RANGES OF POSTERIOR MEANS FOR SOME CLASSES OF NORMAL HIERARCHICAL PRIORS*

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Abstract

We consider the problem of robustness in hierarchical Bayes models. We consider a random vector $\mathbf{X} = (X_1, \ldots, X_p)^t$, the X_i being independently distributed $N(\theta_i, \sigma^2)$ (σ^2 known), while the θ_i are thought to be exchangeable, modelled as i.i.d. $N(\mu, \tau^2)$. The hyperparameter μ is given a noninformative prior distribution $\sqcap(\mu) \equiv 1$ and τ^2 is assumed to be independent of μ having a distribution $G(\tau^2)$ lying in a certain class of distributions $\mathcal{G}$. For several $\mathcal{G}$, including ε–contamination classes and density ratio classes we determine the range of the posterior mean of θ_i as G ranges over $\mathcal{G}$.

Key Words and Phrases: Bayesian Robustness, Classes of Priors, Hierarchical Model, Normal Means.

1980 AMS Subject Classification: Primary 62A15; Secondary 62F15.

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1. INTRODUCTION

1.1 The problem

In order to perform a Bayesian Analysis specification of a prior distribution is needed, but sometimes people can not specify more than a class of priors arising the problem of robustness with respect to this class. Roughly speaking, a class of priors will be robust when Bayesian Analysis for each prior in the class leads to similar results. The problem of robustness with respect to the prior elicitation process have been extensively treated by a lot of statisticians, for a large review of the subject see Berger (1987). Some discussion of the different approaches to the selection of a suitable $\mathcal{G}$ can be also seen in Berger and Berliner (1986), Moreno and Cano (1988), Moreno and Pericchi (1988) and Walley and Pericchi (1988). However, just a few has been done with respect to robustness in the hierarchical Bayes scenario, we can mention Berger and Berliner (1986) and Moreno and Pericchi (1990); the first paper is mainly related with choosing the type II maximum likelihood prior from a class of hierarchical priors while the second one is related to Normal Testing Hypothesis and Likelihood Sets and the uncertainty on the prior is modelled just by ε–contamination classes and in a different way than we do.

Hierarchical Bayes elicitation processes are very convenient because they provide a frame in which a prior can be picked up in several stages allowing the use of structural and subjective prior information and yielding to an elicited prior that would be very hard to be admitted from direct elicitation. In the hierarchical Bayes scenario we consider the Normal Case of Exchangeable Means. We have independent random variables $X_i, i = 1, \ldots, p$; distributed as $N(\theta_i, \sigma^2)$ (σ^2 known), the θ_i being random variables i.i.d. as $N(\mu, \tau^2)$ (this is our structural knowledge) and in a second stage we have to specify our subjective beliefs about the hyperparameters μ and τ^2. However, it is somewhat difficult to subjectively specify second stage prior, so we will specify some classes instead. Along this paper μ will be assumed to have a noninformative prior $\sqcap(\mu) \equiv 1$ whereas τ^2 will be supposed to have a density $G(\tau^2)$ belonging to some class $\mathcal{G}$, being our goal to find out robustness with respect to this class.

The choice $\sqcap(\mu) \equiv 1$ shrinks the posterior mean of θ_i to $\overline{x}$ and it is argued in Berger (1982) that it can be important to use subjective information $G(\tau^2)$ about the amount of shrinkage. One could similarly allow for a more general subjective prior on μ, but it seems to be somewhat less important than utilization of information about τ^2; anyhow the μ known case is handled in a similar way to the studied one yielding similar results and the same is expected in other intermediate cases.

One attractive property of the sort of estimators being used here, from a frequentist viewpoint, is that they are minimax, see Berger and Chen (1987) and Berger and Robert (1988).

1.2 Formulas and Notations

The posterior mean of θ_i will be noted as $\mu_i^G(\mathbf{x})$, being $\mathbf{x} = (x_1, \ldots, x_p)^t$ the observed random vector. Accordingly to Berger (1985) we have

$$\mu_i^G(\mathbf{x}) = x_i - E^{G(\tau^2/\mathbf{x})}\left[\frac{\sigma^2(x_i - \overline{x})}{\sigma^2 + \tau^2}\right], \tag{1.1}$$

where,

$$G(\tau^2/\mathbf{x})\alpha \, \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} G(\tau^2). \tag{1.2}$$

In (1.2) $\overline{x} = \left(\sum_{i=1}^{p} x_i\right)/p$ and $s^2 = \sum_{i=1}^{p}(x_i - \overline{x})^2$. In order to ensure that (1.2) defines a proper density we will suppose that $G(\tau^2)$ is bounded and $p \geq 4$ if we need it.

Let $m(\mathbf{x}/G)$ denote the marginal density in this hierarchical model. Standard calculations lead us to

$$m(\mathbf{x}/G) = \int m(\mathbf{x}/\tau^2) G(\tau^2) d\tau^2, \tag{1.3}$$

where $m(\mathbf{x}/\tau^2)$, the marginal density under the hierarchical prior $\theta_i \sim N(\mu, \tau^2)$ and $\sqcap(\mu) \equiv 1$ is given by

$$m(\mathbf{x}/\tau^2) = \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(2\sqcap)^{p/2}(\sigma^2+\tau^2)^{(p-1)/2}}. \tag{1.4}$$

For priors of the form $G = (1-\varepsilon)G_0 + \varepsilon q$ computations give

$$m(\mathbf{x}/G) = (1-\varepsilon) m(\mathbf{x}/G_0) + \varepsilon m(\mathbf{x}/q), \tag{1.5}$$

and

$$G(\tau^2/\mathbf{x}) = \lambda(\mathbf{x})G_0(\tau^2/\mathbf{x}) + (1-\lambda(\mathbf{x}))q(\tau^2/\mathbf{x}), \tag{1.6}$$

where

$$\lambda(\mathbf{x}) = \frac{(1-\varepsilon)m(\mathbf{x}/G_0)}{m(\mathbf{x}/G)}, \tag{1.7}$$

so the posterior mean is written as

$$\mu_i^G(\mathbf{x}) = \lambda(\mathbf{x})\mu_i^{G_0}(\mathbf{x}) + (1-\lambda(\mathbf{x}))\mu_i^q(\mathbf{x}). \tag{1.8}$$

2. ARBITRARY AND UNIMODAL DISTRIBUTIONS

In a first step one may often be interested in checking if there will be robustness with respect to some general classes of distributions. In this section we choose $\mathcal{G}$ to be the class $\mathcal{G}_A$ of all possible distributions and the class $\mathcal{G}_U$ of unimodal distributions.

The $\mathcal{G}_A$ class was used firstly by Edwards et al. (1963) and the important fact about it is that when robustness is present with respect to this class you should be very comfortable with your inference whatever the true prior is. The $\mathcal{G}_U$ class is a first step in modelling prior uncertainty.

2.1 The $\mathcal{G}_A$ case

By using (1.1) and (1.2) we have that

$$\mu_i^G(\mathbf{x}) = \left\{ x_i - \frac{\sigma^2(x_i-\overline{x}) \int_0^\infty \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} G(\tau^2)d\tau^2}{\int_0^\infty \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} G(\tau^2)d\tau^2} \right\}. \tag{2.1}$$

A standard result due to Sivaganesan and Berger (1987) yields

$$\sup_{G\in\mathcal{G}_A}[\inf]\mu_i^G(\mathbf{x}) = \sup_{\tau^2>0}[\inf]\left\{x_i - \frac{\sigma^2(x_i-\overline{x})}{\sigma^2+\tau^2}\right\}. \tag{2.2}$$

Now the problem is reduced to find out $\sup_{\tau^2>0}[\inf](\sigma^2+\tau^2)^{-1}$ that is $(\sigma^2)^{-1}[0]$. So the interval in which $\mu_i^G(\mathbf{x})$ is ranging is $(\overline{x}, x_i)$ or $(x_i, \overline{x})$ depending on the sign of $(x_i - \overline{x})$.

As a specific example, let $p = 7$, $\sigma^2 = 100$, $\overline{x} = 121$, $s^2 = 762$ and $x_7 = 115$. In this case $\mu_7^G(\mathbf{x})$ will be in the interval (115,121).

2.2 The $\mathcal{G}_U$ case

In this situation we choose $\mathcal{G}$ to be the class

$$\mathcal{G}_U = \left\{ \begin{array}{l} \text{all unimodal distributions with} \\ \text{a fixed mode } \tau_0^2 > 0. \end{array} \right\}. \tag{2.1}$$

Each G in $\mathcal{G}_U$ can be represented as a mixture of uniform densities in the following way

$$G(\tau^2) = \int_0^{\tau^2} \frac{1}{\tau_0^2 - t}\, dF(t), \quad \text{if } \tau^2 < \tau_0^2$$

$$G(\tau^2) = \int_{\tau^2}^{\infty} \frac{1}{t - \tau_0^2}\, dF(t), \quad \text{if } \tau^2 > \tau_0^2,$$

where $F(t)$ is some distribution function with $\int_0^\infty dF(t) = 1$.

Using this representation (2.1) yields after a few algebraic calculations

$$\mu_i^G(\mathbf{x}) = x_i - \frac{\sigma^2(x_i - \overline{x}) \int_0^\infty \frac{1}{\tau_0^2 - t} \left(\int_t^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} d\tau^2 \right) dF(t)}{\int_0^\infty \frac{1}{\tau_0^2 - t} \left(\int_t^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} d\tau^2 \right) dF(t)}, \tag{2.4}$$

being the functions involved in (2.4) defined by continuity when necessary. Easily is derived that

$$\sup_{G \in \mathcal{G}_U} [\inf] \mu_i^G(\mathbf{x}) = \sup_{t>0}[\inf] \left\{ x_i - \frac{\sigma^2(x_i - \overline{x}) \int_t^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} d\tau^2}{\int_t^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} d\tau^2} \right\}. \tag{2.5}$$

Now the problem is derived to find out

$$\sup_{t>0}[\inf] \frac{\int_t^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} d\tau^2}{\int_t^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} d\tau^2},$$

that is (see Lemma A1 of the Appendix for the proof)

$$\frac{\int_{0[\tau_0^2]}^{\tau_0^2[\infty]} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} d\tau^2}{\int_{0[\tau_0^2]}^{\tau_0^2[\infty]} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} d\tau^2}.$$

So the interval in which $\mu_i^G(\mathbf{x})$ is ranging is

$$\left(x_i - \sigma^2(x_i - \overline{x}) \frac{\int_0^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} d\tau^2}{\int_0^{\tau_0^2} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} d\tau^2},\ x_i - \sigma^2(x_i - \overline{x}) \frac{\int_{\tau_0^2}^{\infty} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} d\tau^2}{\int_{\tau_0^2}^{\infty} \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} d\tau^2} \right).$$

By taking $\tau_0^2 = 9$ in the previous example, that will be considered all along this paper, we find out that $\mu_7^G(\mathbf{x})$ is in the interval (117.47, 120.74). All we needed to reach this result was to evaluate numerically the integrals appearing in the above interval when $\tau_0^2 = 9$.

3. ε–CONTAMINATION CLASSES

Along this section we will consider classes of the form

$$\mathcal{G}_\varepsilon = \{G, \text{ such that } G = (1-\varepsilon)G_0 + \varepsilon q,\ q \in \mathcal{Q}\}. \tag{3.1}$$

These classes have been extensively studied in Sivaganesan and Berger (1987) and they mean some uncertainty in an elicited prior G_0, being reflected in ε the amount of probabilistic uncertainty in G_0 and being Q a class of allowable contaminations. For each prior G in $\mathcal{G}_\varepsilon$ we can reach the following formula for $\mu_i^G(\mathbf{x})$ by using the formulas in subsection 1.2

$$\mu_i^G(\mathbf{x}) = \frac{A + x_i \int_0^\infty \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} q(\tau^2) d\tau^2 - \sigma^2(x_i - \overline{x}) \int_0^\infty \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}} q(\tau^2) d\tau^2}{B + \int_0^\infty \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} q(\tau^2) d\tau^2}, \tag{3.2}$$

where,

$$B = \frac{1-\varepsilon}{\varepsilon} \int_0^\infty \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} G_0(\tau^2) d\tau^2, \tag{3.3}$$

and

$$A = x_i B - \sigma^2(x_i - \overline{x})\frac{(1-\varepsilon)}{\varepsilon}\int_0^{\infty}\frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}}G_0(\tau^2)d\tau^2. \tag{3.4}$$

3.1 Choosing $\mathcal{Q}$ as all possible distributions

In this case we will denote $\mathcal{G}_\varepsilon$ as $\mathcal{G}_\varepsilon^A$. From (3.2) it is followed that

$$\sup_{G\in\mathcal{G}_\varepsilon^A}[\inf]\mu_i^G(\mathbf{x}) = \sup_{t>0}[\inf]\frac{A + x_i\frac{\exp\left\{-\frac{s^2}{2(\sigma^2+t)}\right\}}{(\sigma^2+t)^{(p-1)/2}} - \sigma^2(x_i-\overline{x})\frac{\exp\left\{-\frac{s^2}{2(\sigma^2+t)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}}}{B + \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+t)}\right\}}{(\sigma^2+t)^{(p-1)/2}}}. \tag{3.5}$$

Numerical optimization provides us a way to find out the above sup[inf]. Going on with the example in subsections (2.1) and (2.2), for which we take $\varepsilon = 0.1$ and

$$G_0(\tau^2)\alpha\ \frac{\exp\left\{-\frac{7194}{100+\tau^2}\right\}}{(100+\tau^2)^{66}}, \tag{3.6}$$

we get that $\mu_7^G(\mathbf{x})$ is in the interval (120.00, 120.26) being $\mu_7^{G_0}(\mathbf{x}) = 120.18$. We did this election of G_0 because it leads to feasible computational calculations, it is unimodal being its mode $\tau_0^2 = 9$ our guessed one and it is very suitable to represent relative uncertainty as it is showed by some of its quantiles

$$\int_0^9 G_0(\tau^2)d\tau^2 = 0.30, \quad \int_9^{25} G_0(\tau^2)d\tau^2 = 0.48,$$

$$\int_{25}^{\infty} G_0(\tau^2)d\tau^2 = 0.22.$$

3.2 Choosing $\mathcal{Q}$ as all unimodal distributions

In this case we will denote $\mathcal{G}_\varepsilon$ as $\mathcal{G}_\varepsilon^U$, and $\mathcal{Q}$ in (3.1) will be the class of unimodal distributions with some fixed mode τ_0^2. From (3.2) and using the representation of each $q \in \mathcal{Q}$ as a mixture of uniform densities is followed that

$$\mu_i^G(\mathbf{x}) = \frac{A + \int_0^{\infty}\frac{1}{t-\tau_0^2}\left(\int_t^{\tau_0^2}\left(\frac{x_i\exp\left\{\frac{-s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} - \frac{\sigma^2(x_i-\overline{x})\exp\left\{\frac{-s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}}\right)d\tau^2\right)dF(t)}{B + \int_0^{\infty}\frac{1}{t-\tau_0^2}\left(\int_t^{\tau_0^2}\frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}}d\tau^2\right)dF(t)}, \tag{3.9}$$

where $G = (1-\varepsilon)G_0 + \varepsilon q$ and B and A are defined in (3.3) and (3.4) respectively. So

$$\sup_{G\in\mathcal{G}_\varepsilon^U}[\inf]\mu_i^G(\mathbf{x}) = \sup_{t>0}[\inf]\frac{A + \frac{1}{t-\tau_0^2}\int_{\tau_0^2}^{t}\left(\frac{x_i\exp\left\{\frac{-s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}} - \frac{\sigma^2(x_i-\overline{x})\exp\left\{\frac{-s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p+1)/2}}\right)d\tau^2}{B + \frac{1}{t-\tau_0^2}\int_{\tau_0^2}^{t}\frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}}d\tau^2}.$$

We can reach the above sup[inf] by doing numerical optimization. As an illustration we go on with our example taking the same G_0 as in the previous subsection and $\tau_0^2 = 9$, so the class $\mathcal{G}_\varepsilon^U$ will be contained in the class $\mathcal{G}_U$. Now $\mu_7^G(\mathbf{x})$ is in the interval (120.04, 120.23).

4. DENSITY RATIO CLASSES

Along this section $\mathcal{G}$ is denoted as $\mathcal{G}_{\mathrm{DR}}$ and defined as

$$\mathcal{G}_{\mathrm{DR}} = \left\{\begin{array}{l} G \text{ such that } L(\tau^2) \le \alpha G(\tau^2) \le U(\tau^2), \\ \text{for some } \alpha > 0 \end{array}\right\}, \tag{4.1}$$

where L and U are specified nonnegative functions. This class, introduced by DeRobertis and Hartigan (1981), can be viewed as specifying ranges for the ratios of the prior density between any two points: when $L(\tau^2) = 1$ and $U(\tau^2) = k$ $\mathcal{G}_{\mathrm{DR}}$ has all priors with the density ratio between any two points lying in the interval (k^{-1}, k), being this class a good representation of prior uncertainty. Because of (1.2) we can view our model as having just the parameter τ^2, G being the prior and

$$\rho(\tau^2) = \frac{\exp\left\{-\frac{s^2}{2(\sigma^2+\tau^2)}\right\}}{(\sigma^2+\tau^2)^{(p-1)/2}}, \tag{4.2}$$

the likelihood. In that we are looking for

$$\sup_{G\in\mathcal{G}_{\mathrm{DR}}}[\inf]\mu_i^G(\mathbf{x}) = \sup_{G\in\mathcal{G}_{\mathrm{DR}}}[\inf]\left\{x_i - \sigma^2(x_i - \overline{x})E^{G(\tau^2/\mathbf{x})}\left[\frac{1}{\sigma^2+\tau^2}\right]\right\}, \tag{4.3}$$

our problem is reduced to find out

$$\sup_{G\in\mathcal{G}_{\mathrm{DR}}}[\inf]\left\{E^{G(\tau^2/\mathbf{x})}[b(\tau^2)]\right\}, \tag{4.4}$$

where $b(\tau^2) = (\sigma^2 + \tau^2)^{-1}$. According to DeRobertis and Hartigan (1981) we get the above sup and inf as the unique roots of the two following equations

$$\int_{b(\tau^2)<c} (b(\tau^2) - c)\rho(\tau^2)U(\tau^2)d\tau^2 + \int_{b(\tau^2)>c} (b(\tau^2) - c)\rho(\tau^2)L(\tau^2)d\tau^2 = 0, \qquad (4.5)$$

and

$$\int_{b(\tau^2)<c} (b(\tau^2) - c)\rho(\tau^2)L(\tau^2)d\tau^2 + \int_{b(\tau^2)>c} (b(\tau^2) - c)\rho(\tau^2)U(\tau^2)d\tau^2 = 0. \qquad (4.6)$$

These equations can be solved easily by using numerical computation. We did it for $L(\tau^2) = 1$ and $U(\tau^2) = k$ and reached the intervals appearing in table 1 in the next section. Note that for $k = 1$ $\mathcal{G}_{\mathrm{DR}} = \{G^*: G^*(\tau^2) \equiv 1\}$ being $\mu_7^{G^*}(\mathrm{x}) = 117.58$.

All calculations needed along this paper have been done using the package Mathematica.

5. CONCLUSIONS

Conclusions can be in a certain sense independent of the x_i being observed in that for the $\mathcal{G}_A$ class the interval reached will always be $(x_i, \overline{x})$ and when using any other class this interval will be relatively reduced accordingly to figures contained in table 1. Results we have obtained are summarized in the following table

CLASS	INTERVAL
$\mathcal{G}_A$	(115.00 121.00)
$\mathcal{G}_U$	(117.47 120.74)
$\mathcal{G}_\varepsilon^A$	(120.00 120.26)
$\mathcal{G}_\varepsilon^U$	(120.04 120.23)
$\mathcal{G}_{\mathrm{DR}}, k = 2$	(117.18 118.02)
$\mathcal{G}_{\mathrm{DR}}, k = 6$	(116.61 118.72)
$\mathcal{G}_{\mathrm{DR}}, k = 10$	(116.39 119.03)

Table 1. Ranges for the posterior mean under several classes of priors.

From table 1 we can see that a robust inference is achieved for each class but $\mathcal{G}_A$, so if you are comfortable thinking of any of them as a reasonable respresentation of prior uncertainty you could model using it, getting a robust inference. In addition a few specific comments are in order, if you feel that the true prior is close to a specified one, say G_0, you should model through $\mathcal{G}_\varepsilon^A$ or $\mathcal{G}_\varepsilon^U$ (does not matter how G_0 is contaminated) and you will get a very robust inference. Modelling through $\mathcal{G}_U$ a robust inference is reached again although the rate of robustness achieved will be lower and depend on the assessed τ_0^2, the nearer τ_0^2 is from σ^2 the bigger rate of robustness will be achieved, so if you feel that $\mathcal{G}_U$ is a reasonable way to represent prior uncertainty you might model through it although if you could feel comfortable shrinking $\mathcal{G}_U$ to $\mathcal{G}_\varepsilon^U$ a very bigger rate of robustness would be achieved. Eventually, $\mathcal{G}_{\mathrm{DR}}$ classes are very convenient to represent vague prior knowledge and robustness is achieved again using these classes, mainly if $k \leq 6$.

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APPENDIX

Lemma A1

Let $k > 0,\ q \geq 3/2$ and $\tau_0^2 > 0$ be real numbers, then

$$\sup_{t\geq 0}[\inf] \frac{\int_t^{\tau_0^2} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q+1}} dx}{\int_t^{\tau_0^2} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q}} dx} = \frac{\int_{0[\tau_0^2]}^{\tau_0^2[\infty]} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q+1}} dx}{\int_{0[\tau_0^2]}^{\tau_0^2[\infty]} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q}} dx}.$$

Proof.

For the sup part we consider the set $\mathcal{C}$ of all C such that the function $f(t)$ defined as

$$f(t) = C \int_t^{\tau_0^2} \frac{\exp\left\{-\frac{k}{1+x}\right\}}{(1+x)^q} dx - \int_t^{\tau_0^2} \frac{\exp\left\{-\frac{k}{1+x}\right\}}{(1+x)^{q+1}} dx \qquad (A.1)$$

is nonnegative if $t < \tau_0^2$ and nonpositive if $t \geq \tau_0^2$. Obviously $\inf \mathcal{C}$ is the sup we want.

By derivating (A.1) we get

$$f'(t) = -\left[C - \frac{1}{1+t}\right] \frac{\exp\left\{-\frac{k}{1+t}\right\}}{(1+t)^q}, \qquad (A.2)$$

then $f'(0) = -\exp\{-k\}(C-1) > 0$ and $f'(\infty) = -C < 0$, being $f'(t) = 0$ in just one point, say $t*$. In order to satisfy $C \in \mathcal{C}$, $t*$ must lie in the interval $(0, \tau_0^2)$ so $C > \frac{1}{1+\tau_0^2}$, and also $f(0) \geq 0$ is needed so

$$C \geq \frac{\int_0^{\tau_0^2} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q+1}} dx}{\int_0^{\tau_0^2} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q}} dx}. \qquad (A.3)$$

Then $C = \frac{\int_0^{\tau_0^2} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q+1}} dx}{\int_0^{\tau_0^2} \frac{\exp\{-\frac{k}{1+x}\}}{(1+x)^{q}} dx}$ is the small number in $\mathcal{C}$ and so the sup we are looking for. For the inf part an analogous argument is followed.