CONFIDENCE SETS HAVING THE SHAPE OF A HALF SPACE

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SUMMARY: For the problem of estimating the mean of a p-dimensional normal distribution, $p > 1$, confidence regions based on half spaces bounded by a hyperplane having the vector of observations as normal are proposed. Confidence regions with exact probability of coverage are constructed. Tables are provided.

Key words and phrases: Sufficiency, confidence sets, monotone likelihood ratio, half space.

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1. Introduction

Suppose we observe X given by $X = Y + \theta$ where Y is a standard normal random vector in $\mathbb{R}^p$, $p \geq 2$, and θ is an unknown element of $\mathbb{R}^p$. Classical confidence regions of level $1 - \alpha$ for estimating θ are balls having fixed radius and centered at x. A variation from this solution consists in centering a ball having a fixed radius at an estimate $\hat{\theta}$ of θ. This variation, suggested by Stein (1962) has been proven successful by Brown (1966) and Joshi (1967) for $p \geq 3$. An explicit estimate $\hat{\theta}$ has been proposed by Hwang and Casella (1982) for $p \geq 4$. Other confidence regions are the one having the shape of a half space perpendicular to the vector x. An approximate solution of this kind was given by Stein (1962).

In this article, I propose an exact solution to the problem of finding a half space confidence region for the multivariate normal distribution. More precisely, I determine a function h which satisfies the property that a confidence set given in the form $\{\theta \in \mathbb{R}^p\colon\ \theta' x > ||x||\ h(\alpha, ||x||)\}$ covers θ with probability $1 - \alpha$. To a rough approximation, the 50% confidence cut off point of the orthogonal projection of θ on x is found to be $(1 - \text{ med } (\chi^2_{2p})/2x'x)\ x$.

The basic idea is not difficult to describe. Suppose that x is different from zero and take the unit vector $u = x/||x||$ as the basis for a one dimensional vector space $\mathcal{U}$ (say). Denote the orthogonal projection of x and θ on $\mathcal{U}$ by ru and zu respectively. That defines two random variables R and Z ($R = ||X||$ and $Z = \theta' X/||X||$). In terms of (R, Z) the problem consists in solving the equation $P_\theta[Z > h(\alpha, R)] = \alpha$ in h for all $\theta \in \mathbb{R}^p$. The key element in the solution of this problem is the observation that the conditional distribution of R given Z does not depend on θ. Therefore, the original problem is reduced to the one of finding a confidence set for z, of the form $\{z\colon\ z > h(\alpha, r)\}$, based on the conditional distribution of R given that $Z = z$. At this point we might call this method pivotal, claiming that R, in its conditional distribution given that $Z = z$, is the pivotal.

Exact confidence regions for the multivariate normal distribution are presented in section 2. In order to compare the exact solution with the approximation proposed by Stein (1962) some asymptotic results are derived in section 3. Tables are provided in the appendix.

2. Exact Solution

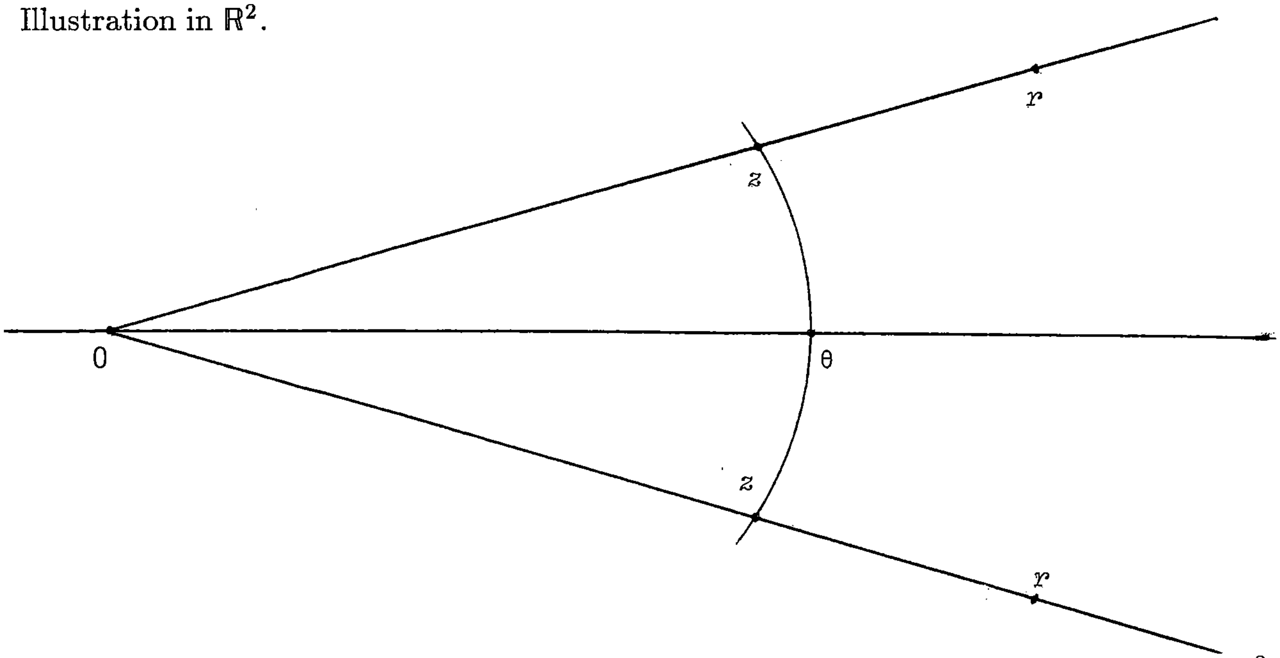
Let X be a p-dimensional normal distribution with mean θ and covariance I, $p \geq 2$. We want to construct confidence regions having the shape of a half space bounded by a hyperplane perpendicular to x. More explicitly, we consider confidence regions of the form $\{\theta \in \mathbb{R}^p\colon\ \theta' x/||x|| > h(\alpha, ||x||)\}$ where $h(\alpha, \cdot)$ is chosen such that the level of coverage equals α. Stein (1962) proposed some approximations for h which are good as long as p is large. In this section I present a function h which gives an exact probability of coverage.

Let $R = ||X||$ and $Z = X'\theta/R$. In the sequel I shall call (R, Z) a "statistic" although Z is not observable. My goal is to find $h(\alpha, \cdot)$ such that $P_\theta[Z > h(\alpha, R)] = 1 - \alpha$ for all $\theta \in \mathbb{R}^p$. In this problem, based on (R, Z), Z is a sufficient statistic and the conditional density of R given that $Z = z$ is given by

$$f_z(r) = k^{-1}(z)\ r^{p-1} \exp -\frac{1}{2}(r-z)^2, \quad r > 0 \tag{2.1}$$

with $k(z) = \int_0^\infty r^{p-1} \exp -\frac{1}{2}(r-z)^2 dr$. The factor r^{p-1} in expression (2.1) comes from the transformation $x \to (r, z)$. This factor can be calculated directly by using jacobiens. The calculations are rather tedious and are omitted. A geometric derivation might be more interesting. Notice that the set of points in $\mathbb{R}^p$ satisfying the relation $x'\theta/||x|| = z$ is a cone so a strip of width dr will have "volume" proportional to $r^{p-1}dr$.

Illustration in $\mathbb{R}^2$.



In this illustration the two line segments leaving from the origin are the points in $\mathbb{R}^2$

satisfying the relation $x'\theta/||x|| = z$. *This set consists of the points* $x \in \mathbb{R}^2$ *such that* $zx/||x||$ *intersects the surface of a sphere of radius* $||\theta||/2$ *centered at* $\theta/2$.

Using the notion of sufficiency and the identity

$$P_\theta[Z < h(\alpha, R)] = E_\theta P[Z < h(\alpha, R)|Z] \tag{2.2}$$

the original problem is transformed to the one of finding confidence regions for z of the form $\{z\colon\ z > h(\alpha, r)\}$ based on the conditional distribution of R given that $Z = z$. Following expression (2.2) I shall require that the expected probability of covering z equal $1 - \alpha$. A simple solution consists in determining h such that the conditional probability of covering z equals $1 - \alpha$ for all z, i.e.

$$P_z[h(\alpha, R) < z|Z = z] = 1 - \alpha \tag{2.3}$$

From expression (2.1), conditionally on $Z = z$, R belongs to a monotone likelihood ratio family. This property implies that $h(\alpha, \cdot)$ is an increasing function (ref. Lehmann 1986). Therefore $z > h(\alpha, r)$ is equivalent to $r < h^{-1}(\alpha, z)$. So, if $\{z\colon\ z > h(\alpha, r)\}$ is a confidence region of level $1 - \alpha$ then we must have $P_z[R \geq h^{-1}(\alpha, z)] = \alpha$ for all $z \in \mathbb{R}$. In particular, if we set $z = h(\alpha, r)$ then we get $P_{h(\alpha,r)}[R \geq r] = \alpha$. Thus, the relation (2.3) holds if and only if

$$\int_r^\infty u^{p-1} \exp -\frac{1}{2}(u - h(\alpha, r))^2 du = \alpha \int_0^\infty u^{p-1} \exp -\frac{1}{2}(u - h(\alpha, r))^2 du. \tag{2.4}$$

In order to solve numerically this equation in h let

$$\varphi(u) = (2\pi)^{-\frac{1}{2}} \exp -u^2/2$$
$$\Phi(u) = \int_{-\infty}^u \varphi(t)dt$$

and

$$q_k(r, h) = \int_r^\infty u^{k-1}\varphi(u)du, \quad k \in \mathbb{N}.$$

With these new notations (2.4) becomes

$$q_p(r, h(\alpha, r)) = \alpha\ q_p(0, h(\alpha, r)). \tag{2.5}$$

Using integration by parts we get the following recursive formula

$$
\begin{aligned}
q_1(r,h) &= \Phi(h-r) \\
q_2(r,h) &= \varphi(h-r) + h\Phi(h-r) \qquad (2.6)\\
q_k(r,h) &= r^{k-2}\varphi(h-r) + hq_{k-1}(r,h) + (k-2)q_{k-2}(r,h) \quad k > 2.
\end{aligned}
$$

Although these formulas are exact I do not recommend to use them when h is negative because of the cancellations. For any $r > 0$, $f(h) = q_p(r,h)/q_p(0,h) - \alpha$ is increasing in h. We need to find the zero of f. This can be done numerically by trial and error or by any other method. In section 3, first step approximations are proposed. Tables are provided in the appendix.

3. Approximations

In this section I describe the behaviour of the function $h(\alpha, r)$ when r is either close to zero or large, α fixed. I also consider approximations when p is large. In order to apply some asymptotic results I shall use the following lemma which is a simple application of the bounded convergence theorem.

Lemma 1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables having densities $g_n(\cdot) = K_n^{-1} f_n(\cdot)$. If there exists two functions f and h such that $K = \int f > 0$, $\int h < \infty$, $f_n \underset{n\to\infty}{\longrightarrow} f$ and $f_n \leq h$ for all n then $X_n \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} X$ where X is a random variable having density function $g = K^{-1}f$.

Let us now study the behaviour of R given that $Z = z$ when p and z vary. Recall that the conditional density of R given that $Z = z$ is

$$g_z(r) = K^{-1}(z) r^{p-1} \exp -\frac{1}{2}(r-z)^2, \qquad r > 0.$$

Lemma 2.

(a) $U_z = -zR \underset{z\to-\infty}{\overset{\mathcal{L}}{\longrightarrow}} \Gamma(p,1)$,

(b) $U_z = R - z \underset{z\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} N(0,1)$.

Proof:

(a) U_z has density $g_z(\cdot) = K^{-1}(z)f_z(\cdot)$ with $f_z(u) = u^{p-1}\exp -\frac{1}{2}(u^2z^{-2} + 2u),\ u > 0$, $f_z(u) \le h(u) = u^{p-1}\exp -u,\ u > 0$ for all $z < 0$ and $f_z \underset{z\to-\infty}{\longrightarrow} h$.

(b) U_z has density $g_z(\cdot) = K^{-1}(z)f_z(\cdot)$ with $f_z(u) = (1 + uz^{-1})^{p-1}\exp -u^2/2,\ u > -z$, $f_z(u) \le (1 + |u|)^{p-1}f(u)$ with $f(u) = \exp -u^2/2$ for all $z > 0$ and $f_z \underset{z\to\infty}{\longrightarrow} f$.

Corollary. $h(\alpha, r) \approx r - c_1(\alpha) - c_2(\alpha)/2r$ with $P[N(0,1) > c_1(\alpha)] = \alpha$ and $P[\chi^2_{2p} > c_2(\alpha)] = \alpha$.

Let $n = p - 1$.

Lemma 3.

(a) If $z = c\sqrt{n} + d$ then $U_z = R - z - a\sqrt{n} \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} N(\frac{-a^2}{1+a^2}, \frac{1}{1+a^2})$ where $2a = \sqrt{4 + c^2} - c$

(b) If $z = cn$ with $c > 0$ then $U_z = R - z \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} N(\frac{1}{c}, 1)$

(c) If $z = -cn$ with $c > 0$ then $U_z = (\frac{1}{c} - R)c\sqrt{n} \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} N(0,1)$

Proof:

(a) U_z has density $g_n(\cdot) = K^{-1}(n)f_n(\cdot)$ with $f_n(u) = (1 + a(u+d)/\sqrt{n})^n \exp -\{a\sqrt{n}(u + d) + u^2/2\}, u > -(d + \sqrt{n}/a)$. Using the inequality $1 + x \le e^x\ \forall x$ we get $f_n(u) \le \exp -u^2/2\ \forall u$ and $f_n \to f$ with $f(u) = \exp -\frac{1}{2}\{a^2(u+d)^2 + u^2\}$.

(b) U_z has density $g_n(\cdot) = K^{-1}(n)f_n(\cdot)$ with $f_n(u) = (1 + u/nc)^n \exp -u^2/2,\ u > -nc$. We have $f_n(u) \le f(u) = \exp -\frac{1}{2}(u^2 - 2u/c)\ \forall u$ and $f_n \underset{n\to\infty}{\longrightarrow} f$.

(c) U_z has density $g_n(\cdot) = K^{-1}(n)f_n(\cdot)$ with $f_n(u) = \{(1 - u/\sqrt{n})\exp(u/\sqrt{n})\}^n \exp -\frac{1}{2c^2}$ $(1 - u/\sqrt{n})^2,\ u < \sqrt{n},\ f_n(u) \le (1 + u^2/4)^{-1}\ \forall u$ and $f_n \to f$ where $f(u) = \exp -u^2/2$. ■

Remark: A direct verification shows that any of the approximations proposed in lemma 3 may be absorbed into the following one

$$\frac{R(R - z) - n}{\sqrt{n + R^2}} \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} N(0,1)$$

which recaptures the approximation proposed by Stein (1962).

Corollary. $h(\alpha,r) \approx r-(p-1)/r-c(\alpha)\sqrt{1+(p-1)/r^2}$ when p is large and $P[N(0,1) > c(\alpha)] = \alpha$.

Tables show that the approximations are fair. They do good where they are expected to do so and they are reasonable everywhere. The precision is much better for $\alpha = 0.5$. Notice that when r is near zero a small difference in the evaluation of $h(\alpha,r)$ does not really affect the coverage level. However, when r is large we need more precision. The approximation given for large p behaves like the first approximation when r is either small or large (as we should expect). The approximation for large p might be better than the first approximation when α is either small or large but computation showed that the first approximation does always better when $\alpha = 0.5$. For instance, when $p = 20$ the difference between the first approximation and the exact result is always smaller than 0.1.

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Critical values $h(\alpha, r)$ and (approximations)* for confidence sets, of level $1-\alpha$, of the form $\{\theta \in \mathbb{R}^p : \theta' x > ||x|| h(\alpha, ||x||)\}$ where X is distributed as a p-dimensional normal with mean θ and covariance I.

$p=2$ r $\alpha \rightarrow$	0.025		0.05		0.50		0.95		0.975	
0.2	-27.7	(-29.61)	-23.6	(-25.17)	-8.12	(-8.19)	-0.99	(0.07)	-0.22	(0.95)
0.4	-13.6	(-15.49)	-11.5	(-13.11)	-3.66	(-3.80)	0.40	(1.16)	0.93	(1.76)
0.6	-8.83	(-10.64)	-7.42	(-8.95)	-2.01	(-2.20)	1.07	(1.65)	1.52	(2.16)
0.8	-6.36	(-8.12)	-5.28	(-6.78)	-1.08	(-1.30)	1.53	(2.00)	1.94	(2.46)
1.0	-4.80	(-6.53)	-3.94	(-5.39)	-0.44	(-0.68)	1.89	(2.29)	2.28	(2.72)
1.2	-3.73	(-5.40)	-2.99	(-4.40)	0.05	(-0.20)	2.21	(2.55)	2.57	(2.96)
1.4	-2.92	(-4.54)	-2.27	(-3.63)	0.45	(0.20)	2.49	(2.79)	2.85	(3.19)
1.6	-2.27	(-3.84)	-1.69	(-3.01)	0.80	(0.55)	2.76	(3.02)	3.10	(3.41)
1.8	-1.73	(-3.25)	-1.20	(-2.48)	1.12	(0.87)	3.01	(3.25)	3.35	(3.63)
2.0	-1.27	(-2.75)	-0.78	(-2.02)	1.40	(1.16)	3.25	(3.47)	3.58	(3.84)
2.2	-0.87	(-2.29)	-0.41	(-1.60)	1.67	(1.44)	3.48	(3.68)	3.81	(4.05)
2.4	-0.51	(-1.88)	-0.08	(-1.22)	1.93	(1.70)	3.70	(3.90)	4.04	(4.26)
2.6	-0.18	(-1.50)	0.23	(-0.87)	2.17	(1.95)	3.93	(4.11)	4.26	(4.47)
2.8	0.12	(-1.15)	0.52	(-0.54)	2.41	(2.20)	4.15	(4.32)	4.47	(4.67)
3.0	0.41	(-0.82)	0.79	(-0.22)	2.64	(2.44)	4.37	(4.53)	4.69	(4.88)
3.2	0.68	(-0.50)	1.05	(0.07)	2.86	(2.68)	4.58	(4.73)	4.91	(5.08)
3.4	0.94	(-0.20)	1.30	(0.36)	3.09	(2.91)	4.80	(4.94)	5.12	(5.29)
3.6	1.19	(0.09)	1.54	(0.64)	3.31	(3.13)	5.01	(5.15)	5.33	(5.49)
3.8	1.43	(0.37)	1.78	(0.91)	3.52	(3.36)	5.22	(5.35)	5.54	(5.70)
4.0	1.66	(0.65)	2.00	(1.17)	3.74	(3.58)	5.43	(5.56)	5.75	(5.90)
4.2	1.89	(0.91)	2.23	(1.43)	3.95	(3.80)	5.64	(5.76)	5.96	(6.10)
4.4	2.12	(1.17)	2.45	(1.68)	4.16	(4.02)	5.85	(5.96)	6.17	(6.30)
4.6	2.34	(1.43)	2.67	(1.92)	4.38	(4.24)	6.06	(6.17)	6.37	(6.51)
4.8	2.56	(1.68)	2.89	(2.17)	4.59	(4.45)	6.26	(6.37)	6.58	(6.71)
5.0	2.77	(1.93)	3.10	(2.41)	4.79	(4.66)	6.47	(6.57)	6.79	(6.91)
5.2	2.99	(2.17)	3.31	(2.64)	5.00	(4.88)	6.67	(6.78)	6.99	(7.11)
5.4	3.20	(2.41)	3.53	(2.88)	5.21	(5.09)	6.88	(6.98)	7.20	(7.32)
5.6	3.41	(2.65)	3.74	(3.11)	5.42	(5.30)	7.09	(7.18)	7.41	(7.52)
5.8	3.62	(2.88)	3.95	(3.34)	5.62	(5.51)	7.29	(7.38)	7.61	(7.72)
6.0	3.83	(3.11)	4.15	(3.56)	5.83	(5.72)	7.50	(7.59)	7.81	(7.92)
6.5	4.35	(3.68)	4.67	(4.13)	6.34	(6.24)	8.01	(8.09)	8.32	(8.42)
7.0	4.87	(4.24)	5.19	(4.68)	6.86	(6.76)	8.51	(8.59)	8.83	(8.93)
7.5	5.38	(4.80)	5.70	(5.22)	7.37	(7.28)	9.02	(9.10)	9.34	(9.43)
8.0	5.89	(5.34)	6.21	(5.76)	7.87	(7.79)	9.53	(9.60)	9.85	(9.93)
8.5	6.41	(5.88)	6.72	(6.30)	8.38	(8.30)	10.0	(10.1)	10.4	(10.4)
9.0	6.91	(6.42)	7.23	(6.83)	8.89	(8.81)	10.5	(10.6)	10.9	(10.9)
9.5	7.42	(6.95)	7.74	(7.36)	9.39	(9.32)	11.0	(11.1)	11.4	(11.4)
10.0	7.93	(7.48)	8.25	(7.88)	9.90	(9.83)	11.6	(11.6)	11.9	(11.9)
12.0	9.95	(9.58)	10.3	(9.60)	11.9	(11.9)	13.6	(13.6)	13.9	(13.9)
14.0	12.0	(11.6)	12.3	(12.0)	13.9	(13.9)	15.6	(15.6)	15.9	(15.9)
16.0	14.0	(13.7)	14.3	(14.1)	15.9	(15.9)	17.6	(17.6)	17.9	(17.9)
18.0	16.0	(15.7)	16.3	(16.1)	17.9	(17.9)	19.6	(19.6)	19.9	(20.0)
20.0	18.0	(17.8)	18.3	(18.1)	20.0	(19.9)	21.6	(21.6)	21.9	(22.0)
22.0	20.0	(19.8)	20.3	(20.1)	22.0	(21.9)	23.6	(23.6)	23.9	(24.0)
24.0	22.0	(21.8)	22.3	(22.2)	24.0	(23.9)	25.6	(25.6)	25.9	(26.0)
26.0	24.0	(23.8)	24.3	(24.2)	26.0	(25.9)	27.6	(27.6)	27.9	(28.0)
28.0	26.0	(25.8)	26.3	(26.2)	28.0	(27.9)	29.6	(29.6)	29.9	(30.0)
30.0	28.0	(27.9)	28.3	(28.2)	30.0	(29.9)	31.6	(31.6)	31.9	(32.0)

*The approximations are based on the formula $h(\alpha, r) \approx r - C_1(\alpha) - C_2(\alpha)/2r$ with $P[N(0,1) > C_1(\alpha)] = \alpha$ and $P[\chi^2_{2p} > C_2(\alpha)] = \alpha$.

$p = 3$ r $\alpha \rightarrow$	0.025		0.05		0.50		0.95		0.975	
0.2	-35.99	(-37.89)	-31.31	(-32.92)	-13.12	(-13.17)	-3.53	(-2.25)	-2.41	(-0.93)
0.4	-17.74	(-19.62)	-15.41	(-16.98)	-6.19	(-6.29)	-1.03	(0.00)	-0.35	(0.81)
0.6	-11.57	(-13.40)	-10.00	(-11.54)	-3.73	(-3.86)	0.03	(0.88)	0.56	(1.53)
0.8	-8.41	(-10.19)	-7.22	(-8.71)	-2.39	(-2.54)	0.69	(1.42)	1.17	(1.99)
1.0	-6.45	(-8.19)	-5.48	(-6.94)	-1.50	(-1.67)	1.19	(1.83)	1.63	(2.34)
1.2	-5.09	(-6.78)	-4.28	(-5.69)	-0.84	(-1.03)	1.60	(2.16)	2.01	(2.65)
1.4	-4.08	(-5.72)	-3.36	(-4.74)	-0.31	(-0.51)	1.96	(2.46)	2.34	(2.92)
1.6	-3.28	(-4.88)	-2.64	(-3.98)	0.13	(-0.07)	2.28	(2.73)	2.65	(3.17)
1.8	-2.63	(-4.17)	-2.04	(-3.34)	0.52	(0.31)	2.57	(2.99)	2.94	(3.42)
2.0	-2.07	(-3.57)	-1.54	(-2.79)	0.86	(0.66)	2.85	(3.24)	3.21	(3.65)
2.2	-1.59	(-3.04)	-1.09	(-2.31)	1.18	(0.98)	3.11	(3.47)	3.46	(3.88)
2.4	-1.16	(-2.57)	-0.70	(-1.87)	1.48	(1.29)	3.37	(3.70)	3.71	(4.10)
2.6	-0.77	(-2.14)	-0.33	(-1.47)	1.76	(1.57)	3.61	(3.93)	3.95	(4.32)
2.8	-0.42	(-1.74)	0.01	(-1.09)	2.03	(1.85)	3.85	(4.15)	4.19	(4.54)
3.0	-0.09	(-1.37)	0.32	(-0.74)	2.29	(2.11)	4.09	(4.37)	4.42	(4.75)
3.2	0.22	(-1.02)	0.61	(-0.41)	2.54	(2.36)	4.32	(4.59)	4.65	(4.97)
3.4	0.51	(-0.69)	0.89	(-0.10)	2.78	(2.61)	4.55	(4.80)	4.88	(5.18)
3.6	0.79	(-0.37)	1.16	(0.21)	3.02	(2.85)	4.77	(5.02)	5.10	(5.39)
3.8	1.05	(-0.06)	1.42	(0.50)	3.25	(3.10)	4.99	(5.23)	5.32	(5.60)
4.0	1.31	(0.23)	1.68	(0.78)	3.48	(3.33)	5.21	(5.44)	5.54	(5.81)
4.2	1.56	(0.52)	1.91	(1.06)	3.71	(3.56)	5.43	(5.65)	5.76	(6.01)
4.4	1.81	(0.80)	2.16	(1.32)	3.93	(3.79)	5.65	(5.86)	5.98	(6.22)
4.6	2.05	(1.07)	2.39	(1.59)	4.15	(4.02)	5.87	(6.07)	6.19	(6.43)
4.8	2.28	(1.33)	2.63	(1.84)	4.37	(4.24)	6.08	(6.27)	6.41	(6.63)
5.0	2.51	(1.60)	2.85	(2.10)	4.59	(4.47)	6.29	(6.48)	6.62	(6.84)
5.2	2.74	(1.85)	3.08	(2.34)	4.81	(4.69)	6.51	(6.69)	6.83	(7.04)
5.4	2.97	(2.10)	3.30	(2.59)	5.02	(4.90)	6.72	(6.89)	7.04	(7.25)
5.6	3.19	(2.35)	3.52	(2.83)	5.24	(5.12)	6.93	(7.10)	7.25	(7.45)
5.8	3.41	(2.59)	3.74	(3.07)	5.45	(5.34)	7.14	(7.30)	7.46	(7.65)
6.0	3.63	(2.84)	3.96	(3.31)	5.66	(5.55)	7.35	(7.51)	7.67	(7.86)
6.2	3.84	(3.07)	4.17	(3.54)	5.87	(5.77)	7.55	(7.71)	7.88	(8.06)
6.4	4.06	(3.31)	4.39	(3.77)	6.08	(5.98)	7.76	(7.92)	8.08	(8.26)
6.6	4.27	(3.55)	4.60	(4.00)	6.29	(6.19)	7.97	(8.12)	8.29	(8.47)
6.8	4.49	(3.78)	4.81	(4.23)	6.50	(6.41)	8.18	(8.32)	8.50	(8.67)
7.0	4.70	(4.01)	5.02	(4.46)	6.71	(6.61)	8.39	(8.53)	8.70	(8.87)
7.5	5.23	(4.58)	5.55	(5.02)	7.23	(7.14)	8.90	(9.04)	9.22	(9.38)
8.0	5.75	(5.14)	6.07	(5.57)	7.75	(7.67)	9.42	(9.54)	9.74	(9.88)
8.5	6.27	(5.69)	6.59	(6.11)	8.26	(8.19)	9.93	(10.05)	10.25	(10.39)
9.0	6.79	(6.24)	7.11	(6.66)	8.78	(8.70)	10.44	(10.55)	10.76	(10.89)
9.5	7.30	(6.78)	7.62	(7.19)	9.29	(9.22)	10.95	(11.06)	11.27	(11.39)
10.0	7.81	(7.32)	8.14	(7.73)	9.80	(9.73)	11.46	(11.56)	11.78	(11.89)
10.5	8.33	(7.85)	8.65	(8.26)	10.31	(10.25)	11.97	(12.07)	12.29	(12.40)
11.0	8.84	(8.38)	9.16	(8.78)	10.82	(10.76)	12.47	(12.57)	12.79	(12.90)
11.5	9.35	(8.91)	9.67	(9.31)	11.33	(11.27)	12.98	(13.07)	13.30	(13.41)
12.0	9.86	(9.44)	10.17	(9.83)	11.83	(11.78)	13.49	(13.58)	13.81	(13.91)
14.0	11.89	(11.52)	12.20	(11.91)	13.86	(13.81)	15.51	(15.59)	15.83	(15.92)
16.0	13.91	(13.59)	14.22	(13.96)	15.87	(15.83)	17.53	(17.59)	17.84	(17.92)
18.0	15.92	(15.64)	16.24	(16.01)	17.89	(17.85)	19.54	(19.60)	19.86	(19.93)
20.0	17.93	(17.68)	18.25	(18.04)	19.90	(19.87)	21.55	(21.60)	21.86	(21.93)
22.0	19.94	(19.71)	20.26	(20.07)	21.91	(21.88)	23.56	(23.61)	23.87	(23.93)
24.0	21.95	(21.74)	22.27	(22.09)	23.92	(23.89)	25.56	(25.61)	25.88	(25.93)
26.0	23.96	(23.76)	24.27	(24.11)	25.92	(25.90)	27.57	(27.61)	27.88	(27.94)
28.0	25.97	(25.78)	26.28	(26.13)	27.93	(27.91)	29.58	(29.62)	29.89	(29.94)
30.0	27.97	(27.80)	28.29	(28.15)	29.93	(29.91)	31.58	(31.62)	31.90	(31.94)

$p=4$ r $\alpha \rightarrow$	0.025		0.05		0.50		0.95		0.975	
0.2	-43.66	(-45.59)	-38.60	(-40.22)	-18.20	(-18.16)	-6.37	(-4.99)	-4.91	(-3.29)
0.4	-21.60	(-23.47)	-19.60	(-20.63)	-8.71	(-8.78)	-2.55	(-1.37)	-1.73	(-0.37)
0.6	-14.14	(-15.97)	-12.43	(-13.97)	-5.42	(-5.52)	-1.05	(-0.03)	-0.42	(0.74)
0.8	-10.34	(-12.11)	-9.04	(-10.54)	-3.67	(-3.79)	-0.15	(0.74)	0.38	(1.40)
1.0	-7.99	(-9.73)	-6.93	(-8.40)	-2.53	(-2.67)	0.49	(1.28)	0.96	(1.87)
1.2	-6.37	(-8.06)	-5.48	(-6.91)	-1.71	(-1.86)	0.99	(1.71)	1.44	(2.25)
1.4	-5.17	(-6.82)	-4.39	(-5.78)	-1.06	(-1.22)	1.42	(2.07)	1.84	(2.58)
1.6	-4.23	(-5.84)	-3.54	(-4.89)	-0.52	(-0.70)	1.80	(2.39)	2.20	(2.88)
1.8	-3.46	(-5.03)	-2.84	(-4.15)	-0.07	(-0.24)	2.14	(2.69)	2.52	(3.15)
2.0	-2.82	(-4.34)	-2.25	(-3.52)	0.34	(0.16)	2.45	(2.96)	2.83	(3.41)
2.2	-2.27	(-3.74)	-1.73	(-2.97)	0.70	(0.53)	2.75	(3.22)	3.11	(3.66)
2.4	-1.78	(-3.21)	-1.27	(-2.48)	1.04	(0.87)	3.03	(3.48)	3.39	(3.91)
2.6	-1.34	(-2.73)	-0.86	(-2.03)	1.36	(1.19)	3.30	(3.72)	3.65	(4.14)
2.8	-0.94	(-2.29)	-0.48	(-1.61)	1.65	(1.49)	3.56	(3.96)	3.91	(4.37)
3.0	-0.57	(-1.88)	-0.13	(-1.23)	1.94	(1.78)	3.81	(4.19)	4.16	(4.60)
3.2	-0.22	(-1.50)	0.19	(-0.87)	2.21	(2.05)	4.06	(4.42)	4.40	(4.82)
3.4	0.10	(-1.14)	0.50	(-0.53)	2.47	(2.32)	4.30	(4.64)	4.64	(5.04)
3.6	0.40	(-0.79)	0.80	(-0.20)	2.73	(2.58)	4.53	(4.87)	4.87	(5.26)
3.8	0.70	(-0.47)	1.08	(0.11)	2.98	(2.83)	4.77	(5.09)	5.11	(5.47)
4.0	0.98	(-0.15)	1.35	(0.42)	3.22	(3.08)	5.00	(5.30)	5.33	(5.69)
4.2	1.24	(0.15)	1.62	(0.71)	3.46	(3.33)	5.23	(5.52)	5.56	(5.90)
4.4	1.51	(0.45)	1.87	(0.99)	3.70	(3.57)	5.45	(5.73)	5.78	(6.11)
4.6	1.77	(0.73)	2.12	(1.27)	3.93	(3.80)	5.67	(5.95)	6.00	(6.32)
4.8	2.01	(1.01)	2.37	(1.54)	4.16	(4.04)	5.90	(6.16)	6.22	(6.53)
5.0	2.26	(1.29)	2.61	(1.80)	4.39	(4.27)	6.12	(6.37)	6.44	(6.74)
5.2	2.50	(1.55)	2.84	(2.06)	4.61	(4.49)	6.33	(6.58)	6.66	(6.95)
5.4	2.73	(1.82)	3.08	(2.32)	4.83	(4.72)	6.55	(6.79)	6.88	(7.16)
5.6	2.96	(2.07)	3.31	(2.57)	5.05	(4.94)	6.77	(7.00)	7.09	(7.37)
5.8	3.20	(2.33)	3.53	(2.82)	5.27	(5.17)	6.98	(7.21)	7.31	(7.57)
6.0	3.42	(2.58)	3.76	(3.06)	5.49	(5.39)	7.20	(7.42)	7.52	(7.78)
6.2	3.65	(2.83)	3.98	(3.30)	5.71	(5.60)	7.41	(7.62)	7.74	(7.98)
6.4	3.87	(3.07)	4.20	(3.54)	5.92	(5.83)	7.62	(7.83)	7.95	(8.19)
6.6	4.09	(3.31)	4.42	(3.78)	6.14	(6.04)	7.84	(8.04)	8.26	(8.39)
6.8	4.31	(3.55)	4.64	(4.01)	6.35	(6.26)	8.05	(8.24)	8.37	(8.60)
7.0	4.53	(3.79)	4.86	(4.25)	6.57	(6.48)	8.26	(8.45)	8.58	(8.80)
7.2	4.75	(4.02)	5.07	(4.48)	6.78	(6.69)	8.47	(8.66)	8.79	(9.01)
7.4	4.96	(4.26)	5.29	(4.71)	6.99	(6.90)	8.68	(8.86)	9.00	(9.21)
7.6	5.18	(4.49)	5.51	(4.93)	7.20	(7.12)	8.88	(9.07)	9.21	(9.42)
7.8	5.39	(4.72)	5.72	(5.16)	7.41	(7.33)	9.09	(9.27)	9.41	(9.62)
8.0	5.60	(4.94)	5.93	(5.39)	7.62	(7.54)	9.30	(9.47)	9.62	(9.82)
8.5	6.13	(5.51)	6.46	(5.94)	8.14	(8.07)	9.82	(9.98)	10.14	(10.33)
9.0	6.66	(6.07)	6.98	(6.49)	8.66	(8.59)	10.34	(10.49)	10.66	(10.84)
9.5	7.18	(6.62)	7.51	(7.04)	9.18	(9.11)	10.85	(11.00)	11.17	(11.35)
10.0	7.70	(7.16)	8.02	(7.58)	9.70	(9.63)	11.37	(11.51)	11.68	(11.85)
10.5	8.22	(7.71)	8.54	(8.12)	10.21	(10.15)	11.88	(12.01)	12.20	(12.36)
11.0	8.74	(8.24)	9.06	(8.65)	10.73	(10.67)	12.39	(12.52)	12.71	(12.86)
11.5	9.25	(8.78)	9.57	(9.18)	11.24	(11.18)	12.90	(13.03)	13.22	(13.37)
12.0	9.77	(9.31)	10.08	(9.71)	11.75	(11.69)	13.41	(13.53)	13.73	(13.87)
12.5	10.28	(9.84)	10.60	(10.23)	12.26	(12.21)	13.92	(14.04)	14.24	(14.37)
13.0	10.79	(10.37)	11.11	(10.76)	12.77	(12.72)	14.43	(14.54)	14.74	(14.88)
13.5	11.30	(10.89)	11.62	(11.28)	13.28	(13.23)	14.93	(15.04)	15.25	(15.38)
14.0	11.81	(11.41)	12.13	(11.80)	13.78	(13.74)	15.44	(15.55)	15.76	(15.88)

$p=4$ r $\alpha \rightarrow$	0.025		0.05		0.50		0.95		0.975	
16.0	13.84	(13.49)	14.16	(13.87)	15.81	(15.77)	17.47	(17.56)	17.78	(17.89)
18.0	15.86	(15.55)	16.18	(15.92)	17.83	(17.80)	19.49	(19.57)	19.80	(19.90)
20.0	17.88	(17.60)	18.20	(17.97)	19.85	(19.82)	21.50	(21.58)	21.82	(21.91)
22.0	19.90	(19.64)	20.21	(20.00)	21.86	(21.83)	23.51	(23.58)	23.83	(23.91)
24.0	21.91	(21.67)	22.23	(22.03)	23.87	(23.85)	25.52	(25.59)	25.84	(25.91)
26.0	23.92	(23.70)	24.24	(24.06)	25.88	(25.86)	27.53	(27.59)	27.85	(27.92)
28.0	25.93	(25.73)	26.25	(26.08)	27.89	(27.87)	29.54	(29.60)	29.86	(29.92)
30.0	27.94	(27.75)	28.25	(28.10)	29.90	(29.88)	31.55	(31.60)	31.86	(31.92)

$p=5$ r $\alpha \rightarrow$	0.025		0.05		0.50		0.95		0.975	
0.2	-51.05	(-52.96)	-45.59	(-47.22)	-23.13	(-23.15)	-9.45	(-8.01)	-7.66	(-5.99)
0.4	-25.29	(-27.16)	-22.55	(-24.13)	-11.22	(-11.28)	-4.15	(-2.88)	-3.18	(-1.72)
0.6	-16.60	(-18.43)	-14.76	(-16.30)	-7.11	(-7.19)	-2.16	(-1.04)	-1.45	(-0.16)
0.8	-12.17	(-13.96)	-10.78	(-12.29)	-4.94	(-5.04)	-1.02	(-0.02)	-0.43	(0.72)
1.0	-9.45	(-11.20)	-8.33	(-9.80)	-3.56	(-3.67)	-0.23	(0.68)	0.29	(1.33)
1.2	-7.59	(-9.29)	-6.64	(-8.07)	-2.56	(-2.69)	0.38	(1.20)	0.86	(1.80)
1.4	-6.20	(-7.87)	-5.38	(-6.78)	-1.80	(-1.94)	0.88	(1.64)	1.33	(2.20)
1.6	-5.14	(-6.76)	-4.40	(-5.77)	-1.17	(-1.32)	1.32	(2.01)	1.74	(2.54)
1.8	-4.27	(-5.85)	-3.60	(-4.93)	-0.65	(-0.80)	1.70	(2.35)	2.11	(2.85)
2.0	-3.54	(-5.08)	-2.93	(-4.22)	-0.18	(-0.34)	2.05	(2.66)	2.44	(3.14)
2.2	-2.91	(-4.41)	-2.35	(-3.61)	0.23	(0.08)	2.38	(2.95)	2.76	(3.42)
2.4	-2.37	(-3.83)	-1.84	(-3.06)	0.61	(0.45)	2.69	(3.22)	3.06	(3.68)
2.6	-1.88	(-3.30)	-1.38	(-2.57)	0.96	(0.80)	2.98	(3.49)	3.35	(3.93)
2.8	-1.44	(-2.82)	-0.96	(-2.11)	1.28	(1.13)	3.26	(3.74)	3.62	(4.17)
3.0	-1.03	(-2.37)	-0.57	(-1.70)	1.59	(1.44)	3.53	(3.99)	3.89	(4.41)
3.2	-0.65	(-1.96)	-0.22	(-1.31)	1.89	(1.74)	3.79	(4.23)	4.14	(4.65)
3.4	-0.30	(-1.57)	0.12	(-0.94)	2.17	(2.03)	4.05	(4.47)	4.40	(4.88)
3.6	0.03	(-1.20)	0.44	(-0.59)	2.44	(2.30)	4.30	(4.70)	4.64	(5.11)
3.8	0.34	(-0.85)	0.74	(-0.25)	2.71	(2.57)	4.54	(4.93)	4.88	(5.33)
4.0	0.64	(-0.52)	1.04	(0.07)	2.96	(2.83)	4.78	(5.15)	5.12	(5.55)
4.2	0.93	(-0.20)	1.32	(0.38)	3.22	(3.09)	5.02	(5.38)	5.36	(5.77)
4.4	1.21	(0.11)	1.59	(0.67)	3.46	(3.34)	5.25	(5.60)	5.59	(5.99)
4.6	1.48	(0.41)	1.85	(0.96)	3.71	(3.58)	5.48	(5.82)	5.82	(6.21)
4.8	1.75	(0.71)	2.11	(1.25)	3.94	(3.83)	5.71	(6.03)	6.05	(6.42)
5.0	2.00	(0.99)	2.36	(1.52)	4.18	(4.07)	5.94	(6.25)	6.27	(6.63)
5.2	2.26	(1.27)	2.61	(1.79)	4.41	(4.30)	6.16	(6.47)	6.50	(6.85)
5.4	2.50	(1.54)	2.85	(2.06)	4.64	(4.54)	6.39	(6.68)	6.72	(7.06)
5.6	2.74	(1.81)	3.09	(2.32)	4.87	(4.77)	6.61	(6.89)	6.94	(7.27)
5.8	2.98	(2.07)	3.32	(2.57)	5.10	(4.99)	6.83	(7.11)	7.16	(7.48)
6.0	3.22	(2.33)	3.56	(2.83)	5.32	(5.22)	7.05	(7.32)	7.37	(7.69)
6.2	3.45	(2.59)	3.79	(3.08)	5.54	(5.45)	7.26	(7.53)	7.59	(7.90)
6.4	3.68	(2.84)	4.02	(3.32)	5.77	(5.67)	7.48	(7.74)	7.81	(8.11)
6.6	3.91	(3.09)	4.25	(3.56)	5.99	(5.89)	7.70	(7.95)	8.02	(8.31)
6.8	4.14	(3.33)	4.47	(3.81)	6.20	(6.11)	7.91	(8.16)	8.24	(8.52)
7.0	4.36	(3.58)	4.70	(4.05)	6.42	(6.33)	8.13	(8.36)	8.45	(8.73)
7.2	4.58	(3.82)	4.92	(4.28)	6.64	(6.55)	8.34	(8.57)	8.67	(8.93)
7.4	4.80	(4.06)	5.14	(4.52)	6.85	(6.77)	8.55	(8.78)	8.88	(9.14)
7.6	5.02	(4.29)	5.36	(4.75)	7.07	(6.99)	8.77	(8.99)	9.09	(9.35)
7.8	5.24	(4.53)	5.57	(4.98)	7.28	(7.20)	8.98	(9.19)	9.30	(9.55)
8.0	5.46	(4.76)	5.79	(5.21)	7.50	(7.42)	9.19	(9.40)	9.51	(9.76)
8.2	5.68	(4.99)	6.01	(5.44)	7.71	(7.63)	9.40	(9.60)	9.72	(9.96)
8.4	5.89	(5.22)	6.22	(5.67)	7.92	(7.84)	9.61	(9.81)	9.93	(10.17)
8.6	6.11	(5.45)	6.43	(5.89)	8.13	(8.06)	9.82	(10.02)	10.14	(10.37)
8.8	6.32	(5.68)	6.65	(6.11)	8.34	(8.27)	10.02	(10.22)	10.35	(10.58)
9.0	6.53	(5.90)	6.86	(6.34)	8.55	(8.48)	10.23	(10.43)	10.56	(10.78)

$p=5$ r $\alpha\rightarrow$	0.025		0.05		0.50		0.95		0.975	
9.5	7.07	(6.46)	7.39	(6.89)	9.08	(9.01)	10.75	(10.94)	11.08	(11.29)
10.0	7.59	(7.02)	7.91	(7.44)	9.60	(9.53)	11.27	(11.45)	11.59	(11.80)
10.5	8.12	(7.56)	8.44	(7.98)	10.12	(10.06)	11.79	(11.96)	12.11	(12.31)
11.0	8.64	(8.11)	8.96	(8.52)	10.63	(10.58)	12.30	(12.47)	12.62	(12.81)
11.5	9.16	(8.65)	9.48	(9.06)	11.15	(11.09)	12.82	(12.97)	13.14	(13.32)
12.0	9.68	(9.19)	10.00	(9.59)	11.67	(11.61)	13.33	(13.48)	13.65	(13.82)
12.5	10.19	(9.72)	10.51	(10.12)	12.18	(12.13)	13.84	(13.99)	14.16	(14.33)
13.0	10.70	(10.25)	11.02	(10.65)	12.69	(12.64)	14.35	(14.49)	14.67	(14.84)
13.5	11.22	(10.78)	11.54	(11.18)	13.20	(13.15)	14.86	(15.00)	15.18	(15.34)
14.0	11.73	(11.31)	12.05	(11.70)	13.71	(13.67)	15.37	(15.50)	15.69	(15.84)
14.5	12.24	(11.83)	12.56	(12.22)	14.22	(14.18)	15.88	(16.01)	16.20	(16.35)
15.0	12.75	(12.36)	13.07	(12.74)	14.73	(14.69)	16.39	(16.51)	16.71	(16.85)
15.5	13.26	(12.88)	13.58	(13.26)	15.24	(15.20)	16.90	(17.02)	17.22	(17.36)
16.0	13.77	(13.40)	14.09	(13.78)	15.75	(15.71)	17.41	(17.52)	17.72	(17.86)
18.0	15.80	(15.47)	16.12	(15.85)	17.78	(17.74)	19.43	(19.54)	19.75	(19.87)
20.0	17.83	(17.53)	18.15	(17.90)	19.80	(19.77)	21.45	(21.55)	21.77	(21.88)
22.0	19.85	(19.57)	20.17	(19.94)	21.82	(21.79)	23.47	(23.56)	23.79	(23.89)
24.0	21.87	(21.61)	22.18	(21.97)	23.83	(23.81)	25.48	(25.56)	25.80	(25.89)
26.0	23.88	(23.65)	24.20	(24.00)	25.85	(25.82)	27.50	(27.57)	27.81	(27.90)
28.0	25.89	(25.67)	26.21	(26.03)	27.86	(27.83)	29.51	(29.57)	29.82	(30.90)
30.0	27.90	(27.70)	28.22	(28.05)	29.87	(29.84)	31.51	(31.58)	31.83	(31.91)