frac{d^j}{d\zeta^j}\xi_2(\zeta)|_{\zeta=\zeta_2} \\
= \ & \sum_{j=0}^{\frac{m}{2}-1}\begin{pmatrix} m-j-2 \\ q-1\end{pmatrix}\frac{(-1)^{j+1}i^j}{j!}\frac{j2^{-(m-j-1)}}{(m-j-2)}w^{-(\frac{m-i}{2}-1)}\frac{d^j}{d\zeta^j}\xi_2(\zeta)|_{\zeta=\zeta_2} \\
& -|z|\frac{d^{m/2-j-1}}{d\zeta^{m/2-j-1}}\phi_2(\zeta)\frac{d^j}{d\zeta^j}\xi_2(\zeta)|_{\zeta=\zeta_2}.
\end{aligned}
$$

An analogous argument gives

$$
\begin{aligned}
\text{Re}\,\{2\pi i \text{Res}(\zeta_2)\} &= \pi w^{-(m-1)/2}\left\{\sum_{k=0}^{\bar{m}} r^{-(n/2+k)}\sin(\frac{n}{2}+k)v'\right. \\
&\quad\cdot\sum_{j=k_2(m)}^{\tilde{k}} w^{k-j+\frac{1}{2}}B_{q,p}(k,j)\frac{(k-j)}{(m/2-(k-j)-1)} \\
&\quad+\sum_{k=0}^{\underline{m}} r^{-(n/2+k+1)}(w\sin(\frac{n}{2}+k+1)v' - |z|\sqrt{w}\cos(\frac{n}{2}+k+1)v') \\
&\quad\left.\cdot\sum_{j=k_1(m)}^{\tilde{k}} w^{k-j+\frac{1}{2}}A_{q,p}(k,j)\frac{(2(k-j)+1}{(m-2(k-j)-3)} - |z|(J_5+J_6)\right\} \\
&= \pi w^{-(m-1)/2}(J_3+J_4-|z|(J_5+J_6)). \qquad (69)
\end{aligned}
$$

Together with (68), this completes the proof.