CHARACTERIZING THE WEAK CONVERGENCE OF STOCHASTIC INTEGRALS

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ABSTRACT

Let (H^n, X^n) be a sequence of càdlàg, adapted processes converging weakly to (H, X). X^n is a good sequence of semimartingales if the above implies that $\int H^n_{s-}\, dX^n_s$ converges weakly to $\int H_{s-}\, dX_s$. We show that the known sufficient conditions for the sequence X^n to be good are also necessary. We further show that if X^n is good, then $\int H^n_{s-}\, dX^n_s$ is also good.

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For $n = 1, 2, \ldots$, let $\Xi_n = (\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t\geq 0}, P^n)$ be a filtered probability space, let H^n be càdlàg and adapted, and let X^n be a càdlàg semimartingale. A fundamental question is: Under what conditions does the convergence in distribution of (H^n, X^n) to (H, X) imply that X is a semimartingale and that $\int_0^t H^n_{s-}\, dX^n_s$ converges in distribution to $\int_0^t H_{s-}\, dX_s$? A slightly more general formulation would put conditions on the sequence X^n alone such that the convergence above holds for all such sequences H^n. A sequence with this property will be called good. To be precise, let $\mathbb{M}^{km}$ denote the real–valued, $k \times m$ matrices, and let $\mathbb{D}_{\mathbb{E}}[0,\infty)$ denote the space of càdlàg, E-valued functions with Skorohod topology.

Definition: For $n = 1, 2, \ldots$, let X^n be an $\mathbb{R}^k$-valued, $(\mathcal{F}^n_t)$-semimartingale, and let the sequence $(X^n)_{n\geq 1}$ converge in distribution in the Skorohod topology to a process X. The sequence $(X^n)_{n\geq 1}$ is said to be good if for any sequence $(H^n)_{n\geq 1}$ of $\mathbb{M}^{km}$-valued, càdlàg processes, H^n $(\mathcal{F}^n_t)$-adapted, such that (H^n, X^n) converges in distribution in the Skorohod topology on $\mathbb{D}_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$ to a process (H, X), there exists a filtration $(\mathcal{F}_t)$ such that H is $(\mathcal{F}_t)$-adapted, X is an $(\mathcal{F}_t)$-semimartingale, and

$$\int_0^t H^n_{s-}\, dX^n_s \quad \Rightarrow \quad \int_0^t H_{s-}\, dX_s.$$

Jakubowski, Mémin and Pagès [1] give a sufficient condition for a sequence $(X^n)_{n\geq 1}$ to be good called "uniform tightness" or "UT". This condition uses the characterization of a semimartingale as a good integrator (see e.g., Protter [4]), and requires that it hold uniformly in n. On Ξ_n, let $\mathcal{H}^n$ denote the set of elementary predictable processes bounded by 1: that is, $\mathcal{H}^n = \{H^n\colon H^n$ has the representation $H^n_t = H^n_0 1_{\{0\}}(t) + \sum_{i=1}^{p-1} H^n_i 1_{[t_i,t_{i+1})}(t)$, with $H^n_i \in \mathcal{F}^n_{t_i}$, $p \in \mathbb{N}$, and $0 = t_0 < t_1 < \ldots < t_p < \infty$, $|H^n_i| \leq 1\}$.

Definition: A sequence of semimartingales $(X^n)_{n\geq 1}$, X^n defined on Ξ_n, satisfies the condition UT if for each $t > 0$ the set $\{\int_0^t H^n_s dX^n_s,\ H^n \in \mathcal{H}^n,\ n \in \mathbb{N}\}$ is stochastically bounded.

Theorem 1. (Jakubowski–Mémin–Pagès). If (H^n, X^n) on Ξ_n converges in distribution to (H, X) in the Skorohod topology and if $(X^n)_{n\geq 1}$ satisfies UT, then there exists a filtration

$(\mathcal{F}_t)$ such that X is an $(\mathcal{F}_t)$-semimartingale and $\int H^n_{s-}\, dX^n_s$ converges in distribution in the Skorohod topology to $\int H_{s-}\, dX_s$. That is, the sequence $(X^n)_{n\geq 1}$ is good.

The condition UT is sometimes difficult to verify in practice. An alternative condition is given in Kurtz and Protter [2]. To "subtract off" the large jumps in a Skorohod continuous manner, define $h_\delta\colon [0,\infty) \to [0,\infty)$ by $h_\delta(r) = (1-\delta/r)^+$, and $J_\delta\colon \mathbb{D}_{\mathbb{R}^m}[0,\infty) \to \mathbb{D}_{\mathbb{R}^m}[0,\infty)$ by

$$J_\delta(x)(t) = \sum_{0\leq s\leq t} h_\delta(|\Delta x_s|)\Delta x_s,$$

where $\Delta x_s = x(s) - x(s-)$. Let $\int_0^t |dA_s|$ denote the total variation of the process A from 0 to t (ω by ω).

<u>Theorem 2</u>. (Kurtz–Protter). Let (H^n, X^n) on Ξ_n converge in distribution to (H,X) on Ξ in the Skorohod topology on $\mathbb{D}_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$. Fix $\delta > 0$ (allowing $\delta = \infty$) and let $X^{n,\delta} = X^n - J_\delta(X^n)$. Then $X^{n,\delta}$ is a semimartingale and let $X^{n,\delta} = M^{n,\delta} + A^{n,\delta}$ be a decomposition of $X^{n,\delta}$ into an $(\mathcal{F}^n_t)$–local martingale and an adapted process of finite variation on compacts. Suppose

(*) For each $\alpha > 0$, there exist stopping times $T^{n,\alpha}$ such that $P(T^{n,\alpha} \leq \alpha) \leq \frac{1}{\alpha}$ and $\sup_n E\{[M^{n,\delta}, M^{n,\delta}]_{t\wedge T^{n,\alpha}} + \int_0^{t\wedge T^{n,\alpha}} |dA^{n,\delta}_s|\} < \infty$.

Then there exists a filtration $(\mathcal{F}_t)$ on Ξ such that H is $(\mathcal{F}_t)$-adapted and X is an $(\mathcal{F}_t)$-semimartingale, and $(H^n, X^n, \int H^n_{s-}\, dX^n_s)$ converges in distribution to $(H, X, \int H_{s-}\, dX_s)$ in the Skorohod topology on $\mathbb{D}_{\mathbb{M}^{km}\times\mathbb{R}^m\times\mathbb{R}^k}[0,\infty)$. That is, $(X^n)_{n\geq 1}$ is good.

It is shown in Kurtz and Protter [2] and also in Mémin and Slominski [3] that UT and (*) are equivalent sufficient conditions for the sequence (X^n) of (vector–valued) semimartingales to be good.

The next theorem, which is the principal result of this note, shows that the sufficient conditions of Jakubowski–Mémin–Pagès and Kurtz–Protter are also necessary.

<u>Theorem 3</u>. Let X^n be a sequence of vector–valued semimartingales on filtered probability spaces Ξ_n. If X^n is a good sequence, then X^n satisfies the condition UT and the condition (*) of Theorem 2.

Proof: Since UT and $(*)$ are equivalent, it suffices to show that UT holds. We treat the case $k = m = 1$ for notational simplicity.

Suppose that $(X^n)_{n\geq 1}$ is a good sequence but that UT does not hold. Then there exists a sequence $(H^n)_{n\geq 1}$, $H^n \in \mathcal{H}_n$, and a sequence c_n tending to ∞ such that for some $\varepsilon > 0$,

$$\liminf_{n\to\infty} P^n\{\int H^n_{s-}\, dX^n_s \geq c_n\} \geq \varepsilon.$$

But this implies

$$\liminf_{n\to\infty} P^n\{\int \frac{1}{c_n} H^n_{s-}\, dX^n_s \geq 1\} \geq \varepsilon \tag{1}$$

as well. Since $|H^n| \leq 1$, we have that $\frac{1}{c_n}H^n$ converges in distribution (uniformly) to the zero process. Since X^n is good by hypothesis, then $\int \frac{1}{c_n} H^n_{s-}\, dX^n_s$ converges in distribution to $\int 0 dX_s = 0$. This contradicts (1), and we have the result. □

We can employ the argument in the previous proof to show that the property of goodness is inherited through stochastic integration.

Theorem 4. Let $(X^n)_{n\geq 1}$ be a sequence of $\mathbb{R}^m$-valued semimartingales, X^n defined on Ξ_n, with $(X^n)_{n\geq 1}$ being good. If H^n defined on Ξ_n are càdlàg, adapted, $\mathbb{M}^{km}$-valued processes, and (H^n, X^n) converges in distribution in the Skorohod topology on $\mathbb{D}_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$, then $Y^n_t = \int_0^t H^n_{s-}\, dX^n_s$ is also a good sequence of semimartingales.

Proof: Let $k = m = 1$. By Theorem 1, it is sufficient to show that (Y^n) satisfies UT. Suppose not. Then, as in the proof of Theorem 3, there exists a sequence $(\tilde{H}^n)$ with $\tilde{H}^n \in \mathcal{H}^n$, a sequence c_n tending to ∞, and $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} P^n\{\int \tilde{H}^n_{s-}\, dY^n_s \geq c_n\} \geq \varepsilon.$$

and equivalently

$$\liminf_{n\to\infty} P^n\{\int \tilde{H}^n_{s-} H^n_{s-}\, dX^n_s \geq c_n\} \geq \varepsilon.$$

which implies

$$\liminf_{n\to\infty} P^n\{\int \frac{1}{c_n}\tilde{H}^n_{s-}H^n_{s-}\ dX^n_s \geq 1\} \geq \varepsilon \tag{2}$$

But, as before, the goodness of (X^n) implies that the stochastic integral in (2) converges to zero contradicting (2) and verifying UT for (Y^n). □

As an application of the preceding, we consider stochastic differential equations. Suppose that for each n, U^n is adapted, càdlàg and X^n is a semimartingale on Ξ_n, $(U^n, X^n) \Rightarrow (U, X)$, and X^n is a good sequence. Let F^n, F be, for example, Lipschitz continuous such that $F^n \to F$ uniformly on compacts, and let Z^n, Z be the unique solutions of

$$Z^n_t = U^n_t + \int_0^t F^n(Z^n_{s-})dX^n_s$$
$$Z_t = U_t + \int_0^t F(Z_{s-})dX_s.$$

Then combining Theorem 4 with Theorem 5.4 of [2] (or similar results in [3] or [5]) yields that Z^n is also a good sequence converging of course to Z. The preceding holds as well for much more general F^n, F; see [2]. In particular by taking $F^n(x) = F(x) = x$, we have that X^n a good sequence implies that $Z^n = \mathcal{E}(\mathcal{X}^n)$ is a good sequence, where $\mathcal{E}(\mathcal{Y})$ denotes the stochastic exponential of a semimartingale Y.

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