SOLUTION OF AN OPTIMIZATION PROBLEM ARISING IN MAXIMUM LIKELIHOOD ESTIMATION OF ORDERED DISTRIBUTIONS

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Abstract

We propose and solve an optimization problem arising in maximum likelihood estimation of ordered distributions. In particular it generalizes a recent result due to Puri and Singh (1988). As an application of our results we obtain a "maximum likelihood" estimate $\hat{F}_n$ of a cumulative distribution function (c.d.f.) F, based on a sample of size n from F, where F is known to satisfy $F_1(x) \leq F(x) \leq F_2(x), \ \forall \ x$, for two given c.d.f.'s F_1 and F_2. It is also shown that $\hat{F}_n$ is strongly consistent.

Key words: Statistical inference under order restrictions, Maximum likelihood estimation, Isotonic regression, Stochastically ordered distributions.

1. Introduction

Frequently one comes across situations where statistical inferences under order restrictions are desirable. The reader may refer for such situations to Barlow et. al. (1972), Robertson and Wright (1974), (1981), Lee and Wolfe (1976), Lee (1981), Dykstra (1982), Feltz and Dykstra (1985), Schoenfeld (1986), Sampson and Whitaker (1987), Robertson et. al. (1988), Puri and Singh (1988), among others. In most of the problems of statistical inference under order restrictions there is an underlying mathematical concept known as 'Isotonic Regression' (see Barlow et. al (1972)). Recently Puri and Singh (1988) have obtained recursive formulas for isotonic regression which can be used without the need of any algorithm. They use these formulas together with a corollary to Proposition 1.1 of Barlow et. al. (1972) (see its page 51) to prove the following theorem.

Theorem 1.1. *Let ϕ be a convex, finite real valued function defined on an interval I of the real line. Let $\beta_1, \beta_2, \ldots, \beta_k$ be k real numbers; $w_i > 0, i = 1, 2, \ldots, k$; $W_j = \sum_{i=1}^{j} w_i, j = 1, 2, \ldots, k$ and*

$$D_1 = \left\{ (x_1, x_2, \ldots, x_k) : x_i \in I, 1 \leq i \leq k; \sum_{i=1}^{j} w_i x_i \leq \beta_j, 1 \leq j \leq k-1; \sum_{i=1}^{k} w_i x_i = \beta_k \right\} \tag{1.1}$$

assumed to be a nonempty set. Then subject to $(x_1, x_2, \ldots, x_k) \in D_1$, the expression

$$\sum_{i=1}^{k} \phi(x_i) w_i \tag{1.2}$$

is minimized at a point $(\tau_1, \tau_2, \ldots, \tau_k) \in D_1$, where τ_i's are given by

$$\tau_1 = \min_{1 \leq i \leq k} [\beta_i / W_i], \tag{1.3}$$

$$\tau_j = \min_{j \leq i \leq k} \left[\left(\beta_i - \sum_{r=1}^{j-1} w_r \tau_r \right) \Big/ \left(\sum_{\ell=j}^{i} w_\ell \right) \right], j = 2, 3, \ldots, k. \tag{1.4}$$

The above minimizing solution is unique if ϕ is strictly convex.

Puri and Singh (1988) use this theorem for finding the "maximum likelihood" estimate (M.L.E.) of a distribution known to dominate stochastically a given distribution. In Section 2 we study a generalization of the optimization problem considered in Theorem 1.1 (see Theorems 2.1 and 2.3) which is found useful for "maximum likelihood" estimation of ordered distributions. As an illustration, in Section 3 we apply Theorem 2.1 for obtaining a M.L.E. of a c.d.f. F, based on a sample of size n from F, where F is apriori known to satisfy $F_1(x) \leq F(x) \leq F_2(x), \quad \forall\, x$, for two given c.d.f.'s F_1 and F_2. Also Theorem 2.1 in turns leads to Theorem 2.3 which may serve as a step towards obtaining the M.L.E.'s of $N \geq 3$ ordered distributions based on N corresponding samples. We end with some concluding remarks in Section 4.

2. An Optimization Problem.

Let ϕ be a convex finite real valued function defined on an interval I of the real line. For $k \geq 2$, let $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}, \beta_1, \beta_2, \ldots, \beta_k$ be real numbers with $\gamma_j \leq \beta_j, j = 1, 2, \ldots, k-1$. Let $w_i > 0, i = 1, 2, \ldots, k$ and

$$D = \Big\{(x_1, x_2, \ldots, x_k) : x_i \in I, 1 \leq i \leq k; \gamma_j \leq \sum_{i=1}^{j} w_i x_i \leq \beta_j, 1 \leq j \leq k-1; \\ \sum_{i=1}^{k} w_i x_i = \beta_k\Big\} \tag{2.1}$$

assumed to be a nonempty set. The problem we consider and solve in this paper is of minimizing the sum (1.2) subject to $(x_1, x_2, \ldots, x_k) \in D$.

We begin with the case where the interval I is closed and the function ϕ is strictly convex. For this case, it can be easily seen that the set D is bounded and closed and since (1.2) is a finite valued convex (and hence continuous) function defined over the k-dimensional cube I^k containing the set D, a minimizing solution $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of (1.2), subject to $(x_1, x_2, \ldots, x_k) \in D$, exists and belongs to D. In order to determine these α_i's explicitly we need the following four lemmas.

Lemma 2.1. *If for some* $i, 1 \leq i \leq k-1$,

$$\sum_{m=1}^{i} \alpha_m w_m > \gamma_i,$$

then

$$\alpha_i \leq \alpha_{i+1}.$$

Proof. Let $\delta = \sum_{m=1}^{i} \alpha_m w_m - \gamma_i$ and $0 < \varepsilon < \delta$. Note that $(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i - (\varepsilon/w_i), \alpha_{i+1} + (\varepsilon/w_{i+1}), \alpha_{i+2}, \ldots, \alpha_k) \in D$ for sufficiently small ε. Thus we have

$$\sum_{m=1}^{k} \phi(\alpha_m) w_m \leq \left[\sum_{m=1}^{i-1} \phi(\alpha_m) w_m + \phi(\alpha_i - (\varepsilon/w_i)) w_i + \phi(\alpha_{i+1} + (\varepsilon/w_{i+1})) w_{i+1} + \sum_{m=i+2}^{k} \phi(\alpha_m) w_m \right].$$

$$\Longrightarrow \phi(\alpha_i) w_i + \phi(\alpha_{i+1}) w_{i+1} \leq \phi(\alpha_i - (\varepsilon/w_i)) w_i + \phi(\alpha_{i+1} + (\varepsilon/w_{i+1})) w_{i+1}$$

$$\Longrightarrow (w_i/\varepsilon)(\phi(\alpha_i) - \phi(\alpha_i - (\varepsilon/w_i))) \leq (w_{i+1}/\varepsilon)(\phi(\alpha_{i+1} + (\varepsilon/w_{i+1})) - \phi(\alpha_{i+1})). \quad (2.2)$$

Now for any $x \in I$, let

$$\phi'_-(x) = \lim_{h \to 0+} (\phi(x) - \phi(x-h))/h, \quad (2.3)$$

$$\phi'_+(x) = \lim_{h \to 0+} (\phi(x+h) - \phi(x))/h, \quad (2.4)$$

Note that (2.3) and (2.4) are well defined since ϕ is a convex function. Thus from (2.2), by letting $\varepsilon \to 0$, we obtain

$$\phi'_-(\alpha_i) \leq \phi'_+(\alpha_{i+1}). \quad (2.5)$$

We prove by contradiction that (2.5) implies $\alpha_i \le \alpha_{i+1}$. Thus to the contrary, let $\alpha_i > \alpha_{i+1}$. Then the convexity of ϕ implies that

$$\phi'_+(\alpha_{i+1}) \le \phi'_-(\alpha_i), \tag{2.6}$$

so that in view of (2.5), we have

$$\phi'_-(\alpha_i) = \phi'_+(\alpha_{i+1}), \tag{2.7}$$

implying thereby that the function ϕ has a constant derivative in (α_{i+1}, α_i). Consequently ϕ is linear in (α_{i+1}, α_i) which is a contradiction to the strict convexity of ϕ. Hence $\alpha_i \le \alpha_{i+1}$. □

Lemma 2.2. *If for some* $i, 1 \le i \le k-1$, $\sum_{m=1}^{i} w_m \alpha_m < \beta_i$, *then* $\alpha_i \ge \alpha_{i+1}$.

Proof. Let $\delta = \beta_i - \sum_{m=1}^{i} w_m \alpha_m$ and $0 < \varepsilon < \delta$, then it is easy to see that

$$(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i + (\varepsilon/w_i), \alpha_{i+1} - (\varepsilon/w_{i+1}), \alpha_{i+2}, \ldots, \alpha_k) \in D,$$

for sufficiently small ε.

Since $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a minimizing solution of (1.2) subject to $(x_1, x_2, \ldots, x_k) \in D$, we have

$$\phi(\alpha_i + (\varepsilon/w_i))w_i + \phi(\alpha_{i+1} - (\varepsilon/w_{i+1}))w_{i+1} \ge \phi(\alpha_i)w_i + \phi(\alpha_{i+1})w_{i+1}$$

or equivalently

$$(w_i/\varepsilon)(\phi(\alpha_i + (\varepsilon/w_i)) - \phi(\alpha_i)) \ge (w_{i+1}/\varepsilon)(\phi(\alpha_{i+1}) - \phi(\alpha_{i+1} - (\varepsilon/w_{i+1}))).$$

Letting $\varepsilon \to 0$ yields

$$\phi'_+(\alpha_i) \ge \phi'_-(\alpha_{i+1}).$$

The rest of the proof, being similar to that in Lemma 2.1, is omitted. □

Lemma 2.3. *For* $1 \leq J \leq k-1$,

$$\alpha_1 \leq \max[\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_J/W_J, \beta_{J+1}/W_{J+1}]. \tag{2.8}$$

Proof. Let $J, 1 \leq J \leq k-1$, be fixed. Suppose that

$$\alpha_1 > \max[\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_J/W_J]. \tag{2.9}$$

We assert that $\alpha_i \leq \alpha_{i+1} \ \forall \ 1 \leq i \leq J$, and prove it by induction. Since $\alpha_1 > \gamma_1/W_1$, therefore by Lemma 2.1, $\alpha_1 \leq \alpha_2$, so that the assertion holds for $i = 1$. Suppose now that the assertion is true for $1 \leq i \leq j$, where j is some natural number not exceeding $J-1$. Then we have

$$\sum_{m=1}^{j+1} \alpha_m w_m \geq \alpha_1 \sum_{m=1}^{j+1} w_m = \alpha_1 W_{j+1} > \gamma_{j+1}.$$

Hence by Lemma 2.1, $\alpha_{j+2} \geq \alpha_{j+1}$ and the assertion follows. Now

$$\beta_{J+1} \geq \sum_{\ell=1}^{J+1} \alpha_\ell w_\ell \geq \alpha_1 \sum_{\ell=1}^{J+1} w_\ell = \alpha_1 W_{J+1},$$

which implies $\alpha_1 \leq \beta_{J+1}/W_{J+1}$. This completes the proof of the Lemma. □

Lemma 2.4.

$$\alpha_1 = \min[\beta_1/W_1, \max(\gamma_1/W_1, \beta_2/W_2),$$

$$\max(\gamma_1/W_1, \gamma_2/W_2, \beta_3/W_3), \ldots, \max(\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_{k-1}/W_{k-1}, \beta_k/W_k)]. \tag{2.10}$$

Proof. By definition of $D, \alpha_1 \leq \beta_1/W_1$ and the result that α_1 is less than or equal to the right hand side of (2.10) now follows immediately by using Lemma 2.3. We prove by contradiction that α_1 is greater than or equal to the right hand side of (2.10). Thus we suppose to the contrary that α_1 is strictly less than the right hand side of (2.10). In this case we assert and prove by induction that $\alpha_i \geq \alpha_{i+1}, i = 1, 2, \ldots, k-1$. Since $\alpha_1 < \beta_1/W_1$, therefore by Lemma 2.2, $\alpha_1 \geq \alpha_2$. So the assertion holds for $i = 1$. Assume that the assertion is true for all i satisfying $1 \leq i \leq J$, where J is a natural number not exceeding $k-2$. Now for $1 \leq j \leq J$, we have $\alpha_1 \sum_{\ell=1}^{j} w_\ell \geq \sum_{\ell=1}^{j} \alpha_\ell w_\ell \geq \gamma_j$ and so $\alpha_1 \geq \gamma_j/W_j$. Since we have assumed that

$$\alpha_1 < \max[\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_J/W_J, \beta_{J+1}/W_{J+1}],$$

it follows that $\alpha_1 < \beta_{J+1}/W_{J+1}$ and hence

$$\sum_{m=1}^{J+1} \alpha_m w_m \leq \alpha_1 \sum_{m=1}^{J+1} w_m = \alpha_1 W_{J+1} < \beta_{J+1}.$$

By Lemma 2.2, we have $\alpha_{J+1} \geq \alpha_{J+2}$. This establishes our assertion, namely $\alpha_i \geq \alpha_{i+1}, 1 \leq i \leq k-1$. Hence for every $j, 1 \leq j \leq k-1$, we have $\alpha_1 W_j \geq \sum_{m=1}^{j} \alpha_m w_m \geq \gamma_j$ implying that $\alpha_1 \geq \gamma_j/W_j$. But we have assumed that

$$\alpha_1 < \max[\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_{k-1}/W_{k-1}, \beta_k/W_k],$$

therefore we have $\alpha_1 < \beta_k/W_k$ and consequently $\sum_{m=1}^{k} \alpha_m w_m \leq \alpha_1 \sum_{m=1}^{k} w_m = \alpha_1 W_k < \beta_k$. This contradicts that $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in D$. □

Below we state and prove the Proposition 2.1 which gives the minimizing solution $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of (1.2), for ϕ strictly convex, subject to $(x_1, x_2, \ldots, x_k) \in D$, when I is a closed interval.

Proposition 2.1. *Let ϕ be a strictly convex, finite real valued function defined on a closed interval I of the real line. Then subject to $(x_1, x_2, \ldots, x_k) \in D$, where D is defined by*

(2.1), *the sum* (1.2) *is minimized at a unique point* $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in D$, *where* α_i*'s are given by*

$$\alpha_1 = \min[\beta_1/W_1, \max(\gamma_1/W_1, \beta_2/W_2), \max(\gamma_1/W_1, \gamma_2/W_2, \beta_3/W_3),$$

$$\ldots, \max(\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_{k-1}/W_{k-1}, \beta_k/W_k)], \tag{2.11}$$

$$\alpha_{j+1} = \min\Bigg[\left(\beta_{j+1} - \sum_{i=1}^{j} w_i\alpha_i\right)/w_{j+1}, \max\left(\left(\gamma_{j+1} - \sum_{i=1}^{j} w_i\alpha_i\right)/w_{j+1},\right.$$

$$\left.\left(\beta_{j+2} - \sum_{i=1}^{j} w_i\alpha_i\right)\Big/\sum_{i=j+1}^{j+2} w_i\right), \ldots, \max\left(\left(\gamma_{j+1} - \sum_{i=1}^{j} w_i\alpha_i\right)/w_{j+1},\right.$$

$$\left(\gamma_{j+2} - \sum_{i=1}^{j} w_i\alpha_i\right)\Big/\sum_{i=j+1}^{j+2} w_i, \ldots,$$

$$\left.\left(\gamma_{k-1} - \sum_{i=1}^{j} w_i\alpha_i\right)\Big/\sum_{i=j+1}^{k-1} w_i, \left(\beta_k - \sum_{i=1}^{j} w_i\alpha_i\right)\Big/\sum_{i=j+1}^{k} w_i\right)\Bigg],$$

$$j = 1, 2, \ldots, k-2, \tag{2.12}$$

$$\alpha_k = \left(\beta_k - \sum_{i=1}^{k-1} w_i\alpha_i\right)/w_k. \tag{2.13}$$

Proof. The proof of (2.11) has already been given in Lemma 2.4. Now suppose that for some $j, 1 \leq j \leq k-1, \alpha_1, \alpha_2, \ldots, \alpha_j$ have been determined. Let $m = k - j$ and

$$D' = \{(y_1, y_2, \ldots, y_m) : (\alpha_1, \alpha_2, \ldots, \alpha_j, y_1, y_2, \ldots, y_m) \in D\}, \tag{2.14}$$

which can also be expressed as

$$D' = \left\{(y_1, y_2, \ldots, y_m) : y_i \in I, 1 \leq i \leq m; \gamma'_i \leq \sum_{r=1}^{i} y_r w'_r \leq \beta'_i, 1 \leq i \leq m-1,\right.$$

$$\left.\sum_{r=1}^{m} y_r w'_r = \beta'_m\right\}, \tag{2.15}$$

where

$$\gamma'_i = \gamma_{j+i} - \sum_{r=1}^{j} \alpha_r w_r, \beta'_i = \beta_{j+i} - \sum_{r=1}^{j} \alpha_r w_r, w'_i = w_{j+i}, i = 1, 2, \ldots, m. \tag{2.16}$$

Let

$$W_i' = \sum_{r=1}^{i} w_{j+r}, i = 1, 2, \ldots, k-j. \tag{2.17}$$

Since $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in D$, therefore $(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_k) \in D'$.

Now observe that

$$\sum_{i=1}^{k} \phi(\alpha_i) w_i \geq \sum_{i=1}^{j} \phi(\alpha_i) w_i + \min_{(y_1, y_2, \ldots, y_m) \in D'} \sum_{i=1}^{m} \phi(y_i) w_i' \geq$$
$$\min_{(x_1, x_2, \ldots, x_k) \in D} \sum_{i=1}^{k} \phi(x_i) w_i = \sum_{i=1}^{k} \phi(\alpha_i) w_i.$$

Thus α_{j+i}'s, $i = 1, 2, \ldots, k-j$ satisfy

$$\sum_{i=j+1}^{k} \phi(\alpha_i) w_i = \min_{(y_1, y_2, \ldots, y_m) \in D'} \sum_{i=1}^{m} \phi(y_i) w_i'. \tag{2.18}$$

Now appealing to Lemma 2.4, we have

$$\alpha_{j+1} = \min[\beta_1'/W_1', \max(\gamma_1'/W_1', \beta_2'/W_2'), \ldots, \max(\gamma_1'/W_1', \gamma_2'/W_2', \ldots,$$
$$\gamma_{m-1}'/W_{m-1}', \beta_m'/W_m')].$$

We obtain (2.12) from above by using (2.16). Finally the equation (2.13) follows from the fact that $\sum_{i=1}^{k} \alpha_i w_i = \beta_k$. □

The following is a generalized version of Proposition 2.1 for the case where ϕ need not be 'strictly' convex.

Proposition 2.2. *Let ϕ be a convex function defined on a closed interval I of the real line. Then subject to $(x_1, x_2, \ldots, x_k)$ in D, the sum* (1.2) *is minimized at $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, where α_i's are given by* (2.11) – (2.13).

Proof. Since D is a compact subset of R^k and is contained in I^k, therefore we can find a finite interval $J \subseteq I$, such that D is contained in J^k. Let ψ be a bounded real valued strictly convex function defined on J. Suppose that $|\psi(x)| \leq M, \ \forall\, x$ in J. For any $\varepsilon > 0$, the function $\phi + \varepsilon\psi$ is strictly convex on J and hence by the Theorem 2.1, we have

$$\min_{(x_1,x_2,\ldots,x_k)\in D} \sum_{i=1}^{k} (\phi+\varepsilon\psi)(x_i)w_i = \sum_{i=1}^{k} (\phi+\varepsilon\psi)(\alpha_i)w_i. \tag{2.19}$$

(2.19) implies that for any $(x_1, x_2, \ldots, x_k) \in D$,

$$\sum_{i=1}^{k} \phi(x_i)w_i \geq \sum_{i=1}^{k} \phi(\alpha_i)w_i + \varepsilon \sum_{i=1}^{k} (\psi(\alpha_i) - \psi(x_i))w_i.$$
$$\Longrightarrow \sum_{i=1}^{k} \phi(x_i)w_i \geq \sum_{i=1}^{k} \phi(\alpha_i)w_i - 2M\varepsilon \sum_{i=1}^{k} w_i.$$

This being true for any $\varepsilon > 0$, we have for any $(x_1, x_2, \ldots, x_k) \in D$,

$$\sum_{i=1}^{k} \phi(x_i)w_i \geq \sum_{i=1}^{k} \phi(\alpha_i)w_i.$$ □

The following theorem generalizes Propositions 2.1 and 2.2 to the case where ϕ is defined on an arbitrary interval I of the real line.

Theorem 2.1. *Let ϕ be a convex finite real valued function defined on an interval I of the real line. Then subject to $(x_1, x_2, \ldots, x_k) \in D$, where D is defined by* (2.1) *and assumed to be nonempty, the sum* (1.2) *is minimized at a point $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ where α_i's are given by* (2.11). *The point $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is unique if ϕ is strictly convex.*

Proof. Choose $\{I_n\}$, an increasing sequence of closed intervals such that $I = \bigcup_{n=1}^{\infty} I_n$, and

$$D_n^* = \Bigg\{(x_1, x_2, \ldots, x_k) : x_i \in I_n, 1 \leq i \leq k;$$
$$\gamma_j \leq \sum_{i=1}^{j} w_i x_i \leq \beta_j, 1 \leq j \leq k-1; \sum_{i=1}^{k} w_i x_i = \beta_k\Bigg\},$$

are all nonempty and increasing for $n = 1, 2, \ldots,$ with $D = \bigcup_{n=1}^{\infty} D_n^*$.

By Proposition 2.2, the sum (1.2), subject to $(x_1, x_2, \ldots, x_k) \in D_n^*$, is minimized at a point $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ given by (2.11). Since this point is independent of n, it follows that subject to $(x_1, x_2, \ldots, x_k) \in D$, the sum (1.2) is also minimized at the point $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ given by (2.11). Finally, if ϕ is strictly convex, the uniqueness of the minimizing point $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ follows from the convexity of the set D. □

Remark. We show in the following how Theorem 1.1 can be easily obtained from Theorem 2.1.

Since D_1 given by (1.1) is nonempty, we can find a natural number N_0 such that for $N \geq N_0, D = D(N)$ given by (2.1), with each $\gamma_j = -N$, is non-empty. If we take N_0 sufficiently large, the α_i's of Theorem 2.1 coincide with τ_i's of Theorem 1.1 and hence the sum (1.2) subject to $(x_1, x_2, \ldots, x_k) \in D(N)$, for $N \geq N_0$, is minimized at the point $(\tau_1, \tau_2, \ldots, \tau_k)$. Since $D_1 = \bigcup_{N \geq N_0} D(N)$, it follows that the sum (1.2), subject to $(x_1, x_2, \ldots, x_k) \in D_1$, is again minimized at the point $(\tau_1, \tau_2, \ldots, \tau_k)$.

We obtain the following theorem as a corollary to Theorem 2.1.

Theorem 2.2. *Let ϕ be a convex, finite real valued function defined on an interval I of the real line. Let $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}, M$, be real numbers with $\gamma_i \leq M, 1 \leq i \leq k-1; w_i > 0, i = 1, 2, \ldots, k-1; W_j = \sum_{i=1}^{j} w_i, j = 1, 2, \ldots, k$ and*

$$D_2 = \left\{(x_1, x_2, \ldots, x_k) : x_i \varepsilon I, 1 \leq i \leq k; \gamma_j \leq \sum_{i=1}^{j} w_i x_i \leq M, \text{ for } 1 \leq j \leq k-1; \sum_{i=1}^{k} w_i x_i = M\right\}.$$

Let D_2 be nonempty. Then subject to $(x_1, x_2, \ldots, x_k) \varepsilon D_2$, the sum (1.2) is minimized at a point $(\theta_1, \theta_2, \ldots, \theta_k) \varepsilon D_2$, where θ_j's are given by

$$\theta_1 = \max[\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_{k-1}/W_{k-1}, M/W_k], \tag{2.20}$$

$$\theta_{j+1} = \max\left[\left(\gamma_{j+1} - \sum_{i=1}^{j} \theta_i w_i\right) \Big/ w_{j+1}, \left(\gamma_{j+2} - \sum_{i=1}^{j} \theta_i w_i\right) \Big/ \left(\sum_{i=j+1}^{j+2} w_i\right), \ldots\right.$$

$$\left(\gamma_{k-1} - \sum_{i=1}^{j} \theta_i w_i\right) \Big/ \left(\sum_{i=j+1}^{k-1} w_i\right), \left(M - \sum_{i=1}^{j} \theta_i w_i\right) \Big/ \sum_{i=j+1}^{k} w_i \Bigg],$$

$$j = 1, 2, \ldots, k-2, \tag{2.21}$$

$$\theta_k = \left(M - \sum_{i=1}^{k-1} w_i \theta_i\right) \Big/ w_k. \tag{2.22}$$

The above minimizing solution is unique if ϕ is strictly convex.

Proof. On taking $\beta_i = M, i = 1, 2, \ldots, k$ in Theorem 2.1, we obtain a minimizing solution $(\theta_1, \theta_2, \ldots, \theta_k)$ of (1.2) subject to $(x_1, x_2, \ldots, x_k) \in D_2$. Thus θ_1 is given by

$$\begin{aligned} \theta_1 = & \min[M/W_1, \max(\gamma_1/W_1, M/W_2), \max(\gamma_1/W_1, \gamma_2/W_2, M/W_3), \\ & \ldots, \max(\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_{k-1}/W_{k-1}, M/W_k)]. \end{aligned} \tag{2.23}$$

It is easy to see that the terms inside the braces in (2.23) are monotonically decreasing. Hence

$$\theta_1 = \max[\gamma_1/W_1, \gamma_2/W_2, \ldots, \gamma_{k-1}/W_{k-1}, M/W_k],$$

which is same as (2.20). By using similar arguments we obtain (2.21). Expression (2.22) follows from the equation $\sum_{i=1}^{k} \theta_i w_i = M$. Finally the uniqueness of minimizing solution, when ϕ is strictly convex, also follows from Theorem 2.1. □

In the following theorem we consider another optimization problem, the solution of which has not been explicitly obtained. However Theorems 1.1, 2.1 and 2.2 have been used to obtain the relations between components of the solution.

Theorem 2.3. *Let ϕ be a strictly convex, finite real valued function defined on an interval I of the non-negative half of the real line. Let $w_{ij} > 0, i = 1, 2, \ldots, N, j = 1, 2, \ldots, k$ be real numbers and the set D_3 defined by*

$$D_3 = \Big\{(x_{11}, x_{12}, \ldots, x_{1k}; x_{21}, x_{22}, \ldots, x_{2k}; \ldots; x_{N1}, x_{N2}, \ldots, x_{Nk}) :$$

$$x_{ij} \in I; \sum_{j=1}^{r} w_{ij}x_{ij} \geq \sum_{j=1}^{r} w_{i'j}x_{i'j}, \text{ for } 1 \leq i \leq i' \leq N, r = 1,2,\ldots,k-1;$$

$$\left. \text{and } \sum_{j=1}^{k} w_{ij}x_{ij} = \beta, \text{ for } 1 \leq i \leq N \right\}, \tag{2.24}$$

be nonempty, where $\beta > 0$ *is a given constant. Then the* N *component subvectors of the unique point* $(x_{11}^*, \ldots, x_{1k}^*; \ldots; x_{N1}^*, \ldots, x_{Nk}^*)$ *at which the function*

$$\sum_{i=1}^{N} \sum_{j=1}^{k} w_{ij}\phi(x_{ij}) \tag{2.25}$$

is minimized subject to $(x_{11}, \ldots, x_{1k}; \ldots; x_{N1}, \ldots, x_{Nk}) \in D_3$, *are related in the following way.*

$$x_{11}^* = \max\left[\max_{1\leq i\leq k}\left(\left(\sum_{r=1}^{i} w_{2r}x_{2r}^*\right) / \sum_{r=1}^{i} w_{1r}\right), \beta/\left(\sum_{r=1}^{k} w_{1r}\right)\right], \tag{2.26}$$

$$x_{1j}^* = \max\left[\max_{j\leq i\leq k}\left(\left(\sum_{r=1}^{i} w_{2r}x_{2r}^* - \sum_{r=1}^{j-1} w_{1r}x_{1r}^*\right) / \sum_{r=j}^{i} w_{1r}\right), \left(\beta - \sum_{r=1}^{j-1} w_{1r}x_{1r}^*\right) / \left(\sum_{r=1}^{k} w_{1r}\right)\right], j = 2,3,\ldots,k-1, \tag{2.27}$$

$$x_{1k}^* = \left(\beta - \sum_{r=1}^{k-1} w_{1r}x_{1r}^*\right) / w_{1k}. \tag{2.28}$$

For $2 \leq p \leq N-1$,

$$x_{p1}^* = \min\Bigg[(w_{p-1,1}x_{p-1,1}^*)/w_{p1}; \max\left((w_{p+1,1}x_{p+1,1}^*)/w_{p1}, \left(\sum_{r=1}^{2} w_{p-1,r}x_{p-1,r}^*\right) / \left(\sum_{r=1}^{2} w_{pr}\right)\right); \ldots; \max\Bigg((w_{p+1,1}x_{p+1,1}^*)/w_{p1}, \left(\sum_{r=1}^{2} w_{p+1,r}x_{p+1,r}^*\right) / \left(\sum_{r=1}^{2} w_{pr}\right), \ldots, \left(\sum_{r=1}^{k-1} w_{p+1,r}x_{p+1,r}^*\right) / \left(\sum_{r=1}^{k-1} w_{pr}\right), \beta/\left(\sum_{r=1}^{k} w_{pr}\right)\Bigg)\Bigg], \tag{2.29}$$

$$x_{pj}^* = \min\Bigg[\left(\sum_{r=1}^{j} w_{p-1,r}x_{p-1,r}^* - \sum_{r=1}^{j-1} w_{pr}x_{pr}^*\right)/w_{pj}; \max\Bigg(\left(\sum_{r=1}^{j} w_{p+1,r}x_{p+1,r}^* - \sum_{r=1}^{j-1} w_{pr}x_{pr}^*\right)/w_{pj}, \Bigg(\sum_{r=1}^{j+1} w_{p-1,r}x_{p-1,r}^*$$

$$
-\sum_{r=1}^{j-1} w_{pr}x^*_{pr}\Big)/\Big(\sum_{r=j}^{j+1} w_{pr}\Big)\Big);\ldots;\max\Big(\Big(\sum_{r=1}^{j} w_{p+1,r}x^*_{p+1,r}-
$$

$$
\sum_{r=1}^{j-1} w_{pr}x^*_{pr}\Big)/w_{pj},\ldots,\Big(\sum_{r=1}^{k-1} w_{p+1,r}x^*_{p+1,r}-\sum_{r=1}^{j-1} w_{pr}x^*_{pr}\Big)/\Big(\sum_{r=j}^{k-1} w_{pr}\Big),
$$

$$
\Big(\beta-\sum_{r=1}^{j-1} w_{pr}x^*_{pr}\Big)/\Big(\sum_{r=j}^{k} w_{pr}\Big)\Big)\Big],j=2,3,\ldots,k-1, \tag{2.30}
$$

$$
x^*_{pk}=\Big(\beta-\sum_{r=1}^{k-1} w_{pr}x^*_{pr}\Big)/w_{pk}, \tag{2.31}
$$

$$
x^*_{N1}=\min\Big[\min_{1\le i\le k-1}\Big(\Big(\sum_{r=1}^{i} w_{N-1,r}x^*_{N-1,r}\Big)/\Big(\sum_{r=1}^{i} w_{Nr}\Big)\Big),\beta/\Big(\sum_{r=1}^{k} w_{Nr}\Big)\Big], \tag{2.32}
$$

$$
x^*_{Nj}=\min\Big[\min_{j\le i\le k}\Big(\Big(\sum_{r=1}^{i} w_{N-1,r}x^*_{N-1,r}-\sum_{r=1}^{j-1} w_{Nr}x^*_{Nr}\Big)/\Big(\sum_{r=j}^{i} w_{Nr}\Big)\Big),
$$

$$
\Big(\beta-\sum_{r=1}^{k-1} w_{Nr}x^*_{Nr}\Big)/\Big(\sum_{r=j}^{k} w_{Nr}\Big)\Big],j=2,3,\ldots,k-1, \tag{2.33}
$$

$$
x^*_{Nk}=\Big(\beta-\sum_{r=1}^{k-1} w_{Nr}x^*_{Nr}\Big)/w_{Nk}. \tag{2.34}
$$

Proof. We first assume that I is a finite closed interval. Here (2.25), being a continuous real valued function defined on the compact set D_3, attains its minimum at a point of D_3. Moreover such a point is unique, as D_3 is a convex set and ϕ is strictly convex. We denote this point by $(x^*_{11},\ldots,x^*_{1k};x^*_{21},\ldots,x^*_{2k};\ldots;x^*_{N1},\ldots,x^*_{Nk})$.

Let $1\le p\le N$. Set

$$
D_3^{(p)}=\{(x_{p1},x_{p2},\ldots,x_{pk}):(x^*_{11},x^*_{12},\ldots,x^*_{1k};\ldots;x^*_{p-1,1},\ldots,x^*_{p-1,k};
$$

$$
x_{p1},\ldots,x_{pk};x^*_{p+1,1},\ldots,x^*_{p+1,k};\ldots,x^*_{N1},\ldots,x^*_{Nk})\in D_3\}
$$

Then $D_3^{(p)}$ is nonempty since $(x^*_{p1},\ldots,x^*_{pk})\in D_3^{(p)}$. Note that

$$
D_3^{(1)}=\Big\{(x_{11},\ldots,x_{1k}):x_{1j}\in I,1\le j\le k;\ \sum_{j=1}^{r} w_{1j}x_{1j}\ge\sum_{j=1}^{r} w_{2j}x^*_{2j},1\le r\le k-1;
$$

$$
\text{and }\sum_{j=1}^{k} w_{1j}x_{1j}=\beta\Big\},
$$

and for $2\le p\le N-1$,

$$
D_3^{(p)}=\Big\{(x_{p1},\ldots,x_{pk}):x_{pj}\in I,1\le j\le k;\ \sum_{j=1}^{r} w_{p-1,j}x^*_{p-1,j}\ge\sum_{j=1}^{r} w_{pj}x_{pj}\ge
$$

$$
\sum_{j=1}^{r} w_{p+1,j}x^*_{p+1,j},1\le r\le k-1;\ \text{and }\sum_{j=1}^{k} w_{pj}x_{pj}=\beta\Big\},
$$

and

$$D_3^{(N)} = \Big\{(x_{N1}, \ldots, x_{Nk}) : x_{Nj} \in I, 1 \le j \le k;\ \sum_{j=1}^{r} w_{N-1,j} x^*_{N-1,j} \ge \sum_{j=1}^{r} w_{Nj} x_{Nj},$$
$$1 \le r \le k-1;\ \text{and}\ \sum_{j=1}^{k} w_{Nj} x_{Nj} = \beta\Big\}.$$

We are given that (2.25), subject to $(x_{11}, \ldots, x_{1k}; \ldots; x_{N1}, \ldots, x_{Nk}) \in D_3$, is minimized at $(x^*_{11}, \ldots, x^*_{1k}; \ldots; x^*_{N1}, \ldots, x^*_{Nk})$. Consequently

$$\sum_{i=1}^{N} \sum_{j=1}^{k} w_{ij} \phi(x^*_{ij}) \le \sum_{i=1}^{N} \sum_{j=1}^{k} w_{ij} \phi(x_{ij}),$$

whenever $(x_{11}, \ldots, x_{1k}; \ldots; x_{N1}, \ldots, x_{Nk}) \in D_3$. In particular

$$\sum_i \sum_j w_{ij} \phi(x^*_{ij}) \le \sum_{i \ne p} \sum_j w_{ij} \phi(x^*_{ij}) + \sum_j w_{pj} \phi(x_{pj})$$

for all $(x_{p1}, \ldots, x_{pk}) \in D_3^{(p)}$. Thus

$$\sum_j w_{pj} \phi(x^*_{pj}) \le \sum_j w_{pj} \phi(x_{pj}),$$

for all $(x_{p1}, \ldots, x_{pk}) \in D_3^{(p)}$ implying that $\sum_{j=1}^{k} w_{pj} \phi(x_{pj})$, subject to $(x_{p1}, \ldots, x_{pk}) \in D_3^{(p)}$, is minimized at $(x^*_{p1}, \ldots, x^*_{pk})$. The result now follows on applying Theorems 2.2, 2.1 and 1.1. Finally for any interval I, not necessarily closed and finite, the result follows by using similar arguments as in Theorem 2.1. □

3. Maximum Likelihood Estimation of a c.d.f. F Subject to $F_1(x) \le F(x) \le F_2(x)$.

We consider the problem of maximum likelihood estimation of a c.d.f. F when it is apriori known that $F_1(x) \le F(x) \le F_2(x)$, $\forall\ x$, where F_1 and F_2 are known c.d.f.'s. In a random sample of size n from a population with c.d.f. F, let the ith ordered distinct value v_i occur n_i times, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} n_i = n$. We assume that $F_2(v_1) > 0$. The problem is to obtain a M.L.E. of F subject to $F_1(x) \leq F(x) \leq F_2(x), \quad \forall\, x$, based on our observation vector $(v_i, n_i, i = 1, 2, \ldots, k)$. This is basically a nonparametric problem where it is not clear how to define what could be realistically called a likelihood function. This is because of the absence of a common σ-finite measure dominating every measure induced by F, as it varies subject only to $F_1(x) \leq F(x) \leq F_2(x)$. Instead we follow the method of maximum likelihood as suggested by Scholz (1980) which bypasses the intermediate step of first defining the so called likelihood function. We refer the reader to Scholz (1980) for the necessary definitions and details (see also his examples 3.1 and 3.4 for the nonparametric situations similar to ours). Alternatively we could follow the intuitive argument as discussed in Puri and Singh (1988) and as also applied to our present case. In either case, as it turns out, our problem is reduced essentially to considering only those F's which are of discrete type and have positive jumps only at the observation points $v_1 < v_2 < \ldots < v_k$. As a result, we are required to find those values of

$$p_i = dF(v_i) = F(v_i) - F(v_i-), i = 1, 2, \ldots, k, \tag{3.1}$$

which maximize

$$\prod_{i=1}^{k} (p_i)^{n_i} \tag{3.2}$$

subject to the conditions

$$F_1(v_j) \leq \sum_{i=1}^{j} n_i p_i \leq F_2(v_j), j = 1, 2, \ldots, k. \tag{3.3}$$

This reduces further to the problem of minimizing

$$\sum_{i=1}^{k} n_i(-\log p_i) \tag{3.4}$$

subject to the conditions

$$\begin{aligned} F_1(v_j) \le \sum_{i=1}^{j} p_i n_i \le F_2(v_j), j = 1, 2, \ldots, k-1, \\ \sum_{i=1}^{k} p_i n_i = F_2(v_k). \end{aligned} \tag{3.5}$$

Applying Theorem 2.1, we see that the minimizing solution $(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_k)$ of (3.4) subject to (3.5) is given by

$$\begin{aligned} \hat{p}_1 = \min\Bigg[& F_2(v_1)/n_1, \max(F_1(v_1)/n_1, F_2(v_2)/(n_1+n_2)), \ldots, \\ & \max\Bigg(F_1(v_1)/n_1, F_1(v_2)/(n_1+n_2), \ldots, F_1(v_{k-1})/\sum_{i=1}^{k-1} n_i, \\ & F_2(v_k)/\sum_{i=1}^{k} n_i\Bigg)\Bigg], \end{aligned} \tag{3.6}$$

$$\begin{aligned} \hat{p}_{j+1} = \min\Bigg[& \left(F_2(v_{j+1}) - \sum_{i=1}^{j} n_i \hat{p}_i\right)/n_{j+1}, \max\left(\left(F_1(v_{j+1}) - \sum_{i=1}^{j} n_i \hat{p}_i\right)/n_{j+1},\right. \\ & \left.\left(F_2(v_{j+2}) - \sum_{i=1}^{j} n_i \hat{p}_i\right)/\sum_{i=j+1}^{j+2} n_i\right), \ldots, \max\Bigg(\left(F_1(v_{j+1}) - \sum_{i=1}^{j} n_i \hat{p}_i\right)/n_{j+1}, \\ & \ldots, \left(F_1(v_{k-1}) - \sum_{i=1}^{j} n_i \hat{p}_i\right)/\sum_{i=j+1}^{k-1} n_i, \left(F_2(v_k) - \sum_{i=1}^{j} n_i \hat{p}_i\right)/\sum_{i=j+1}^{k} n_i\Bigg)\Bigg], \\ & j = 1, 2, \ldots, k-2, \end{aligned} \tag{3.7}$$

$$\hat{p}_k = \left(F_2(v_k) - \sum_{i=1}^{k-1} n_i \hat{p}_i\right)/n_k. \tag{3.8}$$

Thus a maximum likelihood estimate of F, subject to $F_1(x) \le F(x) \le F_2(x)$, $\forall\ x$, is given by

$$\hat{F}_n(x) = \begin{cases} F_1(x) & x < v_1 \\ \sum_{j=1}^{m} n_j \hat{p}_j & v_m \le x < v_{m+1}, m = 1, 2, \ldots, k-1, \\ F_2(x) & x \ge v_k, \end{cases} \tag{3.9}$$

where $\hat{p}_j$'s are given by (3.6) and (3.7).

That $\hat{F}_n(x)$ defined by (3.9) is a consistent estimator of $F(x)$ satisfying $F_1(x) \le F(x) \le F_2(x)$, $\forall\ x$, is shown in the following theorem.

Theorem 3.1. *Let $F_1(x) \leq F(x) \leq F_2(x), \ \forall\, x$. Then F_n defined by (3.9) is a consistent estimator of $F(x)$.*

Proof. Let F_n be the usual empirical distribution function based on the random sample of size n from F. Let

$$C_n = \sup_x |F_n(x) - F(x)|. \tag{3.10}$$

We first establish that

$$|F_n(x) - \hat{F}_n(x)| \leq C_n, \quad \forall\, x. \tag{3.11}$$

(3.11) is trivially seen to be true for $x < v_1$ and $x \geq v_k$. Also for $v_1 \leq x < v_2$, we have

$$F_n(x) - \hat{F}_n(x) = \frac{n_1}{n} - n_1\hat{p}_1 = \max\Bigg[(n_1/n) - F_2(v_1),$$
$$\min\left((n_1/n) - F_1(v_1), (n_1/(n_1+n_2))((n_1+n_2)/n - F_2(v_2))\right), \ldots,$$
$$\min\Bigg((n_1/n) - F_1(v_1), (n_1/(n_1+n_2))((n_1+n_2)/n - F_1(v_2)), \ldots,$$
$$\left(n_1/\sum_{i=1}^{k-1} n_i\right)\left(\left(\sum_{i=1}^{k-1} n_i/n\right) - F_1(v_{k-1})\right), (n_1/n)(1 - F_2(v_k))\Bigg)\Bigg].$$

Using the assumption that $F(x) \leq F_2(x), \ \forall\, x$, we have for $v_1 \leq x \leq v_2$,

$$F_n(x) - \hat{F}_n(x) \leq \max\Bigg[(n_1/n) - F(v_1), (n_1/(n_1+n_2))((n_1+n_2)/n - F(v_2)),$$
$$\ldots, \left(n_1/\sum_{i=1}^{k-1} n_i\right)\left(\left(\sum_{i=1}^{k-1} n_i/n\right) - F(v_{k-1})\right), (n_1/n)(1 - F(v_k))\Bigg],$$

which implies that $F_n(x) - \hat{F}_n(x) \leq C_n$, for $v_1 \leq x < v_2$. Likewise, using an induction argument, it is easily seen that

$$F_n(x) - \hat{F}_n(x) \leq C_n, \text{ for } v_i \leq x < v_{i+1}, i = 1, 2, \ldots, k-1. \tag{3.12}$$

Now for $v_1 \leq x < v_2$,

$$\hat{F}_n(x) - F_n(x) = \min\Bigg[(F_2(v_1) - (n_1/n)), \max\Bigg(F_1(v_1) - (n_1/n), (n_1/(n_1+n_2))(F_2(v_2) - (n_1+n_2)/n)\Bigg), \ldots, \max\Bigg(F_1(v_1) - (n_1/n), (n_1/(n_1+n_2))(F_1(v_2) - (n_1+n_2)/n), \ldots, \left(n_1/\sum_{i=1}^{k-1} n_i\right)\left(F_1(v_{k-1}) - \left(\sum_{i=1}^{k-1} n_i/n\right)\right), (n_1/n)(F_2(v_k) - 1)\Bigg)\Bigg].$$

Using the assumption that $F_1(x) \leq F(x)$ and noting that $F_2(v_k) \leq 1$, we have for $v_1 \leq x < v_2$,

$$\hat{F}_n(x) - F_n(x) \leq \max\Bigg[F(v_1) - (n_1/n), (n_1/(n_1+n_2))(F(v_2) - (n_1+n_2)/n), \ldots, \left(n_1/\sum_{i=1}^{k-1} n_i\right)\left(F(v_{k-1}) - \left(\sum_{i=1}^{k-1} n_i/n\right)\right), 0\Bigg],$$

which implies that $\hat{F}_n(x) - F_n(x) \leq C_n$, for $v_1 \leq x < v_2$. As before, using an induction argument, we easily see that

$$\hat{F}_n(x) - F_n(x) \leq C_n, \text{ for } v_i \leq x < v_{i+1}, i = 1, 2, \ldots, k-1. \tag{3.13}$$

Using (3.12) and (3.13), we obtain (3.11). Uniform consistency of F_n now follows from (3.11) by using Glivenko–Cantelli Theorem. □

4. Concluding Remarks

(a) In Theorem 1.1, the set D_1 involves one-sided restrictions on the sums $\sum_{i=1}^{j} w_i x_i, j = 1, 2, \ldots, k-1$. Theorem 2.1 (and also Theorem 2.2) generalize the result of Theorem 1.1 to cover those cases where these sums may instead be subjected to two-sided restrictions (see also the remark following Theorem 2.1). Thus our Theorems 2.1 and 2.2 are applicable to the two-sided analogs of all those one-sided situations discussed in literature (see for instance Barlow, et. al. (1972)). The example of Section 3 is only one such application. Needless to add that these theorems may also be applicable to optimization problems arising in several other areas of Operation Research.

(b) In Theorem 2.3 one may attempt to solve the system of equations (2.26)–(2.34) by using a suitable convergent iterative procedure, such that its solution converges to the minimizing point $(x^*_{11}, \ldots, x^*_{1k}; \ldots; x^*_{N1}, \ldots, x^*_{Nk})$. A satisfactory solution to this problem will help in providing another algorithm for obtaining the M.L.E.'s of $N \geq 3$ ordered distribution functions. It may be mentioned here that the M.L.E.'s of two ordered distributions have been obtained by Brunk et. al. (1966). An algorithm for finding M.L.E.'s of survival functions of two stochastically ordered random variables in the presence of censoring was given by Dykstra (1982). Later Feltz and Dykstra (1985) gave an iterative algorithm, depending on the solutions of pairwise problems, for finding the M.L.E.'s of survival functions of $N \geq 3$ stochastically ordered random variables.

(c) The study of limit distribution of $\hat{F}_n$ given by (3.9) will form a topic of further study and will be reported elsewhere.

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