MINIMAX ESTIMATION OF LOCATION VECTORS IN ELLIPTICAL DISTRIBUTIONS WITH UNKNOWN SCALE PARAMETER[1]

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Abstract

This note gives some generalizations of Berger (1975) concerning minimax estimation of location vectors for nonnormal families of distributions. An unknown scale parameter is involved here and the estimator of it is distributed either as a linear combination of independent chi-squares or as some other random variables in a certain class. Consequently minimax estimators for the location vectors are generalized in the case when the dispersion matrix is only known up to a multiplicative constant.

Key words: Elliptical Distributions, Minimax Estimation.

1. Introduction

This note gives extensions of some results in Berger (1975) in which minimax estimation of location vectors for densities of the form $f((x-\theta)'\Sigma^{-1}(x-\theta))$ with $\Sigma_{p\times p}$ known and $p \geq 3$ was considered and a characterization of such densities was also given.

The model that we use is slightly generalized in that it involves an unknown scale parameter τ^2, specifically, the densities of the random vector $z_{p\times 1}$ is of the form

$$f(z|\theta) = \int_0^\infty \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi\tau^2\nu)^{\frac{1}{2}}} e^{-\frac{(z-\theta)'\Sigma^{-1}(z-\theta)}{2\tau^2\nu}} dF(\nu)$$

where $\theta_{p\times 1}$ and $\tau^2 > 0$ are unknown parameters and Σ is a known positive matrix and $F(\nu)$ is a known c.d.f. on $(0,\infty)$. This includes, for example, the usual normal density, the ϵ-contaminated Normal density and the p-variate elliptical t-distribution (which itself occurs in regression with random regressors) etc. See Muirhead (1982).

In section 2 and 3, generalizations of Theorem 1 and 3 of Berger (1975) are obtained via estimating the unknown scale parameter τ^2 by a random variable w which is distributed as a linear combination of independent chi-squares or other variables in a certain class. Such an estimate of the (variance) τ^2 arises in a number of practical situations, for example, the estimation of the variance of some random effects in a mixed-model, say, a balanced one-way ANOVA,

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \qquad (j = 1, \ldots, J; i = 1, \ldots, I)$$

where $\epsilon_{ij} \sim$ i.i.d. $N(0,\sigma^2), \alpha_i \sim$ i.i.d. $N(0,\tau^2)$. The UMVU estimation $\hat{\tau}^2$ of τ^2 is therefore

$$\hat{\tau}^2 = \frac{1}{J}\left(\frac{S_\alpha^2}{I-1} - \frac{S^2}{I(J-1)}\right)$$

where $S_\alpha^2 = J\Sigma(Y_{i.} - Y_{..})^2, S^2 = \Sigma(Y_{ij} - Y_{i.})^2$. It is clear that $\hat{\tau}^2$ is a linear combination of two independent chi-squares. Another situation that may occur is where we have

$$Z_{ij}(i = 1, \ldots, I; j = 1, \ldots, J) \stackrel{\text{i.i.d.}}{\sim} N(\xi_i, \sigma^2)$$

and we want to estimate σ^2 when there is some prior information available about (ξ_i, σ^2), say, as in Lehmann (1983) the prior assigns $\frac{1}{2\sigma^2}$ the distribution $\Gamma(\alpha, \frac{1}{\lambda})$ (the gamma density) and takes $\xi_1, \ldots, \xi_I$ to be independent of $\frac{1}{2\sigma^2}$ with uniform noninformative prior. So the Bayes estimators for σ^2 for each group (each i) is

$$\hat{\sigma}_i^2 = \frac{\lambda + S_i}{J + 2\alpha - 3}$$

where $S_i = \sum_{j=1}^{J} (Z_{ij} - \overline{Z}_{i.})^2$. A reasonable estimator of σ^2 is the linear combinations of $\hat{\sigma}_i^2$, $i = 1, \ldots, I$.

2. Minimax Estimators

Suppose $f(z|\theta)$ is of the form in (1.1) and we want to estimate θ. Since τ^2 is unknown, we suppose it is estimated by w and $\tau^{-2}w$ is distributed as a linear combination of two independent chi-square r.v.s. Consider the estimation of θ under the loss

$$L(\delta; \theta, \tau^2) = \frac{(\delta - \theta)' Q (\delta - \theta)}{\tau^2} \tag{2.1}$$

where $Q_{p\times p}$ is a known positive definite matrix. When $p \geq 3$, we can take advantage of the Stein effect and obtain a class of estimators which improve upon z in risk, and consequently those estimators are all minimax estimator of θ in that z remains to be minimax in the present problem.

In the known $-\tau^2$ case, Berger's Σ corresponds to our $\tau^2\Sigma$, and Q to our $\tau^2 Q$. Thus, the class of estimators shown by Berger to dominate z in the known $-\tau^2$ case is of the form

$$(I_p - h(\frac{z'\Sigma^{-1}Q^{-1}\Sigma^{-1}z}{\tau^2})Q^{-1}\Delta^{-1})z. \tag{2.2}$$

Since τ^2 is unknown, we replace τ^2 by w, and consider estimators of the form

$$\delta_h(z,w) = (I_p - h(\frac{z'\Delta^{-1}Q^{-1}\Delta^{-1}z}{w})Q^{-1}\Sigma^{-1})z, \tag{2.3}$$

where $\tau^{-2}w \underline{\overset{\text{dist.}}{}} c_1 w_1 + c_2 w_2$ with $w_1 \sim \chi^2_m$ and $w_2 \sim \chi^2_n$, in which it is assumed $c_1 > 0$ (without loss of generality) and χ^2_l is the chi-square r.v. with the degree of freedom l, and the function $h(t)$ is a function from $[0,\infty)$ to $[0,\infty)$ and satisfies the following two requirements:

(i) $h(u)$ is nonincreasing in $u \geq 0$

(2.4)

(ii) $r(u) = uh(u)$ is nondecreasing in $u \geq 0$.

<u>Theorem 2.1.</u> Suppose that $Ez'z$ and $E(z'z)^{-1}$ are finite when $\theta = 0$ and $\tau^2 = 1$. w and $\delta_h(z,w)$ are defined in (2.3). Then $\delta_h(z,w)$ dominates z in risk (and is hence minimax estimator of θ) provided that

$$0 \leq r_1 = \sup_{t\geq 0} r(t) \leq \frac{2}{f E_{\theta=0,\tau^2=1}(z'\Sigma^{-1}z)^{-1}} \tag{2.5}$$

where

$$f = \begin{cases} c_1 m + c_2 n + 2\max(c_1,c_2) & \text{if } c_2 \geq 0 \\ c_1 m + c_2 n + \frac{4c_1^2 - c_2 n}{2c_1} & \text{if } c_2 \leq 0. \end{cases}$$

Before giving the proof, a repeatedly used integration by parts formula is given as follows.

Lemma 2.1. Let $w \sim \tau^2\chi_l^2$. If $g(w)$ is sufficiently regular in w for integration by parts, and if $Ewg(w)$ exists, then

$$Ewg(w) = l\tau^2 Eg(w) + 2\tau^2 Ewg'(w).$$

PROOF OF THEOREM 2.1. Note that

$$\delta_h(\tau^{-1}z, \tau^{-2}w) = \tau\delta_h(z, w)$$

so that for all h (including $h = 0$, yielding $\delta_0(z, w) = z$)

$$L(\delta_h(z, w); \theta, \tau^2) = L(\delta_h(\tau^{-1}z, \tau^{-2}w); \tau^{-1}\theta, 1).$$

That is, the estimation problem for the class of estimators $\delta_h(z, w)$ is invariant under the transformation $z \to \tau^{-1}z, \theta \to \tau^{-1}\theta$. Consequently, we can assume without loss of generality that $\tau^2 = 1$.

As in Berger (1975), let $A_1 \geq A_2 \geq \ldots \geq A_p \geq 0$ be the eigenvalues of ΔQ^{-1}. Then there exists a nonsingular matrix B such that $B'QB = I_p, B'\Sigma^{-1}B = A^{-1}$, where

$$A = \text{ diag } (A_1, \ldots, A_p).$$

Transforming

$$z \to B^{-1}z, \theta \to B^{-1}\theta,$$

yields a "canonical" estimation problem in which the distribution of z has parameter $\Sigma = A$, the loss function in (3) has centering matrix $Q = I_p$, and the estimators $\delta_h(z, w)$ have the form

$$\delta_h(z, w) = (I_p - h(\frac{z'A^{-2}z}{w})A^{-1})z \tag{2.6}$$

Let $\Delta(\theta)$ be the difference in risks between z and $\delta_h(z,w)$, then

$$\Delta(\theta) = E_\theta[(z-\theta)'(z-\theta) - (\delta_h(z,w)-\theta)'(\delta_h(z,w)-\theta)].$$

Note that τ^2 was already assumed to be 1. And we need to show $\Delta(\theta) \geq 0$ for all θ.

Since w and z are independent, the expectations

$$E_\cdot^{(wz)} = E^w E_\cdot^{z|w} = E^w E^z.$$

Let $h_w(t) = h(\frac{t}{w})$. It is clear that $h_w(t)$ (as a function of t) satifies the two requirements in (2.4), that is, $h_w(t)$ nonincreasing in $t \in [0,\infty)$, and that $r_w(t) = th_w(t)$ is nondecreasing in $t \in [0,\infty)$. Since $r_w(t) = wr(\frac{t}{w})$, if $r_1 = \sup_{t\geq 0} r(t)$, then

$$\sup_{t\geq 0} r_w(t) = r_1 w.$$

Now the estimator is

$$\delta_h(z,w) = (I_p - h_w(z'A^{-2}z)A^{-1})z.$$

Therefore we follow the steps on pp. 1320–1322 in Berger (1975) to obtain

$$\Delta(\theta) \geq E^w\left(\int_0^\infty (2(p-2) - \frac{r_1 w}{v})dF(v)\right)T_\theta(w),$$

where

$$T_\theta(w) = \int_0^\infty \int_{R^p} \frac{v^{-\frac{p-2}{2}} h(\frac{z'A^{-2}z}{2}) \exp\{-\frac{1}{2\nu}(z-\theta)'A^{-1}(z-\theta)\}}{(2\pi)|A|^{\frac{1}{2}}} dz dF(v). \tag{2.7}$$

Let $v_1 = \int_0^\infty \frac{1}{\nu} dF(\nu) = E\frac{1}{v}$, then

$$\Delta(\theta) \geq 2(p-2)ET_\theta(w) - r_1 v_1 E w T_\theta(w) \tag{2.8}$$

where the expectation is taken over w. In the following we'll use Lemma 1 for w_1 and w_2. Note $w = c_1 w_1 + c_2 w_2$, and also the subscript θ in $T_\theta(w)$ is suppressed, i.e., $T_\theta(w) = T(w)$, in the following calculations. Thus

$$
\begin{aligned}
EwT(w) &= c_1 Ew_1T(w) + c_2 Ew_2T(w) \\
&= c_1 E^{w_2} E^{w_1} w_1 T(w) + c_2 E^{w_1} E^{w_2} w_2 T(w) \\
&= c_1 E^{w_2}\{mE^{w_1}T(w) + 2c_1 E^{w_1} w_1 T'(w)\} \\
&+ c_2 E^{w_1}\{nE^{w_2}T(w) + 2c_2 E^{w_2} w_2 T'(w)\} \\
&= (c_1 m + c_2 n)ET(w) + 2c_1^2 Ew_1T'(w) + 2c_2^2 Ew_2T'(w) \\
&= (c_1 m + c_2 n)ET(w) + 2c_1 EwT'(w) + 2(c_2 - c_1)c_2 Ew_2T'(w) \qquad (2.9) \\
&= (c_1 m + c_2 n)ET(w) + 2c_2 EwT'(w) + w(c_1 - c_2)c_1 Ew_1T'(w) \qquad (2.10)
\end{aligned}
$$

Note in fact $T(w) = E^v E^{z|v} v \frac{v(\frac{z'A^{-2}z}{w})}{\frac{z'A^{-2}z}{w}}$ where v has *c.d.f.* $F(v)$, and $z|v \sim N(\theta, v^{-1}A)$. Since $z|v$ has normal density, the derivative can be taken inside the integral. So

$$T'(w) = E^\nu E^{z|\nu} \frac{\partial}{\partial w}\left(\frac{r(\frac{z'A^{-2}z}{w})}{\frac{z'A^{-2}z}{w}}\right).$$

But

$$\frac{\partial}{\partial w}\left[\frac{wr(\frac{z'A^{-2}z}{w})}{z'A^{-2}z}\right] = \frac{r(z'A^{-2}z)}{z'A^{-2}z} - \frac{1}{w}r'(\frac{z'A^{-2}z}{w}).$$

So

$$wT'(w) = h(\frac{z'A^{-2}z}{w}) - E^v E^{z|\nu} vr'\left(\frac{z'A^{-2}z}{w}\right)$$

$$EwT'(w) = ET(w) - Er'(\cdot) \qquad (2.11)$$

Also note $c_1 > 0$. There are two cases to consider.

Case (i) $c_2 > 0$. Moreover if $c_2 < c_1$, by (2.9)

$$2(c_2 - c_1)c_2 E w_2 T'(w) < 0.$$

So

$$EwT(w) \leq (c_1 m + c_2 n)ET(w) + 2c_1 EwT'(w).$$

But if $c_2 > c_1$, by (2.10)

$$2(c_1 - c_2)c_1 E w_1 T'(w) < 0$$

so $EwT(w) \leq (c_1 m + c_2 n)ET(w) + 2c_2 EwT'(w)$. Thus in case (i) when $c_2 > 0$,

$$EwT(w) \leq (c_1 m + c_2 n)ET(w) + 2\max(c_1 c_2) EwT'(w).$$

Combining (2.9),

$$EwT(w) \leq (c_1 m + c_2 n)ET(w) + 2\max(c_1, c_2)ET(w) - 2\max(c_1 c_2)ET'(\cdot)$$

so

$$EwT(w) \leq (c_1 m + c_2 n + 2\max(c_1 c_2))ET_\theta(w).$$

By (2.8),

$$\Delta(\theta) \geq 2(p-2)ET_\theta(w) - r_1 v_1 (c_1 m + c_2 n + 2\max(c_1, c_2))ET_\theta(w)$$

$$= [2(p-2) - r_1 v_1 (c_1 m + c_2 n + 2\max(c_1, c_2))]ET_\theta(w)$$

Note the fact when $\theta = 0, v^{-1} z' A^{-1} z \sim \chi^2_p$, which is independent of v. Then

$$E_{\theta=0}(z' A^{-1} z)^{-1} = E \frac{1}{v} \frac{\frac{1}{z' A^{-1} z}}{v} = \frac{v_1}{p-2}$$

so

$$v_1 = (p-2)E_{\theta=0}(z' A^{-1} z)^{-1}.$$

Therefore the conditions in Theorem 1 yields

$$2(p-2) - r_1 v_1 (c_1 m + c_2 n + 2\max(c_1, c_2)) \geq 0$$

so

$$\Delta(\theta) \geq 0.$$

Case (ii). $c_2 < 0$.

By (2.9) and (2.11)

$$\begin{aligned} EwT(w) &= (c_1 m + c_2 n) ET(w) + 2c_1 EwT'(w) + 2(c_2 - c_1) c_2 E w_2 T'(w) \\ &= (c_1 m + c_2 n) ET(w) + 2c_1 ET(w) - 2c_1 E^{\nu} E^{z|\nu} u r'(\cdot) \\ &\quad + 2(c_2 - c_1) c_2 E w_2 T'(w) \end{aligned}$$

Thus

$$EwT(w) \leq (c_1 m + c_2 n) ET(w) + 2c_1 ET(w) + 2(c_2 - c_1) c_2 E w_2 T'(w). \tag{2.12}$$

But

$$E w_2 T'(w) = E^{w_2} w_2 E^{w_1} T'(w) \tag{2.13}$$

while integrations by parts gives

$$E^{w_1} T'(w) = \frac{1}{c_1} \left(f_{\chi^2_m}(w_1) T(w) |_0^{\infty} - \int_0^{\infty} T(w) d f_{\chi^2_m}(w_1) \right) \tag{2.14}$$

where

$$f_{\chi^2_m}(w_1) = \frac{w_1^{\frac{m}{2}-1} e^{-\frac{w_1}{2}}}{\Gamma(\frac{m}{2}) 2^{\frac{m}{2}}},$$

the density of χ^2_m. Calculus gives

$$f'_{\chi^2_m}(w_1) = -\frac{1}{2} f_{\chi^2_m}(w_1) + \frac{1}{2} f_{\chi^2_{m-2}}(w_1) \tag{2.15}$$

Also note $\lim_{w_2 \to \infty} T(w) f_{\chi^2_m}(w_1) = 0$. This is because that $\lim_{w_1 \to \infty} f_{\chi^2_m}(w_1) = 0$, and $h(t)$ is nonincreasing, $h_w(z'A^{-1}z) = \frac{v(\frac{z'A^{-1}z}{w})}{\frac{z'A^{-1}z}{w}}$ is nondecreasing in w_1, Monotone Convergence Theorem implies

$$\lim_{w_1 \to \infty} T(w) = E^v E^{z|v}_{\lim_{w_1 \to \infty}} v \frac{v(\frac{z'A^{-1}z}{w})}{\frac{z'A^{-1}z}{w}} = E^v E^{z|v} v h(0) < \infty.$$

Further $\lim_{w_1 \to 0} f_{\chi^2_m}(w_1) T(w) = \lim_{w_1 \to 0} f_{\chi^2_m}(w_1) T(c_2 w_2) = 0$. So by (2.14)

$$E^{w_1} T'(w) = -\frac{1}{4} \int_0^\infty T(w) d f_{\chi^2_m}(w_1) \tag{2.16}$$

and combining this with (2.15) and (2.13),

$$E w_2 T'(w) \leq \frac{1}{2c_1} E w_2 T(w) - \frac{1}{2c_1} E^{w_1 = \chi^2_{m-2}} E^{w_2} w_2 T(w).$$

So

$$E w_2 T'(w) \leq \frac{1}{2c_1} E w_2 T(w).$$

Lemma 1 again gives

$$E w_2 T'(w) \leq \frac{1}{2c_1} (n E T(w) + 2 c_2 E w_2 T'(w)) \tag{2.17}$$

i.e.

$$2(c_1 - c_2) E w_2 T'(w) \leq \frac{n}{2c_1} E T(w)$$

i.e.

$$E w_2 T'(w) \leq \frac{n}{4 c_1 (c_1 - c_2)} E T(w).$$

Combining this with (2.12),

$$EwT(w) \leq (c_1m + c_2n + 2c_1)ET(w) + w(c_2 - c_1)c_2Ew_2T'(w)$$
$$\leq (c_1m + c_2n + 2c_1)ET(w) + \frac{2(c_2 - c_1)c_2 \cdot n}{4c_1(c_1 - c_2)}ET(w)$$
$$= (c_1m + c_2n + 2c_1 + \frac{(-c_2)n}{2c_1})ET(w).$$

By (2.13),

$$\Delta(\theta) \geq [2(p-2) - r_1v_1(c_1m + c_2n + \frac{4c_1^2 - c_2n}{2c_1})]ET(w)$$

(2.5) then implies

$$\Delta(\theta) \geq 0.$$ Q.E.D.

3. **Other distributions**

For a generalization of Theorem 3 of Berger (1975), consider

$$f(z|\theta) = f(\frac{(z-\theta)'\Sigma^{-1}(z-\theta)}{\tau^2}).$$

The same reduction leads to estimate θ using estimators of the form (2.6).

Theorem 3.1. Let $f((z-\theta)'\Sigma^{-1}(z-\theta)/\tau^2)$ be a density, with respect to Lebesque measure, satisfying the following four conditions:

(i) $E_{\theta=0}z'z < \infty \quad E_{\theta=0}(z'z)^{-2} < \infty$

(ii) The Lebesque measure of all points $\in (0,\infty)$ such that $f(\cdot)$ discontinuous is 0.

(iii) $c = \inf_{s\in U} \frac{\int_s^\infty f(v)du}{f(s)} > 0$, where $U = \{s \notin W : f(s) > 0\}$.

(iv) Regularity conditions for $\frac{\partial}{\partial w}R(w) = \int \frac{\partial}{\partial w}h(\frac{z'A^{-2}z}{2})f(s)dz$, where $s = (z-\theta)'A^{-1}(z-\theta)$.

Define $\delta_h(z,w)$ by (2.3) where $v(\cdot)$ is nondecreasing and $0 \leq v \leq (p-2)\frac{c}{b}$ with

$$b = \begin{cases} c_1m + c_2n + 2\max(c_1c_2) & \text{if } c_2 \geq 0 \\ c_1m + c_2n + \frac{4c_1^2 - c_2n}{2c_1} & \text{if } c_2 \leq 0 \end{cases}$$

Then δ_h is a minimax estimation of θ under loss (3).

PROOF: $\Delta(\theta) = E(|z-\theta|^2 - |\delta_h(zw)-\theta|^2)$. Still $h_w(t) = \frac{r(\frac{t}{w})}{\frac{t}{w}} v_w(t) = th_w(t)$. Follow the steps on pages 1323–1324 in Berger (1975)

$$\Delta_\delta(\theta) \geq E\int h(\frac{z'A^{-2}z}{w})\left[(p-2)\int_{(z-\theta)'A^{-1}(z-\theta)} f(u)du - wr(\frac{z'A^{-2}z}{w})f(z'-\theta)'A^{-1}(z-\theta))\right] dz$$

$$\Delta_\delta(\theta) \geq (p-2)E\int h(\frac{z'A^{-2}z}{w})(\int_s f(u)du)dz - (p-2)\frac{c}{b}E^w wR_\theta(w)$$

where

$$s = (z-\theta)'A^{-1}(z-\theta)$$

$$R_\theta(w) = \int_{R^p} H(\frac{z'A^{-2}z}{w})f(s)dz$$

Note the proof in Theorem 2.1 can be applied to this case directly. So we have

$$EwT(w) \leq bER_\theta(w)$$

where

$$b = \begin{cases} c_1m + c_2n + 2\max(c_1c_2) & \text{if } c_2 \geq 0 \\ c_1m + c_2n + \frac{4c_1^2 - c_2n}{2c_1} & \text{if } c_2 \leq 0 \end{cases}$$

so

$$\Delta(\theta) \geq (p-2)E^w \int h(\frac{z'A^{-2}z}{w})[\int_s f(u)d\nu - \frac{c}{b}f(s)]dz.$$

Condition (iii) clearly implies $\int_s f(u)du - \frac{c}{b}f(s) \geq 0$ so

$$\Delta(\theta) \geq 0. \qquad \text{Q.E.D.}$$

4. **Remarks.** The point to make all the proofs through is that the distribution of w allows certain kind of integration by parts. Let $\mathcal{E}_0$ be the set of all random variables such that

$$E(w-\mu)g(w) = Ea(w)g'(w)$$

where $a(w) \geq 0, \mu = E_\xi w$, as is considered in Hudson (1978), then we can expect the same result for the minimax estimation of θ, since as long as we can obtain an upper bound of $EwT(w)$ of the form:

$$EwT(w) \leq b_o ET(w).$$

Then if $\sup r(t) \leq \frac{2}{b_o E_{\theta=0,\tau^2=1} z'\Sigma^{-1}z}$, then the estimator defined in (2.4) would dominate z in risks. Note our previous $w \underline{\underline{\text{dist}}} c_1\chi^2_m + c_2\chi^2_n$ is not in $\mathcal{E}_o$. The distribution of $c_1\chi^2_m + c_2\chi^2_n$ is considered in Bock and Soloman (1987). As an example, let $w = \frac{1}{w_1}$ and $w_1 \sim \frac{1}{(n-2)n} X^2_n$. So $f_\xi(w) \propto e^{-(\frac{n}{2}-1)\mu w^{-1}} \frac{1}{w^2} e^{-(\frac{n}{2}-1)\log w} = w^{-\frac{n}{2}-1} e^{-(\frac{n}{2}-1)\mu\frac{1}{w}}$ with $\mu = E_\xi W$ and $a(w) = \frac{w^2}{\frac{n}{2}-1}, w > 0$. Then

$$EwT(w) = \int\int\int w^{-\frac{n}{2}-1} e^{-(\frac{n}{2}-1)\mu\frac{1}{w}} h(\frac{z'A^{-1}z}{w}) e^{-\frac{1}{2}(z-\theta)'\frac{A^{-1}}{v}(z-\theta)} c \cdot dwdzdF(v).$$

By Fubini Theorem (Note integrand ≥ 0) and set $\frac{1}{w} = y$

$$EwT(w) = \int\int\int y^{\frac{n}{2}+3} e^{-(\frac{n}{2}-1)\mu y} h(yz'A^{-1}z) e^{-\frac{1}{2}(z-\theta)'\frac{A^{-1}}{v}(z-\theta)} dydzdF(v),$$

which is finite for most choices of h. Hence (Hudson, 1978, (2.3))

$$\begin{aligned} E_\xi wT(w) &= \mu ET(w) + Ea(w)T'(w) \\ &= \mu E(w) + \frac{1}{\frac{n}{2}-1} Ew^2T'(w) \end{aligned}$$

By (2.11) (Here assume derivative can be taken inside the integers sign)

$$Ew^2T'(w) = EwT(w) - Ewr'(\cdot).$$

So

$$EwT(w) = \mu ET(w) + \frac{1}{\frac{n}{2}-1} EwT(w) - \frac{1}{\frac{n}{2}-1} Ewr'(\cdot).$$

Then

$$EwT(w) \leq \frac{\mu(\frac{n}{2}-1)}{\frac{n}{2}-2} ET(w).$$

So

$$\Delta(\theta) \geq 2(p-2)ET(w) - r_1 v_1 b_0 ET(w)$$

$$= ET(w)[2(p-2) - r_1 v_1 b_0] \geq 0$$

$$\text{provided } 0 \leq v_1 \leq \frac{2}{b_0 E_{\mu=0\,\tau^2=1}(z' A^{-1} z)^{-1}}$$

where $b_0 = \frac{\mu(\frac{n}{2}-1)}{\frac{n}{2}-2}$. So the estimators defined by (2.6) with $r_1 = \sup r(A)$ satisfying the above condition would dominate the usual estimator z in risks and hence are minimax estimators.

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