BEHAVIOR OF THE POSTERIOR DISTRIBUTION AND INFERENCES FOR A NORMAL MEAN WITH t PRIOR DISTRIBUTIONS[1]

by

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ABSTRACT

Bayesian inference is considered when both the likelihood and the prior distributions are t-densities. (This scenario naturally arises in robust Bayesian inference for a normal mean.) The behavior of the posterior density, the posterior mean, and the posterior variance is studied, especially when the parameters of the t prior are chosen to be extreme. This provides considerable insight into the robustness of the analysis. Of particular interest is the study of bimodality of the posterior. A sufficient condition for the posterior density being unimodal is also given.

KEY WORDS: Likelihood, prior distribution; posterior density; posterior mean; posterior variance; marginal.

1 Introduction

Let $X_1, X_2, \ldots, X_n$ be independent observations from a $N(\theta, \sigma^2)$ distribution where θ and σ^2 are unknown. The marginal likelihood function for θ, after integrating out σ^2 with respect to the noninformative prior $\sigma^{-2}d\sigma^2$ (proper inverse gamma prior distributions for σ^2 could also be used here) is of form

$$f(\bar{x}-\theta) = \frac{\sqrt{n}K_{n-1}}{s}\left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2}, \tag{1}$$

where $\bar{x}, s^2$ are the usual sample mean and variance, and $K_i = \Gamma[(i+1)/2][\sqrt{i\pi}\Gamma(i/2)]^{-1}$. Since (1) is a t-density, it is natural and desirable for robustness reasons to perform Bayesian analysis with a t-prior, namely

$$g(\theta-\mu) = \frac{K_{m-1}}{\tau}\left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2}. \tag{2}$$

The parameter μ in (2) represents the best prior guess for θ and the scale parameter τ measures the accuracy of μ. The posterior density is given by

$$\pi(\theta|\mathbf{x}) = \frac{f(\bar{x}-\theta)g(\theta-\mu)}{m(\mathbf{x})}, \tag{3}$$

where

$$m(\mathbf{x}) = \int \frac{\sqrt{n}K_{n-1}K_{m-1}}{s(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}\tau(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}}d\theta. \tag{4}$$

The posterior mean becomes, after a change of variables,

$$\delta^\pi(\mathbf{x}) = \bar{x} + \frac{\sqrt{n}K_{n-1}K_{m-1}}{m(\mathbf{x})}\int \frac{\eta}{s(1+\frac{n\eta^2}{(n-1)s^2})^{n/2}\tau(1+\frac{(\eta+\bar{x}-\mu)^2}{(m-1)\tau^2})^{m/2}}d\eta, \tag{5}$$

and the posterior variance is

$$V^\pi(\mathbf{x}) = \frac{\sqrt{n}K_{n-1}K_{m-1}}{m(\mathbf{x})}\int \frac{\theta^2}{s(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}\tau(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}}d\theta - (\delta^\pi(\mathbf{x}))^2, \tag{6}$$

which can also be written as

$$\begin{aligned} V^\pi(\mathbf{x}) &= \frac{(m-1)\tau^2}{m(\mathbf{x})}\int \frac{\sqrt{n}K_{n-1}K_{m-1}}{s(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}\tau(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2-1}}d\theta \\ &\quad -(m-1)\tau^2 - (\mu-\delta^\pi(\mathbf{x}))^2. \end{aligned} \tag{7}$$

Exact calculation of $m(\mathbf{x}), \delta^{\pi}(\mathbf{x})$ and $V^{\pi}(\mathbf{x})$ is discussed in Fan and Berger (1989).

The behavior of $\pi(\theta|\mathbf{x}), \delta^{\pi}(\mathbf{x})$, and $V^{\pi}(\mathbf{x})$ is of interest when the prior parameters are chosen to be extreme, since it provides considerable insight into the robustness of the analysis. The situation where μ and $\bar{x}$ are far apart, in which case that the posterior becomes bimodal, is of particular interest, since it is the case where the information from the data and the prior clash.

Such situations have received considerable study in the literature. Dawid (1973) studied the limiting behavior of a posterior distribution as in (2) with quite general f and g, as the observation tends to infinity. He showed that, for certain forms of f and g, depending on the thickness of their tails, either f or g will in the limit completely dominate the other. In particular, he showed that if one of these components is a normal density and the other a t-density then the normal component will dominate. O'Hagan (1979) proved that the normal density cannot be dominated by any other normal density. O'Hagan (1981) also proved that, under certain circumstances, the posterior variance has a sharp peak before settling down to its limiting value. Meeden and Isaacson (1979) showed that if f belongs to the one parameter exponential family and g has unbounded support, then, under certain flatness conditions on the tail behavior of the prior, the posterior distribution of θ is approximately normally distributed about its mode for large values of the observation. They also showed that the rate at which the posterior mean approaches infinity depends on the tail of the prior. O'Hagan (1987) studied the tail behavior by defining the 'credence' of a density. (For instance, a t density has credence equal to its degrees of freedom plus one.) His results showed that when the information sources between the prior and the datum conflict, whichever information source has greater credence will dominate. He also extended the results to the general case of many sources of information about a single parameter. Other related work in the area of outlier rejection can be found in DeFinetti (1961), Hill (1974) and O'Hagan (1988).

In Section 3, a detailed study of $\pi(\theta|\mathbf{x})$ in (3) is undertaken, with regard to such behavior at extremes. Because we restrict the study to the important situation of (1) and (2), results that are considerably more detailed than the results in the above papers can be obtained. First, we are able to investigate the behavior of $\pi(\theta|\mathbf{x})$ as $|\mu - \bar{x}| \to \infty$ for all n and m;

results such as Dawid (1973), being given for general f and g, require moment conditions that here are satisfied only if $|n-m| > 2$. Included in the investigation are surprising results for the posterior mean and variance, such as determination of situations in which either the likelihood or the prior dominates the posterior, yet $\delta^\pi(\mathbf{x})$ and/or $V^\pi(\mathbf{x})$ do not arise solely from the dominating likelihood or prior.

In analogy to O'Hagan (1981), precise bounds are obtained in section 3.2 on the size of the posterior variance for large $|\mu - \bar{x}|$. Of perhaps most interest is that, for large $|\mu - \bar{x}|$, there exist choices of τ in (2) such that $V^\pi(\mathbf{x}) \cong (\mu - \bar{x})^2/4$. Thus conflict between the likelihood and prior can cause great uncertainty in the posterior, in contrast to the situation with conjugate priors. This result is also important for Bayesian robustness, where ranges of μ and τ in (2) are typically considered. The indication is that overly large ranges will not yield useful answers.

Section 2 considers the question of bimodality of $\pi(\theta|\mathbf{x})$. A useful sufficient condition for unimodality is given in Section 2.1, while Section 2.2 discusses bimodality as $|\mu - \bar{x}| \to \infty$. In this latter situation, a key lemma demonstrating the concentration of mass near the two modes is given.

2 Unimodality and Bimodality of the Posterior

2.1 A Sufficient Condition for Unimodality

The following theorem gives conditions on μ and τ such that the posterior density, $\pi(\theta|\mathbf{x})$, is unimodal.

Theorem 2.1 *a) For $n \geq m$, the posterior is unimodal if*

$$\frac{s^2}{\tau^2} \leq \frac{2n^2(m-1)}{5.1m(n+m)} \text{ and } \frac{|\mu - \bar{x}|}{\tau} < \frac{2}{m}\{(m-1)(n^2 + 0.6mn - 0.95m^2 - 0.4n - 0.2m)\}^{1/2}.$$

b) For $n < m$, the posterior is unimodal if

$$\frac{\tau^2}{s^2} \leq \frac{2m(n-1)}{5.1n^2(n+m)} \text{ and } \frac{|\mu - \bar{x}|}{s} < \frac{2}{n}\{\frac{(n-1)}{n}(m^2 + 0.6mn - 0.95n^2 - 0.4m - 0.2n)\}^{1/2}.$$

Proof: a) A change of variable yields

$$\pi(\theta|\mathbf{x}) \propto \left(1+\frac{\xi^2}{n-1}\right)^{-n/2} \left(1+v(\xi+t)^2\right)^{-m/2},$$

where

$$v = \frac{s^2}{n(m-1)\tau^2} \quad \text{and} \quad t = \frac{\sqrt{n}(\bar{x}-\mu)}{s}. \tag{8}$$

The derivative of the log posterior is proportional to

$$\rho(\xi) = \xi^3 + 2a\xi^2 + b\xi + 2c, \tag{9}$$

where $a = t(n+m/2)/(n+m), b = (mn-m+nv^{-1}+nt^2)/(n+m)$, and $c = \frac{m}{2}(n-1)t/(n+m)$. The posterior is unimodal if the equation $\rho(\xi) = 0$ has only one real root, which (for a cubic) is the case if

$$[8a^3 - 9ab + 27c]^2 + [3b - (2a)^2]^3 > 0. \tag{10}$$

Algebra reduces (10) to

$$\begin{aligned}
&-a^6\frac{p^2}{(n+p)^2}[1-\frac{p^2}{(n+p)^2}-k_1] + a^4[\frac{n}{(n+2p)v}(1-\frac{4p^2}{(n+p)^2}) - \frac{2p(2p+1)}{n+2p}+k_2] \\
&+a^2\{3(\frac{(n-1)(n+4p)p}{(n+p)(n+2p)} - \frac{n}{(n+2p)v})^2 - [1+\frac{3p^2}{(n+p)^2}][\frac{2(n-1)p}{n+2p} \\
&+\frac{n}{(n+2p)v}]^2\} + k_3 > 0,
\end{aligned} \tag{11}$$

where the k_i are all positive and $p = m/2$.

The condition on s^2/τ^2 ensures that the coefficient of a^2 is positive, so that (11) is satisfied if

$$-a^6\frac{p^2}{(n+p)^2}\frac{(n^2+2pn)}{(n+p)^2} + a^4\{\frac{n}{(n+2p)v}[1-\frac{4p^2}{(n+p)^2}] - \frac{2p(2p+1)}{n+2p}\} > 0.$$

Algebra yields that this is satisfied if

$$t^2 < \frac{4}{vm^2}[n^2+mn-\frac{3}{4}m^2 - m(m+1)(n+\frac{m}{2})^2\frac{v}{n}],$$

which, using the condition on $\frac{s^2}{\tau^2}$, will be true if

$$t^2 < \frac{n\tau^2}{s^2}\frac{4(m-1)}{m^2}[n^2+mn-\frac{3}{4}m^2 - \frac{2(m+1)(n+m/2)^2}{5.1(n+m)}].$$

This yields the conclusion.

b) By another change of variables,

$$\pi(\theta|\mathbf{x}) \propto \left(1+\frac{\xi^2}{m-1}\right)^{-m/2}\left(1+v_1(\xi+t_1)^2\right)^{-n/2},$$

where $v_1 = n\tau^2/[(n-1)s^2]$, and $t_1 = (\mu-\bar{x})/\tau$. Following the proof of a) but switching the roles of n and m, and replacing v by v_1, t by t_1 gives the result. □

Corollary 2.1 *For $m = 2$ (the Cauchy prior), $\tau > s$ and $|\mu-\bar{x}|/s \le 7$, the posterior density is unimodal if $n \ge 7$.*

Proof: For $\tau > s$, the posterior is unimodal if

$$\frac{n^2}{5.1(n+2)} \ge 1 \quad \text{and} \quad \frac{|\mu-\bar{x}|}{\tau} < \sqrt{n^2+0.8n-4.2}.$$

The first inequality holds if $n \ge 7$. Since

$$\frac{|\mu-\bar{x}|}{\tau} < \frac{|\mu-\bar{x}|}{s},$$

the condition on $|\mu-\bar{x}|/\tau$ is also satistied for $n \ge 7$. □

2.2 Bimodality as $|\mu-\bar{x}| \to \infty$

2.2.1 Bimodality of the posterior as $|\mu-\bar{x}| \to \infty$

Theorem 2.2 *For large enough $|\mu-\bar{x}|$, the posterior $\pi(\theta|\mathbf{x})$ is bimodal, with modes at $\bar{x}+(s/\sqrt{n})\xi_i$, for $i = 1, 2$, where ξ_1 and ξ_2 are the smallest and largest solutions to $\rho(\xi) = 0$,respectively, where $\rho(\xi)$ is defined by (9).*

Proof: Using the notation in the proof of Theorem 2.1, $\pi(\theta|\mathbf{x})$ is bimodal if

$$(8a^3-9ab+27c)^2+(3b-4a^2)^3 \le 0. \qquad (12)$$

Algebra reduces the left hand side of (12) to

$$\rho_1(t) = a_6t^6+a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0,$$

where t is defined in (8), $a_6 = -m^2/[4(n+m)^2]$ and the a_i's are suitable constants for $i = 0, 1, \ldots, 5$. Since $a_6 < 0$, it is clear that $\rho_1(t) \leq 0$ for large enough $|t|$. This proves the bimodality of $\pi(\theta|\mathbf{x})$ for large $|\mu - \bar{x}|$. When (9) has three different roots with ξ_1 and ξ_2 being the smallest and largest roots, respectively, the modes of $\pi(\theta|\mathbf{x})$ are then at $\bar{x} + (s/\sqrt{n})\xi_i$, for $i = 1, 2$, by the transformation from ξ to θ. □

2.2.2 Modal Concentration of Mass as $|\mu - \bar{x}| \to \infty$

With conjugate priors, the bulk of the mass of the posterior tends to concentrate between the likelihood and the prior. The situation with t-distributions can be quite different when $|\mu - \bar{x}|$ is large; indeed, virtually all of the mass of the posterior then concentrates near $\bar{x}$ (i.e., at the likelihood) or near μ (i.e. at the prior) or both. The following lemma demonstrates this, showing that the mass concentrates in two regions, R_1 and R_2, near $\bar{x}$ and μ, respectively. The lemma is stated in a general form usable for the study of moments in Section 3. The question of concentration of probability mass is answered by setting $i = 0$ in the lemma.

For use in the lemma and Section 3, define

$$V_f = \begin{cases} \frac{n-1}{n-3}\frac{s^2}{n} & \text{if } n > 3 \\ \infty & \text{if } n = 3, \end{cases} \quad \text{and} \quad V_g = \begin{cases} \frac{m-1}{m-3}\tau^2 & \text{if } m > 3 \\ \infty & \text{if } m = 3. \end{cases} \tag{13}$$

These would be the variances of f (if $n \geq 3$) and g (if $m \geq 3$) alone, considered as densities for θ. The variances of f and g do not exist when $n = 2$, $m = 2$, respectively.

Definition 2.1 *(i)* $\phi(x) = O^\star(x)$ *if* $\exists\, c \neq 0$, *and* $c < \infty$ *such that* $\lim_{x\to\infty} \frac{\phi(x)}{x} = c$.

(ii) $\phi(x) = O(x)$ *if* $\exists\, c < \infty$ *such that* $\lim_{x\to\infty} \frac{\phi(x)}{x} \leq c$.

Lemma 2.1 *Define*

$$r = \begin{cases} 5/6 & \textit{if } \max(n,m) \leq 3 \\ \dfrac{\min(n,m) + 2.5}{n+m-1} & \textit{if } \max(n,m) > 3. \end{cases}$$

Let $R_1 = \{\theta : |\theta - \bar{x}| \leq |\mu - \bar{x}|^r\}$, $R_2 = \{\theta : |\theta - \mu| \leq |\mu - \bar{x}|^r\}$, $R_3 = R - (R_1 \cup R_2)$, *and*

$$A_i = \int_{R_1} (\theta - \bar{x})^i f(\bar{x} - \theta) g(\theta - \mu) d\theta,$$

$$
\begin{aligned}
B_i &= \int_{R_2} (\theta-\mu)^i f(\bar{x}-\theta) g(\theta-\mu) d\theta, \\
C_i &= \int_{-\infty}^{\bar{x}-|\mu-\bar{x}|^r} (\theta-\bar{x})^i f(\bar{x}-\theta) g(\theta-\mu) d\theta, \\
D_i &= \int_{\mu+|\mu-\bar{x}|^r}^{\infty} (\theta-\mu)^i f(\bar{x}-\theta) g(\theta-\mu) d\theta, \\
E_i &= \int_{\bar{x}+|\mu-\bar{x}|^r}^{\mu-|\mu-\bar{x}|^r} (\theta-\bar{x})^i f(\bar{x}-\theta) g(\theta-\mu) d\theta,
\end{aligned}
$$

for $i = 0, 1, 2$. Then, as $|\mu - \bar{x}| \to \infty$, for $n \geq 2$ and $m \geq 2$,

i) $A_i = g(\mu - \bar{x})a_i$, $B_i = f(\mu - \bar{x})b_i$, *for $i = 0, 1, 2$, where*

$$
a_0 = \begin{cases} 1 + O^\star(|\mu-\bar{x}|^{-r(n-1)}) + O^\star(|\mu-\bar{x}|^{-2}) & \textit{if } n > 3 \\ 1 + O^\star(|\mu-\bar{x}|^{-r(n-1)}) & \textit{if } n \leq 3, \end{cases}
$$

$$
b_0 = \begin{cases} 1 + O^\star(|\mu-\bar{x}|^{-r(m-1)}) + O^\star(|\mu-\bar{x}|^{-2}) & \textit{if } m > 3 \\ 1 + O^\star(|\mu-\bar{x}|^{-r(m-1)}) & \textit{if } m \leq 3, \end{cases}
$$

$$
a_1 = \begin{cases} O^\star(|\mu-\bar{x}|^{-1}) & \textit{if } n > 3 \\ O^\star(|\mu-\bar{x}|^{-1}\ln|\mu-\bar{x}|) & \textit{if } n = 3 \\ O^\star(|\mu-\bar{x}|^{r-1}) & \textit{if } n = 2, \end{cases}
\quad
b_1 = \begin{cases} O^\star(|\mu-\bar{x}|^{-1}) & \textit{if } m > 3 \\ O^\star(|\mu-\bar{x}|^{-1}\ln|\mu-\bar{x}|) & \textit{if } m = 3 \\ O^\star(|\mu-\bar{x}|^{r-1}) & \textit{if } m = 2, \end{cases}
$$

$$
a_2 = \begin{cases} V_f + O^\star(|\mu-\bar{x}|^{-r(n-3)}) & \textit{if } n > 3 \\ O^\star(\ln|\mu-\bar{x}|) & \textit{if } n = 3 \\ O^\star(|\mu-\bar{x}|^{r}) & \textit{if } n = 2, \end{cases}
\quad
b_2 = \begin{cases} V_g + O^\star(|\mu-\bar{x}|^{-r(m-3)}) & \textit{if } m > 3 \\ O^\star(\ln|\mu-\bar{x}|) & \textit{if } m = 3 \\ O^\star(|\mu-\bar{x}|^{r}) & \textit{if } m = 2, \end{cases}
$$

ii) C_i *and* D_i *are* $= O(|\mu - \bar{x}|^{-r(n+m-1-i)})$, *for $i = 0, 1, 2$; and*

$$
E_i = \begin{cases} O(|\mu-\bar{x}|^{-r(n+m-1)}) & \textit{for } i = 0 \\ O(|\mu-\bar{x}|^{-[r(n+m-i)-1]}) & \textit{for } i = 1, 2. \end{cases}
$$

Proof: See Appendix A. □

3 Extreme Behavior of the Posterior

3.1 Behavior as $\tau \to \infty$, and $\tau \to 0$

For completeness, we first record the behavior of the posterior when the scale parameter, τ, of the prior is chosen large or small. When τ is chosen to be extremely large, the prior tends to a constant noninformative prior and almost all the available information about θ comes from the likelihood $f(\bar{x}-\theta)$. Intuitively, the posterior density will then tend to $f(\bar{x}-\theta)$, the posterior mean to $\bar{x}$ and the posterior variance to V_f (defined in (13)). (That $\bar{x}$ results, even for $n=2$, is somewhat of a surprise, since $f(\bar{x}-\theta)$ has no mean when $n=2$.)

Theorem 3.1 *a)* $\lim_{\tau\to\infty} \dfrac{m(\mathbf{x})}{g(\bar{x}-\mu)} = 1.$

b) $\lim_{\tau\to\infty} \pi(\theta|\mathbf{x}) = f(\bar{x}-\theta).$

c) $\lim_{\tau\to\infty} \delta^{\pi}(\mathbf{x}) = \bar{x}.$

d) $\lim_{\tau\to\infty} V^{\pi}(\mathbf{x}) = V_f$ *for* $n>3$,

$\lim_{\tau\to\infty} V^{\pi}(\mathbf{x}) = \infty$ *for* $n=2$ *or* 3.

Proof: See Appendix B. □

On the other hand, when τ tends to 0 the prior concentrates about μ, so the posterior density then tends to a point mass at μ, the posterior mean to μ and the posterior variance to 0.

Theorem 3.2 *a)* $\lim_{\tau\to 0} m(\mathbf{x}) = f(\bar{x}-\mu)$

b) $\lim_{\tau\to 0} \pi(\theta|\mathbf{x}) = \delta(\mu)$, *where* $\delta(\mu)$ *denotes a point mass at* μ.

c) $\lim_{\tau\to 0} \delta^{\pi}(\mathbf{x}) = \mu$

d) $\lim_{\tau\to 0} V^{\pi}(\mathbf{x}) = 0.$

Proof: See Appendix B. □

3.2 Behavior of the Posterior as $|\mu-\bar{x}|\to\infty$

The most interesting and unusual behavior of $\pi(\theta|\mathbf{x})$ arises when $|\mu-\bar{x}|$ is large, and the posterior is bimodal. In Section 3.2.1, the behavior of $m(\mathbf{x})$ and $\pi(\theta|\mathbf{x})$ themselves is discussed, with the effect of mass concentration near one or both modes being evidenced. In Section 3.2.2, the behavior of $\delta^{\pi}(\mathbf{x})$ and $V^{\pi}(\mathbf{x})$ is considered, and it is shown that, when $|n-m|\le 2$, the posterior probabilistic concentration at the modes does not result in an analogous posterior moment concentration. In Section 3.2.3, the analog of O'Hagan (1981) is developed: an asymptotically attainable upper bound on the magnitude of $V^{\pi}(\mathbf{x})$ is given.

For use in this section, define

$$\gamma_0 = \frac{K_{n-1}(n-1)^{n/2}(s/\sqrt{n})^{n-1}}{K_{m-1}(m-1)^{m/2}\tau^{m-1}},$$

and

$$w(\mu - \bar{x}) = g(\mu - \bar{x})/[f(\mu - \bar{x}) + g(\mu - \bar{x})].$$

The following lemma will be frequently used.

Lemma 3.1 *As $|\mu - \bar{x}| \to \infty$, for $i, j, k \geq 0$ and $i + j \neq 0$,*

$$[w(\mu-\bar{x})]^i[1-w(\mu-\bar{x})]^j(\mu-\bar{x})^k$$
$$= \begin{cases} \gamma_0^j(1+\gamma_0)^{-(i+j)}|\mu-\bar{x}|^k[sgn(\mu-\bar{x})]^k(1+O^\star(|\mu-\bar{x}|^{-2})) & \text{if } n = m \\ \gamma_0^j|\mu-\bar{x}|^{k-(n-m)j}[sgn(\mu-\bar{x})]^k[1+O^\star(|\mu-\bar{x}|^{-(n-m)})+jO^\star(|\mu-\bar{x}|^{-2})] & \text{if } n > m \\ \gamma_0^{-i}|\mu-\bar{x}|^{k-(m-n)i}[sgn(\mu-\bar{x})]^k[1+O^\star(|\mu-\bar{x}|^{-(m-n)})+iO^\star(|\mu-\bar{x}|^{-2})] & \text{if } n < m. \end{cases}$$

Proof: Clearly

$$\begin{aligned} &[w(\mu-\bar{x})]^i[1-w(\mu-\bar{x})]^j(\mu-\bar{x})^k \\ &= [\frac{1}{1+f(\mu-\bar{x})/g(\mu-\bar{x})}]^i[\frac{f(\mu-\bar{x})/g(\mu-\bar{x})}{1+f(\mu-\bar{x})/g(\mu-\bar{x})}]^j(\mu-\bar{x})^k \\ &= \frac{[f(\mu-\bar{x})/g(\mu-\bar{x})]^j(\mu-\bar{x})^k}{[1+f(\mu-\bar{x})/g(\mu-\bar{x})]^{i+j}} \\ &= \frac{\gamma_0^j|\mu-\bar{x}|^{k-(n-m)j}(1+O^\star(|\mu-\bar{x}|^{-2}))^j}{[1+\gamma_0|\mu-\bar{x}|^{-(n-m)}(1+O^\star(|\mu-\bar{x}|^{-2}))]^{i+j}}[sgn(\mu-\bar{x})]^k. \end{aligned}$$

The result for $n \geq m$ is immediate, and follows for $n < m$ by multiplying numerator and denominator by $\gamma_0^{-(i+j)}|\mu-\bar{x}|^{(n-m)(i+j)}$. □

3.2.1 The Marginal and Posterior Densities

As $|\mu - \bar{x}| \to \infty$, the marginal density $m(\mathbf{x})$ and posterior density $\pi(\theta|\mathbf{x})$ are precisely what one would expect given that the posterior mass concentrates near $\bar{x}$ and/or μ. The marginal density is essentially the sum of the likelihood, f, evaluated at μ and the prior, g, evaluated at $\bar{x}$, while the posterior density converges to either f or g, whichever has sharper tail. This is consistent with Dawid's and O'Hagan's results. When f and g have the same degrees of freedom, the posterior converges to a mixture of f and g.

Theorem 3.3 *As* $|\mu-\bar{x}|\to\infty$, $m(\mathbf{x})=(f(\mu-\bar{x})+g(\mu-\bar{x}))(1+o(1))$.

Proof: Assume $n\geq m$ without loss of generality. Following the notation defined in Lemma 2.1,

$$m(\mathbf{x})=A_0+B_0+C_0+D_0+E_0.$$

The results of Lemma 2.1 yield

$$\frac{C_0+D_0+E_0}{A_0+B_0}=\frac{O\left(|\mu-\bar{x}|^{-r(n+m-1)}\right)}{(g(\mu-\bar{x})+f(\mu-\bar{x}))(1+o(1))}.$$

Note that

$$g(\mu-\bar{x})+f(\mu-\bar{x})=O^{\star}(|\mu-\bar{x}|^{-m})+O^{\star}(|\mu-\bar{x}|^{-n}). \tag{14}$$

Thus,

$$\frac{C_0+D_0+E_0}{A_0+B_0}=\frac{O\left(|\mu-\bar{x}|^{-(r(n+m-1)-m)}\right)}{O^{\star}(1)+O^{\star}\left(|\mu-\bar{x}|^{-(n-m)}\right)}=o(1). \tag{15}$$

This shows that

$$\begin{aligned} m(\mathbf{x}) &= (A_0+B_0)(1+o(1)) \\ &= (f(\mu-\bar{x})a_0+g(\mu-\bar{x})b_0)(1+o(1)) \\ &= (f(\mu-\bar{x}))+g(\mu-\bar{x}))(1+o^{\star}(1)), \end{aligned}$$

where

$$o^{\star}(1)=\begin{cases} O^{\star}(|\mu-\bar{x}|^{-r(m-1)})+O^{\star}(|\mu-\bar{x}|^{-2}) & \text{if } m>3 \\ O^{\star}(|\mu-\bar{x}|^{-r(m-1)}) & \text{if } m\leq 3, \end{cases} \tag{16}$$

and $\lim_{|\mu-\bar{x}|\to\infty} o^{\star}(1)=0$. The proof is completed. □

Note also that , due to definition 3.1 and (14),

$$m(\mathbf{x})=O^{\star}(|\mu-\bar{x}|^{-m})+O^{\star}(|\mu-\bar{x}|^{-n}). \tag{17}$$

Theorem 3.4 *a) If* $n>m$, $\pi(\theta|\mathbf{x})\to f(\bar{x}-\theta)$ *in distribution, as* $|\mu-\bar{x}|\to\infty$.

b) If $n<m$, $\pi(\theta|\mathbf{x})\to g(\theta-\mu)$ *in distribution, as* $|\mu-\bar{x}|\to\infty$.

c) If $n=m$, *for any set* $\Omega_{\bar{x},\mu}=\Lambda_{\bar{x}}\cup\Upsilon_{\mu}$, *where*

$$\begin{aligned} \Lambda_{\bar{x}} &= \{(\bar{x}-a_1,\bar{x}+a_2): a_i\geq 0 \text{ for } i=1,2 \text{ and } a_2+a_1<\infty\}, \\ \Upsilon_{\mu} &= \{(\mu-b_1,\mu+b_2): b_i\geq 0 \text{ for } i=1,2 \text{ and } b_2+b_1<\infty\}, \end{aligned}$$

$$\lim_{|\mu-\bar{x}|\to\infty} P^{\pi(\theta|\mathbf{x})}(\Omega_{\bar{x},\mu}) = (1+\gamma_0)^{-1}\, P^{f_0}(\Lambda_0) + \gamma_0(1+\gamma_0)^{-1}\, P^{g_0}(\Upsilon_0),$$

where $f_0 = f(\theta)$ and $g_0 = g(\theta)$ with f, g being defined by (1) and (2) respectively.

Proof: See Appendix C. □

3.2.2 The Posterior Mean and Variance

We first present the two key theorems concerning the posterior mean and variance, respectively. Then we discuss the different cases that arise depending on $|n-m|$.

Theorem 3.5 *Let $\hat{\delta}^{\pi}(\mathbf{x}) = \bar{x}w(\mu-\bar{x}) + \mu(1-w(\mu-\bar{x}))$. Then, for $n, m \geq 2$ and as $|\mu-\bar{x}| \to \infty$,*

$$|\delta^{\pi}(\mathbf{x}) - \hat{\delta}^{\pi}(\mathbf{x})| = o(1).$$

Proof: See Appendix D. □

The behavior of $V^{\pi}(\mathbf{x})$ is given by the following theorem.

Theorem 3.6 *Define*

$$\psi(|\mu-\bar{x}|) = w(\mu-\bar{x})a_2 + (1-w(\mu-\bar{x}))b_2 + w(\mu-\bar{x})(1-w(\mu-\bar{x}))(\mu-\bar{x})^2,$$

where a_2, b_2 are defined in Lemma 2.1. Then for $n, m \geq 2$ and as $|\mu-\bar{x}| \to \infty$,

$$\begin{array}{rcll} |V^{\pi}(\mathbf{x}) - \Psi(|\mu-\bar{x}|)| &=& o(1) & \textit{if } n \neq m \textit{ and } \max(n,m) > 3 \\ |V^{\pi}(\mathbf{x}) - \Psi(|\mu-\bar{x}|)| &=& O(1) & \textit{if } n = m > 3 \\ V^{\pi}(\mathbf{x})/\Psi(|\mu-\bar{x}|) &=& 1+o(1) & \textit{otherwise.} \end{array}$$

If $n = m = 2$, it is furthermore true that

$$\left| V^{\pi}(\mathbf{x}) - \frac{\gamma_0}{(1+\gamma_0)^2}(\mu-\bar{x})^2 \right| = O(1).$$

Proof: See Appendix D. □

The behavior of $\delta^{\pi}(\mathbf{x})$ and $V^{\pi}(\mathbf{x})$ depend dramatically on $|n-m|$. We now specialize the above theorems to the four basic cases. The proofs of the corollaries are given in Appendix D.

Case 1: $|n-m|>2$.

Here the difference between the tail-thickness of f and g is large enough so that not only the posterior density, but also the first two moments of the posterior converge to the analogous features of the density with sharper tail as $|\mu-\bar{x}|\to\infty$. These results are thus similar to those given by Dawid (1973) and O'Hagan (1979).

Corollary 3.1 *i) If $n>m+2$, then, as $|\mu-\bar{x}|\to\infty$,*

$$|\delta^{\pi}(\mathbf{x})-\bar{x}|=o(1) \quad \textit{and} \quad |V^{\pi}(\mathbf{x})-V_f|=o(1).$$

ii) If $n<m-2$, then, as $|\mu-\bar{x}|\to\infty$,

$$|\delta^{\pi}(\mathbf{x})-\mu|=o(1) \quad \textit{and} \quad |V^{\pi}(\mathbf{x})-V_g|=o(1).$$

Case 2: $n=m$.

When $n=m$, we saw in Theorem 3.4 c) that the posterior does not concentrate about one of the modes as $|\mu-\bar{x}|\to\infty$, and can indeed be thought of as a mixture of posteriors corresponding to f and g, separately. One would, therefore, expect $\delta^{\pi}(\mathbf{x})$ to be a nondegenerate weighted average of $\bar{x}$ and μ, and $V^{\pi}(\mathbf{x})$ to be of magnitude $|\mu-\bar{x}|^2$ (since the mean $\delta^{\pi}(\mathbf{x})$ will differ from the concentrations of mass of the posterior by distances of magnitude $|\mu-\bar{x}|$). The following corollary establishes these facts.

Corollary 3.2 *If $|\mu-\bar{x}|\to\infty$ and $n=m$,*

$$|\delta^{\pi}(\mathbf{x})-[(1+\gamma_0)^{-1}\bar{x}+(1+\gamma_0^{-1})^{-1}\mu]|=o(1),$$

and

$$\begin{aligned}
\left|V^{\pi}(\mathbf{x})-\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2\right| &= O(1) \quad \textit{if } n=m\neq 3 \\
V^{\pi}(\mathbf{x})\Big/\left[\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2\right] &= 1+o(1) \quad \textit{if } n=m=3.
\end{aligned}$$

Case 3: $|n-m|=2$.

Here the posterior density converges to the density with sharper tail and the posterior mean tends to the mean of the limiting distribution, but the posterior is not concentrated enough for the posterior variance to converge to the correponding V_f or V_g. Indeed, the posterior variance converges to the variance of the limiting distribution plus a constant.

Corollary 3.3 *i) If $n = m + 2$, then, as $|\mu - \bar{x}| \to \infty$,*

$$|\delta^\pi(\mathbf{x}) - \bar{x}| = o(1) \quad \textit{and} \quad |V^\pi(\mathbf{x}) - (V_f + \gamma_0)| = o(1).$$

ii) If $n = m - 2$, then, as $|\mu - \bar{x}| \to \infty$,

$$|\delta^\pi(\mathbf{x}) - \mu| = o(1) \quad \textit{and} \quad |V^\pi(\mathbf{x}) - (V_g + \gamma_0^{-1})| = o(1).$$

Case 4: $|n - m| = 1$.

Here the posterior concentrates so slowly near a mode that the posterior mean is shifted towards the other mode by a constant, while the posterior variance actually goes to infinity at order $O^\star(|\mu - \bar{x}|)$.

Corollary 3.4 *i) If $n = m + 1$, then, as $|\mu - \bar{x}| \to \infty$,*

$$\begin{aligned} |\delta^\pi(\mathbf{x}) - [\bar{x} + \gamma_0 sgn(\mu - \bar{x})]| &= o(1); \\ \textit{and} \qquad |V^\pi(\mathbf{x}) - (V_f + \gamma_0|\mu - \bar{x}|)| &= O(1) \textit{ if } n > 3 \\ V^\pi(\mathbf{x})/[\gamma_0|\mu - \bar{x}|] &= (1 + o(1)) \textit{ if } n = 3. \end{aligned}$$

ii) If $n = m - 1$, then, as $|\mu - \bar{x}| \to \infty$,

$$\begin{aligned} |\delta^\pi(\mathbf{x}) - [\mu - \gamma_0^{-1} sgn(\mu - \bar{x})]| &= o(1); \\ \textit{and} \qquad |V^\pi(\mathbf{x}) - (V_g + \gamma_0^{-1}|\mu - \bar{x}|)| &= O(1) \textit{ if } m > 3 \\ V^\pi(\mathbf{x})/[\gamma_0^{-1}|\mu - \bar{x}|] &= (1 + o(1)) \textit{ if } m = 3. \end{aligned}$$

3.2.3 The Supremum of the Posterior Variance

Following O'Hagan (1981), it is of particular interest to observe how large $V^\pi(\mathbf{x})$ can be. The surprising answer is that it can be as large as $\frac{1}{4}(\mu - \bar{x})^2$. Intuitively, as $|\mu - \bar{x}| \to \infty$, $V^\pi(\mathbf{x})$ reaches its maximum when the posterior mass is equally divided around μ and $\bar{x}$.

Thus choosing τ in such a way that $w \to 1/2$ will result in $\delta^\pi(\mathbf{x}) \to (\mu + \bar{x})/2$ and $V^\pi(\mathbf{x}) \cong \frac{1}{2}(\delta^\pi(\mathbf{x}) - \bar{x})^2 + \frac{1}{2}(\delta^\pi(\mathbf{x}) - \mu)^2 \cong \frac{1}{4}(\mu - \bar{x})^2$. Thus clashing data and prior information can ressult in very large posterior uncertainty.

Theorem 3.7 *For $m \geq 2$,*

$$i) \lim_{|\mu-\bar{x}|\to\infty} \sup_\tau \frac{V^\pi(\mathbf{x})}{(\mu-\bar{x})^2/4} = 1, \quad if\, n > 3;$$

$$ii) \sup_\tau V^\pi(\mathbf{x}) = \lim_{\tau\to\infty} V^\pi(\mathbf{x}) = \infty, \quad if\, n \leq 3.$$

Proof: Case i) $n > 3$: The following gives the outline of the proof. See Appendix E for the details.

Without loss of generality, assume $\bar{x} = 0$ and $\mu \geq 0$.

I. Proof that $\lim_{|\mu|\to\infty} \sup_\tau \dfrac{V^\pi(\mathbf{x})}{(\mu^2/4)} \geq 1$:

Choose

$$\tau = \tau_\mu^\star = K_0 \mu^{\frac{n-m}{1-m}},$$

where $K_0 = [\dfrac{K_{m-1}(m-1)^{m/2}}{K_{n-1}(n-1)^{n/2}}(\dfrac{\sqrt{n}}{s})^{n-1}]^{1/(1-m)}$. It can be shown that, as $|\mu| \to \infty$,

$$V_{\tau_\mu^\star}^\pi(\mathbf{x}) \geq \frac{\mu^2}{4}(1 + o(1))$$

by the following steps.

Step 1) Define $\epsilon = 1/3$ when $m - n \leq 2, \epsilon = (1-n)/(1-m)$ when $m - n > 2$. Then $\tau_\mu^\star \leq K_0 \mu^{1-\epsilon}$.

Step 2) For $i = 0, 1, 2$, let

$$A_i = \int_{|\theta| \leq \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1 + \frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta,$$

$$B_i = \int_{|\theta-\mu| \leq \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1 + \frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta.$$

Then

$$A_i = k_0 K_{n-1} \mu^{-n}(b_i + o(1)), \text{ where } b_i = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ V_f & i = 2, \end{cases}$$

and

$$B_i = k_0 K_{n-1}\, \mu^{i-n}(1+o(1)), \text{ for } i = 0,1,2,$$

with $k_0 = (n-1)^{n/2}(s/\sqrt{n})^{n-1}$.

Step 3) For $i = 0,1,2,$

$$\int \frac{\theta^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1+(\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)}^2)^{m/2}} d\theta = (A_i + B_i)(1+o(1)). \tag{18}$$

Step 4) Combining Steps 2 and 3 yields, for $\tau = \tau_\mu^\star$,

$$\begin{aligned} m(\mathbf{x}) &= \left[k_0 K_{n-1}\mu^{-n}(1+o(1)) + k_0 K_{n-1}\mu^{-n}(1+o(1))\right](1+o(1)) \\ &= 2k_0 K_{n-1}\mu^{-n}(1+o(1)), \\ \delta^\pi(\mathbf{x}) &= \frac{[k_0 K_{n-1}\mu^{-n} o(1) + k_0 K_{n-1}\mu^{1-n}(1+o(1))](1+o(1))}{2k_0 K_{n-1}\mu^{-n}(1+o(1))} \\ &= \frac{\mu}{2}(1+o(1)), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{m(\mathbf{x})} \int &\frac{\theta^2 \sqrt{n} K_{n-1} K_{m-1} d\theta}{(1+n\theta^2)^{n/2} \tau_\mu^\star (1+(\frac{\theta-\mu}{\tau_\mu^\star})^2)^{m/2}} \\ &= \frac{\left[k_0 K_{n-1}\mu^{-n}(\frac{(n-1)s^2}{n-3} + o(1)) + k_0 K_{n-1}\mu^{2-n}(1+o(1))\right](1+o(1))}{2k_0 K_{n-1}\mu^{-n}(1+o(1))} \\ &= \frac{\mu^2}{2}(1+o(1)). \end{aligned}$$

Using these yields

$$V_{\tau_\mu^\star}^\pi = \frac{\mu^2}{2}(1+o(1)) - \frac{\mu^2}{4}(1+o(1)) = \frac{\mu^2}{4}(1+o(1)).$$

Hence

$$\lim_{|\mu|\to\infty} \sup_\tau \frac{V^\pi(\mathbf{x})}{\mu^2/4} \geq \lim_{|\mu|\to\infty} \frac{V_{\tau_\mu^\star}^\pi(\mathbf{x})}{\mu^2/4} = 1.$$

II. Proof that $\lim_{|\mu|\to\infty} \sup_\tau \frac{V^\pi(\mathbf{x})}{\mu^2/4} \leq 1$:

Step 1) $|\int_{I^c} \theta^i \pi(\theta|\mathbf{x}) d\theta \,| \leq k_i \mu^{r(-n+i+1)+1}$, for $i = 1,2$; and $0 \leq \delta^\pi(\mathbf{x}) \leq \mu$.

Step 2) $\mid \int_{I^c} \theta^2 \pi(\theta|\mathbf{x}) d\theta \mid \leq k_2 |\mu|^{l_2}$; and

$$\mid [\int_I \theta \pi(\theta|\mathbf{x}) d\theta]^2 - (\delta^\pi(\mathbf{x}))^2 \mid \leq p_1 |\mu|^{l_1}, \quad \text{where } l_i < 2, \text{ for } i = 1, 2.$$

Step 3) Define $c = [\int_I \pi(\theta|\mathbf{x}) d\theta]^{-1}$ and

$$\pi^*(\theta|\mathbf{x}) = \begin{cases} c\pi(\theta|\mathbf{x}) & \text{if } \theta \in I \\ 0 & \text{if } \theta \in I^c. \end{cases}$$

Then, defining $V^{\pi^*}(\mathbf{x}) = \int \theta^2 \pi^*(\theta|\mathbf{x}) d\theta - (\int \theta^2 \pi^*(\theta|\mathbf{x}) d\theta)^2$,

$$\lim_{|\mu| \to \infty} \frac{V^\pi(\mathbf{x}) - V^{\pi^*}(\mathbf{x})}{\mu^2} = 0.$$

Step 4) Any distribution on an interval $[0, v]$ has variance less than or equal to $v^2/4$.

Step 5) Combining the above results, we have, for any τ,

$$V^{\pi^*}(\mathbf{x}) \leq \frac{1}{4}(\mu + 2\mu^r)^2.$$

Therefore,

$$\lim_{|\mu| \to \infty} \sup_\tau \frac{V^{\pi^*}(\mathbf{x})}{\mu^2} \leq \frac{1}{4},$$

which yields

$$\lim_{|\mu| \to \infty} \sup_\tau \frac{V^\pi(\mathbf{x})}{\mu^2/4} \leq 1.$$

Case ii) $n \leq 3$: From Theorem 3.1 d), $\lim_{\tau \to \infty} V^\pi(\mathbf{x}) = \infty$ for $n \leq 3$. Therefore, $\sup_\tau V^\pi(\mathbf{x}) = \infty$, for $n \leq 3$. □

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Appendices

Appendix A. Proof of Lemma 2.1.

Without loss of generality assume $\mu > \bar{x}$, and $n \geq m$. For $|\theta - \bar{x}| \leq |\mu - \bar{x}|^r$, and some $x^\star$ between θ and $\bar{x}$,

$$g(\theta - \mu) = g(\bar{x} - \mu) + (\theta - \bar{x})g'(\bar{x} - \mu) + \frac{(\theta - \bar{x})^2}{2} g''(x^\star - \mu). \tag{19}$$

But $(\mu - \bar{x}) - |\mu - \bar{x}|^r \leq (\mu - x^\star) \leq (\mu - \bar{x}) + |\mu - \bar{x}|^r$. Hence, $\lim_{|\mu - \bar{x}| \to \infty} \frac{|\mu - x^\star|}{|\mu - \bar{x}|} = 1$ so that $|\mu - x^\star| = O^\star(|\mu - \bar{x}|)$ uniformly for $\theta \in R_1$. Note that, for $k = [(m-1)\tau^2]^{-1}$,

$$g'(\bar{x} - \mu) = g(\bar{x} - \mu) \frac{-nk(\mu - \bar{x})}{1 + k(\mu - \bar{x})^2},$$

and, uniformly for $\theta \in R_1$,

$$g''(x^\star - \mu) = g(x^\star - \mu) \left[\frac{-nk[1 - (n+1)k(\mu - x^\star)^2]}{(1 + k(\mu - x^\star)^2)^2} \right] = g(\bar{x} - \mu) O^\star(|\mu - \bar{x}|^{-2}).$$

Therefore, as $|\mu - \bar{x}| \to \infty$, (19) implies that, uniformly for $\theta \in R_1$,

$$g(\theta-\mu) = g(\bar{x}-\mu)\left(1-(\theta-\bar{x})\frac{nk(\mu-\bar{x})}{1+k(\mu-\bar{x})^2}+(\theta-\bar{x})^2O^\star(|\mu-\bar{x}|^{-2})\right).$$

Note that

$$\int_{R_1}(\theta-\bar{x})^l f(\theta-\bar{x})d\theta = \begin{cases} c - O^\star(|\mu-\bar{x}|^{-r(n-l-1)}) & \text{if } l < n-1, \text{ and } l \text{ is even} \\ O^\star(\ln|\mu-\bar{x}|) & \text{if } l = n-1, \text{ and } l \text{ is even} \\ O^\star(|\mu-\bar{x}|^{r(l-n+1)}) & \text{if } l \geq n, \text{ and } l \text{ is even} \\ 0 & \text{if } l \text{ is odd,} \end{cases}$$

where $c = \int_{-\infty}^{\infty}(\theta-\bar{x})^l f(\theta-\bar{x})d\theta$ (so that $c = V_f$ if $l=2$ and $c=1$ if $l=0$). Also for some constant $d < \infty$,

$$\int_{R_1}|\theta-\bar{x}|^{2l+1}f(\theta-\bar{x})d\theta = \begin{cases} d - O^\star(|\mu-\bar{x}|^{-r(n-2l-2)}) & \text{if } 2l+1 < n-1 \\ O^\star(\ln|\mu-\bar{x}|) & \text{if } 2l+1 = n-1 \\ O^\star(|\mu-\bar{x}|^{r(2l+2-n)}) & \text{if } 2l+1 \geq n. \end{cases}$$

Thus

$$\begin{aligned} A_0 &= \int_{R_1} f(\theta-\bar{x})g(\theta-\mu)d\theta \\ &= g(\mu-\bar{x})[\int_{R_1} f(\theta-\bar{x})d\theta - \frac{nk(\mu-\bar{x})}{1+k(\mu-\bar{x})^2}\int_{R_1}(\theta-\bar{x})f(\theta-\bar{x})d\theta \\ &\quad + \int_{R_1}(\theta-\bar{x})^2O^\star(|\mu-\bar{x}|^{-2})f(\theta-\bar{x})d\theta] \\ &= g(\mu-\bar{x})a_0, \end{aligned}$$

$$\begin{aligned} A_1 &= \int_{R_1}(\theta-\bar{x})f(\theta-\bar{x})g(\theta-\mu)d\theta \\ &= g(\mu-\bar{x})[\int_{R_1}(\theta-\bar{x})f(\theta-\bar{x})d\theta - \frac{nk(\mu-\bar{x})}{1+k(\mu-\bar{x})^2}\int_{R_1}(\theta-\bar{x})^2f(\theta-\bar{x})d\theta \\ &\quad + \int_{R_1}(\theta-\bar{x})^3O^\star(|\mu-\bar{x}|^{-2})f(\theta-\bar{x})d\theta] \\ &= g(\mu-\bar{x})a_1, \end{aligned}$$

and

$$A_2 = \int_{R_1}(\theta-\bar{x})^2 f(\theta-\bar{x})g(\theta-\mu)d\theta$$

$$
\begin{aligned}
&= g(\mu-\bar{x})[\int_{R_1}(\theta-\bar{x})^2 f(\theta-\bar{x})d\theta - \frac{nk(\mu-\bar{x})}{1+k(\mu-\bar{x})^2}\int_{R_1}(\theta-\bar{x})^3 f(\theta-\bar{x})d\theta \\
&\quad + \int_{R_1}(\theta-\bar{x})^4 O^\star(|\mu-\bar{x}|^{-2})f(\theta-\bar{x})d\theta] \\
&= g(\mu-\bar{x})a_2.
\end{aligned}
$$

This proves the result for A_i. The result for B_i can similarly be established by switching the roles of f and g.

ii) Note that

$$
\begin{aligned}
|C_i| &= |\int_{-\infty}^{\bar{x}-|\mu-\bar{x}|^r}(\theta-\bar{x})^i f(\bar{x}-\theta)g(\theta-\mu)d\theta| \\
&= \int_{|\mu-\bar{x}|^r}^{\infty}\eta^i f(\eta)g(\eta+(\mu-\bar{x}))d\eta \\
&\le c_1\int_{|\mu-\bar{x}|^r}^{\infty}\eta^{i-n-m}d\eta \\
&= O^\star\left(|\mu-\bar{x}|^{-(r(n+m-1-i))}\right).
\end{aligned}
$$

Similarly, $D_i \le O^\star(|\mu-\bar{x}|^{-r(n+m-1-i)})$.

For $r<1$ and $|\mu-\bar{x}|>2^{1/(1-r)}$, we have $\mu-|\mu-\bar{x}|^r \ge \bar{x}+|\mu-\bar{x}|^r$. Therefore, for large $|\mu-\bar{x}|$, $n\ge 2$, and $i=1,2$,

$$
\begin{aligned}
|E_i| &= |\int_{\bar{x}+|\mu-\bar{x}|^r}^{\mu-|\mu-\bar{x}|^r}(\theta-\bar{x})^i f(\bar{x}-\theta)g(\theta-\mu)d\theta| \\
&= \int_{|\mu-\bar{x}|^r}^{\mu-\bar{x}-|\mu-\bar{x}|^r}\eta^i f(\eta)g(\eta-\mu+\bar{x})d\eta \\
&\le c_1 g(|\mu-\bar{x}|^r)\int_{|\mu-\bar{x}|^r}^{\mu-\bar{x}-|\mu-\bar{x}|^r}\eta^{i-n}d\eta \\
&\le O^\star(|\mu-\bar{x}|^{-rm})O^\star(|\mu-\bar{x}|^{r(i-n)})|\mu-\bar{x}| \\
&= O^\star(|\mu-\bar{x}|^{-[r(n+m-i)-1]});
\end{aligned}
$$

and

$$
\begin{aligned}
E_0 &= \int_{|\mu-\bar{x}|^r}^{\mu-\bar{x}-|\mu-\bar{x}|^r} f(\eta)g(\eta-\mu+\bar{x})d\eta \\
&\le O^\star(|\mu-\bar{x}|^{-rm})\int_{|\mu-\bar{x}|^r}^{\mu-\bar{x}-|\mu-\bar{x}|^r}\eta^{-n}d\eta \\
&= O^\star(|\mu-\bar{x}|^{-r(n+m-1)}). \qquad \square
\end{aligned}
$$

Appendix B: Proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1

a) $$\lim_{\tau\to\infty}\frac{m(\mathbf{x})}{g(\bar{x}-\mu)} = \lim_{\tau\to\infty}\int \frac{\sqrt{n}K_{n-1}(1+\frac{(\bar{x}-\mu)^2}{(n-1)\tau^2})^{m/2}}{s(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}}d\theta$$

$$= \int \lim_{\tau\to\infty}\frac{\sqrt{n}K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}}\frac{((m-1)\tau^2+(\bar{x}-\mu)^2)^{m/2}}{((m-1)\tau^2+(\theta+\bar{x}-\mu)^2)^{m/2}}d\theta$$

$$= 1,$$

by the dominated convergence theorem, since the integrand is bounded by $M\cdot\sqrt{n}K_{n-1}/[s(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}]$ which is an integrable function.

b) $$\lim_{\tau\to\infty}\pi(\theta|\mathbf{x}) = \lim_{\tau\to\infty}\frac{\left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2}\left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2}}{\int\left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2}\left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2}d\theta}$$

$$= \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}},$$

by applying the dominated convergence theorem to the denominator (since the integrand is bounded by the integrable function $(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{-n/2}$).

c)

$$\delta^\pi(\mathbf{x}) = \frac{\int\theta\left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2}\left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2}d\theta}{\int\left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2}\left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2}d\theta}.$$

For $n>2$, the integrand of the numerator is bounded by $|\theta|(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{-n/2}$ and that of the denominator is bounded by $(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{-n/2}$; both are integrable with respect to θ. Therefore, the dominated convergence theorem can be applied and gives

$$\lim_{\tau\to\infty}\delta^\pi(\mathbf{x}) = \frac{\sqrt{n}K_{n-1}}{s}\int\frac{\theta}{(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}}d\theta = \bar{x}.$$

If $n=2$,

$$|\delta^\pi(\mathbf{x})-\bar{x}| = \frac{1}{m(\mathbf{x})}\left|\int\frac{\sqrt{n}K_{n-1}K_{m-1}\eta}{s\tau(1+\frac{n}{(n-1)s^2}\eta^2)(1+\frac{(\eta+\bar{x}-\mu)^2}{(m-1)\tau^2})^{m/2}}d\eta\right|.$$

By symmetry,

$$0 = \frac{1}{m(\mathbf{x})}\int\frac{\sqrt{n}K_{n-1}K_{m-1}\eta}{s\tau(1+\frac{n}{(n-1)s^2}\eta^2)(1+\frac{\eta^2}{(m-1)\tau^2})^{m/2}}d\eta.$$

Hence,

$$\begin{aligned}
|\delta^\pi(\mathbf{x}) - \bar{x}| &\le \frac{\sqrt{n}K_{n-1}K_{m-1}}{s\tau m(\mathbf{x})} \int \frac{|\eta|}{(1+\frac{n}{(n-1)s^2}\eta^2)} \left| \frac{1}{(1+\frac{(\eta+\bar{x}-\mu)^2}{(m-1)\tau^2})^{m/2}} - \frac{1}{(1+\frac{\eta^2}{(m-1)\tau^2})^{m/2}} \right| d\eta \\
&= \frac{\sqrt{n}K_{n-1}K_{m-1}}{s\tau m(\mathbf{x})} \int \frac{|\eta|}{(1+\frac{n}{(n-1)s^2}\eta^2)(1+\frac{\eta^2}{(m-1)\tau^2})^{m/2}} \\
&\quad \cdot \left| \frac{1}{\left(1+\frac{2\eta(\bar{x}-\mu)+(\bar{x}-\mu)^2}{(m-1)\tau^2+\eta^2}\right)^{m/2}} - 1 \right| d\eta.
\end{aligned}$$

Now, for some constant K,

$$\left| \frac{1}{\left(1+\frac{2\eta(\bar{x}-\mu)+(\bar{x}-\mu)^2}{(m-1)\tau^2+\eta^2}\right)^{m/2}} - 1 \right| \le \frac{K(2|\eta(\bar{x}-\mu)| + (\bar{x}-\mu)^2)}{(m-1)\tau^2+\eta^2} \quad \text{for large } \tau^2,$$

so, for suitable constants $K^\star, K'$ and K'',

$$\begin{aligned}
|\delta^\pi(\mathbf{x}) - \bar{x}| &\le \frac{K^\star}{\tau m(\mathbf{x})} \frac{1}{(m-1)\tau^2} \int \frac{2\eta^2|\bar{x}-\mu| + |\eta|(\bar{x}-\mu)^2}{(1+\frac{n}{(n-1)s^2}\eta^2)(1+\frac{\eta^2}{(m-1)\tau^2})^{m/2+1}} d\eta \\
&\le \frac{K'}{\tau m(\mathbf{x})} \frac{1}{(m-1)\tau} \int \frac{1}{\tau(1+\frac{\eta^2}{(m-1)\tau^2})^{m/2+1}} d\eta \\
&= \frac{K''}{\tau m(\mathbf{x})(m-1)\tau} \quad \to \quad 0 \quad \text{as } \tau \to \infty,
\end{aligned}$$

since

$$\lim_{\tau\to\infty} \tau m(\mathbf{x}) = \lim_{\tau\to\infty} \frac{m(\mathbf{x})}{g(\bar{x}-\mu)} \frac{K_{m-1}}{(1+\frac{(\bar{x}-\mu)^2}{(m-1)\tau^2})^{m/2}} = K_{m-1}, \tag{20}$$

by Theorem 3.1 a).

d) Note that

$$V^\pi(\mathbf{x}) = \frac{\int (\theta-\delta^\pi(\mathbf{x}))^2 \left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2} \left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2} d\theta}{\int \left(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2} \left(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2}\right)^{-m/2} d\theta}.$$

Following an argument similar to that of c), we get

$$\lim_{\tau\to\infty} V^\pi(\mathbf{x}) = \frac{\sqrt{n}K_{n-1}}{s} \int \frac{(\theta-\bar{x})^2}{(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}} d\theta = \frac{n-1}{n-3}\frac{s^2}{n}.$$

To prove $\lim_{\tau\to\infty} V^\pi(\mathbf{x}) = \infty$ for $n \le 3$, it is sufficient to show that

$$\lim_{\tau\to\infty} E^{\pi(\theta|\mathbf{x})}(\theta^2) = \infty.$$

For $c = \bar{x} + \frac{\sqrt{n-1}s}{\sqrt{n}} < \tau$, and suitable constants K_1 and K^*,

$$\begin{aligned}
\int \frac{\theta^2}{(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}} d\theta &\geq \int_c^\tau \frac{\theta^2}{(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}} d\theta \\
&\geq K_1 \int_c^\tau \frac{\theta^2}{(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}} d\theta \\
&\geq K^* \int_c^\tau \frac{\theta^2}{(\theta-\bar{x})^n} d\theta \\
&\to \infty, \qquad \text{as } \tau \to \infty, \text{ if } n \leq 3.
\end{aligned}$$

Together with (20), this implies that

$$\lim_{\tau\to\infty} E^{\pi(\theta|\mathbf{x})}(\theta^2) = \lim_{\tau\to\infty} \frac{1}{\pi\tau m(\mathbf{x})} \int \frac{\theta^2 K_{n-1}K_{m-1}\sqrt{n}/s}{(1+\frac{n(\theta-\bar{x})^2}{(n-1)s^2})^{n/2}(1+\frac{(\theta-\mu)^2}{(m-1)\tau^2})^{m/2}} d\theta = \infty. \qquad \square$$

Proof of Theorem 3.2

a) Write, by a change of variable,

$$m(\mathbf{x}) = \int \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\tau\eta+\mu-\bar{x})^2}{(n-1)s^2})^{n/2}} \frac{K_{m-1}}{(1+\frac{\eta^2}{m-1})^{m/2}} d\eta,$$

and note that the integrand is bounded by the integrable function $\sqrt{n}K_{n-1}K_{m-1}s^{-1}(1+\frac{\eta^2}{m-1})^{-m/2}$. Hence the dominated convergence theorem can be applied to give

$$\lim_{\tau\to 0} m(\mathbf{x}) = \int \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\mu-\bar{x})^2}{(n-1)s^2})^{n/2}} \frac{K_{m-1}}{(1+\frac{\eta^2}{m-1})^{m/2}} d\eta = \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\mu-\bar{x})^2}{(n-1)s^2})^{n/2}}.$$

b) Let Θ_τ be the random variable with density $\pi(\theta|\mathbf{x})$. To prove that $\pi(\theta|\mathbf{x})$ converges to a point mass at μ, we need to prove that $\Theta_\tau \to \mu$ in probability. Therefore consider, for any $\epsilon > 0$,

$$\begin{aligned}
& P(|\Theta_\tau - \mu| > \epsilon) \\
= \ & m(\mathbf{x})^{-1} \int_{|\theta-\mu|>\epsilon} f(\bar{x}-\theta)g(\theta-\mu)d\theta \\
= \ & \frac{\sqrt{n}K_{n-1}K_{m-1}}{s\, m(\mathbf{x})} \int_{|\theta-\mu|>\epsilon} \left(1+\frac{n(\tau\eta+\mu-\bar{x})^2}{(n-1)s^2}\right)^{-n/2} \left(1+\frac{\eta^2}{m-1}\right)^{-m/2} d\eta \\
= \ & \frac{\sqrt{n}K_{n-1}K_{m-1}}{s\, m(\mathbf{x})} \int I_{(|\eta|>\epsilon/\tau)} \left(1+\frac{n(\tau\eta+\mu-\bar{x})^2}{(n-1)s^2}\right)^{-n/2} \left(1+\frac{\eta^2}{m-1}\right)^{-m/2} d\eta,
\end{aligned}$$

where $I_{(\cdot)}$ is the indicator function. The integrand has limit 0 as $\tau \to 0$, and is bounded by $(1+\frac{\eta^2}{m-1})^{-m/2}$ which is integrable. Applying a) and the dominated convergence theorem gives

$$\lim_{\tau\to 0} P(|\Theta_\tau - \mu| > \epsilon) = 0, \quad \text{for any } \epsilon > 0.$$

c) An argument similar to that of the proof of a) gives

$$\begin{aligned}\lim_{\tau\to 0}\delta^\pi(\mathbf{x}) &= \lim_{\tau\to 0}\frac{\sqrt{n}K_{n-1}K_{m-1}}{m(\mathbf{x})s}\int \frac{\eta\tau+\mu}{(1+\frac{n(\tau\eta+\mu-\bar{x})^2}{(n-1)s^2})^{n/2}(1+\frac{\eta^2}{m-1})^{m/2}}d\eta \\ &= K_{m-1}\int \frac{\mu}{(1+\frac{\eta^2}{m-1})^{m/2}}d\eta \\ &= \mu.\end{aligned}$$

d) Note that a change of variable rewrites (7) as

$$\begin{aligned}V^\pi(\mathbf{x}) &= \frac{(m-1)\tau^2K_{m-1}}{m(\mathbf{x})}\int \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\tau\eta+\mu-\bar{x})^2}{(n-1)s^2})^{n/2}(1+\frac{\eta^2}{m-1})^{m/2-1}}d\eta \\ &\quad -(m-1)\tau^2-(\mu-\delta^\pi(\mathbf{x}))^2, \end{aligned} \tag{21}$$

which together with c) and a) gives the result directly. □

Appendix C: Proof of Theorem 3.4.

a) For any Borel set $A \subset R$,

$$\int_A \pi(\theta|\mathbf{x})d\theta = m(\mathbf{x})^{-1}[\{\int_{A\cap R_1} + \int_{A\cap R_2} + \int_{A\cap R_3}\}f(\bar{x}-\theta)g(\theta-\mu)d\theta],$$

where the R_i are defined in Lemma 2.1. Then

$$\begin{aligned} g(\mu-\bar{x}+|\mu-\bar{x}|^r)\int_{A\cap R_1} f(\bar{x}-\theta)d\theta &\le \int_{A\cap R_1} f(\bar{x}-\theta)g(\theta-\mu)d\theta \\ &\le g(\mu-\bar{x}-|\mu-\bar{x}|^r)\int_{A\cap R_1} f(\bar{x}-\theta)d\theta,\end{aligned}$$

so

$$\int_{A\cap R_1} f(\bar{x}-\theta)g(\theta-\mu)d\theta = g(\mu-\bar{x})(1+o(1))\int_{A\cap R_1} f(\bar{x}-\theta)d\theta.$$

Similarly,

$$\int_{A\cap R_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta = f(\mu-\bar{x})(1+o(1))\int_{A\cap R_2} g(\bar{x}-\theta)d\theta.$$

Also,

$$\begin{aligned}\int_{A\cap R_3} f(\bar{x}-\theta)g(\theta-\mu)d\theta &\le \int_{R_3} f(\bar{x}-\theta)g(\theta-\mu)d\theta \\ &= C_0+D_0+E_0 \\ &= (A_0+B_0)\cdot o(1) \\ &= (f(\mu-\bar{x})+g(\mu-\bar{x}))\cdot o(1),\end{aligned}$$

by (15) and Lemma 2.1. Combining these results together with Theorem 3.3, we have

$$\begin{aligned}&\int_A \pi(\theta|\mathbf{x})d\theta \\ &= \frac{g(\mu-\bar{x})\int_{A\cap R_1} f(\bar{x}-\theta)d\theta + f(\mu-\bar{x})\int_{A\cap R_2} g(\mu-\theta)d\theta}{f(\mu-\bar{x})+g(\mu-\bar{x})}(1+o(1)) \\ &= \left(w(\mu-\bar{x})\int_{A\cap R_1} f(\bar{x}-\theta)d\theta + (1-w(\mu-\bar{x}))\int_{A\cap R_2} g(\mu-\theta)d\theta\right)(1+o(1)).\end{aligned}$$

Note that $w(\mu-\bar{x})\to 1$ by Lemma 3.1, $\int_{A\cap R_2}\le 1$ and $A\cap R_1\to A$ as $\mu\to\infty$. Therefore

$$\int_A \pi(\theta|\mathbf{x})d\theta \to \int_A f(\bar{x}-\theta)d\theta.$$

b) Analogous to a).

c) Assume $\mu>\bar{x}$ without loss of generality. Since $|\Lambda_{\bar{x}}|<\infty$ and $|\Upsilon_\mu|<\infty$, there exists an $N_1>0$ such that for $|\mu-\bar{x}|>N_1$, $\Lambda_{\bar{x}}\cap\Upsilon_\mu=\phi$. Therefore, for $|\mu-\bar{x}|>N_1$,

$$\begin{aligned}P^{\pi(\theta|\mathbf{x})}(\Omega_{\bar{x},\mu}) &= m(\mathbf{x})^{-1}[\int_{\Lambda_{\bar{x}}} f(\bar{x}-\theta)g(\theta-\mu)d\theta + \int_{\Upsilon_\mu} f(\bar{x}-\theta)g(\theta-\mu)d\theta] \\ &= m(\mathbf{x})^{-1}[\int_{\bar{x}-a_1}^{\bar{x}+a_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta + \int_{\mu-b_1}^{\mu+b_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta]. \quad (22)\end{aligned}$$

Note that

$$\begin{aligned}\int_{\bar{x}-a_1}^{\bar{x}+a_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta &\ge g(\bar{x}-\mu-a_1)\int_{\bar{x}-a_1}^{\bar{x}+a_2} f(\bar{x}-\theta)d\theta; \\ \int_{\bar{x}-a_1}^{\bar{x}+a_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta &\le g(\bar{x}-\mu+a_2)\int_{\bar{x}-a_1}^{\bar{x}+a_2} f(\bar{x}-\theta)d\theta.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\bar{x}-a_1}^{\bar{x}+a_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta &= g(\mu-\bar{x})(1+o(1))\int_{-a_1}^{a_2} f(\theta)d\theta \\ &= g(\mu-\bar{x})P^{f_0}(\Lambda_0)(1+o(1)).\end{aligned}$$

Similarly,

$$\int_{\mu-b_1}^{\mu+b_2} f(\bar{x}-\theta)g(\theta-\mu)d\theta = f(\mu-\bar{x})P^{g_0}(\Upsilon_0)(1+o(1)).$$

Combining these and (22) together with Theorem 3.3 yields

$$P^{\pi(\theta|\mathbf{x})}(\Omega_{\bar{x},\mu}) = [w(\mu-\bar{x})P^{f_0}(\Lambda_0) + (1-w(\mu-\bar{x}))P^{g_0}(\Upsilon_0)](1+o(1)).$$

Applying the result of Lemma 3.1 completes the proof. □

Appendix D. Proof of Section 3.2.2 Results.

Proof of Theorem 3.5

For notational simplicity, $w(\mu-\bar{x})$ will be written as w, and, without loss of generality, assume $n \geq m$ and $\mu \geq \bar{x}$.

i) $n > 2$: Note first that

$$\begin{aligned} |\delta^\pi(\mathbf{x}) - \hat{\delta}^\pi(\mathbf{x})| &= |E^{\pi(\theta|\mathbf{x})}(\theta - w\bar{x} - (1-w)\mu)| \\ &= m(\mathbf{x})^{-1}|\int(\theta - w\bar{x} - (1-w)\mu)f(\bar{x}-\theta)g(\theta-\mu)d\theta|. \end{aligned}$$

This integral can be broken into three integrals, denoted $A^\star$, $B^\star$ and $C^\star$ respectively, over R_1, R_2 and R_3. Then

$$\begin{aligned} |C^\star| &= m(\mathbf{x})^{-1}|C_1 + D_1 + E_1 - (\mu-\bar{x})((1-w)C_0 - wD_0 + (1-w)E_0)| \\ &= |\frac{C_1+D_1+E_1}{m(\mathbf{x})} - \frac{(\mu-\bar{x})(1-w)(C_0+D_0+E_0)}{m(\mathbf{x})} + \frac{(\mu-\bar{x})D_0}{m(\mathbf{x})}|. \end{aligned}$$

By Lemma 2.1, (17) and Lemma 3.1,

$$\begin{aligned} |C^\star| &= |O(|\mu-\bar{x}|^{-[r(n+m-2)-m]}) + O(|\mu-\bar{x}|^{-[r(n+m-1)-1-m]}) \\ &\quad + O(|\mu-\bar{x}|^{-[r(n+m-1)-1+n-2m]}) + O(|\mu-\bar{x}|^{-[r(n+m-1)-1-m]})| \\ &\quad \cdot[O^\star(1) + O^\star(|\mu-\bar{x}|^{-(n-m)})]^{-1} \\ &\leq O(|\mu-\bar{x}|^{-[r(n+m-1)-(1+m)]}) \\ &= o(1). \end{aligned}$$

Also,

$$\begin{aligned}|A^\star + B^\star| &= m(\mathbf{x})^{-1}|A_1 + B_1 - (\mu-\bar{x})(1-w)A_0 + (\mu-\bar{x})wB_0| \\ &= m(\mathbf{x})^{-1}|g(\mu-\bar{x})a_1 + f(\mu-\bar{x})b_1 \\ &\quad -(\mu-\bar{x})[(1-w)g(\mu-\bar{x})a_0 - wf(\mu-\bar{x})b_0]| \\ &= |wa_1 + (1-w)b_1 + (\mu-\bar{x})(1-w)w(b_0-a_0)|(1+o^\star(1)),\end{aligned}$$

where $o^\star(1)$ is defined by (16). By Lemma 3.1 and the definition of a_i and b_i, for $i = 0, 1$,

$$wa_1 = \begin{cases} O^\star(|\mu-\bar{x}|^{-1}) = o(1) & \text{if } n > 3 \\ O^\star(|\mu-\bar{x}|^{-1}\ln|\mu-\bar{x}|) = o(1) & \text{if } n = 3, \end{cases}$$

$$(1-w)b_1 = \begin{cases} O^\star(|\mu-\bar{x}|^{-(n-m)-1}) = o(1) & \text{if } m > 3 \\ O^\star(|\mu-\bar{x}|^{-(n-m)-1}\ln|\mu-\bar{x}|) = o(1) & \text{if } m = 3 \\ O^\star(|\mu-\bar{x}|^{-(n-m)-1+r}) = o(1) & \text{if } m = 2, \end{cases}$$

and

$$\begin{aligned}&(\mu-\bar{x})w(1-w)(b_0-a_0) \\ &= \begin{cases} O^\star(|\mu-\bar{x}|^{1-(n-m)})[O^\star(|\mu-\bar{x}|^{-r(m-1)}) + O^\star(|\mu-\bar{x}|^{-2})] & \text{if } m > 3 \\ O^\star(|\mu-\bar{x}|^{1-(n-m)}) \cdot O^\star(|\mu-\bar{x}|^{-r(m-1)}) & \text{if } m \le 3. \end{cases}\end{aligned}$$

Thus, if $n > 2$, $|A^\star + B^\star| = o(1)$. This completes the proof for $\max(n,m) > 2$.

ii) $n = m = 2$:

By Fan and Berger (1989), an exact formula for $\delta^\pi(\mathbf{x})$ is

$$\delta^\pi(\mathbf{x}) = \bar{x} + \frac{sgn(z)\,s}{\sqrt{2}}\left[\frac{2\sqrt{2}\tau/s(1-\sqrt{2}\tau/s)z - |z|(z^2 - 2\tau^2/s^2 + 1)}{(1+\sqrt{2}\tau/s)z^2 + (\sqrt{2}\tau/s)^3 - 2\tau^2/s^2 - \sqrt{2}\tau/s + 1}\right],$$

where $z = \sqrt{2}(\bar{x}-\mu)/s$. Calculation yields, for c, c' independent of $(\bar{x}-\mu)$,

$$\begin{aligned}\delta^\pi(\mathbf{x}) &= \bar{x} - (\bar{x}-\mu)\left[\frac{2(\bar{x}-\mu)^2/s^2 + c}{(1+\sqrt{2}\tau/s)2(\bar{x}-\mu)^2/s^2 + c'}\right] \\ &= \bar{x} - (\bar{x}-\mu)[(1+\sqrt{2}\tau/s)^{-1} + O(|\mu-\bar{x}|^{-2})] \\ &= \left(1 - \frac{1}{1+\gamma_0^{-1}}\right)\bar{x} + \mu\frac{1}{1+\gamma_0^{-1}} + O(|\mu-\bar{x}|^{-1}) \\ &= \frac{1}{1+\gamma_0}\bar{x} + \frac{\gamma_0}{1+\gamma_0}\mu + o(1).\end{aligned}$$

□

Proof of Theorem 3.6

For notational simplicity, $w(\mu - \bar{x})$ will be written as w, and, without loss of generality, assume $n \geq m$ and $\mu \geq \bar{x}$. Note that

$$V^{\pi}(\mathbf{x}) = E^{\pi(\theta|\mathbf{x})}(\theta - \hat{\delta}^{\pi}(\mathbf{x}))^2 - [\hat{\delta}^{\pi}(\mathbf{x}) - \delta^{\pi}(\mathbf{x})]^2. \tag{23}$$

By Theorem 3.5, $|\hat{\delta}^{\pi}(\mathbf{x}) - \delta^{\pi}(\mathbf{x})| = o(1)$. Hence the proof reduces to determining the behavior, as $|\mu - \bar{x}| \to \infty$, of

$$E^{\pi(\theta|\mathbf{x})}(\theta - \hat{\delta}^{\pi}(\mathbf{x}))^2 = m(\mathbf{x})^{-1} \int (\theta - w\bar{x} - (1-w)\mu)^2 f(\bar{x} - \theta) g(\theta - \mu) d\theta.$$

This integral can be broken up into integrals over R_1, R_2 and R_3, defined in Lemma 2.1, to be denoted by A, B and C, respectively. Note first that

$$\begin{aligned}
|C| &= m(\mathbf{x})^{-1} \Big| \int_{-\infty}^{\bar{x} - |\mu - \bar{x}|^r} (\theta - \bar{x} - (\mu - \bar{x})(1-w))^2 f(\bar{x} - \theta) g(\theta - \mu) d\theta \\
&\quad + \int_{\mu + |\mu - \bar{x}|^r}^{\infty} (\theta - \mu + (\mu - \bar{x})w)^2 f(\bar{x} - \theta) g(\theta - \mu) d\theta \\
&\quad + \int_{\bar{x} + |\mu - \bar{x}|^r}^{\mu - |\mu - \bar{x}|^r} (\theta - \bar{x} - (\mu - \bar{x})(1-w))^2 f(\bar{x} - \theta) g(\theta - \mu) d\theta \Big| \\
&= m(\mathbf{x})^{-1} |C_2 + D_2 + E_2 - 2(\mu - \bar{x})((1-w)C_1 - wD_1 + (1-w)E_1) \\
&\quad + (\mu - \bar{x})^2((1-w)^2 C_0 + w^2 D_0 + (1-w)^2 E_0)|.
\end{aligned}$$

Combining the results of Lemmas 2.1, 3.1 and (17) gives

$$\begin{aligned}
|C| &= |O(|\mu - \bar{x}|^{-[r(n+m-3)-m]}) + O(|\mu - \bar{x}|^{-(r(n+m-2)-1-m)}) \\
&\quad + O(|\mu - \bar{x}|^{1-(n-m)-[r(n+m-2)-m]}) + O(|\mu - \bar{x}|^{1-[r(n+m-2)-m]}) \\
&\quad + O(|\mu - \bar{x}|^{1-(n-m)-[r(n+m-1)-1-m]}) + O(|\mu - \bar{x}|^{2-2(n-m)-[r(n+m-1)-m]}) \\
&\quad + O(|\mu - \bar{x}|^{2-[r(n+m-1)-m]})| \left[O^*(1) + O^*(|\mu - \bar{x}|^{-(n-m)})\right]^{-1} \\
&= O(|\mu - \bar{x}|^{-[r(n+m-1)-m-2]}).
\end{aligned} \tag{24}$$

Also,

$$A + B = m(\mathbf{x})^{-1} \int_{R_1} (\theta - \bar{x} - (\mu - \bar{x})(1-w))^2 f(\bar{x} - \theta) g(\theta - \mu) d\theta$$

$$
\begin{aligned}
&+m(\mathbf{x})^{-1}\int_{R_2}(\theta-\mu+(\mu-\bar{x})w)^2 f(\bar{x}-\theta)g(\theta-\mu)d\theta \\
= \;& m(\mathbf{x})^{-1}[A_2+B_2-2(\mu-\bar{x})((1-w)A_1-wB_1) \\
&+(\mu-\bar{x})^2((1-w)^2A_0+w^2B_0)].
\end{aligned}
$$

Applying Lemma 2.1 and Theorem 3.3 gives

$$
\begin{aligned}
A+B \;=\;& m(\mathbf{x})^{-1}\{g(\mu-\bar{x})a_2+f(\mu-\bar{x})b_2 \\
&-2(\mu-\bar{x})[(1-w)g(\mu-\bar{x})a_1-wf(\mu-\bar{x})b_1] \\
&+(\mu-\bar{x})^2[(1-w)^2g(\mu-\bar{x})a_0+w^2f(\mu-\bar{x})b_0]\} \\
=\;& \{wa_2+(1-w)b_2-2(\mu-\bar{x})(1-w)w(a_1-b_1) \\
&+(\mu-\bar{x})^2[(1-w)^2wa_0+w^2(1-w)b_0]\}(1+o^\star(1)), \qquad (25)
\end{aligned}
$$

where $o^\star(1)$ is defined in (16).

i) $\max(n,m)>3$:

a) $m>3$:

For $r=\frac{m+2.5}{n+m-1}$, $C=o(1)$ from (24). Combining the definition of Ψ, (25), Lemmas 2.1 and 3.1 yields

$$
\begin{aligned}
A+B \;=\;& \{\Psi(|\mu-\bar{x}|)-2(\mu-\bar{x})(1-w)wO^\star(|\mu-\bar{x}|^{-1}) \\
&+(\mu-\bar{x})^2w(1-w)^2[O^\star(|\mu-\bar{x}|^{-r(n-1)})+O^\star(|\mu-\bar{x}|^{-2})] \\
&+(\mu-\bar{x})^2w^2(1-w)[O^\star(|\mu-\bar{x}|^{-r(m-1)})+O^\star(|\mu-\bar{x}|^{-2})]\}(1+o^\star(1)) \\
=\;& \{\Psi(|\mu-\bar{x}|)+O^\star(|\mu-\bar{x}|^{-(n-m)})+O^\star(|\mu-\bar{x}|^{2-2(n-m)-r(n-1)}) \\
&+O^\star(|\mu-\bar{x}|^{-2(n-m)})+O^\star(|\mu-\bar{x}|^{2-(n-m)-r(m-1)})+O^\star(|\mu-\bar{x}|^{-(n-m)})\} \\
&\cdot(1+O^\star(|\mu-\bar{x}|^{-r(m-1)})+O^\star(|\mu-\bar{x}|^{-2})).
\end{aligned}
$$

Since $\Psi(|\mu-\bar{x}|)=O^\star(1)+O^\star(|\mu-\bar{x}|^{2-(n-m)})$ (by Lemma 2.1 and Lemma 3.1),

$$
A+B=\begin{cases}\Psi(|\mu-\bar{x}|)+o(1) & \text{if } n\neq m\\ \Psi(|\mu-\bar{x}|)+O(1) & \text{if } n=m.\end{cases}
$$

b) $m\le 3$:

$$
A+B \;=\; \{\Psi(|\mu-\bar{x}|)-2(\mu-\bar{x})(1-w)wO^\star(|\mu-\bar{x}|^{-1+r})
$$

$$
\begin{aligned}
&+(\mu-\bar{x})^2 w(1-w)^2[O^\star(|\mu-\bar{x}|^{-r(n-1)})+O^\star(|\mu-\bar{x}|^{-2})] \\
&+(\mu-\bar{x})^2 w^2(1-w)O^\star(|\mu-\bar{x}|^{-r(m-1)})\}[1+O^\star(|\mu-\bar{x}|^{-r(m-1)})] \\
= \ &\{\Psi(|\mu-\bar{x}|)+O^\star(|\mu-\bar{x}|^{-(n-m)+r})+O^\star(|\mu-\bar{x}|^{2-2(n-m)-r(n-1)}) \\
&+\{O^\star(|\mu-\bar{x}|^{-2(n-m)})+O^\star(|\mu-\bar{x}|^{2-(n-m)+r})\}[1+O^\star(|\mu-\bar{x}|^{-r(m-1)})].
\end{aligned}
$$

In this case, $r=(m+2.5)/(n+m-1)$,

$$
\Psi(|\mu-\bar{x}|)=[V_f+O^\star(|\mu-\bar{x}|^{-(n-m)})b_2+O^\star(|\mu-\bar{x}|^{2-(n-m)})](1+o(1)),
$$

and $n-m=1$, if $n=4, m=3$; and $n-m\geq 2$, otherwise. Therefore,

$$
A+B=\Psi(|\mu-\bar{x}|)+o(1).
$$

Hence, for $\max(n,m)>3$,

$$
E^{\pi(\theta|\mathbf{x})}(\theta-\hat{\delta}^\pi(\mathbf{x}))^2=A+B+C=\begin{cases}\Psi(|\mu-\bar{x}|)+o(1) & \text{if } n\neq m\\ \Psi(|\mu-\bar{x}|)+O(1) & \text{if } n=m.\end{cases}
$$

The results then follow.

ii) $\max(n,m)\leq 3$:

Since $r=5/6$ and $\Psi(|\mu-\bar{x}|)=O^\star(|\mu-\bar{x}|^{2-(n-m)})$ (using Lemmas 2.1 and 3.1), we have $\frac{C}{\Psi(|\mu-\bar{x}|)}=o(1)$. And by (25),

$$
\begin{aligned}
A+B \ = \ &\{\Psi(|\mu-\bar{x}|)+O^\star(|\mu-\bar{x}|^{1-(n-m)+r-1})+O^\star(|\mu-\bar{x}|^{2-2(n-m)-2+r}) \\
&+O^\star(|\mu-\bar{x}|^{2-(n-m)-2+r})\}[1+O^\star(|\mu-\bar{x}|^{-r(m-1)})].
\end{aligned}
$$

Thus, $\dfrac{V^\pi(\mathbf{x})}{\Psi(|\mu-\bar{x}|)}=1+o(1)$.

iii) $n=m=2$:

Note that, by (7), Theorem 3.5, and the exact results given by Fan and Berger (1989),

$$
\begin{aligned}
V^\pi(\mathbf{x}) \ &= \ \frac{\tau s}{\sqrt{2}}\left[\frac{(z^2+2\tau^2/s^2-1)^2+4z^2}{(1+\sqrt{2}\tau/s)z^2+(2\tau^2/s^2)^3-2\tau^2/s^2-\sqrt{2}\tau/s+1}\right] \\
&\quad -\tau^2-(\mu-\delta^\pi(\mathbf{x}))^2 \\
&= \ \frac{\tau s}{\sqrt{2}}\left[\frac{z^2(z^2+2(1+\tau^2/s^2)c_1/z^2)}{(1+\sqrt{2}\tau/s)z^2+c_2}\right]-\tau^2
\end{aligned}
$$

$$
\begin{aligned}
&-(1+\gamma_0)^{-2}(\mu-\bar{x})^2+O(1)+o(1)\\
=\;& \frac{\tau s}{\sqrt{2}}\frac{2(\mu-\bar{x})^2}{s^2(1+\sqrt{2}\tau/s)}(1+O(|\mu-\bar{x}|^{-2}))-\tau^2\\
&-(1+\gamma_0)^{-2}(\mu-\bar{x})^2+O(1)\\
=\;& (1+\gamma_0)^{-1}(\mu-\bar{x})^2-(1+\gamma_0)^{-2}(\mu-\bar{x})^2+O(1)\\
=\;& \frac{\gamma_0}{(1+\gamma_0)^2}(\mu-\bar{x})^2+O(1).
\end{aligned}
$$

□

In the proofs of the following corollaries, we will consider the case $n > m$. For $n < m$, the proofs are analogous.

Proof of Corollary 3.1

To prove the result for $\delta^\pi(\mathbf{x})$, Theorem 3.5 implies that

$$|(\delta^\pi(\mathbf{x})-\bar{x})-(\hat{\delta}^\pi(\mathbf{x})-\bar{x})|=o(1).$$

By the definition of $\hat{\delta}^\pi(\mathbf{x})$ and Lemma 3.1,

$$\hat{\delta}^\pi(\mathbf{x})-\bar{x}=(\mu-\bar{x})(1-w(\mu-\bar{x}))=\gamma_0 O^\star(|\mu-\bar{x}|^{1-(n-m)})(1+o(1)).$$

The conclusion is immediate. To prove the result for $V^\pi(\mathbf{x})$, note that Theorem 3.6 yields

$$|V^\pi(\mathbf{x})-V_f+V_f-\Psi(|\mu-\bar{x}|)|=o(1).$$

Now, by the definition of Ψ and Lemmas 2.1 and 3.1,

$$
\begin{aligned}
&V_f-\Psi(|\mu-\bar{x}|)\\
&= V_f-[w(\mu-\bar{x})a_2+(1-w(\mu-\bar{x}))b_2+w(\mu-\bar{x})(1-w(\mu-\bar{x}))(\mu-\bar{x})^2]\\
&= V_f-[V_f+o(1)+\gamma_0 O^\star(|\mu-\bar{x}|^{-(n-m)})b_2+\gamma_0|\mu-\bar{x}|^{2-(n-m)}](1+O^\star(|\mu-\bar{x}|^{-2}))\\
&= [o(1)+\gamma_0 O^\star(|\mu-\bar{x}|^{-(n-m)})b_2+\gamma_0|\mu-\bar{x}|^{2-(n-m)}](1+o(1))\\
&= o(1),
\end{aligned}
$$

since $n-m>2$ and $O^\star(|\mu-\bar{x}|^{-2})b_2=o(1)$. □

Proof of Corollary 3.2

Using Lemma 3.1,

$$\begin{aligned}\hat{\delta}^{\pi}(\mathbf{x})-(1+\gamma_0)^{-1}\bar{x}-(1+\gamma_0^{-1})^{-1}\mu &= (\mu-\bar{x})\left(\frac{1}{1+\gamma_0}-w(\mu-\bar{x})\right)\\ &= (\mu-\bar{x})\left(\frac{1}{1+\gamma_0}-\frac{1}{1+\gamma_0}(1+O^{\star}(|\mu-\bar{x}|^{-2})\right)\\ &= \frac{1}{1+\gamma_0}O^{\star}(|\mu-\bar{x}|^{-1}))\\ &= o(1).\end{aligned}$$

The result for $\delta^{\pi}(\mathbf{x})$ then follows by Theorem 3.5.

For $n=m=2$, the result for $V^{\pi}(\mathbf{x})$ follows from Lemma 3.2. For $n=m>3$, consider

$$\begin{aligned}&\left|\Psi(|\mu-\bar{x}|)-\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2\right|\\ =&\left|\left(\frac{1}{1+\gamma_0}a_2+\frac{\gamma_0}{1+\gamma_0}b_2+\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2\right)(1+O^{\star}(|\mu-\bar{x}|^{-2}))-\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2\right|\\ =&\left|\frac{1}{1+\gamma_0}V_f+\frac{\gamma_0}{1+\gamma_0}V_g+\frac{\gamma_0}{(1+\gamma_0)^2}O^{\star}(1)+o(1)\right|,\end{aligned}$$

by Lemmas 2.1 and 3.1. The conclusion then follows by Theorem 3.6 .

For $n=m=3$,

$$\begin{aligned}\Psi(|\mu-\bar{x}|) &= \frac{1}{1+\gamma_0}O^{\star}(\ln|\mu-\bar{x}|)+\frac{\gamma_0}{1+\gamma_0}O^{\star}(\ln|\mu-\bar{x}|)\\ &\quad+\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2(1+O^{\star}(|\mu-\bar{x}|^{-2}))\\ &= \frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2(1+o(1)).\end{aligned}$$

Hence,

$$\frac{\frac{\gamma_0}{(1+\gamma_0)^2}|\mu-\bar{x}|^2}{\Psi(|\mu-\bar{x}|)}=1+o(1).$$

The result follows immediately from Theorem 3.6. □

Proof of Corollary 3.3

An argument similar to that of the proof of Corollary 3.1 yields the results. □

Proof of Corollary 3.4

The result for $\delta^\pi(\mathbf{x})$ follows by an argument similar to that of the proof of Corollary 3.1. To prove the results for $V^\pi(\mathbf{x})$, when $n > 3(m > 2)$ and $n - m = 1$, note that

$$\begin{aligned}
\Psi(|\mu - \bar{x}|) &= [V_f - o(1) + O^\star(|\mu - \bar{x}|^{-1})b_2 + \gamma_0|\mu - \bar{x}|](1 + O^\star(|\mu - \bar{x}|^{-1})) \\
&= V_f + \gamma_0|\mu - \bar{x}| + o(1) + O(1),
\end{aligned}$$

by Lemmas 2.1 and 3.1. Hence,

$$\Psi(|\mu - \bar{x}|) - (V_f + \gamma_0|\mu - \bar{x}|) = O(1).$$

The result then follows from Theorem 3.6. If $n = 3, m = 2$,

$$\begin{aligned}
\Psi(|\mu - \bar{x}|) &= [O^\star(\ln|\mu - \bar{x}|) + O^\star(|\mu - \bar{x}|^{-1})O^\star(|\mu - \bar{x}|^{r}) + \gamma_0|\mu - \bar{x}|](1 + O^\star(|\mu - \bar{x}|^{-1})) \\
&= \gamma_0|\mu - \bar{x}|(1 + o(1)).
\end{aligned}$$

Using Theorem 3.6 completes the proof. □

Appendix E. Proof of Case i) in Theorem 3.7.

Proof of Step 1) of I :

i) If $m-n \leq 0$, then $\frac{n-m}{1-m} \leq 0 \leq \frac{2}{3}$, so $\tau_\mu^\star \leq K_0 \mu^{1-\epsilon}$ if $\epsilon = \frac{1}{3}$.

ii) If $0 < m-n \leq 2$ for $n > 3$(so $m \geq 4$), then $0 < \frac{n-m}{1-m} \leq \frac{2}{3}$, and again $\epsilon = 1/3$ works.

iii) If $m-n > 2$, then $\tau_\mu^\star = K_0 \mu^{1-\epsilon}$, where $0 < \epsilon = \frac{1-n}{1-m} < 1$.

Proof of Step 2) of I :

Note first that for $i = 0, 2$, some constant c, and $n > 3$,

$$0 \leq \int_{|\theta| > \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta \leq c\mu^{(1-\frac{1}{2}\epsilon)(-n+i+1)} = o(1),$$

so that

$$\begin{aligned} \int_{|\theta| \leq \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta &= \int \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta - \int_{|\theta| > \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta \\ &= b_i - o(1). \end{aligned} \tag{26}$$

Also,

$$\begin{aligned} &\frac{K_{m-1}}{\tau_\mu^\star (1+\frac{(\mu+\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} \int_{|\theta| \leq \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta \\ &\leq A_i \leq \frac{K_{m-1}}{\tau_\mu^\star (1+\frac{(\mu-\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} \int_{|\theta| \leq \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta, \end{aligned}$$

and

$$\begin{aligned} \frac{K_{m-1}}{\tau_\mu^\star (1+\frac{(\mu\pm\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} &= \frac{K_{m-1} (\tau_\mu^\star)^{m-1} (m-1)^{m/2}}{\mu^m \left(\frac{(m-1)(\tau_\mu^\star)^2}{\mu^2} + (1 \pm \mu^{-\frac{1}{2}\epsilon})^2\right)^{m/2}} \\ &= \frac{K_{m-1} (\tau_\mu^\star)^{m-1} (m-1)^{m/2}}{\mu^m (1+o(1))} \\ &= \frac{k_0 K_{n-1}}{\mu^n} (1+o(1)). \end{aligned}$$

Together with (26), this yields

$$A_i = \frac{k_0 K_{n-1}}{\mu^n (1+o(1))} (b_i - o(1)) = \frac{k_0 K_{n-1}}{\mu^n} (b_i + o(1)).$$

For $i = 1$,

$$A_i = -\int_0^{\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta\sqrt{n}K_{n-1}K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}\tau_\mu^*(1+\frac{(\theta+\mu)^2}{(m-1)(\tau_\mu^*)^2})^{m/2}}d\theta + \int_0^{\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta\sqrt{n}K_{n-1}K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}\tau_\mu^*(1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^*)^2})^{m/2}}d\theta,$$

so that, for some constant c',

$$\begin{aligned}
0 &\le A_i \\
&\le \Big(-\frac{K_{m-1}}{\tau_\mu^*(1+\frac{(\mu+\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^*)^2})^{m/2}} + \frac{K_{m-1}}{\tau_\mu^*(1+\frac{(\mu-\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^*)^2})^{m/2}}\Big)\int_0^{\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta\sqrt{n}K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}}d\theta \\
&= \Big(\frac{k_0K_{n-1}}{\mu^n}(1+o(1)) - \frac{k_0K_{n-1}}{\mu^n}(1+o(1)\Big)\int_0^{\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta\sqrt{n}K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}}d\theta \\
&\le \frac{k_0K_{n-1}}{\mu^n}o(1)\int_0^{\mu^{1-\frac{1}{2}\epsilon}} c'\theta^{-n+1}d\theta \\
&= \frac{k_0K_{n-1}}{\mu^n}o(1).
\end{aligned}$$

Hence $A_i = k_0K_{n-1}\mu^{-n}(b_i + o(1))$.

Note next that

$$\begin{aligned}
\int_{|\theta-\mu|>\mu^{1-\frac{1}{2}\epsilon}} \frac{K_{m-1}}{\tau_\mu^*(1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^*)^2})^{m/2}}d\theta &= \int_{|t|>\mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^*} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2}}dt \\
\le \quad & \int_{|t|>K_0^{-1}\mu^{\frac{1}{2}\epsilon}} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2}}dt = o(1); \qquad (27)
\end{aligned}$$

$$\int_{|\theta-\mu|\le\mu^{1-\frac{1}{2}\epsilon}} \frac{K_{m-1}}{\tau_\mu^*(1+(\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^*)^2})^{m/2}}d\theta = 1 - o(1); \qquad (28)$$

$$\int_{|\theta-\mu|\le\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta K_{m-1}}{\tau_\mu^*(1+(\frac{\theta-\mu}{\tau_\mu^*})^2)^{m/2}}d\theta = \mu, \text{ (by symmetry)}; \qquad (29)$$

$$\begin{aligned}
&\int_{|\theta-\mu|\le\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^2K_{m-1}}{\tau_\mu^*(1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^*)^2})^{m/2}}d\theta \\
= &\int_{|t|\le\mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^*} \frac{K_{m-1}(\mu^2+2\tau_\mu^*\mu t+(\tau_\mu^*)^2t^2)}{(1+\frac{t^2}{m-1})^{m/2}}dt
\end{aligned}$$

$$
\begin{aligned}
&= \mu^2 \int_{|t|\le \mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^\star} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2}}dt + 2\tau_\mu^\star \mu \int_{|t|\le \mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^\star} \frac{tK_{m-1}}{(1+\frac{t^2}{m-1})^{m/2}}dt \\
&\quad +(\tau_\mu^\star)^2 \left(\int_{|t|\le \mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^\star} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2-1}}dt - \int_{|t|\le \mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^\star} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2}}dt \right) \\
&= (\mu^2 - (\tau_\mu^\star)^2) \int_{|t|\le \mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^\star} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2}}dt + (\tau_\mu^\star)^2 \int_{|t|\le \mu^{1-\frac{1}{2}\epsilon}/\tau_\mu^\star} \frac{K_{m-1}}{(1+\frac{t^2}{m-1})^{m/2-1}}dt \\
&\le (\mu^2 - (\tau_\mu^\star)^2)(1-o(1)) + 2(\tau_\mu^\star)^2 (\frac{\mu^{1-\frac{1}{2}\epsilon}}{\tau_\mu^\star}) \\
&= (\mu^2 - (\tau_\mu^\star)^2)(1+o(1)) + 2\tau_\mu^\star \mu^{1-\frac{1}{2}\epsilon} \\
&\le \mu^2 \left[(1 - \frac{(\tau_\mu^\star)^2}{\mu^2})(1+o(1)) + 2K_0 \mu^{-\frac{3}{2}\epsilon} \right] \\
&= \mu^2 (1+o(1)). \qquad (30)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\mu+\mu^{1-\frac{1}{2}\epsilon})^2}{(n-1)s^2})^{n/2}} \int_{|\theta-\mu|\le \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i K_{m-1}}{\tau_\mu^\star (1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta \le B_i \\
&\le \frac{\sqrt{n}K_{n-1}}{s(1+\frac{n(\mu-\mu^{1-\frac{1}{2}\epsilon})^2}{(n-1)s^2})^{n/2}} \int_{|\theta-\mu|\le \mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i K_{m-1}}{\tau_\mu^\star (1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta,
\end{aligned}
$$

and

$$
\frac{\sqrt{n}K_{n-1}}{s(1+\frac{(\mu\pm\mu^{1-\frac{1}{2}\epsilon})^2}{(n-1)s^2})^{n/2}} = \frac{k_0 K_{n-1}}{\mu^n}(1+o(1)).
$$

Combining this with (27), (28), (29) and (30) yields $B_i = k_0 K_{n-1}\mu^i \mu^{-n}(1+o(1))$, for $i = 0, 1, 2$.

Proof of Step 3) of I :

To prove this , we show that

$$
\frac{\left(\int_{-\infty}^{-\mu^{1-\frac{1}{2}\epsilon}} + \int_{\mu^{1-\frac{1}{2}\epsilon}}^{\mu-\mu^{1-\frac{1}{2}\epsilon}} + \int_{\mu+\mu^{1-\frac{1}{2}\epsilon}}^{\infty} \right) \frac{\theta^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta}{A_i + B_i} = o(1).
$$

From Step 2) of I, we have $A_i \ge 0$ and $B_i = k_0 K_{n-1}\mu^i \mu^{-n}(1+o(1))$. Hence

$$
A_i + B_i \ge \frac{k_0 K_{n-1}\mu^i}{\mu^n}(1+o(1)).
$$

Using $n > 3$, and letting c_1, c_2, and c_3 denote general constants,

$$\begin{aligned}
&\int_{-\infty}^{-\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta \\
\leq\ & \int_{-\infty}^{-\mu^{1-\frac{1}{2}\epsilon}} \frac{|\theta|^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta \\
\leq\ & \frac{K_{m-1}}{\tau_\mu^\star \left(1+\frac{(\mu+\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^\star)^2}\right)^{m/2}} \int_{-\infty}^{-\mu^{1-\frac{1}{2}\epsilon}} \frac{|\theta|^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta \\
\leq\ & \frac{K_{m-1}(m-1)^{m/2}(\tau_\mu^\star)^{m-1}}{\mu^m} \int_{\mu^{1-\frac{1}{2}\epsilon}}^{\infty} c'\, \theta^{-n+i} d\theta \\
=\ & \frac{c'\, K_0^{m-1} K_{m-1}(m-1)^{m/2} \mu^{-n+m}}{(n-i-1)\mu^m} (\mu^{1-\frac{1}{2}\epsilon})^{-n+i+1} \\
=\ & c_1 \mu^{-n+i} \mu^{\alpha_1},
\end{aligned}$$

where $\alpha_1 = -n + \frac{n}{2}\epsilon + 1 - \frac{1}{2}\epsilon i - \frac{1}{2}\epsilon < 0$. Similarly,

$$\begin{aligned}
&\int_{\mu^{1-\frac{1}{2}\epsilon}}^{\mu-\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1} K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2} \tau_\mu^\star (1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}} d\theta \\
\leq\ & \frac{K_{m-1}}{\tau_\mu^\star \left(1+\frac{(\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^\star)^2}\right)^{m/2}} \int_{\mu^{1-\frac{1}{2}\epsilon}}^{\mu-\mu^{1-\frac{1}{2}\epsilon}} \frac{\theta^i \sqrt{n} K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}} d\theta \\
\leq\ & \frac{K_{m-1}(m-1)^{m/2}(\tau_\mu^\star)^{m-1} c'}{(n-i-1)\mu^{m-m\epsilon/2}} [(\mu^{1-\frac{1}{2}\epsilon})^{-n+i+1} - (\mu-\mu^{1-\frac{1}{2}\epsilon})^{-n+i+1}] \\
\leq\ & c_2 \mu^{-m+m\epsilon/2} \mu^{(\frac{n-m}{1-m})(m-1)} \mu^{(1-\frac{1}{2}\epsilon)(-n+i+1)} \\
=\ & c_2 \mu^{-n+i} \mu^{\alpha_2},
\end{aligned}$$

where $\alpha_2 = -n + \frac{1}{2}(n+m)\epsilon - \frac{1}{2}\epsilon i - \frac{1}{2}\epsilon + 1$. Now, when $m - n \leq 2$, $\epsilon = 1/3$, so that

$$\begin{aligned}
\alpha_2 &= -n + n/6 + m/6 + 1 - i/6 - 1/6 \\
&\leq -n + n/6 + (n+2)/6 + 1 - i/6 - 1/6 \\
&< 0;
\end{aligned}$$

when $m - n > 2$, $\epsilon = \frac{1-n}{1-m}$, so that

$$\alpha_2 = -n + \frac{1}{2}(n+m)(\frac{1-n}{1-m}) + 1 - \frac{1}{2}\epsilon i - \frac{1}{2}\epsilon$$

$$\begin{aligned} &\le (1-n) + \frac{1}{2}(n+m)(\frac{1-n}{1-m}) \\ &= (1-m/2+n/2)(\frac{1-n}{1-m}) \\ &< 0. \end{aligned}$$

Also,

$$\begin{aligned} &\int_{\mu+\mu^{1-\frac{1}{2}\epsilon}}^{\infty} \frac{\theta^i\sqrt{n}K_{n-1}K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}\tau_\mu^\star(1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}}d\theta \\ \le \quad & \frac{K_{m-1}}{\tau_\mu^\star\left(1+\frac{(\mu^{1-\frac{1}{2}\epsilon})^2}{(m-1)(\tau_\mu^\star)^2}\right)^{m/2}} \int_{\mu+\mu^{1-\frac{1}{2}\epsilon}}^{\infty} \frac{\theta^i\sqrt{n}K_{n-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}}d\theta \\ \le \quad & \frac{K_{m-1}(m-1)^{m/2}(\tau_\mu^\star)^{m-1}c'}{(n-i-1)\mu^{-m+m\epsilon/2}}(\mu+\mu^{1-\frac{1}{2}\epsilon})^{-n+i+1} \\ \le \quad & c_3\mu^{-m+m\epsilon/2}\mu^{(\frac{n-m}{1-m})(m-1)}\mu^{-n+i+1} \\ = \quad & c_3\mu^{-n+i}\mu^{\alpha_3}, \end{aligned}$$

where $\alpha_3 = -n + m\epsilon/2 + 1 < \alpha_2 < 0$. Therefore,

$$\frac{\left(\int_{-\infty}^{-\mu^{1-\frac{1}{2}\epsilon}} + \int_{\mu^{1-\frac{1}{2}\epsilon}}^{\mu-\mu^{1-\frac{1}{2}\epsilon}} + \int_{\mu+\mu^{1-\frac{1}{2}\epsilon}}^{\infty}\right) \frac{\theta^i\sqrt{n}K_{n-1}K_{m-1}}{s(1+\frac{n\theta^2}{(n-1)s^2})^{n/2}\tau_\mu^\star(1+\frac{(\theta-\mu)^2}{(m-1)(\tau_\mu^\star)^2})^{m/2}}d\theta}{A_i+B_i} \le \sum_{j=1}^{j=3} c_j\mu^{\alpha_j}$$

which is $o(1)$ since $\alpha_j < 0$ for $j = 1, 2, 3$. This proves (18).

Proof of Step 1) of II :

Note that

$$\pi(\theta|\mathbf{x}) = \frac{\left(s^2+\frac{n\theta^2}{n-1}\right)^{-n/2}\left(\tau^2+\frac{(\theta-\mu)^2}{m-1}\right)^{-m/2}}{\int\left(s^2+\frac{n\theta^2}{n-1}\right)^{-n/2}\left(\tau^2+\frac{(\theta-\mu)^2}{m-1}\right)^{-m/2}d\theta}.$$

Now

$$\begin{aligned} &\int \frac{1}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta \ge \int_0^1 \frac{1}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta \\ \ge \quad & \frac{1}{(\tau^2+\frac{\mu^2}{m-1})^{m/2}}\int_0^1 \frac{1}{(s^2+\frac{n\theta^2}{n-1})^{n/2}}d\theta \ge \frac{k}{(\tau^2+\frac{\mu^2}{m-1})^{m/2}}. \end{aligned} \tag{31}$$

Also,

$$\begin{aligned}
&\left|\int_{I^c} \frac{\theta}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta\right| \\
=& \left|\left\{\int_{-\infty}^{-\mu^r}+\int_{\mu+\mu^r}^{\infty}\right\}\frac{\theta}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta\right| \\
\leq& \left|-\int_{\mu^r}^{\infty}\frac{\theta}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta\right|+\left|\int_{\mu+\mu^r}^{\infty}\frac{\theta}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta\right| \\
\leq& \frac{[(n-1)/n]^{n/2}}{(\tau^2+\frac{(\mu+\mu^r)^2}{m-1})^{m/2}}\frac{\mu^{r(-n+2)}}{n-2}+\frac{[(n-1)/n]^{n/2}}{(\tau^2+\frac{\mu^{2r}}{m-1})^{m/2}}\frac{(\mu+\mu^r)^{(-n+2)}}{n-2} \\
\leq& \frac{c_1}{(\tau^2+\frac{\mu^{2r}}{m-1})^{m/2}}\mu^{r(-n+2)} \qquad (32)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{I^c} \frac{\theta^2}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta \\
=& \left\{\int_{-\infty}^{-\mu^r}+\int_{\mu+\mu^r}^{\infty}\right\}\frac{\theta^2}{(s^2+\frac{n\theta^2}{n-1})^{n/2}(\tau^2+\frac{(\theta-\mu)^2}{m-1})^{m/2}}d\theta \\
\leq& \frac{[(n-1)/n]^{n/2}}{(\tau^2+\frac{(\mu+\mu^r)^2}{m-1})^{m/2}}\int_{-\infty}^{-\mu^r}\theta^{-n+2}d\theta+\frac{[(n-1)/n]^{n/2}}{(\tau^2+\frac{\mu^{2r}}{m-1})^{m/2}}\int_{\mu+\mu^r}^{\infty}\theta^{-n+2}d\theta \\
\leq& \frac{c_2}{(\tau^2+\frac{\mu^{2r}}{m-1})^{m/2}}\mu^{r(-n+3)}. \qquad (33)
\end{aligned}$$

Combining (31), (32) and (33) yields

$$\left|\int_{I^c}\theta^i\pi(\theta|\mathbf{x})d\theta\right| \leq k_i\frac{(\tau^2+\frac{\mu^2}{m-1})^{m/2}}{(\tau^2+\frac{\mu^{2r}}{m-1})^{m/2}}\mu^{r(-n+i+1)}, \quad \text{for } i=1,2.$$

But $\left((\tau^2+\frac{\mu^2}{m-1})/(\tau^2+\frac{\mu^{2r}}{m-1})\right)^{m/2} \leq \mu^{m(1-r)} = \mu$, since $\mu \geq 1$ and $r<1$. This proves that

$$\left|\int_{I^c}\theta^i\pi(\theta|\mathbf{x})d\theta\right| \leq k_i\mu^{r(-n+i+1)+1}, \quad \text{for } i=1,2.$$

And, by the fact that $0 \leq \delta^\pi(\mathbf{x}) \leq \mu$.

Proof of Step 2) of II :

By Step 1) of II,

$$\int_{I^c}\theta^2\pi(\theta|\mathbf{x})d\theta \leq k_2|\mu|^{l_2},$$

where $l_2 = r(-n+3)+1 < 1$ since $r > 0$ and $n > 3$. Also,

$$
\begin{aligned}
& | \quad [\int_I \theta\pi(\theta|\mathbf{x})d\theta]^2 - (\delta^\pi(\mathbf{x}))^2 \, | \\
= & \; | \int_I \theta\pi(\theta|\mathbf{x})d\theta - \int \theta\pi(\theta|\mathbf{x})d\theta \, | \; | \int_I \theta\pi(\theta|\mathbf{x})d\theta + \int \theta\pi(\theta|\mathbf{x})d\theta \, | \\
= & \; | \int_{I^c} \theta\pi(\theta|\mathbf{x})d\theta \, | \; | 2\int \theta\pi(\theta|\mathbf{x})d\theta - \int_{I^c} \theta\pi(\theta|\mathbf{x})d\theta \, | \\
\leq & \; | \int_{I^c} \theta\pi(\theta|\mathbf{x})d\theta \, | \; |2\delta^\pi(\mathbf{x})\, | + | \int_{I^c} \theta\pi(\theta|\mathbf{x})d\theta \, |^2 \\
\leq & \; 2k_1\mu^{r(-n+2)+1}\mu + k_1^2\mu^{2r(-n+2)+2} \qquad \text{by Step 1) of II,} \\
\leq & \; p_1|\mu|^{l_1}
\end{aligned}
$$

where $l_1 = r(-n+2)+2 < 2$ since $r > 0$ and $n > 3$.

Proof of Step 3) of II :

$$
\begin{aligned}
& \frac{\int_{I^c} \left(s^2 + \frac{n\theta^2}{n-1}\right)^{-n/2} \left(\tau^2 + \frac{(\theta-\mu)^2}{m-1}\right)^{-m/2} d\theta}{\int_I \left(s^2 + \frac{n\theta^2}{n-1}\right)^{-n/2} \left(\tau^2 + \frac{(\theta-\mu)^2}{m-1}\right)^{m/2} d\theta} \\
= & \; \frac{\left\{\int_{-\infty}^{-\mu^r} + \int_{\mu+\mu^r}^{\infty}\right\} \left(s^2 + \frac{n\theta^2}{n-1}\right)^{-n/2} \left(\tau^2 + \frac{(\theta-\mu)^2}{m-1}\right)^{-m/2} d\theta}{\int_{-\mu^r}^{\mu+\mu^r} \left(s^2 + \frac{n\theta^2}{n-1}\right)^{-n/2} \left(\tau^2 + \frac{(\theta-\mu)^2}{m-1}\right)^{-m/2} d\theta} \\
\leq & \; \frac{\left\{\int_{-\infty}^{-\mu^r} + \int_{\mu+\mu^r}^{\infty}\right\} \left(s^2 + \frac{n\theta^2}{n-1}\right)^{-n/2} \left(\tau^2 + \frac{(\theta-\mu)^2}{m-1}\right)^{-m/2} d\theta}{\int_0^1 \left(s^2 + \frac{n\theta^2}{n-1}\right)^{-n/2} \left(\tau^2 + \frac{(\theta-\mu)^2}{m-1}\right)^{-m/2} d\theta} \\
\leq & \; p_0 \frac{(\tau^2 + \frac{\mu^2}{m-1})^{m/2}}{(\tau^2 + \frac{\mu^{2r}}{m-1})^{m/2}} \mu^{r(-n+1)} \\
\leq & \; p_0\mu^l,
\end{aligned}
$$

where $l = r(-n+1)+1$. For $n > 3$, note that $l < (1-\frac{1}{m})(-3+1)+1 = -1+2/m \leq 0$.

Hence,

$$
\frac{\int_{I^c} \pi(\theta)l_{\mathbf{x}}(\theta)d\theta}{\int_I \pi(\theta)l_{\mathbf{x}}(\theta)d\theta} < p_0\mu^l, \quad \text{where } l < 0.
$$

This implies that, for any $\epsilon > 0$, there exists an $M > 0$ such that, when $|\mu| \geq M$,

$$
\frac{\int_{I^c} \pi(\theta)l_{\mathbf{x}}(\theta)d\theta}{\int_I \pi(\theta)l_{\mathbf{x}}(\theta)d\theta} < \epsilon.
$$

Thus, when $|\mu| \geq M$,

$$(1-\epsilon)\int \pi(\theta)l_{\mathbf{x}}(\theta)d\theta < \int_I \pi(\theta)l_{\mathbf{x}}(\theta)d\theta \leq \int \pi(\theta)l_{\mathbf{x}}(\theta)d\theta.$$

But

$$\int \theta^2\pi(\theta|\mathbf{x})d\theta - \int \theta^2\pi^\star(\theta|\mathbf{x})d\theta = \frac{\int \theta^2\pi(\theta)l_{\mathbf{x}}(\theta)d\theta}{\int \pi(\theta)l_{\mathbf{x}}(\theta)d\theta} - \frac{\int_I \theta^2\pi(\theta)l_{\mathbf{x}}(\theta)d\theta}{\int_I \pi(\theta)l_{\mathbf{x}}(\theta)d\theta},$$

so that, for $|\mu| \geq M$,

$$\begin{aligned}
\int \theta^2\pi(\theta)l_{\mathbf{x}}(\theta)d\theta - \frac{\int_I \theta^2\pi(\theta|\mathbf{x})d\theta}{1-\epsilon} &\leq \int \theta^2\pi(\theta)l_{\mathbf{x}}(\theta)d\theta - \int_I \theta^2\pi^\star(\theta|\mathbf{x})d\theta \\
&\leq \int \theta^2\pi(\theta)l_{\mathbf{x}}(\theta)d\theta - \int_I \theta^2\pi(\theta|\mathbf{x})d\theta.
\end{aligned}$$

Since ϵ was arbitrary, it follows that

$$\begin{aligned}
\lim_{|\mu|\to\infty} \left|\int \theta^2\pi(\theta|\mathbf{x})d\theta - \int \theta^2\pi^\star(\theta|\mathbf{x})d\theta\right| &= \lim_{|\mu|\to\infty} \left|\int \theta^2\pi(\theta|\mathbf{x})d\theta - \int_I \theta^2\pi(\theta|\mathbf{x})d\theta\right| \\
&= \lim_{|\mu|\to\infty} \left|\int_{I^c} \theta^2\pi(\theta|\mathbf{x})d\theta\right|.
\end{aligned}$$

Similarly,

$$[\int_I \theta\pi(\theta|\mathbf{x})d\theta]^2 - (\delta^\pi(\mathbf{x}))^2 \leq [\int \theta\pi^\star(\theta|\mathbf{x})d\theta]^2 - (\delta^\pi(\mathbf{x}))^2 \leq \frac{[\int_I \theta\pi(\theta|\mathbf{x})d\theta]^2}{(1-\epsilon)^2} - (\delta^\pi(\mathbf{x}))^2,$$

so that

$$\lim_{|\mu|\to\infty} \left|\, [\int \theta\pi^\star(\theta|\mathbf{x})d\theta\,]^2 - (\delta^\pi(\mathbf{x}))^2 \,\right| = \lim_{|\mu|\to\infty} \left|\, [\int \theta\pi(\theta|\mathbf{x})d\theta\,]^2 - (\delta^\pi(\mathbf{x}))^2 \,\right|.$$

Therefore,

$$\begin{aligned}
&\lim_{|\mu|\to\infty} \frac{|\, V^\pi(\mathbf{x}) - V^{\pi^\star}(\mathbf{x}) \,|}{\mu^2} \\
= &\lim_{|\mu|\to\infty} \frac{|\int \theta^2\pi(\theta|\mathbf{x})d\theta - (\delta^\pi(\mathbf{x}))^2 - \int \theta^2\pi^\star(\theta|\mathbf{x})d\theta + [\int \theta\pi^\star(\theta|\mathbf{x})d\theta]^2|}{\mu^2} \\
\leq &\lim_{|\mu|\to\infty} \left(\frac{|\int \theta^2\pi(\theta|\mathbf{x})d\theta - \int \theta^2\pi^\star(\theta|\mathbf{x})d\theta|}{\mu^2} + \frac{|[\int \theta\pi^\star(\theta|\mathbf{x})d\theta]^2 - (\delta^\pi(\mathbf{x}))^2|}{\mu^2}\right) \\
= &\lim_{|\mu|\to\infty} \left(\frac{\int_{I^c} \theta^2\pi(\theta|\mathbf{x})d\theta}{\mu^2} + \frac{|\,[\int_I \theta^2\pi(\theta|\mathbf{x})d\theta\,]^2 - (\delta^\pi(\mathbf{x}))^2\,|}{\mu^2}\right) \\
= &\ 0.
\end{aligned}$$

Proof of Step 4) of II :

Let ξ be any probability measure on $[0, v]$, and note that $\int_0^v x(v-x)d\xi(x) \geq 0$. Hence

$$\int_0^v x^2\, d\xi(x) \leq v \int_0^v x\, d\xi(x),$$

and

$$Variance = \int_0^v x^2\, d\xi(x) - \left(\int_0^v x\, d\xi(x)\right)^2 \leq v \int_0^v x\, d\xi(x) - \left(\int_0^v x\, d\xi(x)\right)^2 = vy - y^2,$$

where $y = \int_0^v x\, d\xi(x)$. Clearly $0 \leq y \leq v$, so that $vy - y^2$ is maximized at $y = v/2$, proving the result. $\square$