ON EMPIRICAL BAYES TEST PROCEDURES FOR UNIFORM DISTRIBUTIONS

by

TaChen Liang
Wayne State University

Technical Report #91-25C

Department of Statistics
Purdue University

May 1991

ON EMPIRICAL BAYES TEST PROCEDURES FOR UNIFORM DISTRIBUTIONS*

TaChen Liang
Wayne State University
Department of Mathematics
Detroit, MI 48202
USA

ABSTRACT

This paper deals with the empirical Bayes test procedure of Gupta and Hsiao (1983), say d_n^{GH}, for testing $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ for the uniform distributions $U(0, \theta)$, $\theta > 0$. Two aspects are studied. We first investigate the convergence rate associated with the empirical Bayes test procedure d_n^{GH}. Various convergence rates are established according to the types of the prior distributions. Secondly, an improved smoothed version of d_n^{GH} is constructed. The smoothed version is shown to be at least as good as d_n^{GH} in terms of the Bayes risks.

AMS 1980 Subject Classification: 62C12.

Key Words and Phrases: Bayes rule; empirical Bayes; asymptotically optimal; convergence rate; monotone decision problem.

* This research was supported in part by NSF Grants DMS–8923071 and DMS–8717799 at Purdue University.

1. Introduction

Suppose that the random variable X has a uniform distribution with pdf $f(x|\theta) = \theta^{-1}I_{(0,\theta)}(x)$, $\theta > 0$, and that θ is a realization of a random variable Θ having a prior distribution G over the interval $(0, \infty)$. Consider the problem of testing $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$, where θ_0 is a known positive constant. For each $i = 0, 1$, let i denote the action deciding in favor of H_i. For the parameter θ and the action i, the loss function is defined as

$$L(\theta, i) = (1-i)(\theta_0 - \theta)I_{(0,\theta_0)}(\theta) + i(\theta - \theta_0)I_{[\theta_0,\infty)}(\theta). \tag{1.1}$$

A decision rule d is defined to be a mapping from the sample space $\mathcal{X}$ of X into the interval $[0, 1]$ such that $d(x)$ is the probability of taking action 0 when $X = x$ is observed. Let D be the class of all decision rules. For each $d \in D$, let $r(G, d)$ denote the associated Bayes risk. Then, $r(G) = \inf_{d \in D} r(G, d)$ is the minimum Bayes risk among all the decision rules in the class D. A decision rule, say d_G, such that $r(G, d_G) = r(G)$ is called a Bayes rule.

According to the precedingly described statistical model, the Bayes rule d_G can be obtained as follows.

Let $\varphi_G(x) \equiv E[\Theta|X = x]$ denote the posterior mean of Θ given $X = x$. From Fox (1978),

$$\varphi_G(x) = [1 - F_G(x)]/f_G(x) + x, \qquad x \geq 0, \tag{1.2}$$

where $f_G(x) = \int f(x|\theta)dG(\theta) = \int_x^\infty \theta^{-1}dG(\theta)$ is the marginal pdf of the random variable X and $F_G(x)$ is the corresponding cdf. Note that the pdf $f_G(x)$ is nonincreasing in x for $x \geq 0$. The Bayes rule d_G is:

$$d_G(x) = \begin{cases} 1 & \text{if } \varphi_G(x) \geq \theta_0; \\ 0 & \text{otherwise;} \end{cases} \tag{1.3.a}$$

or equivalently,

$$d_G(x) = \begin{cases} 1 & \text{if either } x \geq \theta_0 \text{ or } (0 \leq x < \theta_0 \text{ and } H(x) \geq 0); \\ 0 & \text{otherwise;} \end{cases} \tag{1.3.b}$$

where $H(x) = 1 - F_G(x) + f_G(x)(x - \theta_0)$.

The minimum Bayes risk is:

$$r(G) = \int_0^\infty [-H(x)] d_G(x) dx + c \tag{1.4}$$

where $c = \int_{x=0}^\infty \int_{\theta=\max(\theta_0,x)}^\infty (\theta - \theta_0) f(x|\theta) dG(\theta) dx$.

Since the class of densities $\{f(x|\theta)|\theta > 0\}$ has monotone likelihood ratio in x, the posterior mean $\varphi_G(x)$ is nondecreasing in x. From (1.3.a), it can be seen that the Bayes rules d_G is a monotone decision rule. That is, $d_G(x)$ is nondecreasing in x.

When the prior distribution G is unknown, this testing problem has been studied by Gupta and Hsiao (1983), Van Houwelingen (1987) and Liang (1990), respectively, through the empirical Bayes approach of Robbins (1956, 1964). Motivated by the monotonicity properties of the posterior mean $\varphi_G(x)$ and the Bayes rule d_G, Van Houwelingen (1987) and Liang (1990) have, respectively, studied monotone empirical Bayes test procedures based on certain nondecreasing empirical Bayes estimators of the posterior mean. Under some regularity conditions, the corresponding rates of convergence of their empirical Bayes test procedures have also been investigated. Gupta and Hsiao (1983) have studied an empirical Bayes test procedure which mimics the form of (1.3.b). However, the proposed empirical Bayes test procedure is not monotone; also, the corresponding rate of convergence was not investigated.

In this paper, we deal with the empirical Bayes test procedure of Gupta and Hsiao (1983), say d_n^{GH}. Two aspects are studied. First, we investigate the asymptotic optimality of the empirical Bayes test procedure d_n^{GH}. Various convergence rates are established according to the types of the prior distributions G. Secondly, we propose an improved smoothed version of the empirical Bayes test procedure d_n^{GH} for the concerned testing problem. This smoothed version is shown to be at least as good as d_n^{GH} in terms of the Bayes risks.

2. Asymptotic Optimality of d_n^{GH}

In the empirical Bayes framework, let $X_1, \ldots, X_n$ denote the n past random observations. It is assumed that $X_1, \ldots, X_n$ be iid with the marginal pdf $f_G(x)$. Let $\underline{X}_n = (X_1, \ldots, X_n)$ denote the n past observations and let $X_{n+1} = X$ denote the present random observation.

Let $\{\alpha_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$. For each $n = 1, 2, \ldots$ and $x \geq 0$, define

$$\begin{cases} F_n(x) = n^{-1} \sum_{j=1}^{n} I_{[0,x]}(X_j); \\ f_n(x) = [F_n(x+\alpha_n) - F_n(x)]/\alpha_n. \end{cases} \tag{2.1}$$

Motivated by (1.3.b), Gupta and Hsiao (1983) proposed an empirical Bayes test procedure d_n^{GH}, which is given as follows: For each $x \geq 0$, let $H_n(x) = 1 - F_n(x) + f_n(x)(x - \theta_0)$. Then

$$d_n^{GH}(x) = \begin{cases} 1 & \text{if either } x \geq \theta_0 \text{ or } (0 \leq x < \theta_0 \text{ and } H_n(x) \geq 0); \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Let $r(G, d_n^{GH})$ denote the conditional Bayes risk (conditional on $\underline{X}_n$) of the empirical Bayes test procedure d_n^{GH}, and let $E[r(G, d_n^{GH})]$ be the associated overall Bayes risk. Then,

$$r(G, d_n^{GH}) = \int_{x=0}^{\infty} [-H(x)] d_n^{GH}(x) dx + c, \tag{2.3}$$

and

$$E[r(G, d_n^{GH})] = \int_{x=0}^{\infty} [-H(x)] E[d_n^{GH}(x)] dx + c, \tag{2.4}$$

where the expectation $E[d_n^{GH}(x)]$ is taken with respect to $\underline{X}_n$. Since $r(G)$ is the minimum Bayes risk, $r(G, d_n^{GH}) - r(G) \geq 0$, and therefore $E[r(G, d_n^{GH})] - r(G) \geq 0$. The nonnegative differences $r(G, d_n^{GH}) - r(G)$ and $E[r(G, d_n^{GH})] - r(G)$ can be used as measures of the optimality of the empirical Bayes test procedure d_n^{GH}; see Van Houwelingen (1987) and Liang (1990). In this paper, we are only concerned with the difference $E[r(G, d_n^{GH})] - r(G)$.

Definition 2.1 (a) A sequence of empirical Bayes test procedures $\{d_n\}_{n=1}^{\infty}$ is said to be asymptotically optimal relative to the prior distribution G if $E[r(G, d_n)] - r(G) \to 0$ as $n \to \infty$.

(b) A sequence of empirical Bayes test procedure $\{d_n\}_{n=1}^{\infty}$ is asymptotically optimal of order β_n relative to the prior distribution G if $E[r(G, d_n)] - r(G) = O(\beta_n)$, where $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \beta_n = 0$.

Gupta and Hsiao (1983) have proved that if $n\alpha_n \to \infty$ as $n \to \infty$, then $E[r(G, d_n^{GH})] - r(G) \to 0$ as $n \to \infty$. Hence, the sequence of the empirical Bayes test procedures $\{d_n^{GH}\}_{n=1}^{\infty}$

defined through (2.1) and (2.2) is asymptotically optimal. However, the associated rate of convergence was not investigated. In the following, we will investigate the convergence rate of the empirical Bayes test procedure d_n^{GH} according to the type of the prior distribution G.

Let $\mathcal{C}$ be the class of all distributions over the interval $(0, \infty)$. Two subclasses of $\mathcal{C}$ are considered:

$$\mathcal{C}_1 = \{G \in \mathcal{C} | f_G(0) < \infty\};$$

$$\mathcal{C}_2 = \{G \in \mathcal{C} | G \text{ is a step function such that the number of discontinuity points of } G \text{ in the interval } (0, \theta_0) \text{ is finite}\}$$

Note that $\mathcal{C}_2 \subsetneq \mathcal{C}_1$.

Let $A = \{0 \leq x < \theta_0 | H(x) > 0\}$ and $B = \{0 \leq x < \theta_0 | H(x) < 0\}$.

Define

$$a = \begin{cases} \inf A & \text{if } A \neq \phi; \\ \theta_0 & \text{if } A = \phi; \end{cases} \tag{2.5}$$

$$b = \begin{cases} \sup B & \text{if } B \neq \phi; \\ 0 & \text{if } B = \phi; \end{cases} \tag{2.6}$$

where ϕ denote the empty set. Since $f_G(x)$ is nonincreasing in x for $x \geq 0$, $H(x)$ is nondecreasing in x for $0 \leq x < \theta_0$ and therefore $0 \leq b \leq a \leq \theta_0$.

From (1.4) and (2.4), also by the definitions of a and b given in (2.5) and (2.6), respectively, it follows that

$$0 \leq E[r(G, d_n^{GH})] - r(G) = I + II, \tag{2.7}$$

where

$$I = \int_0^b [-H(x)] P\{d_n^{GH}(x) = 1\} dx,$$

$$II = \int_a^{\theta_0} H(x) P\{d_n^{GH}(x) = 0\} dx.$$

Note that $I = 0$ if $B = \phi$ and $II = 0$ if $A = \phi$. Without loss of generality, we may assume that $0 < b \leq a < \theta_0$.

2.1 Asymptotic Optimality over $\mathcal{C}_1$

For $0 < x < b$, $H(x) < 0$. By (2.2) and Markov's inequality, it follows that

$$\begin{aligned} P\{d_n^{GH}(x) = 1\} &= P\{H_n(x) - H(x) \geq -H(x)\} \\ &\leq E[|H_n(x) - H(x)|]/(-H(x)). \end{aligned} \tag{2.8}$$

From the definitions of the functions $H_n(x)$ and $H(x)$, applying the triangular inequality, we can obtain the following inequality:

$$\begin{aligned} &E[|H_n(x) - H(x)|] \\ \leq &E[|F_n(x) - F_G(x)|] + (\theta_0 - x)E[|f_n(x) - f_G(x)|]. \end{aligned} \tag{2.9}$$

Now

$$E[|F_n(x) - F_G(x)|] \leq [\text{Var }(F_n(x))]^{1/2} \leq n^{-1/2}. \tag{2.10}$$

Also,

$$\begin{aligned} &E[|f_n(x) - f_G(x)|] \\ \leq &E[|f_n(x) - Ef_n(x)|] + |Ef_n(x) - f_G(x)|. \end{aligned} \tag{2.11}$$

From (2.1), it follows that

$$\begin{aligned} E[|f_n(x) - Ef_n(x)|] &\leq (\text{Var }(f_n(x)))^{1/2} \\ &\leq [\frac{F_G(x+\alpha_n) - F_G(x)}{n\alpha_n^2}]^{1/2} \\ &\leq \left(\frac{f_G(x)}{n\alpha_n}\right)^{1/2} \end{aligned} \tag{2.12}$$

where the last inequality is obtained due to the fact that $f_G(x)$ is nonincreasing in x for $x \geq 0$.

From (2.1) again,

$$|Ef_n(x) - f_G(x)| \leq f_G(x) - f_G(x + \alpha_n). \tag{2.13}$$

Combining (2.8)–(2.13), we obtain: for $G \in \mathcal{C}_1$,

$$\begin{aligned} I &\leq \int_0^b n^{-1/2}dx + \int_0^b \theta_0[\frac{f_G(x)}{n\alpha_n}]^{1/2}dx + \int_0^b \theta_0[f_G(x) - f_G(x+\alpha_n)]dx \\ &\leq O(n^{-1/2}) + O((n\alpha_n)^{-1/2}) + \int_0^{\alpha_n} \theta_0 f_G(0)dx \\ &= O(n^{-1/2}) + O((n\alpha_n)^{-1/2}) + O(\alpha_n). \end{aligned} \tag{2.14}$$

For $a < x < \theta_0$, define

$$H(x,n) = 1 - F_G(x) + [F_G(x+\alpha_n) - F_G(x)](x-\theta_0)/\alpha_n.$$

Note that $H(x,n) \geq H(x) > 0$ for $a < x < \theta_0$. Analogous to the preceding discussion, we can obtain the following results:

For $a < x < \theta_0$,

$$\begin{aligned} P\{d_n^{GH}(x) = 0\} &= P\{H_n(x) - H(x,n) \leq -H(x,n)\} \\ &\leq E[|H_n(x) - H(x,n)|]/H(x,n). \end{aligned}$$

Also

$$\begin{aligned} E[|H_n(x) - H(x,n)|] &\leq E[|F_n(x) - F_G(x)|] + E[|f_n(x) - Ef_n(x)|] \\ &\leq n^{-1/2} + \left(\frac{f_G(x)}{n\alpha_n}\right)^{1/2}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} II &\leq \int_a^{\theta_0} E[|H_n(x) - H(x,n)|]dx \\ &\leq O(n^{-1/2}) + O((n\alpha_n)^{-1/2}). \end{aligned} \tag{2.15}$$

We summarize the above result as a Theorem as follows.

Theorem 2.1. Let $\alpha_n = n^{-1/3}$. Then, the sequence of the empirical Bayes test procedures $\{d_n^{GH}\}_{n=1}^{\infty}$ has the following asymptotic optimality:

$$E[r(G, d_n^{GH})] - r(G) = O(\alpha_n) \text{ for all } G \in \mathcal{C}_1.$$

2.2. Asymptotic Optimality over $\mathcal{C}_2$

For each $G \in \mathcal{C}_2$, let $\{c_1, \ldots, c_m\}$ be the set of discontinuity points of the prior distribution G in the interval $(0, \theta_0)$, where $0 < c_1 < \ldots < c_m < \theta_0$. Straightforward computation shows that the function $H(x)$ is constant on each of the intervals $[c_{i-1}, c_i)$, $i = 1, \ldots, m+1$, where $c_0 = 0$ and $c_{m+1} = \theta_0$. We have the following lemmas.

Lemma 2.1. For each $G \in \mathcal{C}_2$ having the set of discontinuity points $\{c_1 < \ldots < c_m\}$ contained in the interval $(0, \theta_0)$, the following results hold:

(a) If $b > 0$, then $b = c_{i^*}$ for some $i^* = 1, \ldots, m+1$, and $\sup_{x \in B} H(x) = H(c_{i^*-1}) < 0$.

(b) If $a < \theta_0$, then $a = c_{j^*}$ for some $j^* = 0, 1, \ldots, m$, and $\inf_{x \in A} H(x) = H(c_{j^*}) > 0$.

Proof: Straightforward computations will lead us to the results and hence the detail is omitted.

In the following, we still assume that $0 < b \leq a < \theta_0$. Since we are interested in the asymptotic optimality, we consider the case where n is sufficiently large so that $b - \alpha_n > 0$.

Let

$$W_n(x) = \alpha_n[1 - F_G(x)] + (x - \theta_0)[F_G(x + \alpha_n) - F_G(x)].$$

Lemma 2.2. For $G \in \mathcal{C}_2$, let i^* and j^* be the numbers implicitly defined in Lemma 2.1. Then, the following hold.

(a) For $0 < x < b - \alpha_n$, $W_n(x) \leq \alpha_n H(c_{i^*-1}) < 0$.

(b) For $a < x < \theta_0$, $W_n(x) \geq \alpha_n H(c_{j^*}) > 0$.

Proof: (a) For $0 < x < b - \alpha_n$,

$$\begin{aligned} W_n(x) &\leq \alpha_n[1 - F_G(x + \alpha_n) + f_G(x + \alpha_n)(x + \alpha_n - \theta_0)] \\ &= \alpha_n H(x + \alpha_n) \\ &\leq \alpha_n H(c_{i^*-1}) \\ &< 0. \end{aligned}$$

(b) Similarly, for $a < x < \theta_0$

$$\begin{aligned} W_n(x) &\geq \alpha_n[1 - F_G(x) + f_G(x)(x - \theta_0)] \\ &= \alpha_n H(x) \\ &\geq \alpha_n H(c_{j^*}) \\ &> 0. \end{aligned}$$

Lemma 2.3. Under the assumption of Lemma 2.1, we have the following results:

(a) For $0 < x < b - \alpha_n$,

$$P\{d_n^{GH}(x) = 1\} = O(\exp(-nH^2(c_{i^*-1})/2)) \\ + O(\exp(-n\alpha_n H^2(c_{i^*-1})/(16\theta_0^2 f_G(0)))).$$

(b) For $a < x < \theta_0$

$$P\{d_n^{GH}(x) = 0\} = O(\exp(-nH^2(c_{j^*})/2)) \\ + O(\exp(-n\alpha_n H^2(c_{j^*})/(16\theta_0^2 f_G(0)))).$$

Proof: We prove part (a) only. The proof for part (b) is similar, and hence the detail is omitted.

Let $R_n(x) = F_n(x) - F_G(x)$. For $0 < x < b - \alpha_n$, by (2.1) and (2.2), we can obtain,

$$\begin{aligned} &P\{d_n^{GH}(x) = 1\} \\ &= P\{\alpha_n R_n(x) + [R_n(x+\alpha_n) - R_n(x)](\theta_0 - x) \leq W_n(x)\} \qquad (2.16) \\ &\leq P\{\alpha_n R_n(x) \leq W_n(x)/2\} + P\{R_n(x+\alpha_n) - R_n(x) \leq W_n(x)(\theta_0 - x)^{-1}/2\}. \end{aligned}$$

By Lemma 2.2.(a) and Theorem 1 of Hoeffding (1963),

$$P\{\alpha_n R_n(x) \leq W_n(x)/2\} \leq \exp(-nH^2(c_{i^*-1})/2). \qquad (2.17)$$

Also, by Lemma 2.2.(a) and Bernstein's inequality (see Ibragimov and Linnik (1971), page 169),

$$\begin{aligned} &P\{R_n(x+\alpha_n) - R_n(x) \leq W_n(x)(\theta_0 - x)^{-1}/2\} \\ &\leq \exp(-nW_n^2(x)/[16(\theta_0 - x)^2(F_G(x+\alpha_n) - F_G(x))(1 - F_G(x+\alpha_n) + F_G(x))]) \\ &\leq \exp(-n\alpha_n^2 H^2(c_{i^*-1})/[16\theta_0^2 f_G(0)\alpha_n]) \qquad (2.18) \\ &= \exp(-n\alpha_n H^2(c_{i^*-1})/[16\theta_0^2 f_G(0)]). \end{aligned}$$

The proof of part (a) is then complete after combining (2.16)–(2.18) together.

We conclude the following asymptotic optimality of the empirical Bayes test procedure d_n^{GH} over the class $\mathcal{C}_2$.

Theorem 2.2. Let $\{d_n^{GH}\}_{n=1}^{\infty}$ be the sequence of empirical Bayes test procedures constructed through (2.1) and (2.2) with $\alpha_n = n^{-1}(\log\ n)^{1+\varepsilon}$ for some $\varepsilon > 0$. Then, $\{d_n^{GH}\}_{n=1}^{\infty}$ has the following asymptotic optimality:

$$E[r(G, d_n^{GH})] - r(G) = O(\alpha_n) \text{ for all } G \in \mathcal{C}_2.$$

Proof: First, we have

$$\begin{aligned}
0 &\leq E[r(G, d_n^{GH})] - r(G) \\
&= \int_0^{b-\alpha_n} [-H(x)]P\{d_n^{GH}(x) = 1\}dx + \int_{b-\alpha_n}^{b} [-H(x)]P\{d_n^{GH}(x) = 1\}dx \\
&\quad + \int_a^{\theta_0} H(x)P\{d_n^{GH}(x) = 0\}dx,
\end{aligned} \tag{2.19}$$

where

$$\int_{b-\alpha_n}^{b} [-H(x)]P\{d_n^{GH}(x) = 1\}dx = O(\alpha_n). \tag{2.20}$$

By Lemma 2.3(a) and the definition of α_n for sufficiently large n, as $0 < x < b - \alpha_n$,

$$P\{d_n^{GH}(x) = 1\} = O(n^{-1}),$$

which is independent of x. Hence,

$$\int_0^{b-\alpha_n} [-H(x)]P\{d_n^{GH}(x) = 1\}dx = O(n^{-1}). \tag{2.21}$$

Similarly, we can obtain that

$$\int_a^{\theta_0} H(x)P\{d_n^{GH}(x) = 0\}dx = O(n^{-1}). \tag{2.22}$$

Therefore, the results of this Theorem follows from (2.19)–(2.22).

3. An Improved Empirical Bayes Test Procedure

Since the loss function $L(\theta, i)$ in (1.1) is linear and the class of densities of uniform distributions $\{f(x|\theta)|\theta > 0\}$ has monotone likelihood ratio in x, the decision problem under study is a monotone decision problem. Recall that the class of all monotone decision procedures is essentially complete, see Berger (1985). However, the empirical Bayes test procedure d_n^{GH} may not be monotone, and therefore may be inadmissible. In the following, we consider a method to monotonize d_n^{GH} to obtain an improved empirical Bayes test procedure.

For the given past observations $\underline{X}_n$, let $B_n = \{0 \leq x < \theta_0 | d_n^{GH}(x) = 0\}$ and let $b_n = \int_{B_n} dx$. It is clear that $0 \leq b_n \leq \theta_0$. We define an empirical Bayes test procedure d_n^* as follows:

$$d_n^*(x) = \begin{cases} 1 & \text{if } x \geq b_n; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

By (3.1), it is easy to see that this empirical Bayes test procedure d_n^* is monotone. The following theorem shows the superiority of the empirical Bayes test procedure d_n^*.

Theorem 3.1. For any prior distribution G and each n,

(a) $r(G, d_n^*) \leq r(G, d_n^{GH})$, and hence,

(b) $E[r(G, d_n^*)] \leq E[r(G, d_n^{GH})]$.

Proof: Since part (b) is a straight consequence of part (a), it suffices to prove part (a) only. First, we present certain useful preliminary results.

Let $A_n = \{0 \leq x < \theta_0 | d_n^{GH}(x) = 1\}, B_n^* = [0, b_n)$ and $A_n^* = [b_n, \theta_0]$. Note that $A_n \sqcup B_n = A_n^* \sqcup B_n^* = [0, \theta_0)$. By the definition of b_n and B_n^*,

$$\int_{B_n} dx = \int_{B_n^*} dx = b_n.$$

Hence,

$$\int_{B_n \cap B_n^*} dx + \int_{B_n \cap A_n^*} dx = \int_{B_n^* \cap B_n} dx + \int_{B_n^* \cap A_n} dx,$$

which implies that

$$\int_{B_n \cap A_n^*} dx = \int_{B_n^* \cap A_n} dx. \tag{3.2}$$

By the definitions of the sets A_n, B_n, A_n^* and B_n^*,

$$d_n^{GH}(x) - d_n^*(x) = \begin{cases} 1 & \text{if } x \in B_n^* \cap A_n; \\ -1 & \text{if } x \in A_n^* \cap B_n; \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Let $q_1 = \inf_{x \in B_n^* \cap A_n} [-H(x)]$, $q_2 = \sup_{x \in A_n^* \cap B_n} [-H(x)]$. Note that for $x \in B_n^* \cap A_n$ and $y \in A_n^* \cap B_n$, $x \le b_n \le y$, which is obtained due to the definitions of B_n^* and A_n^*. Since $H(x)$ is nondecreasing in x for $x \in [0, \theta_0]$, we can obtain

$$q_1 \ge -H(b_n) \ge q_2. \tag{3.4}$$

Now from (2.3), combining together with (3.2)–(3.4) it follows that

$$\begin{aligned} r(G, d_n^{GH}) - r(G, d_n^*) &= \int_{B_n^* \cap A_n} [-H(x)]dx - \int_{A_n^* \cap B_n} [-H(x)]dx \\ &\ge \int_{B_n^* \cap A_n} q_1 dx - \int_{A_n^* \cap B_n} q_2 dx \\ &\ge 0. \end{aligned}$$

Therefore, the proof of this theorem is complete.

References

Berger, J. O. (1985). *Statistical Decision Theory and Bayesian Analysis.* Second Edition, Springer-Verlag, New York.

Fox, R. J. (1978). Solution to empirical Bayes squared error loss estimation problem. *Ann. Statist.*, **6**, 846–854.

Gupta, S. S. and Hsiao, P. (1983). Empirical Bayes rules for selecting good populations. *J. Statist. Plann. Inference*, **8**, 87–101.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, **58**, 13–30.

Ibragimov, I. A. and Linnik, Yu. V. (1971). *Independent and Stationary Sequences of Random Variables.* Wolters-Noordhoff Publishing Groningen.

Liang, T. (1990). On the convergence rates of a monotone empirical Bayes test for uniform distributions. In *J. Statist. Plann. Inference*, **26**, 25–34.

Robbins, H. (1956). An empirical Bayes approach to statistics. *Proc. Third Berkeley Symp. Math. Statist. Probab.*, **1**, 157–163.

Robbins, H. (1964). The empirical Bayes approach to statistical decision problem. *Ann. Math. Statist.*, **35**, 1–20.

Van Houwelingen, J. C. (1987). Monotone empirical Bayes test for uniform distributions using the maximum likelihood estimator of a decreasing density. *Ann. Statist.*, **15**, 875–879.