SELECTION OF THE BEST WITH A PRELIMINARY TEST FOR LOCATION-SCALE PARAMETRIC MODELS *

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Dedicated to the late Professor P. R. Krishnaiah

ABSTRACT

This paper deals with the problem of selecting the best population from among $k(\geq 2)$ populations which can be described by location-scale parametric models. New selection procedures are proposed for selecting the unique best in terms of the largest location parameter. The procedures include preliminary tests which allow the experimenter to have an option to not select if the statistical evidence is not significant. Two probabilities, the probability to make a selection and the probability of a correct selection, are controlled by these selection procedures. Applications to the normal mean models are considered. Comparisons between the proposed selection procedures and certain earlier existing procedures are also made. Finally, a two-stage procedure for the normal means problem is considered.

Key Words and Phrases: best population; correct selection; preliminary test; indifference zone; nonselection zone; preference zone; location-scale parametric model; two-stage procedure.

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1. INTRODUCTION

The problem of selecting the best population from among $k(\geq 2)$ populations has been studied extensively. Many selection procedures have been derived for different selection goals by several authors. Among them, Bechhofer (1954) introduced the indifference zone approach for selecting the normal population with the largest mean. In his approach, the determination of the sample size depends entirely on the indifference zone assumption. Also, the probability of a correct selection depends on the unknown parameters and is analogous to the power of a test. However, in this formulation, a probability that is analogous to the probability of type-I error of a test was not taken into consideration. Bechhofer's procedure forces the experimenter to make a selection, and often that procedure is not used in practical applications because of the lack of a statistical test for the homogeneity of the parameters as stated in Simon (1977). It should be pointed out that in another approach called the subset selection this drawback, namely, the assumption of indifference zone, is not there. This approach due to Gupta (1956, 1965) as well as Bechhofer's approach are both discussed in detail in the monograph by Gupta and Panchapakesan (1979).

Based on the preceding reasoning, Bishop and Pirie (1979) introduced a selection procedure in which a test of homogeneity was conducted. The procedure allows the experimenter to have the option not to make a selection if the statistical evidence is not significant. Later, Chen (1985) proposed a modified selection procedure for the problem of selecting the best normal population. He considered a preliminary test based on the sampled spacing between the largest and the second largest order statistics. If the statistical evidence of the preliminary test is not significant, the experimenter decides not to make a selection. Otherwise, he or she selects the population yielding the largest sample mean value as the best population. The sample size is determined to control both the probability of type-I error for the preliminary test and the probability of a correct selection. Analogous to Chen (1985), Chen and Mithongtae (1986) proposed selection procedures for two-parameter exponential distribution models. However, both of their procedures cannot be applied to a case where the common scale parameter is unknown. Later, Leu and Liang (1990) proposed selection procedures which improve the results of Chen and Mithongtae (1986) and discussed the case where Chen and Mithongtae's procedures cannot be applied.

In this paper, we generalize the problem proposed by Chen (1985) to location-scale parametric models. Selection procedures based on one-sample are derived according to whether the common scale parameter is known or unknown. Exact sample sizes are determined to control both the probability of type-I error and the probability of a correct selection. Applications to the normal model cases are considered. Comparisons between the proposed selection procedures and certain earlier existing procedures are also made. Finally, a two-stage procedure for the normal means problem is considered.

2. FORMULATION OF THE PROBLEM

Let $\pi_1, \ldots, \pi_k$ denote $k(\geq 2)$ independent location-scale parametric models which have absolutely continuous cumulative distribution functions (cdf) $G(\frac{x-\theta_1}{\sigma}), \ldots, G(\frac{x-\theta_k}{\sigma})$, respectively, where $\sigma > 0$, $-\infty < \theta_i < \infty$, $i = 1, \ldots, k$ and $-\infty < x < \infty$. Let $\underline{\theta} = (\theta_1, \ldots, \theta_k)$ and let $\theta_{[1]} \leq \ldots \leq \theta_{[k]}$ denote the ordered values of $\theta_1, \ldots, \theta_k$. It is assumed that the exact pairing between the ordered parameters and the unordered parameters is unknown. The population associated with the largest location parameter $\theta_{[k]}$ is called the best population. Assume that the experimenter is interested in the selection of the best population. Let

$$\Omega = \{(\underline{\theta}, \sigma) | \underline{\theta} = (\theta_1, \ldots, \theta_k), \; -\infty < \theta_i < \infty, \; \sigma > 0\}$$

be the parameter space. We partition the parameter space into the following three subspaces:

the preference zone: $\Omega(PZ) = \{(\underline{\theta}, \sigma) \in \Omega | \frac{\theta_{[k]} - \theta_{[k-1]}}{\sigma} \geq \delta, \; \delta > 0\}$,

the nonselection zone: $\Omega(NZ) = \{(\underline{\theta}, \sigma) \in \Omega | \theta_{[k-1]} = \theta_{[k]}\}$,

the indifference zone: $\Omega(IZ) = \Omega - \Omega(PZ) - \Omega(NZ)$,

where δ is a known positive constant. Note that our $\Omega(IZ)$ is different from the indifference zone of Bechhofer (1954).

Denote the event of a correct selection by CS and the event of a selection by S. The goal is to develop a selection procedure R to select a single best population with a minimum sample size from each of the k populations such that the following probability requirements are satisfied:

$$P_{(\underline{\theta}, \sigma)}(S|R) \leq \alpha \text{ for all } (\underline{\theta}, \sigma) \in \Omega(NZ) \tag{2.1}$$

and

$$P_{(\underline{\theta}, \sigma)}(CS|R) \geq P^* \text{ for all } (\underline{\theta}, \sigma) \in \Omega(PZ) \tag{2.2}$$

where $\alpha \in (0, 1)$ and $P^* \in (1/k, 1)$ are preassigned probability levels.

The selection procedure R depends on whether the common scale parameter σ is known or unknown.

3. SELECTION PROCEDURE WITH KNOWN SCALE PARAMETER

Let $X_{ij},\ j = 1,\dots,n$ be n independent observations from population π_i, where $\pi_i \sim G(\frac{x-\theta_i}{\sigma}),\ i = 1,\dots,k$, respectively. Let $Y_i = Y(X_{i1},\dots,X_{in})$ be an appropriate statistic for θ_i. We assume that Y_i has the cdf $F_n(\frac{y-\theta_i}{\sigma})$. Also let $Y_{[1]} \le \dots \le Y_{[k]}$ denote the order statistics of $Y_1,\dots,Y_k$. When σ is known, we propose a selection procedure as follows:

R_1: Select the population yielding $Y_{[k]}$ as the best population if $Y_{[k]} - Y_{[k-1]} > \lambda(n,\alpha)\sigma$; otherwise, do not make a selection, where n and $\lambda(n,\alpha)$ are chosen to satisfy the probability requirements (2.1) and (2.2).

For the given rule R_1, we need to investigate the supremum of $P_{(\underline{\theta},\sigma)}(S|R_1)$ for $(\underline{\theta},\sigma) \in \Omega(NZ)$ and the infimum of $P_{(\underline{\theta},\sigma)}(CS|R_1)$ for $(\underline{\theta},\sigma) \in \Omega(PZ)$. We need to make two assumptions here:

(A) The probability density function (pdf) f_n of F_n is log-concave.

(B) For each $\delta > 0$ and any positive integer m, $\int_{-\infty}^{\infty} F_n^m(y+\delta)dF_n(y)$ is strictly increasing in n, and tends to 1 as $n \to \infty$.

These two assumptions are appropriate for many applications. First, we consider a lemma derived by Kim (1986).

Lemma 3.1. Assume that $\log f_n(y)$ is concave. Then for any fixed $c > 0$, $P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > c\}$ is non-increasing in $\theta_{[1]}$ and hence

$$P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > c\} \le P_{(\underline{\theta}^0,\sigma)}\{|Y_{(k)} - Y_{(k-1)}| > c\}$$

for all $(\underline{\theta},\sigma) \in \Omega$, where $\underline{\theta}^0 = (\theta_1^0,\dots,\theta_k^0), \theta_{[i]}^0 = -\infty,\ i = 1,\dots,k-2, \theta_{[i]}^0 = \theta_{[i]}$ for $i = k-1,k$, and $Y_{(i)}$ is the statistic associated with parameter $\theta_{[i]},\ i = 1,\dots,k$.

In the sequel, we let $Z_i = \frac{Y_{(i)}-\theta_{[i]}}{\sigma},\ i = 1,\dots,k$ and let $H_n(t)$ be the cdf of $Z_1 - Z_2$.

Then

$$H_n(t) = \int_{-\infty}^{\infty} F_n(y+t)dF_n(y). \tag{3.1}$$

We note that $H_n(t)$ is a symmetric distribution function and hence $H_n(-t) = 1 - H_n(t)$.

By using Lemma 3.1, we have the following theorem:

<u>Theorem 3.2</u>. $\sup\limits_{\Omega(NZ)} P_{(\underline{\theta},\sigma)}(S|R_1) = 2H_n(-\lambda(n,\alpha)).$ (3.2)

Proof: By Assumption (A) and Lemma 3.1, we have

$$\begin{aligned} P_{(\underline{\theta},\sigma)}(S|R_1) &= P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > \lambda(n,\alpha)\sigma\} \\ &\le P_{(\underline{\theta}^0,\sigma)}\{|Y_{(k)} - Y_{(k-1)}| > \lambda(n,\alpha)\sigma\}. \end{aligned}$$

Hence

$$\begin{aligned} \sup_{\Omega(NZ)} P_{(\underline{\theta},\sigma)}(S|R_1) &= P\{|Z_k - Z_{k-1}| > \lambda(n,\alpha)\} \\ &= 2H_n(-\lambda(n,\alpha)). \end{aligned}$$ □

In order to satisfy the probability requirement (2.1), we may let $2H_n(-\lambda(n,\alpha)) = \alpha$. That is,

$$\lambda(n,\alpha) = -H_n^{-1}(\alpha/2). \tag{3.3}$$

<u>Remark 3.1</u>. $\lambda(n,\alpha) \ge 0$, since H_n is symmetric.

<u>Lemma 3.3</u>. The $\lambda(n,\alpha)$ defined by (3.3) has the properties: $\lambda(n,\alpha)$ is decreasing in n and $\lambda(n,\alpha) \to 0$ as $n \to \infty$, for each fixed α, $0 < \alpha < 1$.

Proof: For $n_1 \ge n_2$, if $\lambda(n_1,\alpha) > \lambda(n_2,\alpha)$, then by Assumption (B) we have

$$\begin{aligned} \alpha/2 &= H_{n_1}(-\lambda(n_1,\alpha)) = 1 - H_{n_1}(\lambda(n_1,\alpha)) \\ &\le 1 - H_{n_1}(t) < 1 - H_{n_2}(t) \le 1 - H_{n_2}(\lambda(n_2,\alpha)) \\ &= H_{n_2}(-\lambda(n_2,\alpha)) = \alpha/2, \text{ for } \lambda(n_1,\alpha) > t > \lambda(n_2,\alpha). \end{aligned}$$

This is a contradiction and hence $\lambda(n_1, \alpha) \leq \lambda(n_2, \alpha)$.

If $\lim_{n \to \infty} \lambda(n, \alpha) = c > 0$, then $\lambda(n, \alpha) \geq c$ for all n, since $\lambda(n, \alpha)$ is decreasing in n for each fixed α. Thus, by the definition of $\lambda(n, \alpha)$, $1 - \frac{\alpha}{2} = H_n(\lambda(n, \alpha)) \geq H_n(c)$.

By Assumption (B), as $c > 0, H_n(c) \to 1$ as $n \to \infty$. Therefore, for n sufficiently large, $H_n(c) \geq 1 - \frac{\alpha}{4} > H_n(\lambda(n, \alpha))$, which is a contradiction. Hence, we have $\lambda(n, \alpha) \to 0$ as $n \to \infty$. □

We next evaluate the infimum of $P_{(\underline{\theta}, \sigma)}(CS|R_1)$ over $(\underline{\theta}, \sigma) \in \Omega(PZ)$.

Theorem 3.4. The infimum of $P_{(\underline{\theta}, \sigma)}(CS|R_1)$ over $\Omega(PZ)$ occurs at the configuration $\theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta\sigma$ and

$$\inf_{\Omega(PZ)} P_{(\underline{\theta}, \sigma)}(CS|R_1) = \int_{-\infty}^{\infty} F_n^{k-1}(y + \delta - \lambda(n, \alpha)) dF_n(y). \tag{3.4}$$

Proof:

$$\begin{aligned} P_{(\underline{\theta}, \sigma)}(CS|R_1) &= P_{(\underline{\theta}, \sigma)}\{Y_{[k]} - Y_{[k-1]} > \lambda(n, \alpha)\sigma,\ Y_{(k)} = Y_{[k]}\} \\ &= P_{(\underline{\theta}, \sigma)}\{Z_j < Z_k + \frac{\theta_{[k]} - \theta_{[j]}}{\sigma} - \lambda(n, \alpha),\ j = 1, \ldots, k-1\} \\ &\geq P_{(\underline{\theta}, \sigma)}\{Z_j < Z_k + \delta - \lambda(n, \alpha),\ j = 1, \ldots, k-1\}. \end{aligned} \tag{3.5}$$

The equality in (3.5) holds when $\theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta\sigma$. Hence

$$\begin{aligned} \inf_{\Omega(PZ)} P_{(\underline{\theta}, \sigma)}(CS|R_1) &= P_{(\underline{\theta}, \sigma)}\{Z_j < Z_k + \delta - \lambda(n, \alpha),\ j = 1, \ldots, k-1\} \\ &= \int_{-\infty}^{\infty} F_n^{k-1}(y + \delta - \lambda(n, \alpha)) dF_n(y). \end{aligned}$$ □

Remark 3.2.

(1) Since $\lambda(n, \alpha) \to 0$ as $n \to \infty$ for each $0 < \alpha < 1$, hence, by Assumption (B), we have, for each fixed $\delta > 0$, $\int_{-\infty}^{\infty} F_n^{k-1}(y + \delta - \lambda(n, \alpha)) dF_n(y) \to 1$ as $n \to \infty$.

(2) In order to satisfy the probability requirement (2.2), we may let the right hand side of (3.4) equal P^*. In practice, n is chosen to be the smallest integer such that (2.1) and (2.2) are satisfied.

4. SELECTION PROCEDURE WITH UNKNOWN SCALE PARAMETER

When the scale parameter σ is unknown, it is assumed that $Y_i = Y(X_{i1}, \ldots, X_{in})$ is a complete sufficient statistic for the parameter θ_i for each fixed $\sigma > 0$. Let $T_i = T(X_{i1}, \ldots, X_{in})$ be a nonnegative function of $X_{i1}, \ldots, X_{in}$, such that $T(X_{i1} - a, \ldots, X_{in} - a) = T(X_{i1}, \ldots, X_{in})$ and $T(cX_1, \ldots, cX_n) = cT(X_1, \ldots, X_n)$ for any $c > 0$. Also, let $S = S(T_1, \ldots, T_k)$ be a nonnegative function of $T_1, \ldots, T_k$ such that $S(cT_1, \ldots, cT_k) = cS(T_1, \ldots, T_k)$ for any $c > 0$. Since Y_i is a complete sufficient statistic for θ_i and the distribution of T_i is independent of θ_i, hence T_i is independent of Y_i. Therefore, S is independent of $(Y_1, \ldots, Y_k)$. Also, by the preceding assumption, the distribution of $W = S/\sigma$ is independent of the parameters $\theta_1, \ldots, \theta_k$ and σ. Hence, we propose a selection procedure R_2 as follows:

R_2: Select the population yielding $Y_{[k]}$ as the best population if $Y_{[k]} - Y_{[k-1]} > \tilde{\lambda}(n, \alpha)S$; otherwise, do not make a selection, where n and $\tilde{\lambda}(n, \alpha)$ are chosen to satisfy the probability requirements (2.1) and (2.2).

First, we evaluate the supremum of $P_{(\underline{\theta},\sigma)}(S|R_2)$ over $(\underline{\theta}, \sigma) \in \Omega(NZ)$. In the following, let $G(w)$ denote the cdf of $W = S/\sigma$.

Theorem 4.1.

$$\sup_{\Omega(NZ)} P_{(\underline{\theta},\sigma)}(S|R_2) = 2\int_0^\infty \int_{-\infty}^\infty F_n(y - \tilde{\lambda}(n, \alpha)w)dF_n(y)dG(w). \tag{4.1}$$

Proof:

$$P_{(\underline{\theta},\sigma)}(S|R_2) = P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > \tilde{\lambda}(n,\alpha)S\}$$

$$= E\{P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > \tilde{\lambda}(n,\alpha)\sigma W|W\}\}$$

$$= \int_0^\infty P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > \tilde{\lambda}(n,\alpha)\sigma w\}dG(w).$$

By Theorem 3.2, we have

$$\sup_{\Omega(NZ)} P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > \tilde{\lambda}(n,\alpha)\sigma w\} = 2H_n(-\tilde{\lambda}(n,\alpha)w).$$

Hence

$$\sup_{\Omega(NZ)} P_{(\underline{\theta},\sigma)}(S|R_2) = \int_0^\infty 2H_n(-\tilde{\lambda}(n,\alpha)w)dG(w)$$

$$= 2\int_0^\infty \int_{-\infty}^\infty F_n(y - \tilde{\lambda}(n,\alpha)w)dF_n(y)dG(w). \qquad \square$$

For each $0 < \alpha < 1$, let $\tilde{\lambda}(n,\alpha)$ be the solution of the equation

$$\int_0^\infty H_n(-\tilde{\lambda}(n,\alpha)w)dG(w) = \alpha/2. \tag{4.2}$$

Then the probability requirement (2.1) is satisfied.

Remark 4.1. $\tilde{\lambda}(n,\alpha)$ is nonnegative.

Lemma 4.2. The $\tilde{\lambda}(n,\alpha)$ defined by (4.2) has the properties: $\tilde{\lambda}(n,\alpha)$ is decreasing in n and $\tilde{\lambda}(n,\alpha) \to 0$ as $n \to \infty$.

Proof: If $n_1 \geq n_2$ and $\tilde{\lambda}(n_1,\alpha) > \tilde{\lambda}(n_2,\alpha)$, then

$$\alpha/2 = \int_0^\infty H_{n_1}(-\tilde{\lambda}(n_1,\alpha)w)dG(w) = 1 - \int_0^\infty H_{n_1}(\tilde{\lambda}(n_1,\alpha)w)dG(w)$$

$$< 1 - \int_0^\infty H_{n_2}(\tilde{\lambda}(n_2,\alpha)w)dG(w) = \int_0^\infty H_{n_2}(-\tilde{\lambda}(n_2,\alpha)w)dG(w) = \alpha/2,$$

which is a contradiction. Hence $\tilde{\lambda}(n,\alpha)$ is decreasing in n for each fixed α, $0 < \alpha < 1$. Moreover, if $\lim_{n\to\infty} \tilde{\lambda}(n,\alpha) = c > 0$, then

$$1 - \alpha/2 = \int_0^\infty H_n(\tilde{\lambda}(n,\alpha)w)dG(w) \geq \int_0^\infty H_n(c)dG(w).$$

However, as $c > 0$, by Assumption (B), $\int_0^\infty H_n(c)dG(w) \to 1$ as $n \to \infty$, which leads to a contradiction. Hence, $\tilde{\lambda}(n, \alpha) \to 0$ as $n \to \infty$. □

Now we evaluate the infimum of $P_{(\underline{\theta},\sigma)}(CS|R_2)$ over $(\underline{\theta}, \sigma) \in \Omega(PZ)$.

Theorem 4.3. The infimum of $P_{(\underline{\theta},\sigma)}(CS|R_2)$ over $\Omega(PZ)$ occurs at the configuration $\theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta\sigma$ and

$$\inf_{\Omega(PZ)} P_{(\underline{\theta},\sigma)}(CS|R_2) = \int_0^\infty \int_{-\infty}^\infty F_n^{k-1}(y + \delta - \tilde{\lambda}(n, \alpha)w)dF_n(y)dG(w). \tag{4.3}$$

Proof:

$$\begin{aligned}
P_{(\underline{\theta},\sigma)}(CS|R_2) &= P_{(\underline{\theta},\sigma)}\{Y_{[k]} - Y_{[k-1]} > \tilde{\lambda}(n, \alpha)S, Y_{(k)} = Y_{[k]}\} \\
&= P_{(\underline{\theta},\sigma)}\{Z_j < Z_k + \frac{\theta_{[k]} - \theta_{[j]}}{\sigma} - \tilde{\lambda}(n, \alpha)W,\ j = 1, \ldots, k-1\} \\
&\geq P\{Z_j < Z_k + \delta - \tilde{\lambda}(n, \alpha)W,\ j = 1, \ldots, k-1\} \qquad (4.4) \\
&= \int_0^\infty \int_{-\infty}^\infty F_n^{k-1}(y + \delta - \tilde{\lambda}(n, \alpha)w)dF_n(y)dG(w).
\end{aligned}$$

The equality in (4.4) holds when $\theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta\sigma$. □

Remark 4.2.

(1) For fixed $w > 0$ and $0 < \alpha < 1$, we have $\tilde{\lambda}(n, \alpha)w \to 0$ as $n \to \infty$. Now by Assumption (B), we have

$$\int_0^\infty \int_{-\infty}^\infty F_n^{k-1}(y + \delta - \tilde{\lambda}(n, \alpha)w)dF_n(y)dG(w) \to 1 \text{ as } n \to \infty.$$

(2) In order to satisfy the probability requirement (2.2), we may choose the smallest n such that

$$\int_0^\infty \int_{-\infty}^\infty F_n^{k-1}(y + \delta - \tilde{\lambda}(n, \alpha)w)dF_n(y)dG(w) \geq P^*. \tag{4.5}$$

5. NORMAL MODEL CASE

In this section, it is assumed that the populations $\pi_1, \ldots \pi_k$ have normal distributions with means $\theta_1, \ldots, \theta_k$, respectively, and a common variance σ^2. Let $X_{ij},\ j = 1, \ldots, n$, be independent samples from $\pi_i,\ i = 1, \ldots, k$ and $Y_i = \overline{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij},\ i = 1, \ldots, k$. Then $Y_i \sim N(\theta_i,\ \frac{\sigma^2}{n})$ and $F_n(x) = \Phi(\sqrt{n}x)$ where Φ is the cdf of the standard normal. One can easily check that Assumptions (A) and (B) are satisfied. Also, we have $H_n(t) = \Phi(\sqrt{\frac{n}{2}}t)$. We discuss both cases when the scale parameter σ is known or unknown.

(a) σ known case:

It follows from (3.3) that

$$\lambda(n, \alpha) = \sqrt{\frac{2}{n}} z_{\alpha/2} \tag{5.1}$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution. Thus, the selection procedure is:

R_1: Select the population yielding $Y_{[k]}$ as the best population if $Y_{[k]} - Y_{[k-1]} > \sqrt{\frac{2}{n}} z_{\alpha/2}\sigma$; otherwise, do not make a selection.

This is the same procedure as the one proposed by Chen (1985).

From equation (3.4), we have

$$\inf_{\Omega(PZ)} P_{(\underline{\theta},\sigma)}(CS|R_1) = \int_{-\infty}^{\infty} \Phi^{k-1}(y + \sqrt{n}\delta - \sqrt{2}z_{\alpha/2}) d\Phi(y).$$

For given k, P^* and α, let d be the solution to the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}(y + d) d\Phi(y) = P^*. \tag{5.2}$$

Then the sample size required to satisfy the probability requirements (2.1) and (2.2) is given by

$$n = < (\frac{d + \sqrt{2}z_{\alpha/2}}{\delta})^2 >,$$

where $< x >$ is the smallest integer not less than x. The solutions of d-values in equation (5.2) can be found in Bechhofer (1954), Gupta (1963) and Gupta, Nagel and Panchapakesan (1973).

(b) σ unknown case:

Let $T_i^2 = \sum_{j=1}^{n}(X_{ij} - Y_i)^2$ and $S^2 = \sum_{i=1}^{k} T_i^2/v$, where $v = k(n-1)$. Then $vS^2/\sigma^2 = vW^2$ follows a χ^2-distribution with ν degrees of freedom. We denote the distribution of the nonnegative random variable W by G.

For $\alpha \in (0,1)$, we determine $\tilde{\lambda}(n,\alpha)$ by solving the equation

$$\int_0^\infty H_n(-\tilde{\lambda}(n,\alpha)w)dG(w) = \alpha/2.$$

However, we can solve an easy equation in this case. Because

$$\int_0^\infty H_n(-\tilde{\lambda}(n,\alpha)w)dG(w) = P\{Z \le -\tilde{\lambda}(n,\alpha)W\}$$

where $Z \sim N(0, \frac{2}{n})$ and $vW^2 \sim \chi_v^2$ are independent, thus

$$\int_0^\infty H_n(-\tilde{\lambda}(n,\alpha)w)dG(w) = P\{\sqrt{\frac{n}{2}}\frac{Z}{W} \le -\sqrt{\frac{n}{2}}\tilde{\lambda}(n,\alpha)\}$$

where $\sqrt{\frac{n}{2}}\frac{Z}{W} \sim t_v$, the t-distribution with degrees of freedom v. Hence $\tilde{\lambda}(n,\alpha) = \sqrt{\frac{2}{n}}\, t_{v,\alpha/2}$, where $t_{v,\alpha/2}$ is the upper $\alpha/2$ quantile of the t_v distribution.

Furthermore, from (4.5), n is the smallest integer such that

$$\int_0^\infty \int_{-\infty}^\infty \Phi^{k-1}(y + \sqrt{n}\delta - \sqrt{2}t_{v,\alpha/2}w)d\Phi(y)dG(w) \ge P^*.$$

Remark: Although this result is the same as that of Chen (1985), the concept is different. Chen's method should specify the ratio δ^*/σ in advance. However, we need not make this assumption in our approach.

6. A TWO-STAGE PROCEDURE WHEN SCALE PARAMETER IS UNKNOWN

If we partition the parameter space Ω into three parts as follows.

$$\tilde{\Omega}(PZ) = \{(\underline{\theta}, \sigma) \in \Omega | \theta_{[k]} - \theta_{[k-1]} \geq \delta^*,\ \delta^* > 0\}$$

$$\tilde{\Omega}(NZ) = \{(\underline{\theta}, \sigma) \in \Omega | \theta_{[k-1]} = \theta_{[k]}\}$$

and

$$\tilde{\Omega}(IZ) = \Omega - \tilde{\Omega}(PZ) - \tilde{\Omega}(NZ).$$

If σ were known, we can take $\delta = \delta^*/\sigma$, then the result is the same as that of Section 3. When σ is unknown, a single-stage procedure does not exist for this problem. In the following, we consider the normal case only. Analogous to that of Bechhofer, Dunnett and Sobel (1954), a two-stage selection procedure R_3 is proposed as follows:

(i) Take an initial sample of size n_0 from each of the k populations, say $X_{i1}, \ldots, X_{in_0}$, $i = 1, \ldots, k$.

Let $Y_i(n_0) = \frac{1}{n_0} \sum_{j=1}^{n_0} X_{ij}$ and $S_0^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_0} (X_{ij} - Y_i(n_0))^2 / v_0$, $v_0 = k(n_0 - 1)$, and $W_0 = S_0/\sigma$.

(ii) Define $N = \max\{n_0, < \frac{h^2 S_0^2}{\delta^{*2}} >\}$, where $h > \sqrt{2} t_{v_0, \alpha/2}$ is determined by

$$\int_0^\infty \int_{-\infty}^\infty \Phi^{k-1}(y + (h - \sqrt{2} t_{v_0,\alpha/2}) w) d\Phi(y) dG_0(w), \tag{6.1}$$

where $G_0(w)$ is the cdf of W_0.

(iii) If necessary, take $N - n_0$ additional observations from each of the k populations and compute

$$Y_i(N) = \frac{1}{N} \sum_{j=1}^{N} X_{ij},\ i = 1, \ldots, k.$$

(iv) The selection rule R_3 is defined by:

R_3: Select the population yielding $Y_{[k]}(N)$ as the best population if $Y_{[k]}(N)-Y_{[k-1]}(N) > \frac{\lambda}{\sqrt{N}}S_0$; otherwise do not make a selection; here λ is chosen to satisfy the probability requirement (2.1).

For the procedure R_3 defined above, we have the following result:

Theorem 6.1.

$$\sup_{\tilde{\Omega}(NZ)} P_{(\underline{\theta},\sigma)}(S|R_3) = P\{|T| > \frac{\lambda}{\sqrt{2}}\}, \tag{6.2}$$

where T follows a Student's t-distribution with v_0 degrees of freedom.

Proof: Let $A_n = \{N = n\}$, then $A_{n_0} = \{W_0^2 \leq \frac{n_0 \delta^{*^2}}{h^2\sigma^2}\}$ and

$$A_n = \{\frac{(n-1)\delta^{*^2}}{h^2\sigma^2} < W_0^2 \leq \frac{n\delta^{*^2}}{h^2\sigma^2}\} \text{ if } n > n_0.$$

Thus,

$$\begin{aligned} P_{(\underline{\theta},\sigma)}(S|R_3) &= P_{(\underline{\theta},\sigma)}\{Y_{[k]}(N) - Y_{[k-1]}(N) > \frac{\lambda}{\sqrt{N}}S_0\} \\ &= \sum_{n=n_0}^{\infty} \int_{A_n} P_{(\underline{\theta},\sigma)}\{\sqrt{N}\frac{Y_{[k]}(N)}{\sigma} - \sqrt{N}\frac{Y_{[k-1]}(N)}{\sigma} > \lambda w\} dG_0(w) \\ &\leq \sum_{n=n_0}^{\infty} \int_{A_n} P_{(\underline{\theta},\sigma)}\{|\sqrt{N}\frac{Y_{(k)}(N)}{\sigma} - \sqrt{N}\frac{Y_{(k-1)}(N)}{\sigma}| > \lambda w\} dG_0(w). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sup_{\tilde{\Omega}(NZ)} P_{(\underline{\theta},\sigma)}(S|R_3) &= \sum_{n=n_0}^{\infty} \int_{A_n} 2\Phi(-\frac{\lambda w}{\sqrt{2}}) dG_0(w) \\ &= \int_0^{\infty} 2\Phi(-\frac{\lambda w}{\sqrt{2}}) dG_0(w) \\ &= P\{|T| > \frac{\lambda}{\sqrt{2}}\}. \end{aligned}$$ □

In order to satisfy the probability requirement (2.1), we set $P\{|T| > \frac{\lambda}{\sqrt{2}}\} = \alpha$. Then $\lambda = \sqrt{2}t_{v_0,\alpha/2}$.

Now, we evaluate the infimum of $P_{(\underline{\theta},\sigma)}(CS|R_3)$ over $(\underline{\theta},\sigma) \in \tilde{\Omega}(PZ)$.

Theorem 6.2. The infimum of $P_{(\underline{\theta},\, \sigma)}(CS|R_3)$ over $\tilde{\Omega}(PZ)$ occurs at the configuration $\theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta^*$ and

$$\inf_{\tilde{\Omega}(PZ)} P_{(\underline{\theta},\sigma)}(CS|R_3) \geq \int_0^\infty \int_{-\infty}^\infty \Phi^{k-1}(y + (h - \sqrt{2}t_{v_0,\alpha/2})w)d\Phi(y)dG_0(w). \tag{6.3}$$

Proof:

$$\begin{aligned}
P_{(\underline{\theta},\, \sigma)}(CS|R_3) &= P_{(\underline{\theta},\, \sigma)}\{Y_{(j)}(N) < Y_{(k)}(N) - \frac{\lambda}{\sqrt{N}}S_0,\ j = 1, \ldots, k-1\} \\
&= P_{(\underline{\theta},\, \sigma)}\{Z_j < Z_k + \frac{\sqrt{N}(\theta_{[k]} - \theta_{[j]})}{\sigma} - \lambda W_0,\ j = 1, \ldots, k-1\} \\
&\geq P_{(\underline{\theta},\, \sigma)}\{Z_j < Z_k + \frac{\sqrt{N}\ \delta^*}{\sigma} - \lambda W_0,\ j = 1, \ldots, k-1\} \qquad (6.4) \\
&\geq P_{(\underline{\theta},\, \sigma)}\{Z_j < Z_k + (h - \lambda)W_0,\ j = 1, \ldots, k-1\},\ \text{since } \sqrt{N} \geq \frac{hS_0}{\delta^*} \\
&= \int_0^\infty \int_{-\infty}^\infty \Phi^{k-1}(y + (h - \lambda)w)d\Phi(y)dG_0(w). \qquad (6.5)
\end{aligned}$$

The equality in (6.4) holds when $\theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta^*$. The λ in (6.5) is $\sqrt{2}t_{v_0}, \alpha/2$.

□

In order to satisfy the probability requirement (2.2), let d be the solution of

$$\int_0^\infty \int_{-\infty}^\infty \Phi^{k-1}(y + dw)d\Phi(y)dG_0(w) = P^*. \tag{6.6}$$

Then $h = \sqrt{2}t_{v_0,\ \alpha/2} + d$.

BIBLIOGRAPHY

Bechhofer, R.E. (1954). A single-sample multiple-decision procedure for ranking means of normal populations with known variances. *Ann. Math. Statist.*, **25**, 16–39.

Bechhofer, R.E., Dunnett, C.W. and Sobel, M. (1954). A two-sample multiple decision procedure for ranking means of normal populations with a common unknown variance. *Biometrika* **41**, 170–176.

Bishop, T.A. and Pirie, W.R. (1979). A class of selection rules with options to not select. *Technical Report No. 196*, Department of Statistics, The Ohio State University, Columbus, Ohio.

Chen, H.J. (1985). A new selection procedure for selecting the best population with a preliminary test (with discussion). *The Frontiers of Modern Statistical Inference Procedures.* American Sciences Press, Inc., Columbus, Ohio, 99–117.

Chen, H.J. and Mithongtae, J. (1986). Selection of the best exponential distribution with a preliminary test. *American Journal of Mathematical and Management Sciences,* **Vol. 6**, 219–249.

Gupta, S.S. (1956). On a decision rule for a problem in ranking means. *Mimeograph Series No. 150,* Institute of Statistics, University of North Carolina, Chapel Hill, North Carolina.

Gupta, S.S. (1963). Probability integrals of the multivariate normal and multivariate *t*. *Ann. Math. Statist.*, **34**, 792–828.

Gupta, S.S. (1963). Bibliography on the multivariate normal integrals and related topics. *Ann. Math. Statist.*, **34**, 829–838.

Gupta, S.S. (1965). On some multiple decision (selection and ranking) rules. *Technometrics* **7**, 225–245.

Gupta, S.S., Nagel, K. and Panchapakesan, S. (1973). On the order statistics from equally correlated normal random variables. *Biometrika* **60**, 403–413.

Gupta, S.S. and Panchapakesan, S. (1979). *Multiple Decision Procedures: Theory and Methodology of Selecting and Ranking Populations.* John Wiley and Sons, Inc., New York.

Kim, W.C. (1986). A lower confidence bound on the probability of a correct selection. *J. Amer. Statist. Assoc.*, **81**, 1012–1017.

Leu, L.Y. and Liang, T. (1990). Selection of the best with a preliminary test for two-parameter exponential distributions. To appear in *Communications in Statistics – Theory and Methods.*

Simon, R. (1977). Adaptive treatment assignment methods and clinical trials. *Biometrics*, **33**, 743–749.

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This paper deals with the problem of selecting the best population from among $k(\geq 2)$ populations which are location-scale models. New selection procedures are proposed for selecting the unique best in terms of the largest location parameter. The procedures include preliminary tests which allow the experimenter to have an option to not select if the statistical evidence is not significant. Two probabilities, the probability to make a selection and the probability of a correct selection, are controlled by these selection procedures. Applications to the normal mean models are considered. Comparisons between the proposed selection procedures and certain earlier existing procedures are also made. Finally, a two-stage procedure for the normal means problem is considered.

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