ON GENERAL EQUATIONS FOR ORTHOGONAL POLYNOMIALS

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Abstract

We derive equations for orthonormal polynomials with respect to an arbitrary (probability) measure on the interval $[-1,1]$, which generalize the equation $(1-x^2)U_{n-1}^2(x)+T_n^2(x)=1$ for the Chebyshev polynomials of the first (T_n) and second kind (U_n). The results are established using general equivalence theorems of optimal design theory for weighted polynomial regression. As special cases new equations are derived for the Legendre-, Chebyshev-, Ultraspherical and Jacobi polynomials.

1. Introduction

Consider the orthogonal polynomials

$$
\begin{aligned}
T_n(x) &= \cos\,(n \text{ arc cos } x) \\
U_n(x) &= \frac{\sin\,((n+1) \text{ arc cos} x)}{\sin\,(\text{arc cos } x)}
\end{aligned}
\tag{1.1}
$$

where $x \in [-1,1]$ and $n \in \mathbb{N}_0$. $T_n(x)$ and $U_n(x)$ are known as Chebyshev polynomials of the first and second kind respectively (see for example Szegö (1959)) and are orthogonal with respect to the measures $(1-x^2)^{-1/2}dx$ and $(1-x^2)^{1/2}dx$. From the trigonometric representation (1.1) and the identity $\cos^2\Theta + \sin^2\Theta = 1$ we can readily derive the equation

$$
(1-x^2)U_{n-1}^2(x) + T_n^2(x) = 1 \qquad \text{for all } x \in \mathbb{R} \tag{1.2}
$$

In this paper we will give similar results for orthogonal polynomials with respect to an arbitrary (probability) measure on the interval $[-1,1]$. To this end we will use the theory of optimal design for polynomial regression. In this theory the Chebyshev polynomials $U_n(x)$ and $T_n(x)$ play a particular role, because its zeros give the support of the optimal design with respect to many optimality criteria (see for example Kiefer (1959), Hoel (1965) and Studden (1968)). The theory of optimal design is briefly described in section 2. In section 3 we consider the case of polynomial regression and show that every probability measure on $[-1,1]$ with finite support can be characterized as an optimal design with respect to a special optimality criterion. To this end we will use the theory of canonical moment which was introduced in the context of optimal design theory by Studden (1980, 1982a, 1982b). The results of section 2 and 3 are used in section 4 to derive equations of the type (1.2) for polynomials orthonormal with respect to an arbitrary (probability) measure on $[-1,1]$. Section 5 deals with the Jacobi polynomials orthogonal with respect to the measure $(1-x)^\alpha(1+x)^\beta dx$ $(\alpha > -1, \beta > -1)$, and we obtain some identities for these polynomials which generalize (1.2). Finally in section 6 equations are derived for the Legendre- and Chebyshev polynomials.

2. Theory of Optimal Design

In this section we will give some basic results of optimal design theory. For shortness we will not give any statistical interpretation or application of the derived results. The interested reader is referred to the books of Fedorov (1972), Silvey (1980) or Pazman (1985). Let $m \in \mathbb{N}_0$, $f_e(x) = (f_{eo}(x), \dots, f_{ee}(x))^T$ denote a vector of $e+1$ real valued functions on $[-1,1]$ $(e = 0, \dots, m)$. A design ξ is a probability measure on $[-1,1]$ and the matrix

$$(2.1) \qquad M_e(\xi) = \int_{-1}^{1} f_e(x) f_e^T(x) d\xi(x) \qquad e = 0, \dots, m$$

is called the information matrix of the design ξ. General optimal design theory deals with the maximization or minimization of functionals of the information matrix $M_e(\xi)$. In this paper we are interested in the functional

$$(2.2) \qquad \Phi(\xi) = \sum_{e=0}^{m} \alpha_e \log(c_e^T M_e^{-1}(\xi) c_e)$$

where the $c_e \in \mathbb{R}^{e+1}$ are given real valued vectors and the α_e are given real numbers such that there exists a design (probability measure) ξ_α on $[-1,1]$ which minimizes $\Phi(\xi)$. Conditions on the α_e which imply the existence of such a design ξ_α are considered in section 3. The following theorem gives an equivalent condition for the minimization problem of (2.2). The proof can be performed by standard arguments of optimal design theory (see Silvey (1980)).

Theorem 2.1 A design ξ_α on $[-1,1]$ minimizes (2.2) if and only if

$$\sum_{e=0}^{m} \alpha_e \frac{[c_e^T M_e^{-1}(\xi_\alpha) f_e(x)]^2}{c_e^T M_e^{-1}(\xi_\alpha) c_e} \leq 1 \qquad \text{for all } x \in [-1,1] \tag{2.3}$$

In (2.3) we have equality if and only if x is a support point of ξ_α.

In what follows let $\gamma_e = (c_e^T M_e^{-1}(\xi_\alpha) c_e)^{-1/2}$, $d_e = \gamma_e M_e^{-1}(\xi) c_e$ and $c_e = (0,\ldots,0,1)^T$ then we have the following corollary for "nested" models $f_e(x)$.

Corollary 2.2. If $f_{ei}(x) = f^{(i)}(x)$ $(i = 0,\ldots,e)$ for all $e = 0,\ldots,m$, $c_e = (0,\ldots,0,1)^T$, then the "polynomials" $\{d_e^T f_e(x)\}_{e=0,\ldots,m}$ form an orthonormal system with respect to the measure $d\xi_\alpha(x)$.

Proof: By the definition of $M_e(\xi)$ we have for $e = 0,\ldots,m$

$$\begin{aligned}\int_{-1}^{1} (d_e^T f_e(x)) f_e^T(x) d\xi_\alpha(x) &= d_e^T \int_{-1}^{1} f_e(x) f_e^T(x) d\xi_\alpha(x) \\ &= d_e^T M_e^{-1}(\xi_\alpha) d_e = \gamma_e c_e^T = (0,\ldots,0,\gamma_e)\end{aligned}$$

and

$$\int_{-1}^{1} (d_e^T f_e(x))^2 d\xi_\alpha(x) = d_e^T M_e^{-1}(\xi_\alpha) d_e = \gamma_c^2 c_e^T M_e^{-1}(\xi_\alpha) c_e = 1.$$

Note that for the vectors $c_e^T = (0,\ldots,0,1)$ the optimality criterion (2.2) reduces (provided $f_{ei}(x) = f^{(i)}(x)$) to

$$\Phi(\xi) = \sum_{e=0}^{m} \alpha_e \log \frac{\det M_{e-1}(\xi)}{\det M_e(\xi)} \tag{2.4}$$

and the equivalent condition in Theorem 2.1 is given by

$$\sum_{e=0}^{m} \alpha_e (d_e^T f_e(x))^2 \leq 1 \qquad \text{for all } x \in [-1,1] \tag{2.5}$$

where $d_e = \gamma_e M_e^{-1}(\xi_\alpha) c_e$ $(e = 0,\ldots,m)$.

3. Polynomial Regression and the Theory of Canonical Moments

In this section we will consider the special functions

$$f_{ei}(x) = x^i(1-x)^a(1+x)^b = g_{ei}(x) \cdot (1-x)^a \ (1+x)^b \tag{3.1}$$

where $a, b \in \{0, \frac{1}{2}\}$, $g_{ei}(x) = x^i$ and $i = 0, \ldots, e$, $e = 0, \ldots, n$. These functions correspond to some optimal design problems for weighted polynomial regression (see Studden 1982b). A very useful tool for the determination of designs minimizing or maximizing functionals depending on the determinants of the information matrices is the theory of canonical moments. We will give a brief introduction and state some of the main results which are needed later. The interested reader is referred to the work of Skibinsky (1967, 1968, 1969, 1986), Studden (1980, 1982a, 1982b, 1989), Lau and Studden (1985, 1988), Lau (1983, 1988) and Dette (1990, 1991).

Let ξ denote a probability measure on $[-1, 1]$ with moments $c_i = \int_{-1}^{1} x^i d\xi(x)$. The canonical moments are defined as follows. For a given set of moments $c_0, c_1, \ldots, c_{i-1}$ let c_i^+ denote the maximum of the i-th moment $\int_{-1}^{1} x^i d\mu(x)$ over the set of all probability measures μ having the given moments $c_0, c_1, \ldots, c_{i-1}$. Similarly let c_i^- denote the corresponding minimum. The canonical moments are defined by

$$p_i = \frac{c_i - c_i^-}{c_i^+ - c_i^-} \quad i = 1, 2, \ldots \ .$$

Note that $0 \le p_i \le 1$ and that the canonical moments are left undefined whenever $c_i^+ = c_i^-$. If i is the first index for which this equality holds, then $0 < p_k < 1$, $k = 1, \ldots, i-2$ and p_{i-1} must have the value 0 or 1 (see Skibinsky (1986), section 1). As an example consider the Jacobi measure with density $(1-x)^\alpha(1+x)^\beta$ $(\alpha > -1, \beta > -1)$. For this measure we have (see Skibinsky (1969))

$$\begin{aligned} p_{2k} &= \frac{k}{\alpha + \beta + 2k + 1} \qquad k \ge 1 \\ p_{2k-1} &= \frac{\beta + k}{\alpha + \beta + 2k} \qquad k \ge 1 \end{aligned} \tag{3.2}$$

The uniform measure ($\alpha = \beta = 0$) has $p_{2k-1} = \frac{1}{2}$ $(k \ge 1)$ and $p_{2k} = k/(2k+1)$. The arc-sin distribution has $p_k = \frac{1}{2}$ for all k $(\alpha = \beta = -\frac{1}{2})$.

In what follows we will denote the information matrices corresponding to the models given in (3.1) by ($g_e(x) = (1, x, \ldots, x^e)^T$)

$$\underline{M}_{2e}(\xi) = \int_{-1}^{1} g_e(x) g_e^T(x) d\xi(x) = \begin{bmatrix} c_0 & c_1 & \cdots & c_e \\ c_1 & c_2 & \cdots & c_{e+1} \\ \vdots & \vdots & & \vdots \\ c_e & c_{e+1} & \cdots & c_{2e} \end{bmatrix} \quad (a = b = 0)$$

$$\underline{M}_{2e+1}(\xi) = \int_{-1}^{1} g_e(x) g_e^T(x)(1+x) d\xi(x) = \begin{bmatrix} c_0 + c_1 & c_1 + c_2 & \cdots & c_e + c_{e+1} \\ c_1 + c_2 & c_2 + c_3 & \cdots & c_{e+1} + c_{e+2} \\ \vdots & \vdots & & \vdots \\ c_e + c_{e+1} & c_{e+1} + c_{e+2} & \cdots & c_{2e} + c_{2e+1} \end{bmatrix} \quad (a = 0, b = \frac{1}{2})$$

$$\overline{M}_{2e}(\xi) = \int_{-1}^{1} g_{e-1}(x) g_{e-1}^T(x)(1-x^2) d\xi(x) = \begin{bmatrix} c_0 - c_2 & c_1 - c_3 & \cdots & c_{e-1} - c_{e+1} \\ c_1 - c_3 & c_2 - c_4 & \cdots & c_e - c_{e+2} \\ \vdots & \vdots & & \vdots \\ c_{e-1} - c_{e+1} & c_e - c_{e+2} & \cdots & c_{2e-2} - c_{2e} \end{bmatrix} \quad (a = b = \frac{1}{2})$$

$$\overline{M}_{2e+1}(\xi) = \int_{-1}^{1} g_e(x) g_e^T(x)(1-x) d\xi(x) = \begin{bmatrix} c_0 - c_1 & c_1 - c_2 & \cdots & c_e - c_{e+1} \\ c_1 - c_2 & c_2 - c_3 & \cdots & c_{e+1} - c_{e+2} \\ \vdots & \vdots & & \vdots \\ c_e - c_{e+1} & c_{e+1} - c_{e+2} & \cdots & c_{2e} - c_{2e+1} \end{bmatrix} \quad (a = \frac{1}{2}, b = 0)$$

The corresponding determinants are denoted by $\underline{D}_{2e}(\xi)$, $\underline{D}_{2e+1}(\xi)$, $\overline{D}_{2e}(\xi)$ and $\overline{D}_{2e+1}(\xi)$ and can be easily expressed in terms of canonical moments (see Skibinsky (1968), Studden (1982b)).

Theorem 3.1. Let ξ denote a probability measure with canonical moments $p_1, p_2, \ldots$, $q_j = 1 - p_j$ ($j \geq 1$), $\zeta_0 = 1$, $\gamma_0 = 1$, $\zeta_1 = p_1$, $\gamma_1 = q_1$, $\zeta_j = q_{j-1} p_j$, $\gamma_j = p_{j-1} q_j$ ($j \geq 2$), then we have

$$\underline{D}_{2e}(\xi) = 2^{e(e+1)} \prod_{i=1}^{e} (\zeta_{2i-1}\zeta_{2i})^{e+1-i}, \qquad \underline{D}_{2e+1}(\xi) = 2^{(e+1)^2} \prod_{i=1}^{e} (\zeta_{2i}\zeta_{2i+1})^{e+1-i}$$

$$\overline{D}_{2e}(\xi) = 2^{e(e+1)} \prod_{i=1}^{e} (\gamma_{2i-1}\gamma_{2i})^{e+1-i}, \qquad \overline{D}_{2e+1}(\xi) = 2^{(e+1)^2} \prod_{i=0}^{e} (\gamma_{2i}\gamma_{2i+1})^{e+1-i}$$

In what follows we are interested in the minimization of functionals of a similar form given in (2.4). To be precise, let $m = 2n$, $\alpha_0 = 0$, $\alpha_j = \beta_j$ $(j = 1, \ldots, n)$, $\alpha_{n+1+j} = \gamma_j$ $(j = 0, \ldots, n-1)$, $c_e = (0, \ldots, 0, 1)^T \in \mathbb{R}^{e+1}$ $(e = 1, \ldots, n)$ and $c_{n+1+e} = (0, \ldots, 0, 1) \in \mathbb{R}^{e+1}$ $(e = 0, \ldots, n-1)$ where $\sum_{j=1}^{n} \beta_j + \sum_{j=0}^{n-1} \gamma_j = 1$. The functions f_{ei} in (2.2) are defined by

$$f_{ei}(x) = x^i \qquad i = 0, \ldots, e \quad , \qquad e = 1, \ldots, n$$

$$f_{n+1+e,i}(x) = \sqrt{1-x}\; x^i \qquad i = 0, \ldots, e \quad , \qquad e = 0, \ldots, n-1$$

and the functional Φ in (2.2) reduces in this case to $(\overline{D}_{-1}(\xi) = \underline{D}_0(\xi) = 1)$

$$\underline{\Phi}_{2n}(\xi) = \sum_{e=1}^{n} \beta_e \log \frac{\underline{D}_{2e-2}(\xi)}{\underline{D}_{2e}(\xi)} + \sum_{e=0}^{n-1} \gamma_e \log \frac{\overline{D}_{2e-1}(\xi)}{\overline{D}_{2e+1}(\xi)}$$

which is a sum of two functionals of a similar form given in (2.4). In the same way we will define the following functionals

$$\underline{\Phi}_{2n+1}(\xi) = \sum_{e=1}^{n} \beta_e \log \frac{\underline{D}_{2e-2}(\xi)}{\underline{D}_{2e}(\xi)} + \sum_{e=0}^{n} \gamma_e \log \frac{\underline{D}_{2e-1}(\xi)}{\underline{D}_{2e+1}(\xi)}$$

$$\overline{\Phi}_{2n+2}(\xi) = \sum_{e=0}^{n} \beta_e \log \frac{\overline{D}_{2e}(\xi)}{\overline{D}_{2e+2}(\xi)} + \sum_{e=0}^{n} \gamma_e \log \frac{\overline{D}_{2e-1}(\xi)}{\overline{D}_{2e+1}(\xi)}$$

$$\overline{\Phi}_{2n+1}(\xi) = \sum_{e=1}^{n} \beta_e \log \frac{\underline{D}_{2e-2}(\xi)}{\underline{D}_{2e}(\xi)} + \sum_{e=0}^{n} \gamma_e \log \frac{\overline{D}_{2e-1}(\xi)}{\overline{D}_{2e+1}(\xi)}$$

By an application of Theorem 3.1 we can determine the designs minimizing $\underline{\Phi}_{2n}$, $\overline{\Phi}_{2n+2}$, $\underline{\Phi}_{2n+1}$ and $\overline{\Phi}_{2n+1}$ in terms of canonical moments. The calculations are straightforward and therefore omitted.

Theorem 3.2. Let $\sigma_i = \sum_{e=i}^{n} \beta_e$ and $\tau_i = \sum_{e=i}^{n} \gamma_e$ (for the functional $\underline{\Phi}_{2n}$ define $\gamma_n = 0$), then we have the following.

a) If $\sigma_i + \tau_i > 0$, $\sigma_i + \tau_{i-1} > 0$, $i = 1, \ldots, n$, the probability measure which minimizes $\underline{\Phi}_{2n}$ is unique and has canonical moments

$$p_{2i} = \frac{\sigma_i + \tau_i}{\sigma_i + \tau_i + \sigma_{i+1} + \tau_i} \quad i = 1, \ldots, n-1, \quad p_{2n} = 1$$
$$p_{2i-1} = \frac{\sigma_i + \tau_i}{\sigma_i + \tau_i + \sigma_i + \tau_{i-1}} \quad i = 1, \ldots, n$$

b) If $\sigma_i + \tau_i > 0$, $\sigma_i + \tau_{i-1} > 0$ $i = 1, \ldots, n$, the probability measure which minimizes $\underline{\Phi}_{2n+1}$ is unique and has canonical moments.

$$p_{2i} = \frac{\sigma_i + \tau_i}{\sigma_i + \tau_i + \sigma_{i+1} + \tau_i} \quad i = 1, \ldots, n$$
$$p_{2i-1} = \frac{\sigma_i + \tau_{i-1}}{\sigma_i + \tau_{i-1} + \tau_i + \sigma_i} \quad i = 1, \ldots, n \quad p_{2n+1} = 1$$

c) If $\sigma_{i-1} + \tau_{i-1} > 0$, $\sigma_{i-1} + \tau_i > 0$ $i = 1, \ldots, n+1$, the probability measure which minimizes $\overline{\Phi}_{2n+2}$ is unique and has canonical moments

$$p_{2i} = \frac{\sigma_i + \tau_i}{\sigma_{i-1} + \tau_i + \sigma_i + \tau_i} \quad i = 1, \ldots, n \quad p_{2n+2} = 0$$
$$p_{2i-1} = \frac{\sigma_{i-1} + \tau_i}{\sigma_{i-1} + \tau_i + \sigma_{i-1} + \tau_{i-1}} \quad i = 1, \ldots, n+1$$

d) If $\sigma_i + \tau_i > 0$, $\sigma_{i+1} + \tau_i > 0$ $i = 1, \ldots, n$, the probability measure which minimizes $\overline{\Phi}_{2n+1}$ is unique and has canonical moments

$$p_{2i} = \frac{\sigma_i + \tau_i}{\sigma_i + \tau_i + \sigma_{i+1} + \tau_i} \quad i = 1, \ldots n$$
$$p_{2i-1} = \frac{\sigma_i + \tau_i}{\sigma_i + \tau_{i-1} + \sigma_i + \tau_i} \quad i = 1, \ldots n, \quad p_{2n+1} = 0$$

The probability measure corresponding to the sequence in a) is supported by $n-1$ points in the interior of $[-1, 1]$ and by the boundary points -1 and 1. Measures corresponding to the sequences in b) or d) are supported at n interior point of $[-1, 1]$ and the boundary point 1 or -1 respectively, while the measure with canonical moments given in c) has $n+1$ support points in the interior of $[-1,1]$ (see Skibinsky (1986)). The interior support points of the measures in a), b), d) and c) are the zeros of the orthogonal polynomials with respect to the measures $(1-x^2)d\xi$, $(1-x)d\xi(x)$, $(1+x)d\xi(x)$ and $d\xi(x)$ where ξ is the measure which minimizes $\underline{\Phi}_{2n}$, $\underline{\Phi}_{2n+1}$, $\overline{\Phi}_{2n+1}$ and $\overline{\Phi}_{2n+2}$ respectively (see Skibinsky (1986) or Karlin and Studden (1966)). Theorem 3.2 defines four maps $\underline{f}_{2n}$, $\underline{f}_{2n+1}$, $\overline{f}_{2n+2}$ and $\overline{f}_{2n+1}$ from the (convex) sets of "weights"

$$\underline{W}_{2n} = \{(\beta_1,\ldots,\beta_n,\gamma_0,\ldots,\gamma_{n-1}) \mid \sigma_1+\tau_0 = 1, \sigma_i+\tau_i > 0,\ \sigma_i+\tau_{i-1} > 0,\ i=1,\ldots,n\}$$

$$\underline{W}_{2n+1} = \{(\beta_1,\ldots,\beta_n,\gamma_0,\ldots,\gamma_n) \mid \sigma_1+\tau_0 = 1, \sigma_i+\tau_i > 0,\ \sigma_i+\tau_{i-1} > 0,\ i=1,\ldots,n\}$$

$$\overline{W}_{2n+2} = \{(\beta_0,\ldots,\beta_n,\gamma_0,\ldots,\gamma_n) \mid \sigma_1+\tau_0 = 1, \sigma_i+\tau_i > 0,\ \sigma_i+\tau_{i+1} > 0, i=0,\ldots,n\}$$

$$\overline{W}_{2n+1} = \{(\beta_1,\ldots,\beta_n,\gamma_0,\ldots,\gamma_n) \mid \sigma_1+\tau_0 = 1, \sigma_i+\tau_i > 0,\ \sigma_{i+1}+\tau_i > 0,\ i=1,\ldots,n\}$$

on the four corresponding sets

$$\underline{\Xi}_{2n} = \{\xi \mid \xi \in \Xi, \#\ \text{supp}\ (\xi) = n+1,\ \text{supp}\ (\xi) \cap [-1,1] = \{-1,1\}\},$$

$$\underline{\Xi}_{2n+1} = \{\xi \mid \xi \in \Xi, \#\ \text{supp}\ (\xi) = n+1,\ \text{supp}\ (\xi) \cap [-1,1] = \{1\}\},$$

$$\overline{\Xi}_{2n+2} = \{\xi \mid \xi \in \Xi, \#\ \text{supp}\ (\xi) = n+1,\ \text{supp}\ (\xi) \subset (-1,1)\},$$

$$\overline{\Xi}_{2n+1} = \{\xi \mid \xi \in \Xi, \#\ \text{supp}\ (\xi) = n+1,\ \text{supp}\ (\xi) \cap [-1,1] = \{-1\}\},$$

where Ξ denotes the set of all probability measures on $[-1,1]$. The image of a weight vector $(\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n)$ under the maps $\underline{f}_{2n}$, $\underline{f}_{2n+1}$, $\overline{f}_{2n+2}$, $\overline{f}_{2n+1}$ is defined as the probability measure which minimizes the functionals $\underline{\Phi}_{2n}$, $\underline{\Phi}_{2n+1}$, $\overline{\Phi}_{2n+2}$ and $\overline{\Phi}_{2n+1}$ respectively (the canonical moments of the corresponding measure are given in Theorem 3.2). If ξ is a probability measure on $[-1,1]$ with canonical moments $p_1, p_2, p_3, \ldots$, then we have the following Lemma (see Skibinsky (1986)).

Lemma 3.3. Let ξ denote a probability measure on $[-1,1]$ with canonical moments $p_1, p_2, \ldots$, then

$$\xi \in \underline{\Xi}_{2n} \iff p_i \in (0,1)\ \ i=1,\ldots,2n-1,\ p_{2n}=1$$

$$\xi \in \underline{\Xi}_{2n+1} \iff p_i \in (0,1)\ \ i=1,\ldots,2n,\ p_{2n+1}=1$$

$$\xi \in \overline{\Xi}_{2n+2} \iff p_i \in (0,1)\ \ i=1,\ldots,2n+1,\ p_{2n+2}=0$$

$$\xi \in \overline{\Xi}_{2n+1} \iff p_i \in (0,1)\ \ i=1,\ldots,2n,\ p_{2n+1}=0.$$

Note that by the conditions on τ_i and σ_i of Theorem 3.2 the design which minimizes $\underline{\Phi}_{2n}$, $\underline{\Phi}_{2n+1}$, $\overline{\Phi}_{2n+2}$, $\overline{\Phi}_{2n+1}$ is in fact an element of the set $\underline{\Xi}_{2n}$, $\underline{\Xi}_{2n+1}$, $\overline{\Xi}_{2n+2}$, $\overline{\Xi}_{2n+1}$ respectively. The following Theorem shows that the maps $\underline{f}_{2n}, \underline{f}_{2n+1}$, $\overline{f}_{2n+2}$ and $\overline{f}_{2n+1}$ are one to one and gives an explicit representation of the inverse maps.

Theorem 3.4. The maps $\underline{f}_{2n}: \underline{W}_{2n} \to \underline{\Xi}_{2n}$, $\underline{f}_{2n+1}: \underline{W}_{2n+1} \to \underline{\Xi}_{2n+1}$, $\overline{f}_{2n+2}: \overline{W}_{2n+2} \to \overline{\Xi}_{2n+2}$, and $\overline{f}_{2n+1}: \overline{W}_{2n+1} \to \overline{\Xi}_{2n+1}$, defined by Theorem 3.2 are one to one. Moreover, if $p_1, p_2, \ldots$ denote the sequence of canonical moments of a probability measure ξ on $[-1,1]$ we have the following representations of the inverse maps.

a) If $\xi \in \underline{\Xi}_{2n}$, then $\underline{f}_{2n}^{-1}(\xi) = (\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_{n-1})$ where (note that $q_{2n} = \overline{D}_{2n}(\xi) = 0$ if $\xi \in \underline{\Xi}_{2n}$)

$$\beta_e = \frac{p_{2e-1}}{q_{2e-1}} \prod_{j=1}^{e-1} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}(1 - \frac{q_{2e}}{p_{2e}}) = \frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] \quad e = 1, \ldots, n$$

$$\gamma_e = \prod_{j=1}^{e} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}(1 - \frac{p_{2e+1}}{q_{2e+1}}) = \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)}\right] \quad e = 0, \ldots, n-1$$

b) If $\xi \in \underline{\Xi}_{2n+1}$, then $\underline{f}_{2n+1}^{-1}(\xi) = (\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n)$ where (note that $q_{2n+1} = \overline{D}_{2n+1}(\xi) = 0$ if $\xi \in \underline{\Xi}_{2n+1}$)

$$\beta_e = \frac{q_{2e-1}}{p_{2e-1}} \prod_{j=1}^{e-1} \frac{q_{2j-1}q_{2j}}{p_{2j-1}p_{2j}}(1 - \frac{q_{2e}}{p_{2e}}) = \frac{\overline{D}_{2e-1}(\xi)}{\underline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] \quad e = 1 \ldots, n$$

$$\gamma_e = \prod_{j=1}^{e} \frac{q_{2j-1}q_{2j}}{p_{2j-1}p_{2j}}(1 - \frac{q_{2e+1}}{p_{2e+1}}) = \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\overline{D}_{2e-1}(\xi)}{\underline{D}_{2e-1}(\xi)} - \frac{\overline{D}_{2e+1}(\xi)}{\underline{D}_{2e+1}(\xi)}\right] \quad e = 1, \ldots, n$$

c) If $\xi \in \overline{\Xi}_{2n+2}$, then $\overline{f}_{2n+2}^{-1}(\xi) = (\beta_0, \ldots, \beta_n, \gamma_0, \ldots, \gamma_n)$ where (note that $p_{2n+2} = \underline{D}_{2n+2}(\xi) = 0$ if $\xi \in \overline{\Xi}_{2n+2}$)

$$\beta_e = \frac{p_{2e+1}}{q_{2e+1}} \prod_{j=1}^{e} \frac{p_{2j-1}p_{2j}}{q_{2j-1}q_{2j}}(1 - \frac{p_{2e+2}}{q_{2e+2}}) = \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)} \left[\frac{\underline{D}_{2e}(\xi)}{\overline{D}_{2e}(\xi)} - \frac{\underline{D}_{2e+2}(\xi)}{\overline{D}_{2e+2}(\xi)}\right] \quad e = 0, \ldots, n$$

$$\gamma_e = \prod_{j=1}^{e} \frac{p_{2j-1}p_{2j}}{q_{2j-1}q_{2j}}(1 - \frac{p_{2e+1}}{q_{2e+1}}) = \frac{\underline{D}_{2e}(\xi)}{\overline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)}\right] \quad e = 0, \ldots, n$$

d) If $\xi \in \overline{\Xi}_{2n+1}$, then $\overline{f}^{\,-1}_{2n+1}(\xi) = (\beta_1, \ldots, \beta_n, \gamma_0, \ldots, \gamma_n)$ where (note that $p_{2n+1} = \underline{D}_{2n+1}(\xi) = 0$ if $\xi \in \overline{\Xi}_{2n+1}$)

$$\beta_e = \frac{p_{2e-1}}{q_{2e-1}} \prod_{j=1}^{e-1} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}(1 - \frac{p_{2e}}{q_{2e}}) = \frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] \quad e = 1, \ldots, n$$

$$\gamma_e = \prod_{j=1}^{e} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}(1 - \frac{p_{2e+1}}{q_{2e+1}}) = \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)}\right] \quad e = 0, \ldots, n$$

Proof: We will only consider the case a) all other cases are treated similarly. Let $\xi \in \underline{\Xi}_{2n}$, ξ is uniquely determined by its canonical moments which are given in Lemma 3.3 by $p_1, p_2, \ldots, p_{2n-1}, 1$ ($p_i \in (0,1)$ $i \le 2n-1$). Because $\underline{f}_{2n}$ maps the set $\underline{W}_{2n}$ in $\underline{\Xi}_{2n}$, we have to determine a unique vector $(\beta_1, \ldots, \beta_n, \gamma_0, \ldots, \gamma_{n-1})$ for which ξ minimizes $\underline{\Phi}_{2n}$. The canonical moments of a probability measure ξ minimizing $\underline{\Phi}_{2n}$ are given by Theorem 3.2 a) and we obtain the equations ($\sigma_{n+1} = 0$).

$$\frac{\sigma_i + \tau_i}{\sigma_i + \tau_i + \sigma_{i+1} + \tau_i} = p_{2i} \qquad i = 1, \ldots, n-1$$
$$\frac{\sigma_i + \tau_i}{\sigma_i + \tau_i + \sigma_i + \tau_{i-1}} = p_{2i-1} \qquad i = 1, \ldots, n$$

which yield

$$\sigma_{i+1} + \tau_i = \frac{q_{2i}}{p_{2i}}(\sigma_i + \tau_i) \qquad i = 1, \ldots, n-1$$
$$\sigma_i + \tau_i = \frac{p_{2i-1}}{q_{2i-1}}(\sigma_i + \tau_{i-1}) \qquad i = 1, \ldots, n \quad .$$

Observing $\sigma_1 + \tau_0 = \sum_{e=1}^{n} \beta_e + \sum_{e=0}^{n} \gamma_e = 1$ we obtain

$$(3.3) \qquad \begin{cases} \sigma_{i+1} + \tau_i = \prod_{j=1}^{i} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}} & i = 1, \ldots, n-1 \\ \sigma_i + \tau_i = (\prod_{j=1}^{i-1} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}) \frac{p_{2i-1}}{q_{2i-1}} & i = 1, \ldots, n \quad . \end{cases}$$

From the definition of σ_e and τ_e we have

$$(3.4) \qquad \begin{cases} \beta_e = \sigma_e - \sigma_{e+1} = \frac{p_{2e-1}}{q_{2e-1}} \prod_{j=1}^{e-1} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}(1 - \frac{q_{2e}}{p_{2e}}) & e = 1, \ldots, n \\ \gamma_e = \tau_e - \tau_{e+1} = \prod_{j=1}^{e} \frac{p_{2j-1}q_{2j}}{q_{2j-1}p_{2j}}(1 - \frac{p_{2e+1}}{q_{2e+1}}) & e = 0, \ldots, n-1 \end{cases}$$

which shows that the given probability measure ξ minimizes $\underline{\Phi}_{2n}$ only for the weights in (3.4) (note that $\sigma_i+\tau_i > 0$ and $\sigma_i+\tau_{i-1} > 0$ by (3.3) and Lemma 3.2 ($\xi \in \Xi_{2n}$)). Therefore the inverse map of $\underline{f}_{2n}$ exists and is defined by (3.4). The assertion of the theorem now follows observing ($\underline{D}_{-1}(\xi) = \overline{D}_{-1}(\xi) = \underline{D}_0(\xi) = \overline{D}_0(\xi) = 1$)

$$p_e = \frac{\underline{D}_e(\xi)\overline{D}_{e-2}(\xi)}{\underline{D}_{e-1}(\xi)\overline{D}_{e-1}(\xi)}, \quad q_e = \frac{\underline{D}_{e-2}(\xi)\overline{D}_e(\xi)}{\underline{D}_{e-1}(\xi)\overline{D}_{e-1}(\xi)}, \quad e \geq 1$$

(see for example Lau (1983)), which proves the representation of the weights β_e and γ_e in terms of the determinants $\overline{D}_e(\xi)$ and $\underline{D}_e(\xi)$.

4. Equations for General Orthogonal Polynomials

In this section we will use the results of section 2 and 3 to derive equations for the polynomials

$$P_n(x) = \frac{1}{\sqrt{\underline{D}_{2n}(\xi)\underline{D}_{2n-2}(\xi)}} \cdot \begin{vmatrix} 1 & c_1 & \cdots & c_{n-1} & 1 \\ c_1 & c_2 & \cdots & c_n & x \\ \vdots & \vdots & & \vdots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} & x^n \end{vmatrix}$$

$$R_n(x) = \frac{1}{\sqrt{\underline{D}_{2n+1}(\xi)\underline{D}_{2n-1}(\xi)}} \begin{vmatrix} 1+c_1 & c_1+c_2 & \cdots & c_{n-1}+c_n & 1 \\ c_1+c_2 & c_2+c_3 & \cdots & c_n+c_{n+1} & x \\ \vdots & \vdots & & \vdots & \vdots \\ c_n+c_{n+1} & c_{n+1}+c_{n+2} & \cdots & c_{2n-1}+c_{2n} & x^n \end{vmatrix}$$

$$Q_n(x) = \frac{1}{\sqrt{\overline{D}_{2n+2}(\xi)\overline{D}_{2n}(\xi)}} \begin{vmatrix} 1-c_2 & c_1-c_3 & \cdots & c_{n-1}-c_{n+1} & 1 \\ c_1-c_3 & c_2-c_4 & \cdots & c_n-c_{n+2} & x \\ \vdots & \vdots & & \vdots & \vdots \\ c_n-c_{n+2} & c_{n+1}-c_{n+3} & \cdots & c_{2n-1}-c_{2n+1} & x^n \end{vmatrix}$$

$$S_n(x) = \frac{1}{\sqrt{\overline{D}_{2n+1}(\xi)\overline{D}_{2n-1}(\xi)}} \begin{vmatrix} 1-c_1 & c_1-c_2 & \cdots & c_{n-1}-c_n & 1 \\ c_1-c_2 & c_2-c_3 & \cdots & c_n-c_{n+1} & x \\ \vdots & \vdots & & \vdots & \vdots \\ c_n-c_{n+1} & c_{n+1}-c_{n+2} & \cdots & c_{2n-1}-c_{2n} & x^n \end{vmatrix}$$

where $c_i = \int_{-1}^{1} x^i d\xi(x)$ and ξ is an arbitrary probability measure on $[-1,1]$. It is well known (see Karlin and Shapley (1953), Szegö (1959) or Karlin and Studden (1966)) that the polynomials $\{P_n(x)\}_{n\geq 0}$, $\{R_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{S_n(x)\}_{n\geq 0}$ are orthonormal with respect to the measures $d\xi(x)$, $(1+x)d\xi(x)$, $(1-x^2)d\xi(x)$ and $(1-x)d\xi(x)$, respectively. Whenever we will consider these polynomials, we assume that they are well defined by the measure ξ. This means, that all determinants $\underline{D}_1(\xi)$, $\overline{D}_1(\xi)$, $\underline{D}_2(\xi)$, $\overline{D}_2(\xi), \ldots, \underline{D}_{2n+2}(\xi)$, $\overline{D}_{2n+2}(\xi)$ are positive, which is equivalent to the fact that the point $(c_0, c_1, \ldots, c_{2n+2})^T$ is an interior point of the corresponding moment space (see Karlin and Shapley (1953), p. 57). It can be shown (see Karlin and Shapley (1953) p. 59) that the "maxi- and minimizing" moments c_e^+ and c_e^- are given by

$$c_e^+ = c_e + \frac{\overline{D}_e(\xi)}{\overline{D}_{e-2}(\xi)} \qquad \text{and} \qquad c_e^- = c_e - \frac{\underline{D}_e(\xi)}{\underline{D}_{e-2}(\xi)}. \tag{4.1}$$

In what follows we will need analogues of the determinants $\underline{D}_e(\xi)$ and $\overline{D}_e(\xi)$, where the moment of highest order is replaced by c_e^+ or c_e^-. To this end define

$$\underline{D}_{2n}^+(\xi) = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & c_{2n-1} \\ c_n & c_{n+1} & \cdots & c_{2n-1} & c_{2n}^+ \end{vmatrix},$$

$$\underline{D}_{2n+1}^+(\xi) = \begin{vmatrix} 1+c_1 & c_1+c_2 & \cdots & c_{n-1}+c_n & c_n+c_{n+1} \\ c_1+c_2 & c_2+c_3 & \cdots & c_n+c_{n+1} & c_{n+1}+c_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1}+c_n & c_n+c_{n+1} & \cdots & c_{2n-2}+c_{2n-1} & c_{2n-1}+c_{2n} \\ c_n+c_{n+1} & c_{n+1}+c_{n+2} & \cdots & c_{2n-1}+c_{2n} & c_{2n}+c_{2n+1}^+ \end{vmatrix}$$

$$\overline{D}_{2n+1}^-(\xi) = \begin{vmatrix} 1-c_1 & c_1-c_2 & \cdots & c_{n-1}-c_n & c_n-c_{n+1} \\ c_1-c_2 & c_2-c_3 & \cdots & c_n-c_{n+1} & c_{n+1}-c_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1}-c_n & c_n-c_{n+1} & \cdots & c_{2n-2}-c_{2n-1} & c_{2n-1}-c_{2n} \\ c_n-c_{n+1} & c_{n+1}-c_{n+2} & \cdots & c_{2n-1}-c_{2n} & c_{2n}-c_{2n+1}^- \end{vmatrix}$$

$$\overline{D}_{2n}^-(\xi) = \begin{vmatrix} 1-c_2 & c_1-c_3 & \cdots & c_{n-2}-c_n & c_n-c_{n+1} \\ c_1-c_3 & c_2-c_4 & \cdots & c_{n-1}-c_{n+1} & c_n-c_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-2}-c_n & c_{n-1}-c_{n+1} & \cdots & c_{2n-4}-c_{2n-3} & c_{2n-3}-c_{2n-1} \\ c_{n-1}-c_{n+1} & c_n-c_{n+2} & \cdots & c_{2n-3}-c_{2n-2} & c_{2n-2}-c_{2n}^- \end{vmatrix}$$

We are now able to state the main theorem of this paper.

Theorem 4.1. Let ξ denote a probability measure on $[-1,1]$, the orthonormal polynomials $Q_n(x)$, $R_n(x)$, $P_n(x)$ and $S_n(x)$ with respect to the measures $(1-x^2)d\xi(x)$, $(1+x)d\xi(x)$, $d\xi(x)$ and $(1-x)d\xi(x)$ satisfy the following equations

a)
$$\begin{aligned}
&\sum_{e=1}^{n-1} \frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] P_e^2(x) \\
&+ \frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n-2}(\xi)\underline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n-2}(\xi)\underline{D}^{+}_{2n}(\xi)} P_n^2(x) \\
&+ (1-x)\sum_{e=1}^{n-1} \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)}\right] S_e^2(x) \\
&= 1 + (x^2-1)\frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}^{+}_{2n}(\xi)} Q_{n-1}^2(x)
\end{aligned}$$

b)
$$\begin{aligned}
&\sum_{e=1}^{n} \frac{\overline{D}_{2e-1}(\xi)}{\underline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] P_e^2(x) \\
&+ (1+x)\frac{\overline{D}_{2n-1}(\xi)\overline{D}_{2n}(\xi)\underline{D}_{2n+1}(\xi)}{\underline{D}_{2n-1}(\xi)\underline{D}_{2n}(\xi)\underline{D}^{+}_{2n+1}(\xi)} R_n^2(x) \\
&+ (1+x)\sum_{e=0}^{n-1} \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\overline{D}_{2e-1}(\xi)}{\underline{D}_{2e-1}(\xi)} - \frac{\overline{D}_{2e+1}(\xi)}{\underline{D}_{2e+1}(\xi)}\right] R_e^2(x) \\
&= 1 + (x-1)\frac{\overline{D}_{2n}(\xi)\overline{D}_{2n+1}(\xi)}{\underline{D}_{2n}(\xi)\underline{D}^{+}_{2n+1}(\xi)} S_n^2(x)
\end{aligned}$$

c)
$$\begin{aligned}
&(1-x^2)\sum_{e=0}^{n-1} \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)} \left[\frac{\underline{D}_{2e}(\xi)}{\overline{D}_{2e}(\xi)} - \frac{\underline{D}_{2e+2}(\xi)}{\overline{D}_{2e+2}(\xi)}\right] Q_e^2(x) \\
&+ (1-x^2)\frac{\underline{D}_{2n}(\xi)\underline{D}_{2n+1}(\xi)\overline{D}_{2n+2}(\xi)}{\overline{D}_{2n}(\xi)\overline{D}_{2n+1}(\xi)\overline{D}^{-}_{2n+2}(\xi)} Q_n^2(x) \\
&+ (1-x)\sum_{e=0}^{n} \frac{\underline{D}_{2e}(\xi)}{\overline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)}\right] S_e^2(x) \\
&= 1 - \frac{\underline{D}_{2n+1}(\xi)\underline{D}_{2n+2}(\xi)}{\overline{D}_{2n+1}(\xi)\overline{D}^{-}_{2n+2}(\xi)} P_{n+1}^2(x)
\end{aligned}$$

d)
$$\begin{aligned}
&\sum_{e=1}^{n} \frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] P_e^2(x) \\
&+ (1-x)\sum_{e=0}^{n-1} \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)}\right] S_e^2(x)
\end{aligned}$$

$$+ (1-x)\frac{\overline{D}_{2n}(\xi)\underline{D}_{2n-1}(\xi)\overline{D}_{2n+1}(\xi)}{\underline{D}_{2n}(\xi)\overline{D}_{2n-1}(\xi)\overline{D}^{-}_{2n+1}(\xi)}S_n^2(x)$$
$$= 1-(1+x)\frac{\overline{D}_{2n}(\xi)\underline{D}_{2n+1}(\xi)}{\underline{D}_{2n}(\xi)\overline{D}^{-}_{2n+1}(\xi)}R_n^2(x)$$

Proof: We will only give a proof of the equation a) all other cases are treated in the same way. Let $p_1, p_2, \dots$ denote the sequence of the canonical moments of the given probability measure ξ. In order to guarantee a correct definition of the polynomials we have to assume $\underline{D}_e > 0$, $\overline{D}_e > 0$ $(e = 1, \dots, 2n)$ which is equivalent to the condition $p_i \in (0,1)$ $(i = 1, \dots, 2n)$ by Theorem 3.1. In what follows let $\tilde{\xi}$ denote a probability measure with the same canonical moments as ξ up to the order $2n-1$ and $p_{2n} = 1$ (i.e. $\tilde{\xi}$ has the canonical moments $p_1, p_2, \dots, p_{2n-1}, 1$ where $p_i \in (0,1)$ $(i = 1, \dots, 2n-1)$) and let $\{\tilde{P}_e(x)\}_{e=0}^n$, $\{\tilde{S}_e(x)\}_{e=0}^{n-1}$ denote the orthonormal polynomials with respect to the measures $d\tilde{\xi}(x)$ and $(1-x)d\tilde{\xi}(x)$ respectively. By Theorem 3.4 a) $\tilde{\xi}$ minimizes the functional $\underline{\Phi}_{2n}(\xi)$ for the given vector of weights

$$(4.2) \qquad \begin{cases} \tilde{\beta}_e = \dfrac{\underline{D}_{2e-1}(\tilde{\xi})}{\overline{D}_{2e-1}(\tilde{\xi})}\left[\dfrac{\overline{D}_{2e-2}(\tilde{\xi})}{\underline{D}_{2e-2}(\tilde{\xi})} - \dfrac{\overline{D}_{2e}(\tilde{\xi})}{\underline{D}_{2e}(\tilde{\xi})}\right] & e = 1, \dots, n \\ \tilde{\gamma}_e = \dfrac{\overline{D}_{2e}(\tilde{\xi})}{\underline{D}_{2e}(\tilde{\xi})}\left[\dfrac{\underline{D}_{2e-1}(\tilde{\xi})}{\overline{D}_{2e-1}(\tilde{\xi})} - \dfrac{\underline{D}_{2e+1}(\tilde{\xi})}{\overline{D}_{2e+1}(\tilde{\xi})}\right] & e = 0, \dots, n-1 \end{cases}$$

From the equivalence Theorem 2.1, (2.4) and (2.5) we obtain an equivalent condition for $\tilde{\xi}$

$$(4.3) \qquad \sum_{e=1}^{n} \tilde{\beta}_e (d_e^T g_e(x))^2 + \sum_{e=0}^{n-1} \gamma_e (\tilde{d}_e^T h_e(x))^2 \le 1 \quad \text{for all} \quad x \in [-1,1]$$

where $g_e(x) = (1, x, \dots, x^e)^T$, $h_e(x) = \sqrt{1-x}(1, x, \dots, x^e)^T$, $d_e = \sqrt{\frac{\underline{D}_{2e}(\tilde{\xi})}{\underline{D}_{2e-2}(\tilde{\xi})}}\ \underline{M}_{2e}^{-1}(\xi)c_e$, $\tilde{d}_e = \sqrt{\frac{\overline{D}_{2e+1}(\tilde{\xi})}{\overline{D}_{2e-1}(\tilde{\xi})}}\overline{M}_{2e+1}(\tilde{\xi})c_e$ and $c_e = (0, \dots, 0, 1)^T \in \mathrm{R}^{e+1}$. By Corollary 2.2 the "polynomials" $(d_e^T g_e(x))$ and $(\tilde{d}_e^T h_e(x))$ are orthonormal with respect to the measure $d\tilde{\xi}(x)$ and thus (4.3) reduces to

$$(4.4) \qquad \sum_{e=1}^{n} \tilde{\beta}_e \tilde{P}_e^2(x) + \sum_{e=0}^{n-1} \tilde{\gamma}_e (1-x)\tilde{S}_e^2(x) \le 1 \quad \text{for all } x \in [-1,1]$$

An application of Theorem 2.1 shows that we have equality in (4.4) only for the support points of the minimizing measure $\tilde{\xi}$. A consideration of the Stieltjes transform of $\tilde{\xi}$ (see Wall (1948) or Studden (1980, 1982b)) yields that the support of $\tilde{\xi}$ is given by the zeros of the polynomial $(1-x^2)\tilde{Q}_{n-1}(x)$ where $\tilde{Q}_{n-1}(x)$ is the $(n-1)$-th orthogonal polynomial with respect to the measure $(1-x^2)d\tilde{\xi}(x)$, i.e.

$$\tilde{Q}_{n-1}(x) = \begin{vmatrix} 1-c_2 & \cdots & c_{n-2}-c_n & 1 \\ c_1-c_3 & \cdots & c_{n-1}-c_{n+1} & x \\ \vdots & & \vdots & \vdots \\ c_{n-1}-c_{n+1} & \cdots & c_{2n-3}-c_{2n-1} & x^{n-1} \end{vmatrix}$$

Note that this property results from the fact that $\tilde{\xi}$ has the terminating sequence $p_1, p_2, \ldots, p_{2n-1}, 1$ and that there exists a recursive relationship for the polynomials $\tilde{Q}_e(x)/\underline{D}_{2e-2}(\tilde{\xi})$ in terms of the canonical moments of the probability measure $\tilde{\xi}$ (see Studden (1982b)). Because the polynomials $\sum_{e=1}^{n} \tilde{\beta}_e \tilde{P}_e^2(x) + (1-x)\sum_{e=0}^{n-1} \tilde{\gamma}_e \tilde{S}_e^2(x) - 1$ and $(1-x^2)\tilde{Q}_{n-1}^2(x)$ are less or equal than 0 in $[-1,1]$ and exactly equal 0 at $n-1$ points in the interior of $[-1,1]$ (the roots of $Q_{n-1}(x)$) and at -1 and 1, they can only differ by a factor. Comparing the leading coefficients of $\tilde{P}_n^2(x)$ and $-x^2\tilde{Q}_{n-1}^2(x)$ we obtain from (4.4) the equation

$$(4.5)\quad \sum_{e=1}^{n} \tilde{\beta}_e \tilde{P}_e^2(x) + (1-x)\sum_{e=0}^{n-1} \tilde{\gamma}_e \tilde{S}_e^2(x) = 1 - (1-x^2)\tilde{Q}_{n-1}^2(x) \frac{\underline{D}_{2n-1}(\tilde{\xi})}{\overline{D}_{2n-1}(\tilde{\xi})\overline{D}_{2n-2}(\tilde{\xi})\underline{D}_{2n}(\tilde{\xi})}$$

The moments and canonical moments of $\tilde{\xi}$ up to the order $2n-1$ coincide with the corresponding quantities of ξ. Thus we have

$$(4.6)\qquad \begin{cases} \underline{D}_{2e-1}(\tilde{\xi}) = \underline{D}_{2e-1}(\xi) & e = 1, \ldots, n \\ \underline{D}_{2e}(\tilde{\xi}) = \underline{D}_{2e}(\xi) & e = 1, \ldots, n-1, \qquad \underline{D}_{2n}(\tilde{\xi}) = 0 \\ \overline{D}_{2e}(\tilde{\xi}) = \overline{D}_{2e}(\tilde{\xi}) & e = 0, \ldots, n-1 \end{cases}$$

and for the polynomials $P_e(x)$, $S_e(x)$ orthonormal with respect to the measures $d\xi(x)$, $(1-x)d\xi(x)$

$$(4.7)\qquad \begin{cases} P_e^2(x) = \tilde{P}_e^2(x) & e = 1, \ldots, n-1, \qquad P_n^2(x) = \dfrac{\underline{D}_{2n}(\tilde{\xi})}{\underline{D}_{2n}(\xi)} \tilde{P}_n^2(x) \\ S_e^2(x) = \tilde{S}_e^2(x) & e = 0, \ldots, n-1. \end{cases}$$

Finally we have from the definition of $\tilde{Q}_{n-1}(x)$ and $Q_{n-1}(x)$

$$\tilde{Q}_{n-1}^2(x) = \overline{D}_{2n}(\xi) \cdot \overline{D}_{2n-2}(\xi) Q_{n-1}^2(x). \tag{4.8}$$

Observing the equation (4.5), the definition of the weights (4.2) (note that $\overline{D}_{2n}(\tilde{\xi}) = 0$), (4.6), (4.7) and (4.8) it follows that

$$\begin{aligned}
&\sum_{e=1}^{n-1} \frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \right] P_e^2(x) + \frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n-2}(\xi)\underline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n-2}(\xi)\underline{D}_{2n}(\tilde{\xi})} P_n^2(x) \\
&\quad + (1-x) \sum_{e=0}^{n-1} \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{D}_{2e+1}(\xi)} \right] S_e^2(x) \\
&\quad = 1 + (x^2-1) \frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n}(\tilde{\xi})} Q_{n-1}^2(x)
\end{aligned}$$

and the assertion a) of Theorem 4.1 now follows from the fact $\underline{D}_{2n}(\tilde{\xi}) = \underline{D}_{2n}^{+}(\xi)$ which can be shown by Theorem 3.1, the definition of the canonical moments and the representation of c_e^+ and c_e^- given in (4.1).

If the probability measure ξ is symmetric on $[-1, 1]$, all canonical moments of odd order are $\frac{1}{2}$ (see Studden (1982b) or Lau (1983)). In this case we have by Theorem 3.1 $\underline{D}_{2e-1} = \overline{D}_{2e-1}$ for all $e \geq 1$, and the weights γ_e in Theorem 4.2 vanish. We have proved the following Corollary which gives equations of a simpler form for orthonormal polynomials with respect to a symmetric measure on the interval $[-1, 1]$.

Corollary 4.2. Let ξ denote a symmetric probability measure are $[-1, 1]$, the orthonormal polynomials $\{Q_n(x)\}_{n\geq 0}$ $\{S_n(x)\}_{n\geq 0}$, $\{R_n(x)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ satisfy the following equations

a)
$$\begin{aligned}
&\sum_{e=1}^{n-1} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \right] P_e^2(x) + \frac{\overline{D}_{2n-2}(\xi)\underline{D}_{2n}(\xi)}{\underline{D}_{2n-2}(\xi)\underline{D}_{2n}^{+}(\xi)} P_n^2(x) \\
&\quad = 1 + (x^2-1) \frac{\overline{D}_{2n}(\xi)}{\underline{D}_{2n}^{+}(\xi)} Q_{n-1}^2(x)
\end{aligned}$$

b)
$$1 - \sum_{e=1}^{n} \left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)} - \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)} \right] P_e^2(x) = \frac{\overline{D}_{2n}(\xi)}{\underline{D}_{2n}(\xi)} \left[\frac{1+x}{2} R_n^2(x) + \frac{1-x}{2} S_n^2(x) \right]$$

c) $$(1-x^2)\left\{\sum_{e=1}^{n-1}\left[\frac{\underline{D}_{2e}(\xi)}{\overline{\overline{D}}_{2e}(\xi)}+\frac{\underline{D}_{2e+2}(\xi)}{\overline{\overline{D}}_{2e+2}(\xi)}\right]Q_e^2(x)+\frac{\underline{D}_{2n}(\xi)\overline{D}_{2n+2}(\xi)}{\overline{\overline{D}}_{2n}(\xi)\overline{D}^{-}_{2n+2}(\xi)}Q_n^2(x)\right\}$$
$$=1-\frac{\underline{D}_{2n+2}(\xi)}{\overline{\overline{D}}^{-}_{2n+2}(\xi)}P_{n+1}^2(x)$$

5. Equations for the Jacobi Polynomials

In this section we will derive some equations for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which are defined as the orthogonal polynomials on $[-1,1]$ with respect to the measure $(1-x)^\alpha(1+x)^\beta dx$ $(\alpha>-1,\beta>-1)$ with leading coefficient

(5.1) $$k_n=\binom{2n+\alpha+\beta}{n}\frac{1}{2^n}=\frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\frac{1}{2^n}$$

where $\Gamma(x)$ denotes the Gamma function (see Szegö p. 58-98). In what follows let $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ denote the Beta-function we have the following theorem.

Theorem 5.1. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the following equations $(\alpha>-1,\beta>-1)$

a) $$(\alpha-\beta)\frac{1-x}{2}\sum_{e=0}^{n-1}\frac{\alpha+\beta+2e+2}{(\alpha+\beta+1)^2}\left[\frac{\beta(\alpha+1,e+1)}{\beta(\alpha+\beta+1,e+1)}P_e^{(\alpha+1,\beta)}(x)\right]^2$$
$$+\left[\frac{n}{\alpha+\beta+1}\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+1,n)}P_n^{(\alpha,\beta)}(x)\right]^2$$
$$-\sum_{e=0}^{n-1}\frac{\alpha+\beta+2e+1}{\alpha+\beta+1}\left[\frac{\beta(\alpha+1,e)}{\beta(\alpha+\beta+1,e)}P_e^{(\alpha,\beta)}(x)\right]^2$$
$$=\frac{x^2-1}{4}\left[\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+2,n)}P_{n-1}^{(\alpha+1,\beta+1)}(x)\right]^2$$

b) $$(\beta-\alpha)\frac{1+x}{2}\sum_{e=0}^{n-1}\frac{\alpha+\beta+2e+2}{(\alpha+\beta+1)^2}\left[\frac{\beta(\beta+1,e+1)}{\beta(\alpha+\beta+1,e+1)}P_e^{(\alpha,\beta+1)}(x)\right]^2$$
$$+\frac{1+x}{2}\left[\frac{\beta(\beta+1,n)}{\beta(\alpha+\beta+2,n)}P_n^{(\alpha,\beta+1)}(x)\right]^2$$
$$+\frac{1-x}{2}\left[\frac{\beta(\beta+1,n)}{\beta(\alpha+\beta+2,n)}P_n^{(\alpha+1,\beta)}(x)\right]^2$$
$$=\sum_{e=0}^{n}\frac{\alpha+\beta+2e+1}{\alpha+\beta+1}\left[\frac{\beta(\beta+1,e)}{\beta(\alpha+\beta+1,e)}P_e^{(\alpha,\beta)}(x)\right]^2$$

c)

$$
\begin{aligned}
&(\alpha+\beta+1)\frac{1-x^2}{4}\sum_{e=0}^{n-1}(\alpha+\beta+2e+3)\left[\beta(\alpha+1,e+1)P_e^{(\alpha+1,\beta+1)}(x)\right]^2 \\
&+\frac{1-x^2}{4}\left[(\alpha+\beta+n+2)\beta(\alpha+1,n+1)P_n^{(\alpha+1,\beta+1)}(x)\right]^2 \\
&+(\alpha-\beta)\frac{1-x}{2}\sum_{e=0}^{n}(\alpha+\beta+2e+2)\left[\beta(\alpha+1,e+1)P_e^{(\alpha+1,\beta)}(x)\right]^2 \\
&=1-\left[(n+1)\beta(\alpha+1,n+1)P_{n+1}^{(\alpha,\beta)}(x)\right]^2
\end{aligned}
$$

d)

$$
\begin{aligned}
&(\alpha-\beta)\frac{1-x}{2}\sum_{e=0}^{n-1}\frac{\alpha+\beta+2e+2}{(\alpha+\beta+1)^2}\left[\frac{\beta(\alpha+1,e+1)}{\beta(\alpha+\beta+1,e+1)}P_e^{(\alpha+1,\beta)}(x)\right]^2 \\
&+\frac{1-x}{2}\left[\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+2,n)}P_n^{(\alpha+1,\beta)}(x)\right]^2 \\
&+\frac{1+x}{2}\left[\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+2,n)}P_n^{(\alpha,\beta+1)}(x)\right]^2 \\
&=\sum_{e=0}^{n}\frac{\alpha+\beta+2e+1}{\alpha+\beta+1}\left[\frac{\beta(\alpha+1,e)}{\beta(\alpha+\beta+1,e)}P_e^{(\alpha,\beta)}(x)\right]^2
\end{aligned}
$$

Proof: We will only give a proof of equation a) all other cases are treated similarly. For the application of Theorem 4.1 we have to calculate the determinants $\underline{D}_{2e-1}(\xi)$, $\overline{D}_{2e-1}(\xi)$, $\underline{D}_{2e}(\xi)$, $\overline{D}_{2e}(\xi)$, $\underline{D}_{2n}^{+}(\xi)$, where ξ is the probability measure with density proportional to $(1-x)^{\alpha}(1+x)^{\beta}dx$. The canonical moments of ξ are given by (3.2) and by an application of Theorem 3.1 be obtain for the factors of the polynomials $P_e^2(x)$ in Theorem 4.1.

$$
\begin{aligned}
\overline{\beta}_e &= \frac{\underline{D}_{2e-1}(\xi)}{\overline{D}_{2e-1}(\xi)}\left[\frac{\overline{D}_{2e-2}(\xi)}{\underline{D}_{2e-2}(\xi)}-\frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e}(\xi)}\right] \\
&= -\frac{\Gamma(\beta+e+1)\Gamma(\alpha+1)\Gamma(\alpha+\beta+e+1)}{\Gamma(\alpha+e+1)\Gamma(\beta+1)\Gamma(e+1)\Gamma(\alpha+\beta+1)} \qquad e=1,\ldots,n-1
\end{aligned}
$$

$$
\begin{aligned}
\overline{\beta}_n &= \frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n-2}(\xi)\underline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n-2}(\xi)\underline{D}_{2n}^{+}(\xi)} \\
&= \frac{\Gamma(\beta+n+1)\Gamma(\alpha+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+1)\Gamma(n)\Gamma(\alpha+\beta+2)}\,\frac{n}{\alpha+\beta+2n+1}
\end{aligned}
$$

and for the factors of the polynomials $S_e^2(x)$

$$\overline{\gamma}_e = \frac{\overline{D}_{2e}(\xi)}{\underline{D}_{2e(\xi)}}\left[\frac{\underline{D}_{2e-1}(\xi)}{\overline{\overline{D}}_{2e-1}(\xi)} - \frac{\underline{D}_{2e+1}(\xi)}{\overline{\overline{D}}_{2e+1}(\xi)}\right]$$
$$= \frac{\Gamma(\beta+e+1)\Gamma(\alpha+1)\Gamma(\alpha+\beta+e+2)}{\Gamma(\alpha+e+2)\Gamma(\beta+1)\Gamma(\alpha+\beta+2)\Gamma(e+1)}(\alpha-\beta)$$

Because the leading coefficients of $P_e^2(x)$ and $S_e^2(x)$ are given by $\frac{\underline{D}_{2e-2}(\xi)}{\underline{D}_{2e}(\xi)}$ and $\frac{\overline{D}_{2e-1}(\xi)}{\overline{D}_{2e+1}(\xi)}$ we obtain by Theorem 3.1 that the leading coefficients of the polynomials $\overline{\beta}_e P_e^2(x)(e = 1,\dots,n-1)$, $\overline{\beta}_n P_n^2(x)$ and $\overline{\gamma}_e S_e^2(x)$ $(e=0,\dots,n-1)$ are given by

$$-\frac{1}{2^{2e}}\left[\frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2e+1)}{\Gamma(e+1)\Gamma(\alpha+e+1)\Gamma(\alpha+\beta+1)}\right]^2 \cdot \frac{\alpha+\beta+2e+1}{\alpha+\beta+1} \quad (e=1,\dots,n-1),$$
$$\frac{1}{2^{2n}}\left[\frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2n+1)}{\Gamma(n)\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+2)}\right]^2 \quad \text{and}$$
$$\frac{1}{2^{2e+1}}\left[\frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2e+2)}{\Gamma(e+1)\Gamma(\alpha+e+2)\Gamma(\alpha+\beta+2)}\right]^2 \cdot (\alpha+\beta+2e+2)(\alpha-\beta).$$

By Theorem 4.1 the polynomials $\{P_e(x)\}_{e\geq 0}$, $\{S_e(x)\}_{e\geq 0}$ are orthonormal with respect to the measures $d\xi(x) = k\cdot(1-x)^\alpha(1+x)^\beta dx$ and $(1-x)d\xi(x) = k'\,(1-x)^{\alpha+1}(1+x)^\beta d\xi(x)$ and a comparison of the leading coefficients of $(P_e^{(\alpha,\beta)}(x))^2$ and $\overline{\beta}_e P_e^2(x)$ and $\overline{\gamma}_e S_e^2(x)$ with the leading coefficients of $(P_e^{(\alpha+1,\beta)}(x))^2$ given in (5.1) yields

$$\overline{\beta}_e P_e^2(x) = -\frac{\alpha+\beta+2e+1}{\alpha+\beta+1}\left[\frac{\beta(\alpha+1,e)}{\beta(\alpha+\beta+1,e)}P_e^{(\alpha,\beta)}(x)\right]^2 \qquad e=1,\dots,n-1 \tag{5.2}$$

$$\overline{\beta}_n P_n^2(x) = \left[\frac{n}{\alpha+\beta+1}\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+1,n)}P_n^{(\alpha,\beta)}(x)\right]^2 \tag{5.3}$$

(5.4)
$$\overline{\gamma}_e S_e^2(x) = \frac{1}{2}(\alpha-\beta)\frac{\alpha+\beta+2e+2}{(\alpha+\beta+1)^2}\left[\frac{\beta(\alpha+1,e+1)}{\beta(\alpha+\beta+1,e+1)}P_e^{(\alpha+1,\beta)}(x)\right]^2 \qquad e=0,\dots,n-1$$

By a similar reasoning we obtain for the leading coefficient of $\frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n}^+(\xi)}\,Q_{n-1}^2(x)$

$$\frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n-2}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n}^+(\xi)} = \frac{1}{2^{2n}}\left[\prod_{i=1}^{n} q_{2i-1}^2 \prod_{i=1}^{n-1} p_{2i}^2\right]^{-1} = \frac{1}{2^{2n}}\left[\frac{\Gamma(\alpha+\beta+2n+1)\Gamma(\alpha+1)}{\Gamma(n)\Gamma(\alpha+\beta+2)\Gamma(\alpha+n+1)}\right]^2$$

which yields (note that the polynomials $Q_e(x)$ are orthonormal with respect to the measure $(1-x^2)d\xi(x) = \tilde{k}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx$)

$$\frac{\underline{D}_{2n-1}(\xi)\overline{D}_{2n-2}(\xi)}{\overline{D}_{2n-1}(\xi)\underline{D}_{2n}^{+}(\xi)} Q_{n-1}^2(x) = \frac{1}{4}\left[\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+2,n)} P_{n-1}^{(\alpha+1,\beta+1)}(x)\right]^2 .$$

By an application of Theorem 4.1, (5.2), (5.3) and (5.4) we now obtain

$$\begin{aligned}
&\left[\frac{n}{\alpha+\beta+1}\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+1,n)} P_n^{(\alpha,\beta)}(x)\right]^2 - \sum_{e=1}^{n-1} \frac{\alpha+\beta+2e+1}{\alpha+\beta+1}\left[\frac{\beta(\alpha+1,e)}{\beta(\alpha+\beta+1,e)} P_e^{(\alpha,\beta)}(x)\right]^2 \\
&\quad + (\alpha-\beta)\frac{1-x}{2}\sum_{e=0}^{n-1}\frac{\alpha+\beta+2e+2}{(\alpha+\beta+1)^2}\left[\frac{\beta(\alpha+1,e+1)}{\beta(\alpha+\beta+1,e+1)} P_e^{(\alpha+1,\beta)}(x)\right]^2 \\
&\quad = 1 + \frac{x^2-1}{4}\left[\frac{\beta(\alpha+1,n)}{\beta(\alpha+\beta+2,n)} P_{n-1}^{(\alpha+1,\beta+1)}(x)\right]^2
\end{aligned}$$

which completes the proof of the theorem (in the case a)).

In the following we will apply this theorem to derive equations for measures with canonical moments of odd (or even) order equal 1/2. It is well known that the measure ξ is symmetric if and only if $p_{2i-1} = \frac{1}{2}$ $i = 1, 2, \ldots$ (see Studden (1982b)). In the case of Jacobi polynomials this condition yields $\alpha = \beta$ and the equations of Theorem 5.1 reduce to equations for the ultraspherical polynomials (see Szegö (1959) p. 81–86) which are given by ($\alpha > -\frac{1}{2}$)

$$C_n^{(\alpha)}(x) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}{\Gamma(2\alpha)\Gamma(n+\alpha+\frac{1}{2})} P_n^{(\alpha-1/2,\alpha-1/2)}(x). \tag{5.5}$$

By an application of this formula we obtain the following Corollary.

Corollary 5.2. For the ultraspherical polynomials defined in (5.5) we have the following equations

a) $$\left[\frac{n}{2\alpha} C_n^{(\alpha)}(x)\right]^2 - \sum_{e=0}^{n-1}\frac{e+\alpha}{\alpha}\left[C_e^{(\alpha)}(x)\right]^2 = \left[C_{n-1}^{(\alpha+1)}(x)\right]^2 (x^2-1)$$

b) $$\begin{aligned}
&\frac{1+x}{2}\left[\frac{\beta(\alpha+\frac{1}{2},n)}{\beta(2\alpha+1,n)} P_n^{(\alpha-1/2,\alpha+1/2)}(x)\right]^2 + \frac{1-x}{2}\left[\frac{\beta(\alpha+\frac{1}{2},n)}{\beta(2\alpha+1,n)} P_n^{(\alpha+1/2,\alpha-1/2)}(x)\right]^2 \\
&\quad = \sum_{e=0}^{n}\frac{\alpha+e}{\alpha}\left[C_e^{(\alpha)}(x)\right]^2
\end{aligned}$$

c)
$$4\alpha(1-x^2)\sum_{e=0}^{n-1}(e+\alpha+1)\left[\beta(2\alpha+1,e+1)C_e^{(\alpha+1)}(x)\right]^2$$
$$+(1-x^2)\left[(2\alpha+n+1)\beta(2\alpha+1,n+1)C_n^{(\alpha+1)}(x)\right]^2$$
$$+\left[(n+1)\beta(n+1,2\alpha)C_{n+1}^{(\alpha)}(x)\right]^2=1$$

If all canonical moments of odd order are equal $\frac{1}{2}$ we have $\alpha = -1-\beta$ (where $\beta \in (-1,0)$) and obtain the following equations for the Jacobi polynomials $P_n^{(-\beta,\beta)}(x)$

Corollary 5.3. The Jacobi polynomials $P_n^{(-\beta,\beta)}(x)$ satisfy the equations ($\beta \in (-1,0)$)

a)
$$\left[n\beta(-\beta,n)P_n^{(-\beta-1,\beta)}(x)\right]^2-\frac{x^2-1}{4}\left[n\beta(-\beta,n)P_{n-1}^{(-\beta,\beta+1)}(x)\right]^2$$
$$=1+(2\beta+1)\frac{1-x}{2}\sum_{e=0}^{n-1}(2e+1)\left[\beta(-\beta,e+1)P_e^{(-\beta,\beta)}(x)\right]^2$$

b)
$$\frac{1+x}{2}\left[n\beta(\beta+1,n)P_n^{(-\beta-1,\beta+1)}(x)\right]^2+\frac{1-x}{2}\left[n\beta(\beta+1,n)P_n^{(-\beta,\beta)}(x)\right]^2$$
$$=1-(2\beta+1)\frac{1+x}{2}\sum_{e=0}^{n-1}(2e+1)\left[\beta(\beta+1,e+1)P_e^{(-\beta-1,\beta+1)}(x)\right]^2$$

c)
$$\frac{1-x^2}{4}\left[(n+1)\beta(-\beta,n+1)P_n^{(-\beta,\beta+1)}(x)\right]^2+\left[(n+1)\beta(-\beta,n+1)P_{n+1}^{(-\beta-1,\beta)}(x)\right]^2$$
$$=1+(2\beta+1)\frac{1-x}{2}\sum_{e=0}^{n}(2e+1)\left[\beta(-\beta,e+1)P_e^{(-\beta,\beta)}(x)\right]^2$$

d)
$$\frac{1+x}{2}\left[n\beta(-\beta,n)P_n^{(-\beta-1,\beta+1)}(x)\right]^2+\frac{1-x}{2}\left[n\beta(-\beta,n)P_n^{(-\beta,\beta)}(x)\right]^2$$
$$=1+(2\beta+1)\frac{1-x}{2}\sum_{e=0}^{n-1}(2e+1)\left[\beta(-\beta,e+1)P_e^{(-\beta,\beta)}(x)\right]^2$$

6. Equations for Legendre and Chebyshev Polynomials

Let $P_n(x)$ denote the n-th Legendre polynomial which is orthogonal with respect to the lebesgue measure and satisfies $P_n(1) = 1$ (see Szegö (1959)). It is well known, that

$$P_n(x) = C_n^{(\frac{1}{2})}(x), \qquad P_n'(x) = C_{n-1}^{(\frac{3}{2})}(x)$$

(see Szegö (1959) p. 84). From Corollary 5.2 we obtain the following identities for the polynomials $P_n(x)$.

Proposition 6.1. The Legendre polynomials satisfy the following equations

a) $$[nP_n(x)]^2 - \sum_{e=0}^{n-1}(2e+1)P_e^2(x) = [P_n'(x)]^2(x^2-1)$$

b) $$\frac{1+x}{2}[(n+1)P_n^{(0,1)}(x)]^2 + \frac{1-x}{2}[(n+1)P_n^{(1,0)}(x)]^2 = \sum_{e=0}^{n}(2e+1)P_e^2(x)$$

c) $$(1-x^2)\left\{\sum_{e=0}^{n-1}(2e+3)\left[\frac{P_{e+1}'(x)}{(e+1)(e+2)}\right]^2 + \left[\frac{P_{n+1}'(x)}{n+1}\right]^2\right\} + P_{n+1}^2(x) = 1$$

For the Chebyshev polynomials of the first and second kind we obtain from the equations given in Corollary 5.2 ($\alpha = 0, \alpha = 1$) and Corollary 5.3 the following identities which are the generalizations of (1.2) given in the introduction (note that $C_n^{(1)}(x) = U_n(x)$ $\lim_{\alpha \to 0} \frac{1}{\alpha} C_n^{(\alpha)}(x) = C_n^{(0)}(x) = \frac{2}{n} T_n(x)$ and $U_{n+1}'(x) = 2C_n^{(2)}(x)$ (Abramowitz and Stegun (1964)).

Proposition 6.2. For the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ the following identities hold.

a) $$T_n^2(x) = (x^2-1)U_{n-1}^2(x) + 1$$

b) $$\left[\frac{n}{2}U_n(x)\right]^2 - \sum_{e=0}^{n-1}(e+1)U_e^2(x) = (x^2-1)\left[\frac{U_n'(x)}{2}\right]^2$$

c) $(1-x^2)\left\{\sum_{e=0}^{n-1}(e+2)\left[\beta(3,e+1)U'_{e+1}(x)\right]^2+\left[\frac{U'_{n+1}(x)}{(n+1)(n+2)}\right]^2\right\}+\left[\frac{U_{n+1}(x)}{n+2}\right]^2=1$

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