

SELECTING THE FAIREST OF $k(\geq 2)$ m -SIDED DICE

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ABSTRACT

In this paper, we investigate the problem of selecting, from $k(\geq 2)$ m -sided dice, the fairest die. The fairest die is the one corresponding to the smallest (unknown) value of $\theta_i = \sum_{j=1}^m (p_{ij} - \frac{1}{m})^2$, where p_{ij} denotes the j^{th} cell (face) probability for the i^{th} die. The proposed selection procedures are based on Schur-convex functions. The problem is studied in the context of the subset selection approach. For small samples case, a method for finding conservative solutions for the selection constants is given. Large sample approximations have also been provided. A related problem of selecting all good populations is also investigated. A procedure for selecting the die with the greatest bias is also proposed and studied. Tables of constants necessary to carry out the procedure for selecting the fairest die are given.

Key Words and Phrases: Subset selection procedures; multinomial distribution; best population; majorization; Schur-convex; Schur-concave; good populations.

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1. INTRODUCTION

It frequently happens in problems concerned with ranking and selection that, whatever the original formulation or purpose of the experiment, the actual outcome is the rejection of certain processes and the acceptance of the remaining processes as being superior with respect to a desired characteristic. We shall try to formalize this in the special case when the observations are from multinomial distributions. For example, when bets are to be placed on the outcomes of an m -sided die, we are interested in the problem of finding which of the several dice is the fairest. Let $\underline{p} = (p_1, \dots, p_m)$ be an unknown vector, where p_i denote the probability of the i^{th} outcome when we throw an m -sided die. How to characterize and select a fair die is the main concern in this kind of problem.

In practice, a Schur-convex or Schur-concave function of \underline{p} may be appropriate. There are two measures of diversity of a multinomial population which have been commonly used. They are Shannon's entropy and Gini-Simpson index. The notion of the entropy function was introduced by Shannon (1948). The Gini-Simpson index was introduced by Gini (1912) and Simpson (1949). Both these indices are Schur-concave functions of \underline{p} .

Gupta and Huang (1976) have studied the problem of selecting the population with the largest entropy function when $m = 2$. Gupta and Wong (1975) have considered the problem of a selection procedure based on a Schur-concave function for selecting a subset containing the population with the largest entropy. Dudewicz and Van der Meulen (1981) have studied a selection procedure based on a generalized entropy function. More recently, Alam, Mitra, Rizvi, and Saxena (1986) have studied selection procedures based on Shannon's entropy function and Gini-Simpson index using the indifference zone approach. Rizvi, Alam, and Saxena (1987) have also considered a subset selection procedure based on diversity indices.

For $m = 2$, i.e. the binomial case, Sobel and Starr (1975) studied a selection procedure based on the criterion $|p_i - \frac{1}{2}|$. In this paper, we discuss the general case for $m \geq 2$. We may use the criteria $\sum_{i=1}^m (p_i - \frac{1}{m})^2$ or $\max_{1 \leq i \leq m} |p_i - \frac{1}{m}|$. Our main goal is to define (optimal) subset selection procedures based on $\theta = \sum_{i=1}^m (p_i - \frac{1}{m})^2$. Note that θ is a Schur-convex

function and is equivalent to the Gini–Simpson index. It should be pointed out that in our paper we make some improvements for the derivation of the results of Rizvi, Alam, and Saxena (1987). Our proofs are stronger and more general. It should be noted that since the majorization is only a partial order relation, we need to make some assumptions about the parameter space.

Let π_1, \dots, π_k denote k dice with unknown probability vectors $\underline{p}_1, \dots, \underline{p}_k$ respectively, where $\underline{p}_i = (p_{i1}, \dots, p_{im})$, $m \geq 2$, $p_{ij} \geq 0$, $\sum_{j=1}^m p_{ij} = 1$, $i = 1, \dots, k$. We define

$$\theta_i = \varphi(\underline{p}_i) = \sum_{j=1}^m \left(p_{ij} - \frac{1}{m}\right)^2 \quad (1.1)$$

and

$$\Omega = \{\omega = (\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k)\}.$$

Let $\theta_{[1]} \leq \dots \leq \theta_{[k]}$ denote the ordered values of $\theta_1, \dots, \theta_k$. It is assumed that the exact pairing between the ordered parameters θ_i 's and the unordered θ_i 's is unknown. The unknown population associated with the smallest parameter $\theta_{[1]}$ is called the best population. Our goal is to define a selection procedure which selects a non-trivial, non-empty subset of $\{\pi_1, \dots, \pi_k\}$ and satisfies the basic probability requirement, that is,

$$\inf_{\Omega} P(\text{CS}) \geq P^* \quad (1.2)$$

where $k^{-1} < P^* < 1$ and CS stands for a correct selection, that is, the selection of a subset which includes the best population.

In Section 2, we formulate the problem, define the selection procedure, and study its properties. In Section 3, we consider the problem of selecting all good populations. In Section 4, we propose and study a procedure for selecting the die with the greatest bias. Tables of constants $d = d(k, n, m, P^*)$ are provided for $m = 2$ and selected values of k, n and P^* .

2. SELECTING THE FAIREST DIE

Suppose that we have n independent observations from each of the k dice. Let X_{ij} denote the number of outcomes of the j^{th} side in the i^{th} die. Then $\underline{X}_i = (X_{i1}, \dots, X_{im})$

follows as a multinomial distribution with parameters n and $\underline{p}_i = (p_{i1}, \dots, p_{im})$. We will denote it by $\underline{X}_i \sim M(n, \underline{p}_i)$. We are interested in the population associated with the smallest parameter $\theta_{[1]}$, where θ_i is defined by (1.1). A reasonable estimator of θ_i is $Y_i = \varphi(\frac{1}{n} \underline{X}_i)$ and a natural selection procedure R_1 is proposed as follows:

R_1 : Select π_i if and only if $Y_i \leq \min_{1 \leq j \leq k} Y_j + d$, where d is the smallest non-negative number such that the probability requirement (1.2) is satisfied.

In order to find the d -value, which depends on k , n , m , and P^* , we need some lemmas. For the definition of majorization and basic properties, see Marshall and Olkin (1979). In the following, we will use $\underline{x} \leq_m \underline{y}$ to mean that \underline{x} is majorized by \underline{y} .

Lemma 2.1. (Rinott (1973))

Let $\underline{X} \sim M(n, \underline{p})$ and $\phi(\underline{x})$ be a Schur-convex (Schur-concave) function of \underline{x} . Then $E(\phi(\underline{X}))$ is a Schur-convex (Schur-concave) function of \underline{p} .

Lemma 2.2. Let $\underline{X} \sim M(n, \underline{p})$ and $\psi(\underline{x})$ be a Schur-convex (Schur-concave) function of \underline{x} . Then $P\{\psi(\frac{1}{n} \underline{X}) \leq c\}$ is a Schur-concave (Schur-convex) function of \underline{p} . Similarly, $P\{c \leq \psi(\frac{1}{n} \underline{X})\}$ is a Schur-convex (Schur-concave) function of \underline{p} .

Proof. Define $\phi(\underline{x}) = I_{\{\psi(\frac{1}{n} \underline{x}) \leq c\}}$, where I_A is the indicator function of the set A . Then apply Lemma 2.1.

Lemma 2.3. If $\psi(\underline{x})$ is a Schur-convex (Schur-concave) function of \underline{x} and $\underline{X} \sim M(n, \underline{p})$. Then $P\{\psi(\frac{1}{n} \underline{x}) - d \leq \psi(\frac{1}{n} \underline{X})\}$ is a Schur-concave (Schur-convex) function of \underline{x} when \underline{p} is fixed.

Proof. If $\underline{x} \leq_m \underline{y}$, then $\psi(\frac{1}{n} \underline{x}) \leq \psi(\frac{1}{n} \underline{y})$ and

$$\left\{ \psi\left(\frac{1}{n} \underline{y}\right) - d \leq \psi\left(\frac{1}{n} \underline{X}\right) \right\} \subset \left\{ \psi\left(\frac{1}{n} \underline{x}\right) - d \leq \psi\left(\frac{1}{n} \underline{X}\right) \right\}.$$

Hence

$$P\left\{ \psi\left(\frac{1}{n} \underline{y}\right) - d \leq \psi\left(\frac{1}{n} \underline{X}\right) \right\} \leq P\left\{ \psi\left(\frac{1}{n} \underline{x}\right) - d \leq \psi\left(\frac{1}{n} \underline{X}\right) \right\}.$$

Theorem 2.4. $P(\text{CS}|R_1)$ is a Schur-concave function of $\underline{p}_{(1)}$ when all other $\underline{p}_{(i)}$, $i \neq 1$, are kept fixed and is a Schur-convex function of $\underline{p}_{(j)}$, $j \neq 1$, when all other $\underline{p}_{(\ell)}$, $\ell \neq j$,

are kept fixed, where $\underline{p}_{(i)}$ denote the probability vectors corresponding to the unknown population with parameter $\theta_{[i]}$ and statistic $Y_{(i)}$.

$$\begin{aligned} \text{Proof. } P(\text{CS}|R_1) &= P\{Y_{(1)} \leq Y_{(j)} + d, j = 2, \dots, k\} \\ &= E\{P\{Y_{(1)} - d \leq Y_{(j)}, j = 2, \dots, k | Y_{(1)}\}\}. \end{aligned}$$

Now

$$\begin{aligned} &P\left\{Y_{(1)} - d \leq Y_{(j)}, j = 2, \dots, k | Y_{(1)} = \varphi\left(\frac{1}{n} \underline{x}\right)\right\} \\ &= P\left\{\varphi\left(\frac{1}{n} \underline{x}\right) - d \leq Y_{(j)}, j = 2, \dots, k\right\} \\ &= \prod_{j=2}^k P\left\{\varphi\left(\frac{1}{n} \underline{x}\right) - d \leq Y_{(j)}\right\}. \end{aligned} \quad (2.1)$$

By Lemma 2.3, when $\underline{p}_{(j)}$, $j \neq 1$, are fixed, then (2.1) is a product of non-negative Schur-concave functions of \underline{x} and hence a Schur-concave function of \underline{x} . Then, by Lemma 2.1, $P(\text{CS}|R_1)$ is a Schur-concave function of $\underline{p}_{(1)}$. Also, by Lemma 2.2, each term of (2.1) is a Schur-convex function of $\underline{p}_{(j)}$. Hence the result of the other part follows.

Since majorization is only a partial order relation, to simplify the problem, we may assume that there exists some i such that $\underline{p}_i \leq_m \underline{p}_j$, $j = 1, \dots, k$, $j \neq i$. For our problem, this assumption is reasonable because we expect that there exists a fair die. The following theorem provides the main result of this section.

Theorem 2.5. Let $\Omega_1 = \{\underline{w} = (\underline{p}_1, \dots, \underline{p}_k) \in \Omega | \underline{p}_{(1)} \leq_m \underline{p}_{(j)}, j = 2, \dots, k\}$ and $\Omega_0 = \{\underline{w} = (\underline{p}, \dots, \underline{p}) \in \Omega\}$. Then

$$\inf_{\Omega_1} P(\text{CS}|R_1) = \inf_{\Omega_0} P(\text{CS}|R_1). \quad (2.2)$$

Proof. By Theorem 2.4 and the assumption $\underline{p}_{(1)} \leq_m \underline{p}_{(j)}$, $j = 2, \dots, k$, the infimum is attained when $\underline{p}_{(1)} = \dots = \underline{p}_{(k)}$.

Although we have found the relation in (2.2), we still do not know the exact point \underline{p} at which the infimum is attained. For small samples case, we consider a conditional procedure which is similar to the one proposed by Gupta and Huang (1976) to overcome this difficulty.

In the following, we may assume that $\underline{X}_1, \dots, \underline{X}_k$ are i.i.d. because $P(\text{CS}|R_1)$ occurs at Ω_0 . Then

$$\left(\sum_{i=1}^k X_{i1}, \dots, \sum_{i=1}^k X_{im} \right) \sim M(nk, \underline{p}).$$

For $\underline{t} = (t_1, \dots, t_m)$, $0 \leq t_j \leq nk$, $j = 1, \dots, m$ and $\sum_{j=1}^m t_j = nk$, let

$$M(k, d(\underline{t}), \underline{t}, m, n) = \Sigma^* \prod_{i=1}^k \binom{n}{s_{i1}, \dots, s_{im}} \quad (2.3)$$

where Σ^* denotes the summation over the set of all m -tuples (s_{i1}, \dots, s_{im}) such that $0 \leq s_{ij} \leq n$, $i = 1, \dots, k$, $j = 1, \dots, m$, $\sum_{i=1}^k s_{ij} = t_j$, $j = 1, \dots, m$, $\sum_{j=1}^m s_{ij} = n$, $i = 1, \dots, k$ and

$$\varphi\left(\frac{1}{n} \underline{s}_1\right) \leq \min_{2 \leq j \leq k} \varphi\left(\frac{1}{n} \underline{s}_j\right) + d(\underline{t}),$$

for some constant $d(\underline{t})$ depending on \underline{t} . It is easy to prove the following lemma.

Lemma 2.6. Let $M(k, d(\underline{t}), \underline{t}, m, n)$ be defined as in (2.3). Then

$$\begin{aligned} & P \left\{ \varphi\left(\frac{1}{n} \underline{X}_1\right) \leq \min_{2 \leq i \leq k} \varphi\left(\frac{1}{n} \underline{X}_i\right) + d(\underline{t}) \mid \sum_{i=1}^k X_{ij} = t_j, j = 1, \dots, m \right\} \\ &= M(k, d(\underline{t}), \underline{t}, m, n) / \binom{nk}{t_1, \dots, t_m} \end{aligned}$$

is independent of \underline{p} .

Using Lemma 2.6, we have the following result.

Theorem 2.7. For given P^* and each \underline{t} , let $d(\underline{t})$ be the smallest number such that

$$M(k, d(\underline{t}), \underline{t}, m, n) \geq \binom{nk}{t_1, \dots, t_m} P^* \quad (2.4)$$

and let

$$d = \max_{\underline{t}} d(\underline{t}),$$

then

$$\inf_{\Omega_0} P(\text{CS}|R_1) \geq P^*.$$

Proof. $\inf_{\Omega_0} P(\text{CS}|R_1)$

$$\begin{aligned}
&= P \left\{ \varphi \left(\frac{1}{n} \underline{X}_1 \right) \leq \min_{2 \leq i \leq k} \varphi \left(\frac{1}{n} \underline{X}_i \right) + d \right\} \\
&= \sum_{\underline{t}} P \left\{ \varphi \left(\frac{1}{n} \underline{X}_1 \right) \leq \min_{2 \leq i \leq k} \varphi \left(\frac{1}{n} \underline{X}_i \right) + d \mid \sum_{i=1}^k X_{ij} = t_j, j = 1, \dots, m \right\} \\
&\quad P \left\{ \sum_{i=1}^k X_{ij} = t_j, j = 1, \dots, m \right\} \\
&\geq \sum_{\underline{t}} P \left\{ \varphi \left(\frac{1}{n} \underline{X}_1 \right) \leq \min_{2 \leq i \leq k} \varphi \left(\frac{1}{n} \underline{X}_i \right) + d(\underline{t}) \mid \sum_{i=1}^k X_{ij} = t_j, j = 1, \dots, m \right\} \\
&\quad P \left\{ \sum_{i=1}^k X_{ij} = t_j, j = 1, \dots, m \right\} \\
&= \sum_{\underline{t}} P \left\{ \sum_{i=1}^k X_{ij} = t_j, j = 1, \dots, m \right\} M(k, d(\underline{t}), \underline{t}, m, n) / \binom{nk}{t_1, \dots, t_m} \\
&\geq P^*.
\end{aligned}$$

Remark: For small samples (k and n are both small), for given P^* and \underline{t} , we can easily determine the smallest $d(\underline{t})$ satisfying (2.4). From these, we have computed tables of d -values for $m = 2$, $k = 2(1)7$, $n = 2(1)15$, $P^* = 0.75, 0.80, 0.90$ and 0.95 , which are given at the end of the paper.

For large samples, the above computation involves a lot of computation time. Hence, in the following large samples approximations are considered.

We know that $\frac{1}{n} \underline{X}$ is asymptotically multivariate normal with mean vector $\underline{p} = (p_1, \dots, p_m)$ and covariance matrix $\Sigma = (\sigma_{ij})$, where $\sigma_{ii} = \frac{1}{n} p_i(1-p_i)$ and $\sigma_{ij} = -\frac{1}{n} p_i p_j$, $i \neq j$. Then $\sqrt{n}(\varphi(\frac{1}{n} \underline{X}) - \varphi(\underline{p}))$ is asymptotically normal with mean 0 and variance

$$\begin{aligned}
\sigma_n^2(\underline{p}) &= \sum_{i=1}^m p_i(1-p_i) \left(\frac{\partial}{\partial p_i} \varphi(\underline{p}) \right)^2 - 2 \sum_{i < j} p_i p_j \left(\frac{\partial}{\partial p_i} \varphi(\underline{p}) \right) \left(\frac{\partial}{\partial p_j} \varphi(\underline{p}) \right) \\
&= 4 \left[\sum_{i=1}^m p_i^3 - \left(\sum_{i=1}^m p_i^2 \right)^2 \right]. \tag{2.5}
\end{aligned}$$

Theorem 2.8. For large n , we have

$$\inf_{\Omega_1} P(\text{CS}|R_1) \approx \inf_{\underline{p}} \int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{nd}}{\sigma_n(\underline{p})} \right) d\Phi(x), \quad (2.6)$$

where $\Phi(x)$ is the cdf of the standard normal and “ \approx ” means approximately equal.

Proof. Let $Z_i = \sqrt{n}(\varphi(\frac{1}{n} \underline{X}_i) - \varphi(\underline{p}_i))/\sigma_n(\underline{p}_i)$, $i = 1, \dots, k$, then Z_i , $i = 1, \dots, k$ are asymptotically i.i.d. $N(0, 1)$. By Theorem 2.5, we have

$$\begin{aligned} \inf_{\Omega_1} P(\text{CS}|R_1) &= \inf_{\Omega_0} P(\text{CS}|R_1) \\ &= \inf_{\underline{p}} P \left\{ Z_1 \leq Z_j + \frac{\sqrt{nd}}{\sigma_n(\underline{p})}, j = 2, \dots, k \right\} \\ &\approx \inf_{\underline{p}} \int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{nd}}{\sigma_n(\underline{p})} \right) d\Phi(x). \end{aligned}$$

Remark: Rizvi, Alam, and Saxena (1987) pointed out that

$$\sup_{\underline{p}} \sigma_n(\underline{p}) = \sigma_n(\underline{p}^0),$$

where $\underline{p}^0 = (p_0, \dots, p_0, 1 - (m-1)p_0)$ and

$$p_0 = \frac{5m - 2 + (9m^2 - 4m + 4)^{\frac{1}{2}}}{8m(m-1)}. \quad (2.7)$$

Hence the value d can be found by using the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{nd}}{\sigma_n(\underline{p}^0)} \right) d\Phi(x) = P^*. \quad (2.8)$$

The integral (2.8) has been tabulated by Bechhofer (1954), Gupta (1963), and Gupta, Nagel and Panchapakesan (1973).

If we don't make the assumption “ $\underline{p}_{(1)} \leq_m \underline{p}_{(j)}$, $j = 2, \dots, k$ ”, we consider some partial solutions based on some other restrictions. Firstly, we consider the approach suggested by Rizvi, Alam, and Saxena (1987). For convenience, we assume that π_1 is the best population. Let

$$\sum_{r=1}^s \tilde{p}_{[m-r+1]} = \min_{2 \leq i \leq k} \left(\sum_{r=1}^s p_{i[m-r+1]} \right), \quad s = 1, \dots, m, \quad (2.9)$$

where $p_{i[m-r+1]}$ is the $(m-r+1)$ st smallest components of \underline{p}_i . Then we can determine a vector $\tilde{\underline{p}}$ such that $\tilde{\underline{p}} < \underline{p}_i$, $i = 2, \dots, k$, and if there is a $\underline{p} < \underline{p}_i$, $i = 2, \dots, k$, then $\underline{p} < \tilde{\underline{p}}$. Since φ is Schur-convex, we have $\varphi(\tilde{\underline{p}}) \leq \varphi(\underline{p}_i)$, $i = 2, \dots, k$. We will consider the problem under the parameter space

$$\Omega_2 = \{\underline{w} = (\underline{p}_1, \dots, \underline{p}_k) | \varphi(\underline{p}_1) \leq \varphi(\tilde{\underline{p}}) \leq \varphi(\underline{p}_i), i = 2, \dots, k\}.$$

We note that if $\underline{p}_1 < \underline{p}_i$, $i = 2, \dots, k$, then $\underline{p}_1 < \tilde{\underline{p}}$ and hence $\varphi(\underline{p}_1) \leq \varphi(\tilde{\underline{p}})$. Hence the parameter space Ω_2 includes the parameter space considered by Gupta and Wong (1975).

In the following, we will give a clearer proof of Theorem 4.1 in the paper of Rizvi, Alam, and Saxena (1987).

Theorem 2.9. The infimum of $P(\text{CS}|R_1)$ over Ω_2 is attained when $\underline{p}_i = \tilde{\underline{p}}$, $i = 2, \dots, k$ and $\varphi(\underline{p}_1) = \dots = \varphi(\underline{p}_k)$.

Proof. By Theorem 2.4, $P(\text{CS}|R_1)$ is a Schur-convex function of \underline{p}_i , $i = 2, \dots, k$. Let $P(\text{CS}|R_1) = f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k)$, then $f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k) \geq f(\underline{p}_1, \tilde{\underline{p}}, \dots, \tilde{\underline{p}})$. For $\underline{w} = (\underline{p}_1, \dots, \underline{p}_k) \in \Omega_2$, we have $\varphi(\underline{p}_1) \leq \varphi(\tilde{\underline{p}}) \leq \varphi(\underline{p}_i)$, $i = 2, \dots, k$. If $\varphi(\underline{p}_1) < \varphi(\tilde{\underline{p}})$, let $\underline{p}_1 = (p_1, p_2, \dots, p_m)$, $p_1 \leq p_2 \leq \dots \leq p_m$. For $\varepsilon > 0$, consider $\underline{p}_\varepsilon = (p_1 - \varepsilon, p_2, \dots, p_{m-1}, p_m + \varepsilon)$, then $\underline{p}_1 < \underline{p}_\varepsilon$. By Theorem 2.4 again, we have $f(\underline{p}_1, \tilde{\underline{p}}, \dots, \tilde{\underline{p}}) \geq f(\underline{p}_\varepsilon, \tilde{\underline{p}}, \dots, \tilde{\underline{p}})$. Now take $\varepsilon = \frac{(p_1 - p_m) + [(p_m - p_1)^2 + 2(\varphi(\tilde{\underline{p}}) - \varphi(\underline{p}_1))]^{1/2}}{2} > 0$. Then $\varphi(\underline{p}_\varepsilon) = \varphi(\tilde{\underline{p}})$ and $f(\underline{p}_1, \dots, \underline{p}_k) \geq f(\underline{p}_\varepsilon, \tilde{\underline{p}}, \dots, \tilde{\underline{p}})$. This completes the proof of the theorem.

Remark: For large samples approximation, it is easy to see that

$$\inf_{\Omega_2} P(\text{CS}|R_1) \approx \inf_{\underline{p}, \underline{q}} \int_{-\infty}^{\infty} \Phi^{k-1} \left(\frac{\sigma_n(\underline{p})}{\sigma_n(\underline{q})} x + \frac{\sqrt{nd}}{\sigma_n(\underline{q})} \right) d\Phi(x) \quad (2.10)$$

where the infimum on the right side of (2.10) is over all vectors \underline{p} and \underline{q} for which $\varphi(\underline{p}) = \varphi(\underline{q})$.

In the following we will approximate the infimum of the probability of a correct selection under some restrictions.

Theorem 2.10. $\inf_{\Omega} P(\text{CS}|R_1) \approx \inf_{\Omega_0^*} P(\text{CS}|R_1)$, where $\Omega_0^* = \{(\underline{p}, \dots, \underline{p}) \in \Omega_0 | \underline{p} = (p, \dots, p, q), q = 1 - (m-1)p\}$.

Proof. Without loss of generality, we will assume that $\underline{p}_{(i)} = \underline{p}_i$, $i = 1, \dots, k$ and $p_{i1} \leq p_{i2} \leq \dots \leq p_{im}$. We know that

$$\underline{p}_i^{(1)} < \underline{p}_i < \underline{p}_i^{(2)} \quad (2.11)$$

where

$$\underline{p}_i^{(1)} = \left(\frac{1 - p_{im}}{m - 1}, \dots, \frac{1 - p_{im}}{m - 1}, p_{im} \right)$$

and

$$(2.12)$$

$$\underline{p}_i^{(2)} = (p_{i1}, \dots, p_{i1}, 1 - (m - 1)p_{i1}).$$

Let $P(\text{CS}|R_1) = f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k)$. By Theorem 2.4, we have

$$f(\underline{p}_1^{(2)}, \underline{p}_2^{(1)}, \dots, \underline{p}_k^{(1)}) \leq f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k) \leq f(\underline{p}_1^{(1)}, \underline{p}_2^{(2)}, \dots, \underline{p}_k^{(2)}).$$

Let

$$\varphi^*(p) = \varphi\left(\frac{1 - p}{m - 1}, \dots, \frac{1 - p}{m - 1}, p\right) = \frac{(mp - 1)^2}{m(m - 1)}, \quad (2.13)$$

then $\varphi^*(p)$ is a continuous strictly increasing function of p whenever $p > \frac{1}{m}$. Hence there exist \underline{p}_i^* such that

$$\underline{p}_i^{(1)} < \underline{p}_i^* < \underline{p}_i^{(2)}, \quad i = 1, \dots, k,$$

where $\underline{p}_i^* = (p_i, \dots, p_i, q_i)$ and $\varphi(\underline{p}_i^*) = \varphi(\underline{p}_i)$. Moreover,

$$f(\underline{p}_1^{(2)}, \underline{p}_2^{(1)}, \dots, \underline{p}_k^{(1)}) \leq f(\underline{p}_1^*, \underline{p}_2^*, \dots, \underline{p}_k^*) \leq f(\underline{p}_1^{(1)}, \underline{p}_2^{(2)}, \dots, \underline{p}_k^{(2)}).$$

By Theorem 2.4, there exists a $\underline{p}^* = (p, \dots, p, q)$ such that

$$\underline{p}_1^* < \underline{p}^* < \underline{p}_i^*, \quad i = 2, \dots, k,$$

and

$$f(\underline{p}_1^*, \dots, \underline{p}_k^*) \geq f(\underline{p}^*, \dots, \underline{p}^*).$$

If $f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k) \geq f(\underline{p}^*, \underline{p}^*, \dots, \underline{p}^*)$, the result follows. Otherwise

$$\begin{aligned} f(\underline{p}_1^{(2)}, \underline{p}_2^{(1)}, \dots, \underline{p}_k^{(1)}) &\leq f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k) < f(\underline{p}^*, \underline{p}^*, \dots, \underline{p}^*) \\ &\leq f(\underline{p}_1^*, \underline{p}_2^*, \dots, \underline{p}_k^*) \leq f(\underline{p}_1^{(1)}, \underline{p}_2^{(2)}, \dots, \underline{p}_k^{(2)}). \end{aligned}$$

We may tolerate the difference between $f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_k)$ and $f(\underline{p}^*, \underline{p}^*, \dots, \underline{p}^*)$ and still use this as a lower bound. Hence $\inf_{\Omega} P(\text{CS}|R_1) \approx \inf_{\Omega_0^*} P(\text{CS}|R_1)$.

Remark: For large n , we have

$$\begin{aligned} \inf_{\Omega_0^*} P(\text{CS}|R_1) &\approx \inf_{\Omega_0^*} \int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{nd}}{\sigma_n(\underline{p})} \right) d\Phi(x) \\ &= \int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{nd}}{\sigma_n(\underline{p}^0)} \right) d\Phi(x), \end{aligned}$$

where \underline{p}^0 is defined by (2.7).

For $m = 2$, $\underline{p}_i = (p_i, 1 - p_i)$, $i = 1, \dots, k$ and $\varphi(\underline{p}_i) = 2(p_i - \frac{1}{2})^2$. Since we are dealing with the problem concerned with Schur-concave and Schur-convex function, we may assume that $p_i > \frac{1}{2}$, $i = 1, \dots, k$. Hence

$$\varphi(\underline{p}_i) \leq \varphi(\underline{p}_j) \text{ if and only if } p_i < p_j$$

and $\Omega = \Omega_1$ in this case. By Theorem 2.5, we have

$$\inf_{\Omega} P(\text{CS}|R_1) = \inf_{\Omega_0} P(\text{CS}|R_1).$$

Thus, the infimum of $P(\text{CS}|R_1)$ is attained when $p_1 = \dots = p_k = p$. For small samples case, we can solve it by using Theorem 2.7. For large samples case, we solve

$$\int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{nd}}{\sigma_n(\underline{p}^0)} \right) d\Phi(x) = P^*$$

where $p_0 = (2 + \sqrt{2})/4$.

3. SELECTING A SUBSET WHICH CONTAINS ALL GOOD POPULATIONS

Let $\pi_i \sim M(n, \underline{p}_i)$, $\underline{p}_i = (p_{i1}, \dots, p_{im})$, $0 \leq p_{ij} \leq 1$, $\sum_{j=1}^m p_{ij} = 1$, $i = 1, \dots, k$.

We define π_i as a good population if $\varphi(\underline{p}_i) = \sum_{j=1}^m (p_{ij} - \frac{1}{m})^2 \leq \delta$ and a bad population if $\varphi(\underline{p}_i) > \delta$, where $0 < \delta < 1 - \frac{1}{m}$ is prespecified. Our goal is to define a selection procedure which selects a subset of $\{\pi_1, \dots, \pi_k\}$ such that the selected subset contains all

good populations with probability at least P^* . With the same notation as that in Section 2, we propose a natural selection procedure R_2 as follows:

R_2 : Select π_i if and only if $\varphi(\frac{1}{n} \underline{X}_i) \leq c$, where $\delta < c$ is the smallest constant such that

$$\inf_{\Omega} P(\text{CS}|R_2) \geq P^*. \quad (3.1)$$

Let $G = \{\underline{p} = (p_1, \dots, p_m) | 0 \leq p_i \leq 1, \sum_{i=1}^m p_i = 1, \varphi(\underline{p}) \leq \delta\}$ denote the parameter space of good populations. We assume that there are k_1 (unknown) good populations, $1 \leq k_1 \leq k$. Without loss of generality, we assume that $\underline{p}_1, \dots, \underline{p}_{k_1} \in G$. Then we have the following Lemma:

Lemma 3.1. Let $\underline{X} \sim M(n, \underline{p})$ and $g(\underline{p}) = P\{\varphi(\frac{1}{n} \underline{X}) \leq c\}$, $c > 0$. Then

$$\inf_{\underline{p} \in G} g(\underline{p}) = \inf_{\underline{p} \in G_0} g(\underline{p}), \quad (3.2)$$

where $G_0 = \{\underline{p} \in G | \varphi(\underline{p}) = \delta\}$.

Proof. For $\underline{p} \in G$, if $\varphi(\underline{p}) < \delta$, we take

$$\varepsilon = \frac{(p_1 - p_m) + \sqrt{(p_m - p_1)^2 + 2(\delta - \varphi(\underline{p}))}}{2} > 0,$$

then

$$\underline{p}_m < \underline{p}_\varepsilon \text{ and } \varphi(\underline{p}_\varepsilon) = \delta,$$

where

$$\underline{p}_\varepsilon = (p_1 - \varepsilon, p_2, \dots, p_{m-1}, p_m + \varepsilon), \quad p_1 \leq p_2 \leq \dots \leq p_m.$$

By Lemma 2.2, $g(\underline{p})$ is a Schur-concave function of \underline{p} . Hence

$$g(\underline{p}) \geq g(\underline{p}_\varepsilon).$$

This completes the proof of the lemma.

In order to overcome the difficulty of partial order relation, we consider the parameter space G_1 defined by

$$G_1 = \{\underline{p} \in G | \underline{p}_m < \underline{p}_\delta, \varphi(\underline{p}_\delta) \leq \delta\},$$

where \underline{p}_δ is known or unknown.

For the case when \underline{p}_δ is known, we have $g(\underline{p}) \geq g(\underline{p}_\delta)$ for all $\underline{p} \in G_1$. Hence

$$\inf_{\underline{p} \in G_1} g(\underline{p}) = g(\underline{p}_\delta). \quad (3.3)$$

Further, we have the following result:

Theorem 3.2. Under the parameter space G_1 , we have

$$\inf P(\text{CS}|R_2) \geq (g(\underline{p}_\delta))^k.$$

The value c can be taken as the solution to the equation

$$g(\underline{p}_\delta) = P^{*1/k}. \quad (3.4)$$

If \underline{p}_δ is unknown, by using the same arguments as that in the proof of Lemma 3.1, we may assume that $\varphi(\underline{p}_\delta) = \delta$. For large n , we have

$$\begin{aligned} g(\underline{p}_\delta) &= P\{\varphi(\frac{1}{n} \underline{X}) \leq c\} \\ &\approx \Phi\left(\frac{\sqrt{n}(c - \delta)}{\sigma_n(\underline{p}_\delta)}\right). \end{aligned}$$

Hence the value c can be taken as the solution to the equation

$$\Phi\left(\frac{\sqrt{n}(c - \delta)}{\sigma_n(\underline{p}^0)}\right) = P^{*1/k}, \quad (3.5)$$

where \underline{p}^0 is defined in (2.7).

Also, for each $\underline{p} = (p_1, \dots, p_m)$, $p_1 \leq p_2 \leq \dots \leq p_m$, we have $\underline{p}^{(1)} \leq \underline{p} \leq \underline{p}^{(2)}$, where $\underline{p}^{(i)}$, $i = 1, 2$ are defined as in (2.12). Given $\Delta > 0$, we define

$$G_\Delta = \{\underline{p} \in G \mid |\varphi(\underline{p}^{(2)}) - \varphi(\underline{p})| \leq \Delta\}.$$

Then we have the following result:

Lemma 3.3. $\inf_{\underline{p} \in G_\Delta} g(\underline{p}) \geq \inf_{\varphi^*(p) \leq \delta + \Delta} g^*(p)$, where $\varphi^*(p) = \varphi\left(\frac{1-p}{m-1}, \dots, \frac{1-p}{m-1}, p\right)$ and $g^*(p) = g\left(\frac{1-p}{m-1}, \dots, \frac{1-p}{m-1}, p\right)$.

Proof. For $\underline{p} \in G_\Delta$, we may assume that $\varphi(\underline{p}) = \delta$. Since $\underline{p}^{(1)} < \underline{p} < \underline{p}^{(2)}$, we have

$$g(\underline{p}) \geq g(\underline{p}^{(2)}) \text{ and } \varphi(\underline{p}) \leq \varphi(\underline{p}^{(2)}).$$

Further, $|\varphi(\underline{p}^{(2)}) - \varphi(\underline{p})| \leq \Delta$, so $\varphi(\underline{p}^{(2)}) \leq \delta + \Delta$.

Remark: g is a Schur-concave function, hence

$$\inf_{\underline{p} \in G_\Delta} g(\underline{p}) \geq g^*(p), \quad (3.6)$$

where $p = \frac{1}{m} + \sqrt{\frac{m-1}{m}(\delta + \Delta)}$.

Theorem 3.4. Under the parameter space G_Δ , we have

$$\inf P(\text{CS}|R_2) \geq (g^*(p))^k,$$

where $p = \frac{1}{m} + \sqrt{\frac{m-1}{m}(\delta + \Delta)}$. The value c can be taken as the solution to the equation $g^*(p) = P^{*1/k}$.

For large samples, we have the following result:

Theorem 3.5. For large n , under the parameter space G_Δ , we have

$$\inf P(\text{CS}|R_2) \approx \Phi\left(\sqrt{n} \frac{(c - \delta)}{\sigma_n(\tilde{p})}\right)^k,$$

where $\sigma_n^2(\tilde{p})$ is defined in (2.5) and $\tilde{p} = \left(\frac{1-p}{m-1}, \dots, \frac{1-p}{m-1}, p\right)$, $p = \frac{1}{m} + \sqrt{\frac{m-1}{m}(\delta + \Delta)}$.

Proof. We may assume that $\varphi(\underline{p}) = \delta$. Then

$$g(\underline{p}) \approx \Phi\left(\sqrt{n} \frac{(c - \delta)}{\sigma_n(\underline{p})}\right).$$

Under $\varphi(\underline{p}) = \delta$, $\sigma_n^2(\underline{p}) = 4 \left[\sum_{i=1}^m p_i^3 - \left(\frac{1}{m} + \delta\right)^2 \right]$ is a Schur-convex function. Furthermore, $\sigma_n^2(\underline{p}) = \sigma_n^2(\underline{p}^*)$ for some $\underline{p}^* = (p, \dots, p, q)$. As a function of q , $\sigma_n^2(\underline{p}^*)$ is increasing in q .

Thus

$$\sup_{\underline{p} \in G_\Delta} \sigma_n^2(\underline{p}) \leq \sigma_n^2(\tilde{p}),$$

where $\tilde{\underline{p}} = \left(\frac{1-p}{m-1}, \dots, \frac{1-p}{m-1}, p\right)$, $p = \frac{1}{m} + \sqrt{\frac{m-1}{m}(\delta + \Delta)}$.

When $m = 2$, $\varphi(\underline{p}_i) \leq \delta$ if and only if $p_i \leq \frac{1}{2} + \sqrt{\frac{1}{2}\delta}$. (Note that we assume that $p_i > \frac{1}{2}$ again). Hence, $\inf_{\underline{p} \in G} g(\underline{p}) = g(\tilde{\underline{p}}^0)$, where $\tilde{\underline{p}}^0 = \left(\frac{1}{2} - \sqrt{\frac{1}{2}\delta}, \frac{1}{2} + \sqrt{\frac{1}{2}\delta}\right)$. Moreover, $\inf P(\text{CS}|R_2) = (g(\tilde{\underline{p}}^0))^k$.

4. SELECTING THE DIE WITH THE GREATEST BIAS

In this section, let $\theta_i = \varphi(\underline{p}_i)$ be as defined by (1.1). We are now interested in the largest parameter $\theta_{[k]}$, that is, we wish to select the die with the greatest bias. Following the same notation as that in Section 2, we propose a natural selection procedure R_3 as follows:

R_3 : Select π_i if and only if $Y_i \geq \max_{1 \leq j \leq k} Y_j - \tilde{d}$, where \tilde{d} is the smallest non-negative number such that the probability requirement (1.2) is satisfied and where, as before, $Y_i = \varphi(\frac{1}{n} \underline{X}_i)$.

Analogous to the proof of Lemma 2.3, we have the following result.

Lemma 4.1. If $\psi(\underline{x})$ is a Schur-convex (Schur-concave) function of \underline{x} and $\underline{X} \sim M(n, \underline{p})$. Then $P\{\psi(\frac{1}{n} \underline{X}) \leq d + \psi(\frac{1}{n} \underline{x})\}$ is a Schur-convex (Schur-concave) function of \underline{x} when \underline{p} is fixed.

If we define

$$\Omega_3 = \{\underline{w} = (\underline{p}_1, \dots, \underline{p}_k) \in \Omega \mid \underline{p}_{(i)} < \underline{p}_{(k)}, j = 1, \dots, k-1\}. \quad (4.1)$$

Analogous to Theorem 2.4 and Theorem 2.5, we have the following results:

Theorem 4.2. $P(\text{CS}|R_3)$ is a Schur-convex function of $\underline{p}_{(k)}$ when all other $\underline{p}_{(i)}$, $i \neq k$, are kept fixed and is a Schur-concave function of $\underline{p}_{(j)}$, $j \neq k$, when all other $\underline{p}_{(\ell)}$, $\ell \neq j$, are kept fixed.

Theorem 4.3. $\inf_{\Omega_3} P(\text{CS}|R_3) = \inf_{\Omega_0} P(\text{CS}|R_3)$.

For $\underline{t} = (t_1, \dots, t_m)$, $0 \leq t_j \leq nk$, $j = 1, \dots, m$ and $\sum_{j=1}^m t_j = nk$, let

$$\tilde{M}(k, \tilde{d}(\underline{t}), \underline{t}, m, n) = \Sigma^{**} \prod_{i=1}^k \binom{n}{s_{i1}, \dots, s_{im}} \quad (4.2)$$

where Σ^{**} denotes the summation over the set of all m -tuples (s_{i1}, \dots, s_{im}) such that $0 \leq s_{ij} \leq n$, $i = 1, \dots, k$, $j = 1, \dots, m$, $\sum_{i=1}^k s_{ij} = t_j$, $j = 1, \dots, m$, $\sum_{j=1}^m s_{ij} = n$, $i = 1, \dots, k$ and

$$\varphi\left(\frac{1}{n} \underline{s}_k\right) \geq \max_{1 \leq j \leq k-1} \varphi\left(\frac{1}{n} \underline{s}_j\right) - \tilde{d}(\underline{t}),$$

for some constant $\tilde{d}(\underline{t})$ depending on \underline{t} . Analogous to Theorem 2.7, we have the following result:

Theorem 4.4. For given P^* and each \underline{t} , let $\tilde{d}(\underline{t})$ be the smallest number such that

$$\tilde{M}(k, \tilde{d}(\underline{t}), \underline{t}, m, n) \geq \binom{nk}{t_1, \dots, t_m} P^* \quad (4.3)$$

and let

$$\tilde{d} = \max_{\underline{t}} \tilde{d}(\underline{t}),$$

then

$$\inf_{\Omega_0} P(\text{CS}|R_3) \geq P^*.$$

For large samples approximation, we have the following result:

Theorem 4.5. For large n , we have

$$\inf_{\Omega_3} P(\text{CS}|R_3) \approx \inf_{\underline{p}} \int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{n\tilde{d}}}{\sigma_n(\underline{p})} \right) d\Phi(x). \quad (4.4)$$

Remark: The value \tilde{d} can be found by using the equation (2.8) when d is replaced by \tilde{d} .

If we don't make the assumption " $\underline{p}_{(i)} \leq_m \underline{p}_{(k)}$, $i = 1, \dots, k-1$ ", we consider some partial solutions based on some other restrictions. For convenience, we assume that π_k is the best population. Let

$$\sum_{r=1}^s \tilde{p}_{[m-r+1]} = \max_{1 \leq i \leq k-1} \left(\sum_{r=1}^s p_{i[m-r+1]} \right), \quad s = 1, \dots, m, \quad (4.5)$$

where $p_{i[m-r+1]}$ is the $(m-r+1)$ st smallest components of \underline{p}_i . Then we can determine a vector $\tilde{\underline{p}}$ such that $\underline{p}_i < \tilde{\underline{p}}$, $i = 1, \dots, k-1$, and if there is a \underline{p} such that $\underline{p}_i < \underline{p}$, $i = 1, \dots, k-1$, then $\tilde{\underline{p}} < \underline{p}$. Since φ is Schur-convex, we have $\varphi(\underline{p}_i) \leq \varphi(\tilde{\underline{p}})$, $i = 1, \dots, k-1$. We will consider the problem under the parameter space

$$\Omega_4 = \{\underline{w} = (\underline{p}_1, \dots, \underline{p}_k) | \varphi(\underline{p}_i) \leq \varphi(\tilde{\underline{p}}) \leq \varphi(\underline{p}_k), i = 1, \dots, k-1\}. \quad (4.6)$$

We note that if $\underline{p}_i < \underline{p}_k$, $i = 1, \dots, k-1$, then $\tilde{\underline{p}} < \underline{p}_k$ and hence $\varphi(\tilde{\underline{p}}) \leq \varphi(\underline{p}_k)$. Hence the parameter space Ω_4 includes the parameter space Ω_3 . Analogous to Theorem 2.9, we have the following result:

Theorem 4.6. The infimum of $P(\text{CS}|R_3)$ over Ω_4 is attained when $\underline{p}_i = \tilde{\underline{p}}$, $i = 1, \dots, k-1$ and $\varphi(\underline{p}_1) = \dots = \varphi(\underline{p}_k)$.

Proof. The only difference is replaced $\underline{p}_\varepsilon$ by $\tilde{\underline{p}}_\varepsilon$, where

$$\tilde{\underline{p}}_\varepsilon = (p_1 + \varepsilon, p_2, \dots, p_{m-1}, p_m - \varepsilon)$$

and

$$\varepsilon = \frac{(p_m - p_1) - [(p_m - p_1)^2 - 2(\varphi(\underline{p}_k) - \varphi(\tilde{\underline{p}}))]^{1/2}}{2}.$$

Note that $\varepsilon > 0$ provided that $2\varepsilon < p_m - p_1$.

Remark: For large samples approximation, we have

$$\inf_{\Omega_4} P(\text{CS}|R_3) \approx \inf_{\underline{p}, \underline{q}} \int_{-\infty}^{\infty} \Phi^{k-1} \left(\frac{\sigma_n(\underline{p})}{\sigma_n(\underline{q})} x + \frac{\sqrt{n}\tilde{d}}{\sigma_n(\underline{q})} \right) d\Phi(x) \quad (4.7)$$

where the infimum on the right side of (4.7) is over all vectors \underline{p} and \underline{q} for which $\varphi(\underline{p}) = \varphi(\underline{q})$.

Analogous to Theorem 2.10, if we tolerate some loss, we may have the following result.

Theorem 4.7. $\inf_{\Omega} P(\text{CS}|R_3) \approx \inf_{\Omega_0^*} P(\text{CS}|R_3)$.

Remark: For large n , we have

$$\inf_{\Omega_0^*} P(\text{CS}|R_3) \approx \int_{-\infty}^{\infty} \Phi^{k-1} \left(x + \frac{\sqrt{n}\tilde{d}}{\sigma_n(\underline{p}^0)} \right) d\Phi(x),$$

where \underline{p}^0 is defined by (2.7).

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Table I. Table of d -values for the procedure R_1 .

m=2		$k =$	2	3	4	5	6	7
n= 2	$p^* = .75$.5000	.5000	.5000	.5000	.5000	.5000
	.80		.5000	.5000	.5000	.5000	.5000	.5000
	.90		.5000	.5000	.5000	.5000	.5000	.5000
	.95		.5000	.5000	.5000	.5000	.5000	.5000
3	.75		.4444	.4444	.4444	.4444	.4444	.4444
	.80		.4444	.4444	.4444	.4444	.4444	.4444
	.90		.4444	.4444	.4444	.4444	.4444	.4444
	.95		.4444	.4444	.4444	.4444	.4444	.4444
4	.75		.3750	.3750	.5000	.5000	.5000	.5000
	.80		.5000	.5000	.5000	.5000	.5000	.5000
	.90		.5000	.5000	.5000	.5000	.5000	.5000
	.95		.5000	.5000	.5000	.5000	.5000	.5000
5	.75		.3200	.4800	.4800	.4800	.4800	.4800
	.80		.4800	.4800	.4800	.4800	.4800	.4800
	.90		.4800	.4800	.4800	.4800	.4800	.4800
	.95		.4800	.4800	.4800	.4800	.4800	.4800
6	.75		.2778	.4444	.4444	.4444	.4444	.4444
	.80		.4444	.4444	.4444	.4444	.4444	.4444
	.90		.4444	.4444	.4444	.5000	.5000	.5000
	.95		.5000	.5000	.5000	.5000	.5000	.5000
7	.75		.2449	.4082	.4082	.4082	.4082	.4082
	.80		.4082	.4082	.4082	.4082	.4082	.4082
	.90		.4082	.4082	.4898	.4898	.4898	.4898
	.95		.4898	.4898	.4898	.4898	.4898	.4898
8	.75		.2500	.3750	.3750	.3750	.3750	.3750
	.80		.3750	.3750	.3750	.3750	.3750	.3750
	.90		.4688	.4688	.4688	.4688	.4688	.4688
	.95		.4688	.4688	.4688	.4688	.4688	.5000
9	.75		.2469	.3457	.3457	.3457	.3457	.3457
	.80		.3457	.3457	.3457	.3457	.3457	.3457
	.90		.4444	.4444	.4444	.4444	.4444	.4444
	.95		.4444	.4444	.4444	.4938	.4938	.4938
10	.75		.2400	.3200	.3200	.3200	.3200	.3200
	.80		.3200	.3200	.3200	.3200	.3200	.3200
	.90		.4200	.4200	.4200	.4200	.4200	.4200
	.95		.4200	.4200	.4800	.4800	.4800	.4800
11	.75		.2314	.2975	.2975	.2975	.2975	.2975
	.80		.2975	.2975	.2975	.2975	.2975	.2975
	.90		.3967	.3967	.3967	.3967	.3967	.3967
	.95		.3967	.3967	.4628	.4628	.4628	.4628

Table I (continued).

m=2		k=	2	3	4	5	6	7
n= 12	P*=.75		.2222	.2778	.2778	.2778	.2917	.2917
	.80		.2778	.2778	.2917	.2917	.2917	.2917
	.90		.3750	.3750	.3750	.3750	.3750	.3750
	.95		.3750	.3750	.4444	.4444	.4444	.4444
13	.75		.2130	.2604	.2604	.2604	.2840	.2840
	.80		.2604	.2604	.2840	.2840	.2840	.2840
	.90		.3550	.3550	.3550	.3550	.3550	.3550
	.95		.3550	.3550	.4260	.4260	.4260	.4260
14	.75		.2041	.2449	.2449	.2449	.2755	.2755
	.80		.2449	.2449	.2755	.2755	.2755	.2755
	.90		.3367	.3367	.3367	.3367	.3367	.3367
	.95		.3367	.3367	.4082	.4082	.4082	.4082
15	.75		.1956	.2311	.2311	.2311	.2667	.2677
	.80		.2311	.2311	.2667	.2667	.2667	.2677
	.90		.3200	.3200	.3200	.3200	.3200	.3200
	.95		.3200	.3200	.3911	.3911	.3911	.3911

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