WONG-ZAKAI CORRECTIONS, RANDOM EVOLUTIONS, AND SIMULATION SCHEMES FOR SDE's

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ABSTRACT

A general weak limit theorem for solutions of stochastic differential equations driven by arbitrary semimartingales is applied to give a unified treatment of limit theorems for random evolutions and consistency results for numerical schemes for stochastic differential equations. The asymptotic distribution of the error in an Euler scheme is studied. The Wong-Zakai correction in the random evolution limit arises through an integration by parts.

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Abstract

A general weak limit theorem for solutions of stochastic differential equations driven by arbitrary semimartingales is applied to give a unified treatment of limit theorems for random evolutions and consistency results for numerical schemes for stochastic differential equations. The asymptotic distribution of the error in an Euler scheme is studied. The Wong-Zakai correction in the random evolution limit arises through an integration by parts.

1 Introduction

For $n = 1, 2, ...$, let Y_n be an $\mathbb{R}^m$-valued semimartingale with respect to a filtration $\{\mathcal{F}_t^n\}$, that is a cadlag process such that Y_n can be written as $Y_n = M_n + A_n$ where M_n is an $\{\mathcal{F}_t^n\}$-local martingale and A_n has sample paths of finite variation. Suppose that $\{Y_n\}$ converges in distribution in the Skorohod topology to a process Y. We will say that the sequence is *good* if for every sequence $\{X_n\}$ of cadlag $k \times m$-matrix-valued processes such that X_n is $\{\mathcal{F}_t^n\}$-adapted and $(X_n, Y_n) \Rightarrow (X, Y)$, one has $\int X_n dY_n \Rightarrow \int X dY$. Note that the stochastic integrals are the usual semimartingale integrals given by

$$\int_0^t X dY = \int_0^t X(s-)dY(s) \tag{1.1}$$
$$= \lim \sum X(t_k)(Y(t_{k+1}) - Y(t_k))$$

where $\{t_k\}$ is a partition of $[0,t]$ and the limit is in probability and is taken as $\max_k |t_{k+1} - t_k| \to 0$. Jakubowski, Mémin, and Pages (1989) and Kurtz and Protter (1991a) give equivalent sets of sufficient conditions for "goodness". See also Mémin and Słomiński (1990). Kurtz and Protter (1991b) show that the conditions are in fact necessary. In particular, to state these conditions define $h_\delta : [0,\infty) \to [0,\infty)$ by $h_\delta(r) = (1 - \delta/r)^+$ and $J_\delta : D_{\mathbb{R}^m}[0,\infty) \to D_{\mathbb{R}^m}[0,\infty)$ by

$$J_\delta(x)(t) = \sum_{s \le t} h_\delta(|x(s) - x(s-)|)(x(s) - x(s-)). \tag{1.2}$$

The mapping $x \to (x, J_\delta(x))$ is a continuous mapping of $D_{\mathbb{R}^m}[0,\infty) \to D_{\mathbb{R}^m \times \mathbb{R}^m}[0,\infty)$ (see Lemma 2.1 of Kurtz and Protter (1991a)), and hence $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_{\mathbb{M}^{km} \times \mathbb{R}^m}[0,\infty)$ implies that $(X_n, Y_n, Y_n^\delta, J_\delta(Y_n)) \Rightarrow (X, Y, Y^\delta, J_\delta(Y))$ in $D_{\mathbb{M}^{km} \times \mathbb{R}^{3m}}[0,\infty)$ where $Y_n^\delta = Y_n - J_\delta(Y_n)$ and $Y^\delta = Y - J_\delta(Y)$. The following is Theorem 2.2 of Kurtz and Protter (1991a).

Theorem 1.1 *For each n, let (X_n, Y_n) be an $\{\mathcal{F}_t^n\}$-adapted process with sample paths in $D_{\mathbb{M}^{km} \times \mathbb{R}^m}[0,\infty)$, and let Y_n be an $\{\mathcal{F}_t^n\}$-semimartingale. Fix $\delta > 0$ (allowing $\delta = \infty$), and define $Y_n^\delta = Y_n - J_\delta(Y_n)$. (Note that Y_n^δ will also be a semimartingale.) Let $Y_n^\delta = M_n^\delta + A_n^\delta$ be a decomposition of Y_n^δ into an $\{\mathcal{F}_t^n\}$-local martingale and a process with finite variation. Suppose*

C1 For each $\alpha > 0$, there exist stopping times $\{\tau_n^\alpha\}$ such that $P\{\tau_n^\alpha \le \alpha\} \le \frac{1}{\alpha}$ and $\sup_n E[[M_n^\delta]_{t\wedge\tau_n^\alpha} + T_{t\wedge\tau_n^\alpha}(A_n^\delta)] < \infty$. ($T_t(A)$ denotes the total variation of A and $[M]$ denotes the quadratic variation of M.)

If $(X_n, Y_n) \Rightarrow (X, Y)$ in the Skorohod topology on $D_{\mathbb{M}^{km} \times \mathbb{R}^m}[0,\infty)$, then Y is a semimartingale with respect to a filtration to which X and Y are adapted, and $(X_n, Y_n, \int X_n dY_n) \Rightarrow (X, Y, \int X dY)$ in the Skorohod topology on $D_{\mathbb{M}^{km} \times \mathbb{R}^m \times \mathbb{R}^k}[0,\infty)$. If $(X_n Y_n) \to (X, Y)$ in probability, then the triple converges in probability.

In other words, if $\{Y_n\}$ is a sequence of semimartingales converging in distribution and satisfies C1, then $\{Y_n\}$ is good. This condition is shown to be necessary for goodness in Kurtz and Protter (1991b).

The following is a consequence of goodness.

Proposition 1.2 *Let $f : \mathbb{R}^k \to \mathbb{M}^{km}$ be bounded and continuous, for each n, let (U_n, Y_n) be an $\{\mathcal{F}_t^n\}$-adapted process in $D_{\mathbb{R}^k \times \mathbb{R}^m}[0,\infty)$, and let $\{Y_n\}$ be a good sequence of semimartingales with $(U_n, Y_n) \Rightarrow (U, Y)$. If for each n, X_n is a solution of*

$$X_n(t) = U_n(t) + \int_0^t f(X_n(s-))dY_n(s), \tag{1.3}$$

then $\{(X_n, Y_n)\}$ is relatively compact (in the sense of convergence in distribution) and any limit point (U, X, Y) satisfies

$$X(t) = U(t) + \int_0^t f(X(s-))dY(s). \tag{1.4}$$

(More general results can be found in Słomiński (1989) and Kurtz and Protter (1991a).)

It is well-known from the work of Wong and Zakai (1965) that the conclusion of Proposition 1.2 fails for many natural sequences $\{Y_n\}$ approximating Brownian motion.

Example 1.3 Let $\{\xi_k\}$ be independent and identically distributed with mean zero and variance σ^2. Define

$$W_n^a(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k. \tag{1.5}$$

Then it is easy to check that $\{W_n^a\}$ is a good sequence which, by the Donsker invariance principle, converges in distribution to σW where W is standard Brownian motion. □

Example 1.4 Let W be standard Brownian motion and let W_n^b satisfy $W_n^b(0) = 0$ and

$$\frac{d}{dt} W_n^b(t) = n\Big(W\Big(\frac{[nt]+1}{n}\Big) - W\Big(\frac{[nt]}{n}\Big)\Big). \tag{1.6}$$

Then $W_n^b \Rightarrow W$, but $\{W_n^b\}$ is not good (e.g., $\int_0^t W_n^b dW_n^b \Rightarrow \int_0^t W dW + \frac{1}{2}t$). Since the sample paths of W_n^b have finite variation it is a semimartingale, but in the canonical decomposition of W_n^b into a local martingale plus a finite variation process, the local martingale is zero and $T_t(W_n^b) = O(\sqrt{n})$. (The decomposition used in Theorem 1.1 does not need to be the canonical decomposition, but, of course, the counter example ensures that no decomposition will satisfy C1.) □

Example 1.5 Let $\{\xi_k, k \geq 0\}$ be a finite, irreducible Markov chain with transition matrix $P = ((p_{ij}))$. Let $\pi = (\pi_1, \dots, \pi_M)$ give the stationary distribution, and let f be a function satisfying

$$\sum_m f(m)\pi_m = 0. \tag{1.7}$$

Define

$$W_n^c(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} f(\xi_k). \tag{1.8}$$

Letting $Pg(i) \equiv \sum_j g(j)p_{ij}$, by (1.7) there exists a function h such that $Ph - h = f$. Substituting in (1.8), we obtain

$$\begin{aligned} W_n^c(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Ph(\xi_k) - h(\xi_k)) \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Ph(\xi_{k-1}) - h(\xi_k)) \\ &\quad + \frac{1}{\sqrt{n}} \Big(Ph(\xi_{[nt]}) - Ph(\xi_0)\Big) \\ &\equiv Y_n(t) + Z_n(t). \end{aligned} \tag{1.9}$$

Then $\{Y_n\}$ is a sequence of martingales satisfying C1 and converging in distribution to σW for some σ and $Z_n \Rightarrow 0$. (See Kurtz and Protter (1991a) Example 5.9 for details.) The sequence $\{W_n^c\}$, however, in general fails to satisfy C1. In particular, $\int_0^t Z_n dW_n^c \Rightarrow \alpha t$ where $\alpha = -2\Sigma\pi_i p_{ij}(Ph(j) - h(i))^2$. □

Note that W_n^b can also be written as the sum of a martingale and a process Z_n^b such that $Z_n^b \Rightarrow 0$. Specifically, $W_n^b(t) = W\left(\frac{[nt]+1}{n}\right) + \left(W_n^b(t) - W\left(\frac{[nt]+1}{n}\right)\right) \equiv Y_n^b + Z_n^b$. All these processes are adapted to the filtration given by $\mathcal{F}_t^n = \sigma\left(W(s) : s \leq \frac{[nt]+1}{n}\right)$ and Y_n^b is an $\{\mathcal{F}_t^n\}$-martingale. With these last two examples in mind consider the equation

$$\begin{aligned} X_n(t) &= X_n(0) + \textstyle\int_0^t F(X_n(s-))dW_n(s) \\ &= X_n(0) + \textstyle\int_0^t F(X_n(s-))dY_n(s) \\ &\quad + \textstyle\int_0^t F(X_n(s-))dZ_n(s). \end{aligned} \tag{1.10}$$

We have the following extension of the classical results of Wong and Zakai (1965).

Theorem 1.6 *Let Y_n and Z_n be $\{\mathcal{F}_t^n\}$-semimartingales, and let $X_n(0)$ be $\mathcal{F}_0^n$-measurable. Let $F : \mathbb{R}^k \to \mathbb{M}^{km}$ in (1.10) be bounded and have bounded and continuous first and second order derivatives. Define $H_n = ((H_n^{\beta\gamma}))$ and $K_n = ((K_n^{\beta\gamma}))$ by*

$$H_n^{\beta\gamma}(t) = \int_0^t Z_n^\beta(s-)dZ_n^\gamma(s) \tag{1.11}$$

and

$$K_n^{\beta\gamma}(t) = [Y_n^\beta, Z_n^\gamma]_t. \tag{1.12}$$

Suppose that $\{Y_n\}$ and $\{H_n\}$ satisfy C1 and that $(X_n(0), Y_n, Z_n, H_n, K_n) \Rightarrow (X(0), Y, 0, H, K)$. Then $\{(X_n(0), Y_n, Z_n, H_n, K_n, X_n)\}$ is relatively compact, and any limit point $(X(0), Y, 0, H, K, X)$ satisfies

$$\begin{aligned} X(t) &= X(0) + \textstyle\int_0^t F(X(s-))dY(s) \\ &\quad + \textstyle\sum_{\alpha,\beta,\gamma} \int_0^t \partial_\alpha F_\beta(X(s-))F_{\alpha\gamma}(X(s-))d(H^{\gamma\beta}(s) - K^{\gamma\beta}(s)) \end{aligned} \tag{1.13}$$

where ∂_α denotes the partial derivative with respect to the αth variable and F_β denotes the βth column of F.

Proof The theorem follows by integrating the second term on the right of (1.10) by parts and then applying Theorem 1.2. See Kurtz and Protter (1991a), Theorem 5.10. □

2 Random evolutions

We now consider sequences of stochastic ordinary differential equations in $\mathbb{R}^k$ of the form

$$\dot{X}_n(t) = G(X_n(t), \xi(n^2t)) + nH(X_n(t), \xi(n^2t)) \tag{2.1}$$

where ξ is a stochastic process representing the random noise in the system. Models of this type were considered first by Stratonovich (1963, 1967) and Khas'minski (1966) and, in an abstract form, were dubbed random evolutions by Griego and Hersh (1969). Limit theorems for random evolutions are closely related to the results of the previous section.

Let ξ be a continuous time Markov chain with state space $E = \{1, \ldots, m\}$ and intensity matrix Q. We assume that Q is irreducible and hence that there is a unique stationary distribution π. Suppose $\Sigma_\beta H(x,\beta)\pi_\beta = 0$ and define V_n and W_n by

$$V_n^\beta(t) = \int_0^t I_{\{\beta\}}(\xi(n^2s))ds \tag{2.2}$$

and

$$W_n^\beta(t) = n(V_n^\beta(t) - \pi_\beta t) = n\int_0^t (I_{\{\beta\}}(\xi(n^2s)) - \pi_\beta)ds. \tag{2.3}$$

Then (2.1) becomes

$$X_n(t) = X_n(0) + \sum_{\beta=1}^{m} \int_0^t G(X_n(s),\beta)dV_n^\beta(s) + \sum_{\beta=1}^{m} \int_0^t H(X_n(s),\beta)dW_n^\beta(s). \tag{2.4}$$

Let h_β satisfy

$$\sum_{k=1}^{m} q_{jk}h_\beta(k) = I_{\{\beta\}}(j) - \pi_\beta \tag{2.5}$$

(h_β exists by the uniqueness of π), and note that Y_n defined by

$$Y_n^\beta(t) = n\int_0^t (I_{\{\beta\}}(\xi(n^2s)) - \pi_\beta)ds - \tfrac{1}{n}h_\beta(\xi(n^2t)) + \tfrac{1}{n}h_\beta(\xi(0)) \tag{2.6}$$

is a martingale. Define Z_n by

$$Z_n^\beta(t) = \frac{1}{n}h_\beta(\xi(n^2t)) - \frac{1}{n}h_\beta(\xi(0)) \tag{2.7}$$

so that $W_n = Y_n + Z_n$. Let $N_{ij}(t)$ denote the number of transitions of ξ from state i to state j up to time t. Then

$$[Y_n^\beta, Y_n^\gamma]_t = \sum_{ij=1}^{m} \frac{N_{ij}(n^2t)}{n^2}(h_\beta(j) - h_\beta(i))(h_\gamma(j) - h_\gamma(i)) \tag{2.8}$$

$$[Y_n^\beta, Z_n^\gamma]_t = -[Y_n^\beta, Y_n^\gamma]_t \tag{2.9}$$

and

$$\int_0^t Z_n^\beta(s-)dZ_n^\gamma(s) = \sum_{ij=1}^{m} \frac{N_{ij}(n^2t)}{n^2}h_\beta(i)(h_\gamma(j) - h_\gamma(i)) - \frac{1}{n}h_\beta(\xi(0))Z_n^\gamma(t). \tag{2.10}$$

As $n \to \infty$ we obtain

$$[Y_n^\beta, Y_n^\gamma]_t \to C_{\beta\gamma}t \equiv \sum_{ij=1}^{m} \pi_i q_{ij}(h_\beta(j) - h_\beta(i))(h_\gamma(j) - h_\gamma(i))t \tag{2.11}$$

and

$$\int_0^t Z_n^\beta(s-)dZ_n^\gamma(s) \to D_{\beta\gamma}t \equiv \sum_{ij=1}^{m} \pi_i q_{ij}h_\beta(i)(h_\gamma(j) - h_\gamma(i))t. \tag{2.12}$$

The martingale central limit theorem (see, for example, Ethier and Kurtz (1986), Theorem 7.1.4) gives $Y_n \Rightarrow Y$ where Y is a Brownian motion with infinitesimal covariance $C = ((C_{\beta\gamma}))$. Theorem 1.6 gives the following

Theorem 2.1 *Let ξ in (2.1) be a finite Markov chain with state space $E = \{1,\ldots,m\}$ and intensity matrix Q, and let $X_n(0)$ be independent of ξ. Let G be bounded and continuous, and let H be bounded and have bounded and continuous first and second derivatives. Assume that Q is irreducible and that π, $((C_{\beta\gamma}))$ and $((D_{\beta\gamma}))$ are as above. Define $\bar{G} : \mathbb{R}^k \to \mathbb{R}^k$ by $\bar{G}(x) = \Sigma_\beta G(x,\beta)\pi_\beta$ and $H : \mathbb{R}^k \to \mathbb{M}^{km}$ by $H(x) = (H(x,1),\ldots,H(x,m))$. If $X_n(0) \Rightarrow X(0)$, then $\{X_n\}$ is relatively compact and any limit point satisfies*

$$X(t) = X(0) + \int_0^t \bar{G}(X(s))ds + \int_0^t H(X(s-))dY(s) + \sum_{\alpha,\beta,\gamma} \int_0^t \partial_\alpha H(X(s-),\beta)H_\alpha(X(s-),\gamma)(D_{\gamma\beta} + C_{\gamma\beta})ds. \tag{2.13}$$

Remark 2.2 Hersh and Papanicolaou (1972) and Kurtz (1973) prove the above result using functional analytic arguments.

3 Numerical schemes

Let Y be and $\{\mathcal{F}_t\}$-semimartingale in $\mathbb{R}^m$, and let $f : \mathbb{R}^k \to \mathbb{M}^{km}$ be continuous. A number of authors (for example, Mil'shtein (1974), Rumelin (1982), Pardoux and Talay (1985), Talay and Tubaro (1989), Wagner (1989)) have considered analogues of classical numerical schemes as means of simulating solutions of the stochastic differential equation

$$X(t) = X(0) + \int_0^t f(X(s-))dY(s). \tag{3.1}$$

The results discussed above provide a natural approach to checking the consistency of these schemes, and we will see that they are also useful in a more careful study of the error in the scheme.

The simplest numerical scheme is, of course, the Euler scheme. Specifying a mesh $0 = t_0 < t_1 < \cdots$, define X_0 recursively by setting $X_0(0) = X(0)$ and

$$X_0(t_{k+1}) = X_0(t_k) + f(X_0(t_k))\Delta Y(t_k) \tag{3.2}$$

where $\Delta Y(t_k) = Y(t_{k+1}) - Y(t_k)$. If we extend the definition of X_0 to all t by setting $X_0(t) = X_0(t_k)$ for $t_k \le t < t_{k+1}$ and we define Y_0 by $Y_0(t) = Y(t_k)$ for $t_k \le t < t_{k+1}$, then

$$X_0(t) = X_0(0) + \int_0^t f(X_0(s-))dY_0(s). \tag{3.3}$$

Note that if we define $\beta(t) = t_k$ for $t_k \le t < t_{k+1}$, then we can write $Y_0 = Y \circ \beta$. With this observation, consistency for the Euler scheme is a consequence of Proposition 1.2 and the following lemma. Note that the lemma would allow for a mesh determined by $\{\mathcal{F}_t\}$-stopping times.

Lemma 3.1 *Let Y be an $\{\mathcal{F}_t\}$-semimartingale. For each n, let β_n be a nonnegative, nondecreasing process such that for each $u \ge 0$, $\beta_n(u)$ is an $\{\mathcal{F}_t\}$-stopping time. If $\beta_n(u) \to u$ a.s. for each $u \ge 0$, then $Y \circ \beta_n \to Y$ a.s. in the Skorohod topology and $\{Y \circ \beta_n\}$ is a good sequence.*

Proof First observe that $\beta_n(u) \to u$ for each u implies $Y \circ \beta_n \to Y$ by Proposition 3.6.5 of Ethier and Kurtz (1986). To verify goodness for $\{Y \circ \beta_n\}$ it is enough to verify goodness for $\{Y^{\tau_m} \circ \beta_n\}$ ($Y^{\tau_m} \equiv Y(\cdot \wedge \tau_m)$) for some family of stopping times satisfying $P\{\tau_m \le m\} \le \frac{1}{m}$. In particular, let d be the metric for the Skorohod topology given in (3.5.2) of Ethier and Kurtz (1986), and suppose that $(X_n, Y \circ \beta_n) \Rightarrow (X, Y)$. Define X_n^m by setting $X_n^m(t) = X_n(t)$ for $t < \tau_m$ and $X_n^m(t) = X_n(\tau_m-)$ for $t \ge \tau_m$, and define X^m analogously. Then $(X_n^m, Y^{\tau_m} \circ \beta_n) \Rightarrow (X^m, Y^{\tau_m})$. Let

$$\begin{aligned} V_n^m(t) &= \textstyle\int_0^t X_n^m(s-)dY^{\tau_m} \circ \beta_n(s) \\ V^m(t) &= \textstyle\int_0^t X^m(s-)dY^{\tau_m}(s) \end{aligned} \tag{3.4}$$

and

$$V_n(t) = \int_0^t X_n(s-))dY \circ \beta_n(s) \quad V(t) = \int_0^t X(s-))dY(s) \tag{3.5}$$

and observe that

$$\begin{aligned} \overline{\lim}_{n\to\infty} E[d(V_n^m, V_n)] &\le \overline{\lim}_{n\to\infty} E[e^{-\beta_n^{-1}(\tau_m)}] \\ &\le e^{-m} + \tfrac{1}{m}. \end{aligned} \tag{3.6}$$

Consequently, if $V_n^m \Rightarrow V^m$ for each m, we have $V_n \Rightarrow V$ which would give the goodness for $\{Y \circ \beta_n\}$.

Fix $0 < \delta < \infty$, and define Y^δ as in Theorem 1.1. Let $Y^\delta = M^\delta + A^\delta$ be the canonical decomposition of Y^δ (so that the discontinuities of M^δ are bounded by 2δ and the discontinuities of A^δ are bounded by δ). Define $\tau_m = \inf\{t : [M^\delta]_t + T_t(A^\delta) > c_m\}$ where c_m is selected so that $P\{\tau_m \le m\} \le \frac{1}{m}$. Then

$$E[[M^\delta]_{\tau_m} + T_{\tau_m}(A^\delta)] < c_m + 3\delta. \tag{3.7}$$

We can write

$$\begin{aligned} Y^{\tau_m} \circ \beta_n &= M^\delta(\beta_n(\cdot) \wedge \tau_m) + A^\delta(\beta_n(\cdot) \wedge \tau_m) + J_\delta(Y^{\tau_m}) \circ \beta_n \\ &\equiv M_n + A_n + Z_n \end{aligned} \tag{3.8}$$

and since $E[[M_n]_t] \le E[[M^\delta]_{\tau_m}]$ and $E[T_t(A_n)] \le E[T_{\tau_m}(A^\delta)]$, we can apply Theorem 2.7 of Kurtz and Protter (1991a) to conclude that $V_n^m \Rightarrow V^m$. □

There is an alternative approach to representing the approximation given by the Euler scheme as a solution of a stochastic differential equation. Define $\eta(t) = t_k$ for $t_k \le t < t_{k+1}$ (which is the same as β defined above, but the general assumptions that will be placed on the sequence $\{\eta_n\}$ below will be different from the assumptions placed on $\{\beta_n\}$). Let $\tilde{X}_0$ satisfy

$$\tilde{X}_0(t) = X(0) + \int_0^t f(\tilde{X}_0 \circ \eta(s-))dY(s). \tag{3.9}$$

Then $\tilde{X}_0(t_k) = X_0(t_k)$. The consistency of the Euler scheme can also be obtained through the analysis of this equation.

Lemma 3.2 *For each n, let Y_n be an $\mathbb{R}^m$-valued $\{\mathcal{F}_t^n\}$-semimartingale, X_n a cadlag, $\mathbb{M}^{km}$-valued $\{\mathcal{F}_t^n\}$-adapted process, and η_n a right continuous, nondecreasing $\{\mathcal{F}_t^n\}$-adapted process. Suppose that $\eta_n(t) \le t$ and $\eta_n(t) \to t$ for all $t \ge 0$. Assume that $\{Y_n\}$ is a good sequence and that $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$. Then $\int X_n \circ \eta_n dY_n \Rightarrow \int X dY$.*

Proof First observe that for each fixed $\delta > 0$,

$$\int_0 \Big(J_\delta(X_n)\circ\eta_n(s-) - J_\delta(X_n)(s-)\Big)dY_n(s) \Rightarrow 0. \tag{3.10}$$

Consequently, there exist $\delta_n \to 0$ such that

$$\int_0 \Big(J_{\delta_n}(X_n)\circ\eta_n(s-) - J_{\delta_n}(X_n)(s-)\Big)dY_n(s) \Rightarrow 0. \tag{3.11}$$

But the asymptotic continuity of $X_n^{\delta_n}$ implies that $X_n^{\delta_n} - X_n^{\delta_n}\circ\eta_n \Rightarrow 0$, so

$$\int_0 \Big(X_n^{\delta_n}(s-) - X_n^{\delta_n}\circ\eta_n(s-)\Big)dY_n(s) \Rightarrow 0 \tag{3.12}$$

and hence

$$\int_0 \Big(X_n(s-) - X_n\circ\eta_n(s-)\Big)dY_n(s) \Rightarrow 0 \tag{3.13}$$

which gives the lemma. □

Theorem 3.3 *For each n, let Y_n be an $\mathbb{R}^m$-valued $\{\mathcal{F}_t^n\}$-semimartingale and η_n a right continuous, nondecreasing $\{\mathcal{F}_t^n\}$-adapted process. Suppose that $\eta_n(t) \le t$ and $\eta_n(t) \to t$ for all $t \ge 0$. Assume that $\{Y_n\}$ is a good sequence and that $Y_n \Rightarrow Y$. Let $f : \mathbb{R}^k \to \mathbb{M}^{km}$ be bounded and continuous, and let $\tilde{X}_n$ satisfy*

$$\tilde{X}_n(t) = X(0) + \int_0^t f(\tilde{X}_n\circ\eta_n(s-))dY_n(s). \tag{3.14}$$

Then $\{(\tilde{X}_n, Y_n)\}$ is relatively compact and any limit point (X, Y) satisfies

$$X(t) = X(0) + \int_0^t f(X(s-))dY(s). \tag{3.15}$$

If the Y_n are defined on the same sample space as Y, $\sup_{s\le t}|Y_n(s) - Y(s)| \to 0$ in probability for each $t > 0$, and sample path uniqueness holds for the solution of (3.15), then $\sup_{s\le t}|\tilde{X}_n(s) - X(s)| \to 0$ in probability for each $t > 0$.

Remark 3.4 For $Y_n = Y$, $n = 1, 2, \ldots$, and $\eta_n(t) = \tau_k^n$, $\tau_k^n \le t < \tau_{k+1}^n$, for a sequence of stopping times $\{\tau_k^n\}$, this result is a special case of Theorem V.16 of Protter (1990).

Proof The relative compactness follows from Lemma 4.1 and Proposition 4.3 of Kurtz and Protter (1991a). The fact that any limit point satisfies (3.15) then follows from Lemma 3.2. Under the assumptions of the final assertion, we can treat $(\tilde{X}_n, X)$ as a solution of a single system. Using the uniqueness assumption, it follows that $(\tilde{X}_n, X) \Rightarrow (X, X)$ and hence that $\tilde{X}_n - X \Rightarrow 0$ which gives the desired conclusion. □

Note that if Y is a semimartingale and $Y_n = Y$ for all n, then $\{Y_n\}$ is good.

The next theorem gives an approach to the analysis of the error in the Euler scheme.

Theorem 3.5 *Let Y be an $\{\mathcal{F}_t\}$-semimartingale, and suppose that $f = (f_1, \dots, f_m)$ is a bounded and continuously differentiable $k \times m$-matrix-valued function. For each n, let $0 = \tau_0^n < \tau_1^n < \cdots$ be $\{\mathcal{F}_t\}$-stopping times, define $\eta_n(t) = \tau_k^n$, $\tau_n^n \le t < \tau_{k+1}^n$, and let $\tilde{X}_n$ satisfy (3.14). Let $\{\alpha_n\}$ be a positive sequence converging to infinity, set $U_n = \alpha_n(\tilde{X}_n - X)$, and define Z_n by*

$$Z_n^{ij}(t) = \alpha_n \int_0^t (Y_i(s-) - Y_i \circ \eta_n(s-))dY_j(s). \tag{3.16}$$

Suppose that $\{Z_n\}$ is a good sequence with $(Y, Z_n) \Rightarrow (Y, Z)$. Then $U_n \Rightarrow U$ satisfying

$$\begin{aligned} U(t) &= \sum_i \int_0^t \nabla f_i(X(s-))U(s-)dY_i(s) \\ &+ \sum_{ij} \int_0^t \sum_k \partial_k f_i(X(s-)) f_{kj}(X(s-))dZ^{ij}(s). \end{aligned} \tag{3.17}$$

Remark 3.6 a) For a discussion of linear stochastic differential equations, see Protter (1990), p271.
b) Rootzén (1980) gives the asymptotic distribution for the error in certain approximations for stochastic integrals.
c) Suppose

$$Y(t) = \begin{pmatrix} W(t) \\ t \end{pmatrix} \tag{3.18}$$

where W is an $(m-1)$-dimensional standard Brownian motion. Let $\eta_n(t) = \frac{[nt]}{n}$. Then, taking $\alpha_n = \sqrt{n}$, $(Y, Z_n) \Rightarrow (Y, Z)$ where Z is independent of Y, $Z^{im} = Z^{mi} = 0$, and for $1 \le i, j \le m-1$, Z^{ij} are independent mean zero Brownian motions with $E[(Z^{ij}(t))^2] = \frac{1}{2}t$.
d) With $m = 1$, let $1 < \beta < 3$, and let Y be the stable process with generator

$$Af(x) = \int_{-\infty}^{\infty} \left(f(x+y) - f(x) - \frac{y}{1+y^2} f'(x)\right) \frac{1}{|y|^\beta} dy. \tag{3.19}$$

Then, taking $\alpha_n = n^{\frac{1}{\beta - 1}}$, Z is a process with stationary, independent increments and generator

$$A_Z f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(f(x+zy) - f(x) - \frac{zy}{1+(zy)^2} f'(x)\right) \frac{1}{|y|^\beta} dy \mu_\beta(dz). \tag{3.20}$$

where μ_β is the distribution of $Y(1)$.

Proof Under the hypotheses of the theorem, the solution of (3.15) is unique, and $(\tilde{X}_n, X, Y, Z_n) \Rightarrow (X, X, Y, Z)$. For simplicity, assume $k = m = 1$. Then, noting that $\tilde{X}_n(s-) - \tilde{X}_n \circ \eta_n(s-) = f(\tilde{X}_n \circ \eta_n(s-))(Y(s-) - Y \circ \eta_n(s-))$

$$\begin{aligned} U_n(t) &= \int_0^t \alpha_n \Big(f(\tilde{X}_n(s-)) - f(X(s-))\Big) dY \\ &- \int_0^t \alpha_n \Big(f(\tilde{X}_n(s-)) - f(\tilde{X}_n \circ \eta_n(s-))\Big) dY \\ &= \int_0^t \frac{f(\tilde{X}_n(s-)) - f(X(s-))}{\tilde{X}_n(s-) - X(s-)} U_n(s-)dY(s) \\ &- \int_0^t \Big(f(\tilde{X}_n \circ \eta_n(s-) + f(\tilde{X}_n \circ \eta_n(s-))(Y(s-) - Y \circ \eta_n(s-))) \\ &\quad - f(\tilde{X}_n \circ \eta_n(s-))\Big)(Y(s-) - Y \circ \eta_n(s-))^{-1} dZ_n(s) \end{aligned} \tag{3.21}$$

where the integrands are defined in the obvious manner when the denominator vanishes. Let $\tau_n^a = \inf\{t : |U_n(t)| > a\}$. Then $\{U_n(\cdot \wedge \tau_n^a)\}$ is relatively compact, and any limit point will satisfy (3.17) on the time interval $[0, \tau^a]$ where $\tau^a = \inf\{t : |U(t)| > a\}$. But $\tau^a \to \infty$ as $a \to \infty$, so $U_n \Rightarrow U$. □

Bibliography

[1] Ethier, Stewart N. and Kurtz, Thomas G. (1986). *Markov Processes: Characterization and Convergence.* Wiley, New York.

[2] Griego, Richard J. and Hersh, Reuben (1969). Random evolutions, Markov chains, and systems of partial differential equations. *Proc. Nat. Acad. Sci.* USA 62, 305-308.

[3] Hersh, Reuben and Papanicolaou, George C. (1972). Non-commuting random evolutions, and an operator-valued Feynman-Kac formula. *Comm. Pure Appl. Math.* 25, 337-367.

[4] Jakubowski, A., Mémin, J., and Pagès, G. (1989). Convergence en loi des suites d'integrales stochastique sur l'espace D^1 de Skorokhod. *Probab. Th. Rel. Fields* 81, 111-137.

[5] Khas'minskii, R. Z. (1966). A limit theorem for the solutions of differential equations with random right-hand sides. *Theory Probab. Appl.* 11, 390-406.

[6] Kurtz, Thomas G. (1973). A limit theorem for perturbed operator semigroups with applications to random evolutions. *J. Funct. Anal.* 12, 55-67.

[7] Kurtz, Thomas G. and Protter, Philip (1991a). Weak limit theorems for stochastic integrals and stochastic differential equations. Ann. Probab. to appear

[8] Kurtz, Thomas G. and Protter, Philip (1991b). Characterizing weak convergence of stochastic integrals. (in preparation)

[9] Mémin, J. and Slomiński, L. (1990). Condition UT et stabilité en loi des solutions d'équations différentielles stochastiques. (preprint)

[10] Mil'shtein, G. N. (1974). Approximate integration of stochastic differential equations. *Theory Prob. App.* 19, 557-562.

[11] Pardoux, Etienne and Talay, Denis (1985). Discretization and simulation of stochastic differential equations. *Acta Applicandae Math.* 3, 23-47.

[12] Protter, Philip (1990). *Stochastic Integration and Differential Equations: A New Approach.* Springer-Verlag, New York.

[13] Rootzén, Holger (1980). Limit distributions for the error in approximation of stochastic integrals. *Ann. Probab.* 8, 241-251.

[14] Rumelin, Werner (1982). Numerical treatment of stochastic differential equations. *SIAM J. Numer. Anal.* 19, 604-613.

[15] Slomiński, Leszek (1989). Stability of strong solutions of stochastic differential equations. *Stochastic Process. Appl.* 31, 173-202.

[16] Stratonovich, R. L. (1963, 1967). *Topics in the Theory of Random Noise I, II.* Gordon Breach, New York.

[17] Talay, Denis, and Tubaro, L. (1989). Expansion of the global error for numerical schemes solving stochastic differential equations. (preprint).

[18] Wagner, W. (1989). Unbiased monte carlo estimators for functionals of weak solutions of stochastic differential equations, *Stochastics* 28, 1-20.

[19] Wong, Eugene, and Zakai, Moshe (1965). On the convergence of ordinary integrals to stochastic integrals. *Ann. Math. Statist.* 36, 1560-1564.