ON SELECTING A POPULATION CLOSE TO A CONTROL: A NONPARAMETRIC APPROACH

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Abstract

In this paper we investigate the problem of selecting a population close to a control or a standard in a nonparametric setup. The measure of distance between two distribution functions is assumed to be Kolmogrov-Smirnov distance function. The problems of selecting the distribution closest to the control under the indifference zone approach and the subset selection approach are investigated. Asymptotic results on the probability of correct selection are obtained. Finally, results of some simulation studies concerning small sample performance of the procedures is given.

Key Words: Selection and ranking, Nonparametric, Kolmogrov-Smirnov distance function.

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1 Introduction

In many practical situations the goal of the experimenter is to compare two or more populations with a control and select one (or more) which is (are) closest to the control. For example, one may wish to select a simulation method among several simulation methods. Statistical selection and ranking methodology provides useful techniques for solving such problems. Often in practice, especially for the new treatments (methods) or when the observations are expensive, there is very little past data available which could lead us to assume some parametric model. For this reasons we adopt a nonparametric approach.

Selection and ranking problems have been generally formulated using either indifference zone approach due to Bechhofer (1954) or the subset selection approach due to Gupta (1956). In the indifference zone approach, a single population is chosen and is guaranteed to be the best with probability at least P^*. However, in this formulation it is assumed that the "best" population is sufficiently apart from the remaining $k-1$ populations. Subset selection approach requires no such restriction. A random size subset of k populations is selected and is guaranteed to contain the best population. Additionally, the subset selection approach is much more data-dependent *i.e.* the decision rule takes into account the outcome of the experiment.

In the nonparametric setup, considerable amount of work has been done on the problems of selecting the population with the largest αth quantile (or the largest location parameter). Some references are Barlow and Gupta (1969), Gupta and McDonald (1970) , Gupta and Huang (1974), among others. An extensive review of nonparametric procedures is in Desu and Bristol (1986). Recently Gupta and Hande (1990) have solved the problem of selecting a population with the largest functional (or selecting a subset containing a population with the largest functional) of the associated distribution function.

To formulate the problem, let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be the $k+1$ populations. The population

Π_i is associated with the cumulative distribution function F_i over R, for $i = 0, 1, 2, \ldots, k$. We are interested in selecting the population which is closest to Π_0. Problem of selecting the populations when they are compared with the control are considered in the literature by Bechhofer and Turnbull (1978), Roberts (1964), Dunnett (1955), Gupta and Sobel (1958), Gupta and Singh (1979, 1980), among others.

To measure closeness, one finds many measures of distance between two distribution functions. In this paper we use the distance between two distribution functions as the Kolmogrov-Smirnov distance. A vast amount of literature is available for statistical inference based on the Kolmogrov-Smirnov distance. For references see, for example, Billingsley (1968).

We will propose selection procedures under the indifference zone and the subset selection approach. Under some regularity conditions, lower bounds for the probability of correct selection associated with proposed procedures are obtained.

2 Formulation of the Problem

Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ denote k populations with associated distribution functions $F_1, F_2, \ldots, F_k$, respectively. We assume no prior information about the distribution functions. Let the population Π_0 be the control with associated distribution function $F_0(.)$. We define the distance between two distributions F, and G as the Kolmogrov-Smirnov distance function,

$$d(F, G) = \sup_{x \in R} |F(x) - G(x)|.$$

Define $\mu_i = d(F_i, F_0)$ and let $\mu_{[1]} \leq \mu_{[2]} \ldots \leq \mu_{[k]}$ denote the ordered values of $\mu_1, \mu_2, \ldots, \mu_k$. We assume no prior knowledge about the correct pairing of the ordered and unordered μ_i's. Let

$$\Omega = \{\mu : \mu = (\mu_1, \mu_2, \ldots, \mu_k)\}$$

and

$$\mathcal{F} = \{F : F = (F_1, F_2, \ldots, F_k),\ \ F_i \text{ is a continuous distribution function on } R\}.$$

In general, if we allow F to take any value in $\mathcal{F}$ there does not exist a procedure which would satisfy the P^* condition, hence we need to restrict the space of distributions. Let d be the real number in the interval $(0,1)$. Define

$$\Omega' = \Omega(d) = \{\mu : \quad \mu_{[2]} - \mu_{[1]} \geq d\},$$

and

$$\mathcal{F}' = \mathcal{F}(d) = \{F : \sup_{x \in R} |F_i(x) - F_0(x)| = \mu_i; \quad \mu \in \Omega(d)\}.$$

Correct Selection (CS): Selecting the best.

Goal: For given P^* $(1/k < P^* < 1)$, define a procedure R such that for all $n \geq n_o(d)$

$$P_F(CS|R, n) \geq P^* \quad \text{for every } F \in \mathcal{F}(d).$$

Let $X_{i1}, X_{i2}, \ldots, X_{in}$ be the observable independent random variables from population Π_i, for $i = 1, 2, \ldots, k$. Let $F_{in}(.)$ be the empirical distribution from population Π_i *i.e.*

$$F_{in}(x) = n^{-1} \sum_{j=1}^{n} I_{(X_{ij} \leq x)}.$$

Let $T_{in}(x) = \sup_{t \in R} |F_{in}(t) - F_0(t)|$.

Now we propose the following selection procedure:

Procedure R:

Select the population Π_i for which

$$T_{in}(x) = \min_j T_{jn}(x);$$

and in case of ties randomize.

The following theorem insures that there exists an $n_0 = n(d, P^*)$ such that for every $n \geq n_0$ the P^* condition is satisfied.

Theorem 2.1

$$\inf_{F \in \mathcal{F}'} P_F(CS|R,n) \longrightarrow 1 \quad as \quad n \longrightarrow \infty.$$

Proof:

If necessary, making transformations, without loss of generality we can assume that F_i is uniform distribution over the interval $(0,1)$ and $T_{in} = \sup |F_{in}(x) - G_i(x)|$ where $G_i = F_0 F_i^{-1}$, and $\mu_i = \sup_{[0,1]} |t - G_i(t)|$. Let Π_1 be the best population *i.e* $\mu_1 = \mu_{[1]}$ and

$$A_i = \{t : |t - G_i(t)| > \mu_i - \epsilon\} \text{ for } i = 2, \ldots, k \ ,$$

where $0 < \epsilon < d$. It is easy to see that,

$$\begin{aligned} T_{1n} &\leq \sup_t |F_{1n}(t) - t| + \sup_t |t - G_1(t)| \\ &\leq \sup_t |F_{1n}(t) - t| + \mu_1. \end{aligned}$$

And for $i = 2, \ldots, k$,

$$\begin{aligned} T_{in} &\geq \sup_{t \in A_i} |t - G_i(t)| - \sup |F_{in}(t) - t| \\ &\geq \mu_2 - \epsilon - \sup |F_{in} - t|. \end{aligned}$$

Hence

$$\begin{aligned} P_F(CS) &\geq P(\sup_t |F_{1n}(t) - t| + \mu_1 \leq \mu_2 - \epsilon - \sup |F_{in}(t) - t|) \\ &\geq P(\sup_t |F_{1n}(t) - t| < d - \epsilon - \sup |F_{in}(t) - t|). \end{aligned}$$

Note that the last term of the above equation is distribution free. Hence we have

$$\inf_{F \in \mathcal{F}'} P_F(CS) \geq P(\sup_t |F_{1n}(t) - t| < d - \epsilon - \sup |F_{in}(t) - t|).$$

By Glivenko-Cantelli theorem, it follows that for every $i = 1, 2, \ldots, k$, $\sup_t |F_{in}(t) - t| \longrightarrow 0$ almost surely as $n \longrightarrow \infty$. Since d is a positive number the result follows.

In the subset selection approach a random size subset of the k populations is selected. The selected subset of the k populations is guaranteed to contain the best population (population closest to the control) with probability P^*. In this approach we need not assume any restriction on the "parameter space". This approach is due to Gupta (1956). Following Gupta (1956), define

Correct Selection (CS): Selecting a subset containing the best.

Goal: For given P^* find a procedure R such that

$$P_F(CS|R,n) \geq P^* \quad \text{for every } F \in \mathcal{F}. \tag{1}$$

Now we propose Gupta's maximum type procedure.

Procedure R_d:

Select population Π_i if and only if

$$T_{in}(x) \leq \min_j T_{jn}(x) + d,$$

where $d = d(n, k, P^*)$ is a constant in the interval $(0, 1]$ which is determined in advance to satisfy the P^*-condition.

For mathematical convenience we assume, that no population is identical to the control *i.e.* $\min_i \sup_x |F_i(x) - F_0(x)| > 0$. Let

$$\mathcal{F}_s = \{F : \min_i \sup_x |F_i(x) - F_0(x)| > 0\}.$$

The constant d is to be chosen to satisfy the probability requirement (1). We have the following theorem analogous to Theorem 2.2.

Theorem 2.2 *For every $d > 0$;*

$$\inf_{F \in \mathcal{F}} P_F(CS|R_d, n) \longrightarrow 1 \quad as \quad n \longrightarrow \infty.$$

We need to find out the required minimum sample size to satisfy the P^* condition. The analytic expressions for $P_F(CS|R,n)$, $P_F(CS|R_d,n)$ are almost impossible to handle. We will study their behavior for large n.

3 Asymptotic Results

First we give some notation. Let $X_1, X_2, \ldots, X_n$ be the independent observations from a continuos distribution F and let G be another independent continuous distribution. Let $\lambda = d(F,G)$ and $K = \{x : d(F,G) = |F(x) - G(x)|\}$ $K_1 = \{x : F(x) - G(x) = d(F,G)\}$, $K_2 = K - K_1$. Let $F_n(x) = n^{-1}\sum_{i=1}^{n} I_{(X_i \leq x)}$ $T_n = d(F_n, G)$, $Z_n = \sup_{x\in K} |F_n(x) - G(x)|$.

We need the following lemma, which was proved by Raghavachari (1973). Here we give a simpler proof.

Lemma 3.1

$$\sqrt{n}|T_n - Z_n| \longrightarrow 0 \;\; as \;\; n \longrightarrow \infty.$$

Proof: Let $F' = FG^{-1}$, $Y_i = G(X_i)$ for $i = 1, \ldots, k$ and U be the distribution function of uniform random variable. Since

$$\begin{aligned} d(F_n, G) &= \max_{1\leq i\leq n} |i/n - G(X_{(i)})| \\ &= \max_{1\leq i\leq n} |i/n - Y_{(i)}| \\ &= d(F_n', U). \end{aligned}$$

Without loss of generality, assume that $G = U$ and F is a distribution function on the interval $(0, 1)$.

Case 1: Assume that $K = \{t\}$. Without loss of generality assume that $F(t) - t = \lambda = d(F,G)$. Let $B(\delta)$ be a δ neighborhood around t, then

$$\begin{aligned} P(\sqrt{n}|T_n - Z_n| > \epsilon) &= P(\sqrt{n}(T_n - Z_n) > \epsilon) \\ &\leq P(\sqrt{n}(\sup_{B(\delta)} |F_n(x) - G(x)| - Z_n) > \epsilon) \\ &+ P(\sqrt{n}(\sup_{B^c(\delta)} |F_n(x) - G(x)| - Z_n) > \epsilon). \end{aligned}$$

Let $A_n = \{|F_n(x) - G(x)| = F_n(x) - G(x) \text{ for all} x \in B(\delta)\}$, then

$$
\begin{aligned}
& P(\sqrt{n}(\sup_{B(\delta)}|F_n(x)-G(x)|-Z_n)>\epsilon) \\
\leq\ & P(\sqrt{n}(\sup_{B(\delta)}|F_n(x)-G(x)|-Z_n)>\epsilon\cap A_n)+P(A_n^c) \\
\leq\ & P(\sqrt{n}(\sup_{B(\delta)}F_n(x)-F(x)+F(x)-G(x)-F_n(t)+G(t))>\epsilon)+P(A_n^c) \\
\leq\ & P(\sqrt{n}(\sup_{B(\delta)}F_n(x)-F(x)+F(t)-G(t)-F_n(t)+G(t))>\epsilon)+P(A_n^c) \\
=\ & P(\sqrt{n}(\sup_{B(\delta)}F_n(x)-F(x)+F(t)-F_n(t))>\epsilon)+P(A_n^c).
\end{aligned}
$$

Hence

$$
\begin{aligned}
P(\sqrt{n}|T_n-Z_n|>\epsilon) \ \leq\ & P(\sqrt{n}(\sup_{B(\delta)}F_n(x)-F(x)+F(t)-F_n(t))>\epsilon)+P(A_n^c) \\
+\ & P(\sqrt{n}(\sup_{B^c(\delta)}|F_n(x)-G(x)|-Z_n)>\epsilon).
\end{aligned}
$$

Since $\sqrt{n}(F_n(x)-F(x))$ converges in distribution to a brownian bridge $W(x)$ for small fixed $\delta>0$ as $n\longrightarrow\infty$ Also $P(A_n)\longrightarrow 0$ and since $(\sup_{B^c(\delta)}|F_n(x)-G(x)|-Z_n)\longrightarrow c<0$. Hence

$$
\limsup_{n\to\infty}P(\sqrt{n}|T_n-Z_n|>\epsilon)\leq P(\sup_{B(\delta)}W(x)-W(t)>\epsilon)
$$

by letting $\delta\longrightarrow 0$ we have

$$
\limsup_{n\to\infty}P(\sqrt{n}|T_n-Z_n|>\epsilon)=0.
$$

Case 2: Assume $K=[t_1,t_2]$ a closed interval,

$$
P(\sqrt{n}|T_n-Z_n|>\epsilon)=P(\sqrt{n}(\sup_{x\in K^c}|F_n(x)-F(x)|-\sup_{x\in K}|F_n(x)-F(x)|)>\epsilon).
$$

Let $B(\delta,t_1)$ and $B(\delta,t_2)$ be the open neighborhood around t_1 and t_2 respectively. As before we have the following inequality.

$$
\begin{aligned}
P(\sqrt{n}|T_n-Z_n|>\epsilon) \ \leq\ & P(\sqrt{n}(\sup_{B(\delta,t_1)}|F_n(x)-G(x)|-|F_n(t_1)-G(t_1)|>\epsilon)) \\
+\ & P(\sqrt{n}(\sup_{B(\delta,t_2)}|F_n(x)-G(x)|-|F_n(t_1)-G(t_1)|>\epsilon)) \\
+\ & P(\sqrt{n}(\sup_{(B(\delta,t_1)\cup B(\delta,t_2)\cup K)^c}|F_n(x)-G(x)|-|F_n(t_1)-G(t_1)|)>\epsilon).
\end{aligned}
$$

As $n \longrightarrow \infty$ the first two terms can shown to tend to 0 as in Case 1. Since as $n \longrightarrow \infty$ $F_n(x) \longrightarrow F(x)$ uniformly almost surely and since for all $x \in (B(\delta,t_1) \cup B(\delta,t_2) \cup K)^c$, $|F(x) - G(x)| - |F(t) - G(t)| < c < 0$, as $n \longrightarrow \infty$,

$$P(\sqrt{n}(\sup_{(B(\delta,t_1)\cup B(\delta,t_2)\cup K)^c} |F_n(x) - G(x)| - |F_n(t_1) - G(t_1)|) > \epsilon) \longrightarrow 0.$$

This completes the proof of the Case 2. Now we will consider the most general case.

Case 3: Note that K is a compact set on the interval $[0,1]$, hence it can be written as a finite union of closed intervals. Then proof of this case is similar as in Case 2. This completes the proof of the lemma.

Since $X_n(.) = X_n(t) = \sqrt{n}(F_n(t) - F(t))$ converges in distribution to $W(t)$, we have the following corollary.

Corollary 3.1

$$P(\sqrt{n}(T_n - d(F,G)) < \alpha) \longrightarrow P(\sup_{t\in K_1} W(t) < \alpha \ ; \ \sup_{t\in K_2} -W(t) < \alpha),$$

where $W(t)$ is a brownian bridge with $EW(t) = 0$ $EW(t)W(s) = F(s)(1 - F(t))$ for $s \leq t$.

Following is the result in the direction of obtaining $\inf_{F\in\mathcal{F}'} P(CS|R,n)$.

Theorem 3.1 *:*

For large n,

$$\inf_{F\in\mathcal{F}'} P(CS|R,n) \geq \inf_{0\leq\sigma_i\leq\frac{1}{2}} P(\sup_{t\in[0,1]} |W(t)| \leq \min_{2\leq i\leq k} \sigma_i Z_i + \sqrt{n}d) \tag{2}$$

and for every $F \in \mathcal{F}_s$

$$P_F(CS|R(d),n) \geq \inf_{0\leq\sigma_i\leq\frac{1}{2}} P(\sup_{t\in[0,1]} |W(t)| \leq \min_{2\leq i\leq k} \sigma_i Z_i + \sqrt{n}d), \tag{3}$$

where $W(.)$ is a brownian bridge and $Z_2, Z_3, \ldots, Z_k$ are independent standard normal random variables.

Proof: Using Corollary 3.1 for large n

$$P(CS|R,n) \simeq P(H_1(W_1)+\sqrt{n}\mu_{[1]} \le H_i(W_i)+\sqrt{n}\mu_{[i]} \text{ for } i=2,3,\ldots,k),$$

where $W_i(t)$ are independent brownian bridges as defined in the corollary,

$$H_i(W_i) = \max(\sup_{K_{1i}} W_i(t), \sup_{K_{2i}} -W_i(t))$$

and

$$\begin{aligned} K_{1i} = \{t : F_i(t) - F_0(t) &= \sup |F_i(x) - F_0(x)|\} \ , \\ K_{2i} = \{t : F_i(t) - F_0(t) &= -\sup |F_i(x) - F_0(x)|\}. \end{aligned}$$

But if $t_i \in (K_{1i} \cup K_{2i})$ then it follows that,

$$P(H_1(W_1)+\sqrt{n}\mu_{[1]} \le \min_{2\le i\le k} H_i(W_i)+\sqrt{n}\mu_{[i]}) \le P(\sup_{t\in[0,1]} |W(t)| \le \min_{2\le i\le k} W_i(t_i)+\sqrt{n}d).$$

But $W_i(t_i)$ is normal random variable with mean zero and variance $F_i(t)(1-F_i(t))$. This completes the proof of (2). (3) can be proved similarly.

If we assume that Π_1 is population closest to Π_0 and there is only one point in the subset K_1, then we have the following result.

Theorem 3.2 *:*

Under the above assumption, for large n,

$$\begin{aligned} \inf_{F\in\mathcal{F}'} P_F(CS|R,n) &\simeq \inf_{0\le\sigma_i\le\frac{1}{2}} P(\sigma_1 Z_1 \le \min_i \sigma_i Z_i + \sqrt{n}d), && (4) \\ &\simeq P(Z_1 \le \min_i Z_i + 2\sqrt{n}d) && (5) \end{aligned}$$

and for every $F \in \mathcal{F}_s$

$$P(CS|R(d),n) \ge P(Z_1 \le \min_i Z_i + 2\sqrt{n}d), \qquad (6)$$

where $Z_1, Z_2, \ldots, Z_k$ are independent standard normal random variables.

Proof: The first assertion follows from Theorem 2.2. Notice that

$$\begin{aligned} P(\sigma_1 Z_1 \le \min_{2\le i\le k} \sigma_i Z_i + \sqrt{n}d) &= P(\sigma_1 Z_1 \le \min_{2\le i\le k} \sigma_i Z_i + \sqrt{n}d;\ \ \sigma_1 Z_1 - \sqrt{n}d > 0) \\ &+ P(\sigma_1 Z_1 \le \min_{2\le i\le k} \sigma_i Z_i + \sqrt{n}d;\ \ \sigma_1 Z_1 - \sqrt{n}d < 0). \end{aligned}$$

Since the first term on the right is increasing in $\sigma_2, \sigma_3, \ldots, \sigma_k$ and the second term is decreasing in $\sigma_2, \sigma_3, \ldots, \sigma_k$, we have

$$\begin{aligned} P(\sigma_1 Z_1 \le \min_{2\le i\le k} \sigma_i Z_i + \sqrt{n}d) &\ge \inf_{0\le\sigma_1\le\frac{1}{2}} P(\sigma_1 Z_1 \le \min_{2\le i\le k} \frac{1}{2}\ Z_i + \sqrt{n}d;\ \ \sigma_1 Z_1 - \sqrt{n}d < 0) \\ &\ge P(\sigma_1 Z_1 \le \min_{2\le i\le k} \frac{1}{2}\ Z_i + \sqrt{n}d) + P(0 < \sigma_1 Z_1 < \sqrt{n}d) - \frac{1}{2}. \end{aligned}$$

Using a similar argument one can prove that

$$P(\sigma_1 Z_1 \le \min_{2\le i\le k} \frac{1}{2}\ Z_i + \sqrt{n}d) \ge P(\frac{1}{2} Z_1 \le \min_{2\le i\le k} \frac{1}{2}\ Z_i + \sqrt{n}d) + P(\min_{2\le j\le k} \frac{1}{2} Z_j \ge -\sqrt{n}d) - 1.$$

But as $n \longrightarrow \infty$,

$$P(0 < \sigma_1 Z_1 < \sqrt{n}d) - \frac{1}{2} \longrightarrow 0$$

and

$$P(\min_{2\le j\le k} \frac{1}{2} Z_j \ge -\sqrt{n}d) - 1 \longrightarrow 0.$$

This completes the proof of the first assertion. The second assertion can be proved similarly.

Bechhofer (1954), Gupta(1963) have tabulated the values of $d^* = 2\sqrt{n}d$ for several selected values of k and P^* to satisfy the equation

$$P(Z_1 \le \min_j +2\sqrt{n}d) = P^*.$$

In the following section we give some examples and Monte Carlo results.

4 Examples

In this section we give some examples and the Monte Carlo results. This section is designed to study the small sample performance of the procedures developed. Standard error for all the estimates is less that 0.033.

Let $\Pi_0, \Pi_1, \Pi_2, \ldots, \Pi_k$ be the $k+1$ populations. The population Π_i is associated with the cumulative distribution function $F_i(x) = F(x - \mu_i)$. Let $X_{i1}, X_{i2}, \ldots, X_{in}$ be the observable independent random variable from population Π_i. We are interested in selecting the population closest to the control. We assume μ_0 is known.

Example 4.1

In this example we assume that $F_i(.)$ is a logistic distribution for $i = 0, 1, 2, \ldots, k$. We choose a slippage configuration. Let $\mu_1 = \mu_2 = \ldots = \mu_{k-1} = d$ and $\mu_k = \mu_0 = 0$. Let R_1 be the nonparametric selection rule, R_2 be the selection rule which selects the population associated with smallest absolute sample mean and R_3 be the selection rule which selects the population associated with smallest absolute sample median.

Following table gives, n, d, asymptotic probability of correct selection, actual probability of correct selections and probability of correct selection for the rule which selects the population associated with smallest absolute sample mean.

n	d	P^*	$P(CS\|R_1)$	$P(CS\|R_2)$
20	1.88013	0.990	0.993	0.998
20	1.42353	0.950	0.938	0.958
20	1.19713	0.900	0.862	0. 879

Example 4.2

In this example we assume that $F_i(.)$ is a double exponential distribution for $i = 0, 1, 2, \ldots, k$. We choose a slippage configuration. Let $\mu_1 = \mu_2 = \ldots = \mu_{k-1} = d$ and $\mu_k = \mu_0 = 0$.

Following table gives, n, d, asymptotic probability of correct selection, actual probability of correct selections and probability of correct selection for the rule which selects the population associated with smallest absolute sample median.

n	d	P^*	$P(CS\|R_1)$	$P(CS\|R_3)$
20	1.80998	0.990	1.000	1.000
20	1.14937	0.950	0.989	0.984
20	0.87063	0.900	0.918	0.952

Form the above tables we see that the asymptotic aproximation for the probabilty of correct selection for the nonparametric rule works well for the moderate sample size.

References

[1] Barlow, R. E. and Gupta, S. S. (1969) *"Selection procedures for restricted families of probability distribution"*, Annals of Mathematical Statistics 40, 905-917.

[2] Bechhofer, R. E. (1954) *"A single sample multiple decision procedure for ranking means of normal populations with known variances"* , Annals of Mathematical Statistics 25, 16-39.

[3] Bechhofer, R. E. and Turnbull, B. W. (1978) *"Two (k+1) selection procedures for comparing k normal means with specified standard"*, J. Amer. Statist. Assoc. 73, 385-392.

[4] Billingsley, P. (1968) *"Convergence of probability measures"* Wiley, New York.

[5] Desu, M. M. and Bristol, D. R. (1986) *"Rizvi and Sobel type non parametric selection procedures : Review and Extensions"* , AJMMS 6, 87-107.

[6] Gupta, S. S. (1956) *"On a decision rule for a problem of ranking means"* , Mimeograph series No. 150 , Institute of statistics, University of North Carolina, Chapel Hill.

[7] Gupta, S. S. (1963) *"Probability integrals of the multivariate normal and multivariate t"*, Ann. Math. Stat. 34, 829-838.

[8] Gupta, S. S. and Hande, S. N. (1990) "On some nonparametric selection procedures", Technical Report # 90-26, Department of Statistics, Purdue University.

[9] Gupta, S. S. and Huang, D. Y. (1974) *"Non parametric subset selection procedures for the 't' best populations"*, Bulletin of Mathematics , Academia Sinica 2, 377-386.

[10] Gupta, S. S. and McDonald, G. C. (1970) *"On classes of selection procedures based on ranks"* , Nonparametric Techniques in statistical Inference , ed M. L. Puri , Cambridge University Press, London, 419-514.

[11] Gupta, S. S. and Sobel, M. (1958) *"On selecting a subset which contains all populations better than a standard"*, Ann. Math. Statist. 28, 957-967.

[12] Gupta, S. S. and Singh, A. K. (1979) *"On selection rules for treatments versus control problem"*, Proceedings of the 42nd Session of the International Statistical Institute at Manila, Philippines, Contributed papers, 1979, 229-232.

[13] Gupta, S. S. and Singh, A. K. (1980) *"On rules based on sample medians for selection of the largest location parameter"*, Communications in Statistics-Theory and Methods, A9, 1277-1298.

[14] Roberts, C. D. (1964) *"An asymptotic optimal fixed sample size procedures for comparing several experimental categories with a control"*, Ann. Math. Statist. 35, 1571-1575.