BAYESIAN HYPOTHESIS TESTING WITH SYMMETRIC AND UNIMODAL PRIORS

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Bayesian Hypothesis Testing with Symmetric and Unimodal Priors [1]

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Abstract. Lower bounds on Bayes factors in favor of the null hypothesis for some one-sided and two-sided hypothesis tests are developed. These are then applied to derive lower bounds on Bayes factors for univariate and multivariate testing problems. The general conclusion is that, for small P–values, these lower bounds tend to be substantially larger than P–values when the priors satisfy reasonable properties of symmetry and unimodality. These symmetry features are easily specified in the univariate case but can be much harder in multivariate problems.

Key Words. Lower bounds on Bayes factors, star-unimodality, lower bounds on posterior probabilities, multivariate tests.

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Short Title: Bayesian Testing

1 Introduction

1.1 Overview

Bayes factors and posterior probabilities are tools used in Bayesian hypothesis tests. Lower bounds on Bayes factors (and posterior probabilities) in favor of null hypotheses, H_0, have been discussed in Edwards, Lindman and Savage (1963), Dickey (1977), Good (1950, 1958, 1967), Berger (1985), Berger and Sellke (1987) Casella and Berger (1987), Berger and Delampady (1987), Delampady (1989a, 1989b), and Delampady and Berger (1990) among others. The startling feature of these results is that they establish that the Bayes factor and posterior probability of H_0 are generally substantially larger than the P–value. When such is the case, the interpretation of P–values as measures of evidence against H_0 requires great care.

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The interest in lower bounds on Bayes factors from Bayesian and likelihood viewpoints is that they provide bounds on the amount of evidence for the null hypothesis, in a Bayes factor or weighted likelihood ratio sense, that depend only on the general class of priors being considered, and not on a specific prior distribution. Therefore, these lower bounds are very useful when the class considered has appealing properties.

1.2 Notation

A random quantity X, having density (or mass function) $f(x|\theta,\eta)$ is observed. The unknown p-dimensional vector θ is assumed to belong to a space Θ and inferences about θ are considered. η is a nuisance parameter.

In the discussion that follows we shall consider two kinds of hypothesis testing problems. These are, respectively, testing a point null hypothesis,

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0, \tag{1}$$

where θ_0 is a specified quantity, and testing a one-sided hypothesis,

$$H_0 : c'\theta \leq c'\theta_0 \quad \text{versus} \quad H_1 : c'\theta > c'\theta_0, \tag{2}$$

where θ_0 is a specified quantity and c is a specified vector. Most of the interesting problems in analysis of variance and linear regression are of this nature.

We assume that a classical significance test for (1) or (2) is based on a test statistic $T(X)$, large values of which provide evidence against the null hypothesis, H_0. The P–value, or the observed significance level, of data x is then defined to be

$$\alpha \;=\; \sup_{\theta\in\Theta_0} P_\theta\left(T(X) \geq T(x)\right), \tag{3}$$

where Θ_0 is the subset of the parameter space specified by H_0.

Our approach to Problem (1) assumes a prior distribution π on Θ which assigns mass π_0 to θ_0 and $1-\pi_0$ to $\Theta_1 = \{\theta = \theta_0\}^c$. Let the conditional density (with respect to Lebesgue measure or counting measure as appropriate) of π on Θ_1 be g_1. In other words,

$$\pi(\theta) \;=\; \pi_0 I_{\{\theta=\theta_0\}} + (1-\pi_0) g_1(\theta) I_{\Theta_1}. \tag{4}$$

We assume an independent prior with density h on η. The quantities of interest are then
(i) the Bayes factor of H_0 relative to H_1:

$$B^\pi(x) = \frac{\int f(x|\theta_0,\eta)h(\eta)\,d\eta}{\int_{\Theta_1}\left(\int f(x|\theta,\eta)h(\eta)\,d\eta\right)g_1(\theta)\,d\theta};$$

(ii) the posterior probability of H_0:

$$P^\pi(H_0|x) = \left[1+\frac{(1-\pi_0)}{\pi_0}\frac{1}{B^\pi(x)}\right]^{-1}.$$

A similar approach is possible for Problem (2), but we prefer the following: A prior distribution π with density $g(\theta)$ and $P^\pi(H_0)=\pi_0$ is assigned to the space Θ. As before, an independent prior h is assigned to η. Since π_0 is absorbed in g, it is easier to define the posterior probability of H_0 given x:

$$P^\pi(H_0|x) = \frac{\int_{\Theta_0}\left(\int f(x|\theta,\eta)h(\eta)\,d\eta\right)g(\theta)\,d\theta}{\int_{\Theta}\left(\int f(x|\theta,\eta)h(\eta)\,d\eta\right)g(\theta)\,d\theta}.$$

Since η is a nuisance parameter, it can simply be assigned a noninformative prior in the absence of strong subjective input. On the other hand, the quantity of interest, θ, needs to be handled with care. If Γ is a class of prior distributions π under consideration, we will consider the lower bounds

$$\underline{B}_\Gamma(x) = \inf_{\pi\in\Gamma} B^\pi(x),$$

and

$$\underline{P}_\Gamma(H_0|x) = \inf_{\pi\in\Gamma} P^\pi(H_0|x) = \left[1+\frac{(1-\pi_0)}{\pi_0}\cdot\frac{1}{\underline{B}_\Gamma}\right]^{-1}.$$

Note that, once π_0, the prior probability of H_0, has been specified, $\underline{B}_G$ and $\underline{P}_G(H_0|x)$ determine each other. Therefore results can be presented in terms of one of the two quantities. In our discussion we choose $\pi_0 = 1/2$ and present only the lower bounds on the posterior probability.

1.3 Choice of G

Unlike a Bayesian, who might restrict g or g_1 to single densities, a robust Bayesian restricts g and g_1 to classes of densities which consist of all reasonable choices. But any such restrictions require specific subjective input. Of interest to Bayesians and non-Bayesians alike are choices of G which require only general shape specifications concerning G.

Emphasis of this paper will be on deriving lower bounds on Bayes factors for classes of prior densities which are interesting but have not been considered before in this context. In particular, we shall consider the class of star-unimodal densities and symmetric star-unimodal densities. A detailed discussion of the properties of star-unimodal densities can be found in Joag-Dev and Dharmadhikari (1988).

Berger and Sellke (1987), Casella and Berger (1987), Berger and Delampady (1987), and Delampady (1989) consider unimodal symmetric densities for univariate testing problems. For multivariate testing problems Berger and Delampady (1987) and Delampady and Berger (1990) consider unimodal spherically symmetric densities. Note, however, that unimodal spherical symmetry is just one particular generalization of unimodal symmetry to higher dimensions. A more general class is the class of all symmetric star-unimodal densities. In particular, standard densities such as multivariate normal and multivariate Student's t with general form of covariance matrix belong to the class of symmetric star-unimodal densities but not necessarily to that of spherically symmetric densities. A comparison of these two different generalizations of unimodal symmetry will be conducted and the implications of choosing any one particular class will be illustrated.

We shall discuss Problems (1) and (2) separately. In Section 2 Problem (1) will be studied. General results will be derived and illustrated using examples. In Section 3 Problem (2) will be discussed. Some comments and discussion will follow the main results in each of Sections 2 and 3. The proofs of the main results will be given in Appendix.

2 Bounds for Point Null Hypotheses

Consider Problem (1). Since the problem can be expressed in terms of $\theta - \theta_0$, without loss of generality assume that $\theta_0 = 0$. In this section we shall consider the two classes, star-unimodal and symmetric star-unimodal densities, for the choice of g_1 in (4).

To prove the results in this section, we require the following condition on f and h, which states that the marginal posterior density of θ has elliptical contours.

$$\int f(x|\theta,\eta)h(\eta)\,d\eta \;=\; r\left((\theta-\mu)'A^{-1}(\theta-\mu)\right), \tag{5}$$

where r is a nonnegative function decreasing in its argument, and μ and A can depend on x.

2.1 Star-unimodal Densities

Let G denote the class of prior distributions π, on the p-vector θ, of form (4) where g_1 is any star-unimodal density. Then we have the following result for the lower bound on the Bayes factor of H_0 relative to H_1, and hence an equivalent result for the lower bound on the posterior probability of H_0 given the data, x.

Theorem 1. *If the prior density h on η is such that the condition (5) holds, then*

$$\inf_{\pi\in G} B^{\pi}(x) \;=\; \left[\sup_{\beta\geq 0} \frac{p\int_0^1 v^{p-1} r\left((v\beta-1)^2\mu'A^{-1}\mu\right)\,dv}{r\left(\mu'A^{-1}\mu\right)}\right]^{-1}; \tag{6}$$

$$\inf_{\pi\in G} P^{\pi}(H_0|x) \;=\; \left[1+\frac{(1-\pi_0)}{\pi_0}\left(\sup_{\beta\geq 0}\frac{p\int_0^1 v^{p-1} r\left((v\beta-1)^2\mu'A^{-1}\mu\right)}{r\left(\mu'A^{-1}\mu\right)}\right)^{-1}\right]^{-1}. \tag{7}$$

Proof: See Appendix.

Remark: The interesting and important fact to be noted here is that the p-dimensional optimization problem reduces to a univariate optimization calculation.

Example 1. $X \sim N(\theta, I)$ in $p \geq 1$ dimensions. We want to test $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. Since the covariance matrix is assumed to be known, the problem of specifying h does not arise. Then, from (6), the problem reduces to finding $\sup_{\beta\geq 0}\int_0^1 \exp\left(-(v\beta-1)^2x'x/2\right)\,dv$.

For $p = 1$, 2, 3 and 4, $\pi_0 = 1/2$, and selected P-values, the corresponding lower bounds on the posterior probability of H_0 given x (denoted by $\underline{P}_{SU}$) are tabulated in Table 1. Discussion of these lower bounds will be deferred to a later subsection.

Example 2. $X \sim N(\theta, \Sigma)$ in $p \geq 1$ dimensions. The covariance matrix Σ is assumed to be unknown here. A random sample, $X_1, \cdots, X_n$, is observed from this distribution. As in Example 1, we want to test $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. We assign a noninformative prior on Σ, such as $|\Sigma|^{-\alpha}$, where $\alpha > 0$. Note that the sample mean $\overline{X}$ and the sample covariance matrix S are sufficient statistics. Then we see that $\int f(\overline{x}, s|\theta, \Sigma)h(\Sigma)\, d\Sigma$ is simply a p-variate Student's t density which has elliptical contours as required by (5). Therefore (6) and (7) follow with μ and A proportional to $\overline{x}$ and s respectively.

2.2 Symmetric Star-unimodal Densities

In a problem such as in Example 1 or 2 the likelihood function exhibits many symmetry features. Therefore a class G of prior densities which display some of these properties is more appealing and interesting. Let G now denote the class of prior distributions π of form (4) where g_1 is any symmetric star-unimodal density. Note that for $p = 1$ this class reduces to the class of symmetric unimodal densities. We have the following result for the lower bound on the Bayes factor of H_0 relative to H_1.

Theorem 2. *If the prior density h on η is such that the condition (5) holds, then*

$$\inf_{\pi \in G} B^\pi(x) = \left[\sup_{\beta \geq 0} \frac{p \int_0^1 v^{p-1} \left[r\left((v\beta - 1)^2 \mu' A^{-1} \mu\right) + r\left((v\beta + 1)^2 \mu' A^{-1} \mu\right)\right] dv}{2r\left(\mu' A^{-1} \mu\right)}\right]^{-1}. \qquad (8)$$

Proof: See Appendix.

Again, note that the problem reduces to a univariate optimization. Examples 1 and 2 can now be studied with this new class of densities. For $p = 1$, 2, 3 and 4, $\pi_0 = 1/2$, and some selected P-values, the lower bounds on the posterior probability (denoted by $\underline{P}_{SSU}$) obtained in Example 1 are displayed in Table 1. Also included in this table are lower bounds on posterior probabilities ($\underline{P}_{US}$) obtained for the same problem over the class of unimodal spherically symmetric densities. These numerical values were taken from Table 4 of Berger and Delampady (1987).

A comparison of these lower bounds is in order here. Clearly, the class US is contained in the class SSU which in turn is contained in SU. Therefore $\underline{P}_{US} \leq \underline{P}_{SSU} \leq \underline{P}_{SU}$. When $p = 1$, since SSU coincides with US $\underline{P}_{SSU} = \underline{P}_{US}$ and hence the values of $\underline{P}_{SSU}$ are substantially larger than the corresponding P-values, as was noted in Berger and Sellke (1987). Even the values of $\underline{P}_{SU}$ are much larger than the P-values in this case. However, the behaviour changes dramatically for $p > 1$. For $p = 2$, the values corresponding to SU and SSU are still larger than the P-values but not by much. For $p = 3$ these values actually decrease below the corresponding P-values. The reason for

Table 1: Lower bounds for star-unimodal and symmetric star-unimodal priors

	$P-value$ =.01			$P-value$ =.05			$P-value$ =.10		
Dimension=p	$\underline{P}_{SU}$	$\underline{P}_{SSU}$	$\underline{P}_{US}$	$\underline{P}_{SU}$	$\underline{P}_{SSU}$	$\underline{P}_{US}$	$\underline{P}_{SU}$	$\underline{P}_{SSU}$	$\underline{P}_{US}$
hline 1	.058	.109	.109	.174	.29	.29	.254	.39	.39
2	.014	.027	.089	.061	.113	.258	.108	.196	.363
3	.004	.009	.083	.024	.046	.246	.049	.093	.351
4	.003	.003	.078	.019	.020	.239	.044	.047	.344

this behaviour can be understood by examining the proofs of Theorems 1 and 2. Note that the least favourable prior density actually has its support on the real line irrespective of the dimension p of θ. This implies that the class SSU (and hence SU also) becomes simply too large as p increases.

3 Bounds for One-sided Testing

Here we consider Problem (2). In addition, we assume that $X \sim N(\theta, I)$ in p dimensions. The case of unknown covariance matrix can most probably be handled under an assumption similar to (5). As before, we assume, without loss of generality, that $\theta_0 = 0$. Unlike Problem (1), the hypotheses here are on an equal footing. Therefore it is reasonable to assume that $\pi_0 = 1/2$ and assign a symmetric prior distribution to θ. Let G denote the class of all symmetric star-unimodal densities. Then we have the following result for the lower bound on the posterior probability of $H_0 : c'\theta \leq 0$:

Theorem 3. *If $c'x \geq 0$,*

$$\inf_{\pi \in G} P^{\pi}(H_0|x) = \left[1 + \sup_{\beta \leq 0} \frac{\int_0^1 v^{p-1} \exp\left(-(v\beta+1)^2 x'x/2\right) dv}{\int_0^1 v^{p-1} \exp\left(-(v\beta-1)^2 x'x/2\right) dv}\right]^{-1}. \quad (9)$$

Proof: See Appendix.

Remark 1. As in Theorems 1 and 2, the p-variate optimization problem reduces to a univariate maximization calculation.

Remark 2. If $p = 1$, since symmetric star-unimodality is the same as symmetric unimodality, a more general result can be proved as shown in Theorem 3.2 of Casella and Berger (1987).

This result of Casella and Berger shows that the lower bound on the posterior probability coincides with the corresponding P-value. Our interest, however, is to examine the behaviour of the symmetric star-unimodal class of densities in higher dimensions. For $p = 2$ and selected P-values, we have computed the lower bounds on the posterior probabilities of $H_0 : \theta_1 - \theta_2 \leq 0$, where $\theta = (\theta_1, \theta_2)'$. Recall that $X \sim N(\theta, I)$. Therefore the classical test (likelihood ratio test) rejects H_0 when $(X_1 - X_2)^2/2$ is large. Also, this statistic has a χ^2 distribution with one degree of freedom if H_0 is true.

P-value	.01	.05	.10
$\underline{P}$	.0015	.0123	.0344

The lower bounds given above indicate again that the class of symmetric star-unimodal densities is too large, and that it contains densities which are unreasonable as priors. This can be seen by noting that the least favourable densities, at which the above lower bounds are attained, have support on the real line and hence are not two-dimensional priors. Thus, reasonable priors to be considered in these problems need to have restrictions in addition to symmetry and star-unimodality.

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Appendix

Proofs of Theorems 1, 2 and 3 are given here. In all the proofs we make use of the following representation for a star-unimodal distribution (see Dharmadhikari and Joag-Dev, 1988):

Result. *If Y has a p-variate star-unimodal distribution, then $Y = U^{1/p}Z$, where U has a Uniform distribution in $[0,1]$ and Z is a p-dimensional random variable independent of U.*

Proof of Theorem 1. Note, from (5), that

$$B^{\pi}(x) \;=\; \frac{r\left(\mu' A^{-1}\mu\right)}{\int_{\theta\neq 0} r\left((\theta-\mu)' A^{-1}(\theta-\mu)\right) g_1(\theta)\, d\theta}.$$

Hence, we need to compute $E\left[r\left((\theta-\mu)'A^{-1}(\theta-\mu)\right)\right]$, where E denotes the expectation with respect to the distribution which has density g_1. Since g_1 is a star-unimodal density, $\theta = U^{1/p}Z$, where $U \sim U[0,1]$ and Z has some distribution in R^p. Therefore,

$$\begin{aligned} E\left[r\left((\theta-\mu)'A^{-1}(\theta-\mu)\right)\right] &= E\left[\int_0^1 r\left((u^{1/p}Z-\mu)'A^{-1}(u^{1/p}Z-\mu)\right)\,du\right] \\ &= E\left[p\int_0^1 v^{p-1} r\left((vZ-\mu)'A^{-1}(vZ-\mu)\right)\,dv\right] \\ &\equiv E\left[\psi(Z)\right], \end{aligned}$$

where E on the previous line denotes the expectation with respect to the distribution of Z. Now note that, since the distribution of Z is arbitrary,

$$\sup E\left[\psi(Z)\right] = \sup_{z \in R^p} \psi(z).$$

Therefore,

$$\inf_{\pi \in G} B^{\pi}(x) = \left[\sup_{z \in R^p} \frac{p\int_0^1 v^{p-1} r\left((vz-\mu)'A^{-1}(vz-\mu)\right)\,dv}{r\left(\mu' A^{-1}\mu\right)}\right]^{-1}.$$

Since r is a nonnegative decreasing function, to find the supremum required above, we need only maximize $(vz-\mu)'A^{-1}(vz-\mu)$ over $z \in R^p$. We now note that the z that achieves that maximum is $\beta\mu$, where $\beta \geq 0$. □

Proof of Theorem 2. This is very similar to the proof above. Note that any symmetric star-unimodal distribution has the same representation as given in Result above with the added condition that Z be symmetric. Therefore the only change needed in the proof of Theorem 1 is in the evaluation of $\sup E\left[\psi(Z)\right]$. We need to find this supremum over the class of symmetric Z. Note that symmetric distributions in R^p are mixtures of 2-point symmetric distributions. Therefore, we now have,

$$\sup E\left[\psi(Z)\right] = \sup_{z \in R^p} \left[\psi(z) + \psi(-z)\right]/2.$$

Rest of the argument is exactly as in Theorem 1. □

We need the following lemma to prove Theorem 3.

Lemma. *For each $a > 0$, and $b \geq 0$, $\left[\int_0^a x^b \exp\left(-\frac{1}{2}(x+c)^2\right) dx\right] / \left[\int_0^{1/a} x^b \exp\left(-\frac{1}{2}(x-c)^2\right) dx\right]$ is a monotone decreasing function of c.*

Proof of Lemma. Consider any $c_1 < c_2$. Then we shall show that

$$\frac{\int_0^a x^b \exp\left(-\frac{1}{2}(x+c_2)^2\right) dx}{\int_0^a x^b \exp\left(-\frac{1}{2}(x-c_2)^2\right) dx} \leq \frac{\int_0^a y^b \exp\left(-\frac{1}{2}(y+c_1)^2\right) dy}{\int_0^a y^b \exp\left(-\frac{1}{2}(y-c_1)^2\right) dy}.$$

Equivalently, we need to show that,

$$\begin{aligned} &K \\ &\equiv \int_0^a \int_0^a x^b y^b \left\{\exp\left(-\frac{1}{2}\left[(x+c_2)^2 + (y-c_1)^2\right]\right) - \exp\left(-\frac{1}{2}\left[(x-c_2)^2 + (y+c_1)^2\right]\right)\right\} dx\, dy \\ &\leq 0. \end{aligned}$$

Clearly,

$$\begin{aligned} K \;=\; & \int_{a\geq x\geq y\geq 0} x^b y^b \Big\{\exp\left(-\frac{1}{2}\left[(x+c_2)^2 + (y-c_1)^2\right]\right) \\ & - \exp\left(-\frac{1}{2}\left[(x-c_2)^2 + (y+c_1)^2\right]\right)\Big\} dx\, dy \\ & + \int_{a\geq y\geq x\geq 0} x^b y^b \Big\{\exp\left(-\frac{1}{2}\left[(x+c_2)^2 + (y-c_1)^2\right]\right) \\ & - \exp\left(-\frac{1}{2}\left[(x-c_2)^2 + (y+c_1)^2\right]\right)\Big\} dx\, dy, \end{aligned}$$

and since,

$$\begin{aligned} & \int_{a\geq x\geq y\geq 0} x^b y^b \left\{\exp\left(-\frac{1}{2}\left[(x+c_2)^2 + (y-c_1)^2\right]\right) - \exp\left(-\frac{1}{2}\left[(x-c_2)^2 + (y+c_1)^2\right]\right)\right. \\ & = \int_{a\geq y\geq x\geq 0} x^b y^b \Big\{\exp\left(-\frac{1}{2}\left[(x-c_1)^2 + (y+c_2)^2\right]\right) \\ & - \exp\left(-\frac{1}{2}\left[(x+c_1)^2 + (y-c_2)^2\right]\right)\Big\} dx\, dy, \end{aligned}$$

$$\begin{aligned} K \;=\; & \int_{a\geq y\geq x\geq 0} x^b y^b \Big\{\exp\left(-\frac{1}{2}\left[(x+c_2)^2 + (y-c_1)^2\right]\right) + \exp\left(-\frac{1}{2}\left[(x-c_1)^2 + (y+c_2)^2\right]\right) \\ & - \exp\left(-\frac{1}{2}\left[(x-c_2)^2 + (y+c_1)^2\right]\right) - \exp\left(-\frac{1}{2}\left[(x+c_1)^2 + (y-c_2)^2\right]\right)\Big\} dx\, dy \\ =\; & \exp\left(-\frac{1}{2}(x^2+y^2)\right) \int_{a\geq y\geq x\geq 0} x^b y^b \Big\{\exp(-c_2 x + c_1 y) + \exp(c_1 x - c_2 y) \\ & - \exp(c_2 x - c_1 y) - \exp(-c_1 x + c_2 y)\Big\} dx\, dy. \end{aligned}$$

Since $c_1y - c_2x \le c_2y - c_1x$, we get, $\exp(-c_2x + c_1y) \le \exp(-c_1x + c_2y)$ and $\exp(c_1x - c_2y) \le \exp(c_2x - c_1y)$. Therefore $K \le 0$, as required. $\square$

Proof of Theorem 3. From Result above,

$$\begin{aligned}
P^\pi(H_0|x) &= \frac{\int_{\{c'\theta\le 0\}} \exp\left(-(\theta-x)'(\theta-x)/2\right) g(\theta)\,d\theta}{\int_{R^p} \exp\left(-(\theta-x)'(\theta-x)/2\right) g(\theta)\,d\theta} \\
&= \frac{E\left[\exp\left(-(U^{1/p}Z-x)'(U^{1/p}-x)/2\right) I_{\{U^{1/p}c'Z\le 0\}}\right]}{E\left[\exp\left(-(U^{1/p}Z-x)'(U^{1/p}-x)/2\right)\right]} \\
&= \frac{\int_{\{c'z\le 0\}} \int_0^1 v^{p-1}\exp\left(-(vz-x)'(vz-x)/2\right)\,dv\,dF(z)}{\int_{R^p}\int_0^1 v^{p-1}\exp\left(-(vz-x)'(vz-x)/2\right)\,dv\,dF(z)} \\
&= \frac{\int_{R^p}\psi_2(z)\,dF(z)}{\int_{R^p}\psi_1(z)\,dF(z)},
\end{aligned}$$

where F is a symmetric distribution depending on g, and

$$\begin{aligned}
\psi_1(z) &= \int_0^1 v^{p-1}\exp\left(-(vz-x)'(vz-x)/2\right)\,dv, \\
\psi_2(z) &= \psi_1(z) I_{\{c'z\le 0\}}.
\end{aligned}$$

Since, F is a mixture of 2-point symmetric distributions,

$$\begin{aligned}
\inf_{\pi\in G} P^\pi(H_0|x) &= \inf_{z\in R^p}\left[\frac{\psi_2(z)+\psi_2(-z)}{\psi_1(z)+\psi_1(-z)}\right] \\
&= \inf_{z\in R^p}\left[\frac{\psi_1(z)I_{\{c'z\le 0\}}+\psi_1(-z)I_{\{c'z\ge 0\}}}{\psi_1(z)+\psi_1(-z)}\right] \\
&= \min\left\{\inf_{\{z:c'z\le 0\}}\frac{\psi_1(z)}{\psi_1(z)+\psi_1(-z)}, \inf_{\{z:c'z\ge 0\}}\frac{\psi_1(-z)}{\psi_1(z)+\psi_1(-z)}\right\} \\
&= \inf_{\{z:c'z\le 0\}}\frac{\psi_1(z)}{\psi_1(z)+\psi_1(-z)} \\
&= \left[1+\sup_{\{z:c'z\le 0\}}\frac{\psi_1(-z)}{\psi_1(z)}\right]^{-1}.
\end{aligned}$$

The third equality follows from

$$\inf_{\{z:c'z\le 0\}}\frac{\psi_1(z)}{\psi_1(z)+\psi_1(-z)} = \inf_{\{z:c'z\ge 0\}}\frac{\psi_1(-z)}{\psi_1(z)+\psi_1(-z)}.$$

Set $\sigma^2 = (z'z)^{-1}$, $\mu = \frac{z'x}{z'z}$ and $\nu = \mu/\sigma$. Then, we have,

$$\frac{\psi_1(-z)}{\psi_1(z)} = \frac{\int_0^1 v^{p-1}\exp\left(-(vz+x)'(vz+x)/2\right)\,dv}{\int_0^1 v^{p-1}\exp\left(-(vz-x)'(vz-x)/2\right)\,dv}$$

$$= \frac{\int_0^1 v^{p-1} \exp\left(-\frac{1}{2\sigma^2}(v+\mu)^2\right) dv}{\int_0^1 v^{p-1} \exp\left(-\frac{1}{2\sigma^2}(v-\mu)^2\right) dv}$$

$$= \frac{\int_0^{1/\sigma} v^{p-1} \exp\left(-\frac{1}{2}(v+\nu)^2\right) dv}{\int_0^{1/\sigma} v^{p-1} \exp\left(-\frac{1}{2}(v-\nu)^2\right) dv}.$$

From Lemma above, $\psi_1(-z)/\psi_1(z)$ is monotone decreasing in $\nu = z'x/\sqrt{z'z}$ for each fixed value of $z'z$. Then to maximize $\psi_1(-z)/\psi_1(z)$, subject to $c'z \leq 0$, we need only maximize $z'x$ for fixed $z'z$, subject to $c'z \leq 0$. The unrestricted maximum of $z'x$ for fixed $z'z$ is achieved by $z_0 = \beta_0 x$, $\beta_0 \leq 0$. However, since we already have $c'x \leq 0$, it follows that

$$c'z_0 = \beta_0 c'x \leq 0.$$

Therefore, z_0 satisfies the restriction $c'z \leq 0$, thus giving us the restricted maximum also. This implies that we need to maximize

$$\frac{\int_0^1 v^{p-1} \exp\left(-(v\beta+1)^2 x'x/2\right) dv}{\int_0^1 v^{p-1} \exp\left(-(v\beta-1)^2 x'x/2\right) dv},$$

which completes the proof. □

From dasgupta Thu May 7 14:59:00 1992
Received: by pop.stat.purdue.edu (5.61/Purdue_CC)
id AA09734; Thu, 7 May 92 14:58:56 -0500
Date: Thu, 7 May 92 14:58:56 -0500
From: dasgupta (Anirban DasGupta)
Message-Id: <9205071958.AA09734@pop.stat.purdue.edu>
To: nlucas@pop.stat.purdue.edu
Subject: Re: your joint paper
Status: R

yes, thank you.

Regarding Tech report # 90-47

1. remove ~~the~~ Anirban's name from this tech report (original in Brenda's file.
2. Then you can send this paper to list of names in folder + Mary needs a copy for a request ~~that~~ she has
3. put 5 c. in 519
5c mail to Delampady
both sides, front to back

Took DasGupta's name off of the cover page and page 1 as per request. Original cover page attached as Tech report is not in computer file.
B. Williams 5-8-92

```
From nlucas Thu May  7 09:01:45 1992
Received: by pop.stat.purdue.edu (5.61/Purdue_CC)
        id AA05502; Thu, 7 May 92 09:01:39 -0500
Date: Thu, 7 May 92 09:01:39 -0500
From: nlucas (Norma Lucas)
Message-Id: <9205071401.AA05502@pop.stat.purdue.edu>
To: dasgupta@l.cc.purdue.edu
Cc: nlucas@pop.stat.purdue.edu
Status: R
```

Hi Anirban,

I hope everything worked out okay last night.

Regarding paper #90-47 with Delampady, Bayesian Hypothesis .... Priors, is it ready for distribution yet? We have several requests for that paper.

Thanks Norma

Delampady

From dasgupta Thu May 7 12:25:42 1992
Received: by pop.stat.purdue.edu (5.61/Purdue_CC)
id AA07813; Thu, 7 May 92 12:25:41 -0500
Date: Thu, 7 May 92 12:25:41 -0500
From: dasgupta (Anirban DasGupta)
Message-Id: <9205071725.AA07813@pop.stat.purdue.edu>
To: nlucas@pop.stat.purdue.edu
Status: RO

Norma,

If people are asking for this paper, please go ahead and send it to them, but please have the pages that have my name on this work removed before they are sent out.

anirban

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From dasgupta Thu May  7 12:27:26 1992
Received: by pop.stat.purdue.edu (5.61/Purdue_CC)
        id AA07828; Thu, 7 May 92 12:27:20 -0500
Date: Thu, 7 May 92 12:27:20 -0500
From: dasgupta (Anirban DasGupta)
Message-Id: <9205071727.AA07828@pop.stat.purdue.edu>
To: nlucas@pop.stat.purdue.edu
Status: RO
```

What I mean is the first one or two pages that have me listed as a coauthor.

anirbaban

BAYESIAN HYPOTHESIS TESTING WITH SYMMETRIC AND UNIMODAL PRIORS

by

Anirban DasGupta and Mohan Delampady
Purdue University Univ. of British Columbia

Technical Report #90-47

Department of Statistics
Purdue University

August 1990

*

Bayesian Hypothesis Testing with Symmetric and Unimodal Priors [1]

Anirban DasGupta and Mohan Delampady
Purdue University   University of British Columbia

Abstract. Lower bounds on Bayes factors in favor of the null hypothesis for some one-sided and two-sided hypothesis tests are developed. These are then applied to derive lower bounds on Bayes factors for univariate and multivariate testing problems. The general conclusion is that, for small P–values, these lower bounds tend to be substantially larger than P–values when the priors satisfy reasonable properties of symmetry and unimodality. These symmetry features are easily specified in the univariate case but can be much harder in multivariate problems.

Key Words. Lower bounds on Bayes factors, star-unimodality, lower bounds on posterior probabilities, multivariate tests.

1980 AMS Subject Classification: Primary 62A15; Secondary 62F15

Short Title: Bayesian Testing

1 Introduction

1.1 Overview

Bayes factors and posterior probabilities are tools used in Bayesian hypothesis tests. Lower bounds on Bayes factors (and posterior probabilities) in favor of null hypotheses, H_0, have been discussed in Edwards, Lindman and Savage (1963), Dickey (1977), Good (1950, 1958, 1967), Berger (1985), Berger and Sellke (1987) Casella and Berger (1987), Berger and Delampady (1987), Delampady (1989a, 1989b), and Delampady and Berger (1990) among others. The startling feature of these results is that they establish that the Bayes factor and posterior probability of H_0 are generally substantially larger than the P–value. When such is the case, the interpretation of P–values as measures of evidence against H_0 requires great care.

[1]This research was supported by the National Science Foundation (U.S.) Grant DMS-89-230-71, and Natural Sciences and Engineering Research Council (Canada) Grant A9250.

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90-38C
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Mailed 5/8/92 [initials]

Our fax number is (3468) 835418. We have not e-mail yet but you can send us messages to the following one we have borrowed from a friend: a.toval@dia.um.es

When finished send ~~them~~ Juan the above reports.