CONVERGENCE RATES FOR EMPIRICAL BAYES ESTIMATION OF THE SCALE PARAMETER IN A PARETO DISTRIBUTION

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Abstract

Let $f(x|\theta) = \alpha\theta^{\alpha}/x^{\alpha+1} I_{(\theta,\infty)}(x)$ be the pdf of a Pareto distribution with known shape parameter $\alpha > 0$ and unknown scale parameter θ. We study the problem of estimating the scale parameter θ under a squared–error loss through the nonparametric empirical Bayes approach. An empirical Bayes estimator is proposed and the corresponding asymptotic optimality is also investigated. It is shown that under certain weak conditions the proposed empirical Bayes estimator is asymptotically optimal and the associated rate of convergence is of order $O(n^{-2/3})$.

Keywords: Empirical Bayes; Asymptotically optimal; Rate of convergence.

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1. Introduction

Pareto distributions have been used extensively to model various socio–economic data. The reader is referred to Arnold [1] for a general discussion and its applications.

Let X be a random variable having a Pareto distribution with the probability density function given by

$$f(x|\theta) = \alpha\theta^{\alpha}x^{-(\alpha+1)}I_{(\theta,\infty)}(x), \ \alpha > 0, \ \theta > 0, \tag{1.1}$$

where the parameter α is assumed to be known. Consider the problem of estimating the parameter θ under the squared–error loss. Suppose that the parameter θ is a realization of a random parameter Θ which has a prior distribution G over the interval $(0,\infty)$. Then, the Bayes estimator of θ given $X = x$ is $\varphi_G(x)$, the posterior mean of Θ, where

$$\varphi_G(x) = E[\Theta|X = x] = \int \theta f(x|\theta)dG(\theta)/f(x), \tag{1.2}$$

where $f(x) = \int_{\theta=0}^{x} f(x|\theta)dG(\theta)$ is the marginal pdf of X. The Bayes risk of φ_G is

$$R(G,\varphi_G) = E[(\varphi_G(X) - \Theta)^2] \tag{1.3}$$

where the expectation E is computed with respect to (X,Θ).

When the prior distribution G is unknown, Tiwari and Zalkikar [6] studied the estimation problem through the empirical Bayes approach of Robbins [3].

In the empirical Bayes framework, let $X_1,\ldots,X_n$ denote the past data, which are assumed to be iid with the pdf $f(x)$. Let $\varphi_n(X) \equiv \varphi_n(X,X_1,\ldots,X_n)$ be an empirical Bayes estimator of the parameter θ based on the past data $\underset{\sim}{X}_n = (X_1,\ldots,X_n)$ and the present observation X. Let $R(G,\varphi_n|\underset{\sim}{X}_n)$ denote the conditional Bayes risk of φ_n given $\underset{\sim}{X}_n$, and let $R(G,\varphi_n)$ denote the overall Bayes risk of φ_n. That is, $R(G,\varphi_n) = E[R(G,\varphi_n|\underset{\sim}{X}_n)]$ and the expectation E is taken with respect to $\underset{\sim}{X}_n$. Since φ_G is the Bayes estimator, $R(G,\varphi_n|\underset{\sim}{X}_n) - R(G,\varphi_G) \geq 0$ for all $\underset{\sim}{X}_n$ and for all n. Therefore $R(G,\varphi_n) - R(G,\varphi_G) \geq 0$ for all n. The nonnegative difference $R(G,\varphi_n) - R(G,\varphi_G)$ can be used as a measure of the performance of the empirical Bayes estimator φ_n; for example, see Lin [2] and Singh [5].

A sequence of empirical Bayes estimators $\{\varphi_n\}_{n=1}^{\infty}$ is said to be asymptotically optimal if $R(G,\varphi_n) - R(G,\varphi_G) \to 0$ as $n \to \infty$. Moreover, if $R(G,\varphi_n) - R(G,\varphi_G) = O(\alpha_n)$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$, then the sequence of empirical Bayes estimators $\{\varphi_n\}_{n=1}^{\infty}$ is said to be asymptotically optimal having convergence rate of order $\{\alpha_n\}_{n=1}^{\infty}$.

Tiwari and Zalkikar [6] have proposed an empirical Bayes estimator for the parameter θ and investigated the corresponding convergence rate for a class of prior distributions G satisfying the following conditions.

Condition A:

A1. $G(m) = 1$ for some known positive number m.

A2. Let $a^* = \sup\{\theta | G(\theta) = 0\}$. $f(x)$ is decreasing in x for $x \in (a^*, m]$.

Condition B:

B1. $\sup_{x>a^*} f(x) < \infty$.

B2. $E\{(Xf^2(X))^{-1} I(X < m)\} < \infty$.

B3. $E\{F(X)(1 - F(X))(f^2(X))^{-1} I(X < m)\} < \infty$.

B4. $E\{(X^{2\alpha+2} f^2(X))^{-1} I(X \geq m)\} < \infty$. Here, $F(x)$ is the distribution function associated with $f(x)$.

It should be noted that Condition A2 was used by Tiwari and Zalkika [6] in the proof of their Theorem 2.1, though it was not clearly stated as an assumption. They showed that under the precedingly described conditions, the convergence rate of their proposed empirical Bayes estimator is of order $\{n^{-1/2}\}$.

In this paper, we propose an alternative empirical Bayes estimator for this empirical Bayes estimation problem. We also prove that under Condition A the proposed empirical Bayes estimator is asymptotically optimal having convergence rate of order $\{n^{-2/3}\}$. Since we only assume Condition A, the class of prior distributions under study is larger than the class of prior distributions considered in Tiwari and Zalkika [6].

2. The Proposed Empirical Bayes Estimator

First, we give a representation of the posterior mean $\varphi_G(x)$. Under Condition A1, straightforward computation yields the following: As $0 < x \le m$,

$$\varphi_G(x) = x - \frac{\int_{\theta=0}^{x} \theta^{\alpha+1} dF(\theta)}{x^{\alpha+1} f(x)} = x - \frac{M(x)}{x^{\alpha+1} f(x)} \tag{2.1}$$

where $M(x) = \int_0^x \theta^{\alpha+1} dF(\theta)$. Note that $0 \le \varphi_G(x) \le x$, and therefore

$$\frac{M(x)}{x^{\alpha+1} f(x)} \le x. \tag{2.2}$$

As $x > m$,

$$\varphi_G(x) = \frac{m^{\alpha+2} f(m)}{x^{\alpha+1} f(x)} - \frac{\int_0^m \theta^{\alpha+1} dF(\theta)}{x^{\alpha+1} f(x)}. \tag{2.3}$$

Since $f(x) = \frac{1}{x^{\alpha+1}} \int_{\theta=0}^{\min(m,x)} \alpha\theta^{\alpha} dG(\theta)$, hence $x^{\alpha+1} f(x) = m^{\alpha+1} f(m)$ for all $x \ge m$. Therefore, (2.3) can be rewritten as

$$\varphi_G(x) = m - \frac{M(m)}{m^{\alpha+1} f(m)} \doteq \varphi_G(m) \qquad \text{for } x > m. \tag{2.4}$$

Let $h(x) = \int_{\theta=0}^{x} \alpha\theta^{\alpha} dG(\theta)$ for $0 < x \le m$. $h(x)$ is increasing in x and $x^{\alpha+1} f(x) = h(x)$ for $0 < x \le m$.

Let $\{b_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} nb_n = \infty$. For each n and $x > 0$, define

$$f_n(x) = [F_n(x + b_n) - F_n(x)]/b_n$$

where $F_n(x)$ is the empirical distribution based on $\underset{\sim}{X}_n$, and let

$$M_n(x) = \frac{1}{n} \sum_{j=1}^{n} X_j^{\alpha+1} I_{(0,x)}(X_j).$$

Both $f_n(x)$ and $M_n(x)$ are consistent estimators of $f(x)$ and $M(x)$, respectively. Also, $E[M_n(x)] = M(x)$.

We then propose an empirical Bayes estimator $\varphi_n(X)$ given as follows:

$$\begin{aligned}\varphi_n(X) = &\left[\left(X - \frac{M_n(X)}{X^{\alpha+1} f_n(X)}\right) I_{(0,m]}(X) \vee 0\right] \\ &+ \left[\left(m - \frac{M_n(m)}{m^{\alpha+1} f_n(m)}\right) I_{(m,\infty]}(X) \vee 0\right]\end{aligned} \tag{2.5}$$

where $a \vee b = \max(a, b)$, $\frac{0}{0} \equiv 0$ and $\frac{c}{0} = \infty$ for $c > 0$.

3. Asymptotic Optimality

The following analysis is made based on Condition A. Straightforward computation yields the following:

$$\begin{aligned}0 &\leq R(G, \varphi_n) - R(G, \varphi_G) \\ &= E\{(\varphi_n(X) - \varphi_G(X))^2\} \\ &= \int_0^m E[(\varphi_n(x) - \varphi_G(x))^2] f(x)dx \\ &\quad + \int_m^\infty E[(\varphi_n(x) - \varphi_G(x))^2] f(x)dx.\end{aligned} \tag{3.1}$$

From (2.4) and (2.5), for $x \geq m$, $\varphi_n(x) - \varphi_G(x) = \varphi_n(m) - \varphi_G(m)$. Hence,

$$\begin{aligned}&\int_m^\infty E[(\varphi_n(x) - \varphi_G(x))^2] f(x)dx \\ &= \int_m^\infty E[(\varphi_n(m) - \varphi_G(m))^2] f(x)dx \\ &= E[(\varphi_n(m) - \varphi_G(m))^2][1 - F(m)].\end{aligned} \tag{3.2}$$

Thus, it suffices to consider $E[(\varphi_n(x) - \varphi_G(x))^2]$ for $0 < x \leq m$. Note that for each $0 < x \leq m$, $|\varphi_n(x) - \varphi_G(x)| \leq x$. By (2.1) and (2.5), using Lemma of Singh [4], we can obtain the following inequality.

$$\begin{aligned}&E[(\varphi_n(x) - \varphi_G(x))^2] \\ &\leq E\left[\left(\left|\frac{M_n(x)}{x^{\alpha+1} f_n(x)} - \frac{M(x)}{x^{\alpha+1} f(x)}\right| \wedge x\right)^2\right] (a \wedge b = \min(a, b)) \\ &\leq \frac{8}{f^2(x)} E[(M_n(x) - M(x))/x^{\alpha+1}]^2 \\ &+ \frac{8}{f^2(x)} \left[\left(\frac{M(x)}{x^{\alpha+1} f(x)}\right)^2 + \frac{x^2}{2}\right] E[f_n(x) - f(x)]^2.\end{aligned} \tag{3.3}$$

Since $0 \le M_n(x)/x^{\alpha+1} \le 1$ and $M_n(x)$ is an unbiased estimator of $M(x)$, we have

$$E[(M_n(x) - M(x))/x^{\alpha+1}]^2 = \mathrm{Var}(M_n(x)/x^{\alpha+1}) \le \frac{1}{n}. \tag{3.4}$$

Also,

$$E[f_n(x) - f(x)]^2 = \mathrm{Var}(f_n(x)) + [Ef_n(x) - f(x)]^2 \tag{3.5}$$

where

$$\mathrm{Var}(f_n(x)) \le \frac{F(x+b_n) - F(x)}{nb_n^2} \le \frac{f(x)}{nb_n}, \tag{3.6}$$

where the second inequality is obtained due to Condition A2.

Define $h(y) = h(m)$ for all $y \ge m$. Note that $h(y)$ is an increasing function for $y > 0$. Under Condition A2 again, using the relationship that $f(y) = h(y)/y^{\alpha+1}$ for all $y > 0$, we obtain

$$\begin{aligned} 0 &\le f(x) - Ef_n(x) \\ &= f(x) - \frac{1}{b_n}\int_x^{x+b_n} f(y)dy \\ &= \frac{1}{x^{\alpha+1}}\left[h(x) - \frac{1}{b_n}\int_x^{x+b_n} \frac{x^{\alpha+1}}{y^{\alpha+1}} h(y)dy\right] \\ &\le \frac{1}{x^{\alpha+1}}\left[h(x) - \frac{1}{b_n}\int_x^{x+b_n} \frac{x^{\alpha+1}}{y^{\alpha+1}} h(x)dy\right] \\ &= \frac{h(x)}{x^{\alpha+1}}\left[1 - \frac{1}{b_n}\int_x^{x+b_n} \frac{x^{\alpha+1}}{y^{\alpha+1}} dy\right] \\ &= f(x)\left[1 - \frac{x}{\alpha b_n}\left(1 - \left(\frac{x}{x+b_n}\right)^{\alpha}\right)\right]. \end{aligned} \tag{3.7}$$

By Taylor's series expansion, there exists an $x(b_n) \in (x, x+b_n)$ such that

$$1 - \frac{x}{\alpha b_n}\left[1 - \left(\frac{x}{x+b_n}\right)^{\alpha}\right] = \frac{(\alpha+1)b_n}{2x(b_n)} \le \frac{(\alpha+1)b_n}{2x}. \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$[Ef_n(x) - f(x)]^2 \le f^2(x)\frac{(\alpha+1)^2 b_n^2}{4x^2}. \tag{3.9}$$

Since $0 \le \frac{M(x)}{x^{\alpha+1}f(x)} \le x$, we obtain

$$\left[\frac{M(x)}{x^{\alpha+1}f(x)}\right]^2 + \frac{x^2}{2} \le x^2 + \frac{x^2}{2} \le 2x^2. \tag{3.10}$$

Based on the above discussion, we have:

$$\begin{aligned}
&\int_0^m E[(\varphi_n(x)-\varphi_G(x))^2]f(x)dx\\
&\le \int_0^m \frac{8}{nf^2(x)}f(x)dx+\int_0^m \frac{8}{f^2(x)}\times\frac{f(x)}{nb_n}\times 2x^2f(x)dx\\
&\qquad\qquad + \int_0^m \frac{8}{f^2(x)}\times 2x^2\times\frac{(\alpha+1)b_n^2}{4x^2}f^2(x)f(x)dx, \qquad (3.11)\\
&\le \frac{8}{n}\int_0^m \frac{1}{f(m)}dx+\frac{16m^3}{nb_n}+4(\alpha+1)b_n^2\\
&= O(n^{-1})+O\left(\frac{1}{nb_n}\right)+O(b_n^2),
\end{aligned}$$

and

$$\begin{aligned}
&\int_m^\infty E[(\varphi_n(x)-\varphi_G(x))^2]f(x)dx\\
&= E[(\varphi_n(m)-\varphi_G(m))^2][1-F(m)] \qquad (3.12)\\
&= O(n^{-1})+O\left(\frac{1}{nb_n}\right)+O(b_n^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\le R(G,\varphi_n)-R(G,\varphi_G)\\
&= O(n^{-1})+O\left(\frac{1}{nb_n}\right)+O(b_n^2). \qquad (3.13)
\end{aligned}$$

If we choose $b_n=n^{-1/3}$, we obtain

$$0\le R(G,\varphi_n)-R(G,\varphi_G)=O(n^{-2/3}).$$

We summarize the preceding result as a theorem as follows:

Theorem 3.1. Let $\{\varphi_n\}_{n=1}^{\infty}$ be the sequence of empirical Bayes estimators constructed in Section 2. Then, under Conditions A1 and A2,

$$R(G,\varphi_n)-R(G,\varphi_G)=O(n^{-1})+O\left(\frac{1}{nb_n}\right)+O(b_n^2).$$

4. A Lower Bound For $R(G,\varphi_n)-R(G,\varphi_G)$

Throughout this section, we assume that $m=1$ and the prior distribution G is the uniform distribution $\mathcal{U}(0,1)$, and $\alpha>1$. Then,

$$f(x)=\begin{cases}\frac{\alpha}{\alpha+1} & \text{if } 0<x\le 1;\\ \frac{\alpha}{(\alpha+1)x^{\alpha+1}} & \text{if } x>1.\end{cases} \qquad (4.1)$$

Note that $f(x)$ is decreasing in x for $x > 0$. Hence, Conditions A1 and A2 are satisfied. For this prior distribution $G = \mathcal{U}(0,1)$, we claim:

Theorem 4.1. $R(G,\varphi_n) - R(G,\varphi_G) \geq O\left(\frac{1}{nb_n}\right) + O(b_n^2)$.

The lower bound is established based on certain lemmas given below. First, we introduce some results associated with the prior distribution G and some notations.

For $0 < x \leq 1$, when $G = \mathcal{U}(0,1)$ is the prior distribution, $M(x) = \frac{\alpha x^{\alpha+2}}{(\alpha+1)(\alpha+2)} = \frac{x^{\alpha+2}f(x)}{\alpha+2}, \varphi_G(x) = \frac{(\alpha+1)x}{\alpha+2}$.

Let δ be a positive value such that $0 < \delta < \frac{1}{4}$. For each $0 < x \leq 1-\delta$, let $B_n(x) = I\left(\frac{M_n(x)}{x^{\alpha+1}f_n(x)} \leq x\right)$ and let $B_n^c(x)$ be the complement of $B_n(x)$.

Lemma 4.1. $\lim\limits_{n\to\infty} E[B_n^c(x)] = 0$ and $\lim\limits_{n\to\infty} E[B_n(x)] = 1, 0 < x \leq 1-\delta$.

Proof: We consider the case only where n is sufficiently large such that $b_n < \delta$. Hence, for each $x \in (0, 1-\delta]$,

$$E[M_n(x) - x^{\alpha+2}f_n(x)] = M(x) - x^{\alpha+2}f(x),$$

$$\begin{aligned}
\text{Var }(M_n(x) - x^{\alpha+2}f_n(x)) &= \frac{1}{n}\text{ Var }(X_1^{\alpha+1}I_{(0,x)}(X_1) - \frac{x^{\alpha+2}}{b_n}I_{(x,x+b_n]}(X_1)) \\
&\leq \frac{2}{n}\left[\text{Var }(X_1^{\alpha+1}I_{(0,x)}(X_1)) + \text{ Var }\left(\frac{x^{\alpha+2}}{b_n}I_{(x,x+b_n]}(X_1)\right)\right] \\
&\leq \frac{2}{n}\left[1 + \frac{x^{2(\alpha+2)}f(x)}{b_n}\right]
\end{aligned}$$

where $\frac{2}{n}\left[1 + \frac{x^{2(\alpha+2)}f(x)}{b_n}\right] \to 0$ as $n \to \infty$ since $\lim\limits_{n\to\infty} nb_n = \infty$.

Hence, by Chebychev's inequality,

$$\begin{aligned}
E[B_n^c(x)] &= P\left\{\frac{M_n(x)}{x^{\alpha+1}f_n(x)} > x\right\} \\
&= P\left\{[M_n(x) - x^{\alpha+2}f_n(x)] - [M(x) - x^{\alpha+2}f(x)] > \frac{(\alpha+1)x^{\alpha+2}f(x)}{\alpha+2}\right\} \\
&\leq \frac{(\alpha+1)\text{ Var }(M_n(x) - x^{\alpha+2}f_n(x))}{(\alpha+1)^2x^{2(\alpha+2)}f^2(x)}.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} E[B_n^c(x)] = 0$ and $\lim\limits_{n\to\infty} E[B_n(x)] = 1$. □

Lemma 4.2. $\int_0^1 E[(\varphi_n(x) - \varphi_G(x))^2]f(x)dx \geq O\left(\frac{1}{nb_n}\right)$.

Proof: $\int_0^1 E[(\varphi_n(x)-\varphi_G(x))^2]f(x)dx \ge \int_0^{1-\delta} E[(\varphi_n(x)-\varphi_G(x))^2 B_n(x)]f(x)dx$, where for each $x \in (0, 1-\delta]$, for the prior distribution $G = \mathcal{U}(0,1)$, straightforward computation leads to:

$$\begin{aligned} &E[(\varphi_n(x)-\varphi_G(x))^2 B_n(x)] \\ &\quad = E\left[\left(\frac{(\alpha+2)[M_n(x)-M(x)] - x^{\alpha+2}[f_n(x)-f(x)]}{(\alpha+2)x^{\alpha+1}f_n(x)}\right)^2 B_n^2(x)\right] \end{aligned} \tag{4.2}$$

Since $Ef_n(x) = f(x)$, Var $(f_n(x)) = \frac{f(x)(1-b_n f(x))}{nb_n}$ and $\lim\limits_{n\to\infty} b_n = 0$, hence

$$x^{\alpha+2}\sqrt{nb_n}(f_n(x)-f(x)) \underset{d}{\rightarrow} N(0, x^{2(\alpha+2)}f(x)) \text{ as } n \to \infty.$$

Also,

$$\begin{aligned} (\alpha+2)\sqrt{nb_n}(M_n(x)-M(x)) &\underset{P}{\rightarrow} 0 \qquad \text{as } n \to \infty, \\ (\alpha+2)x^{\alpha+1}f_n(x) &\underset{P}{\rightarrow} (\alpha+2)x^{\alpha+1}f(x) \qquad \text{as } n \to \infty, \end{aligned}$$

and $E[B_n(x)] \to 1$ as $n \to \infty$. Hence, by Slutsky's theorem,

$$\begin{aligned} &\sqrt{nb_n}\left[\frac{(\alpha+2)[M_n(x)-M(x)] - x^{\alpha+2}[f_n(x)-f(x)]}{(\alpha+2)x^{\alpha+1}f_n(x)}\right] B_n(x) \\ &\underset{d}{\rightarrow} N\left(0, \frac{x^2}{(\alpha+2)^2 f(x)}\right). \end{aligned} \tag{4.3}$$

Then, by convergence theorem, from (4.2) and (4.3), we have:

$$\lim\ \inf E[nb_n(\varphi_n(x)-\varphi_G(x))^2 B_n(x)] \ge \frac{x^2}{(\alpha+2)^2 f(x)}, \tag{4.4}$$

for all $x \in (0, 1-\delta]$ as $b_n < \delta$. Then, by Fatou's lemma,

$$\begin{aligned} &\lim\ \inf \int_0^{1-\delta} E[nb_n(\varphi_n(x)-\varphi_G(x))^2 B_n(x)]f(x)dx \\ &\ge \int_0^{1-\delta} \{\lim\ \inf E[nb_n(\varphi_n(x)-\varphi_G(x))^2 B_n(x)]\} f(x)dx \\ &\ge \int_0^{1-\delta} \frac{x^2}{(\alpha+2)^2 f(x)} f(x)dx \\ &= \frac{(1-\delta)^3}{3(\alpha+2)^2}. \end{aligned}$$

Hence, we conclude:

$$\int_0^1 E[(\varphi_n(x)-\varphi_G(x))^2]f(x)dx \ge O\left(\frac{1}{nb_n}\right). \qquad \square$$

Lemma 4.3. $E[(\varphi_n(1)-\varphi_G(1))^2] \geq O(b_n^2)$ as $\alpha \geq 1$.

Proof: Let $C_n = b_n/(\alpha+2)$ and $A_n = I(|\varphi_n(1)-\varphi_G(1)| \geq C_n)$. Now,

$$
\begin{aligned}
&E[(\varphi_n(1)-\varphi_G(1))^2] \\
=&E[(\varphi_n(1)-\varphi_G(1))^2 B_n(1)] + E[(\varphi_n(1)-\varphi_G(1))^2 B_n^c(1)],
\end{aligned}
$$

where

$$
E\{[(\varphi_n(1)-\varphi_G(1))^2 B_n^c(1)\} = \left(\frac{\alpha+1}{\alpha+2}\right)^2 P\{B_n^c(1)\},
$$

and

$$
\begin{aligned}
&E[(\varphi_n(1)-\varphi_G(1))^2 B_n(1)] \\
\geq& C_n^2 P\{A_n \cap B_n(1)\} \\
\geq& C_n^2 P\left\{\frac{M_n(1)}{f_n(1)} - \frac{1}{\alpha+2} \geq C_n, B_n(1)\right\} \\
=& C_n^2 P\left\{M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right) \geq 0, B_n(1)\right\}.
\end{aligned}
$$

For n being sufficiently large, $C_n \leq \frac{\alpha+1}{\alpha+2}$ and hence,

$$
\begin{aligned}
&E[(\varphi_n(1)-\varphi_G(1))^2] \\
\geq& C_n^2 P\left\{M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right) \geq 0, B_n(1)\right\} + C_n^2 P\{B_n^c(1)\} \\
\geq& C_n^2 P\left\{M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right) \geq 0\right\} \\
=& C_n^2 P\Bigg\{\left[M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)\right] - E\left[M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)\right] \\
&\qquad \geq -E\left[M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)\right]\Bigg\},
\end{aligned}
$$

where $E\left[M_n(1) - f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)\right]$

$$
\begin{aligned}
&= \frac{\alpha}{(\alpha+1)(\alpha+2)}\left[1 - \frac{1}{b_n}\int_1^{1+b_n}\frac{dx}{x^{\alpha+1}} - \frac{(\alpha+2)C_n}{b_n}\int_1^{1+b_n}\frac{dx}{x^{\alpha+1}}\right] \\
&= \frac{\alpha}{(\alpha+1)(\alpha+2)}\left[1 - \frac{1}{b_n}\int_1^{1+b_n}\frac{dx}{x^{\alpha+1}} - \int_1^{1+b_n}\frac{dx}{x^{\alpha+1}}\right] \quad (\text{since } (\alpha+2)C_n = b_n) \\
&\geq \frac{\alpha}{(\alpha+1)(\alpha+2)}\left[1 - \frac{1}{b_n}\int_1^{1+b_n}\frac{dx}{x^2} - \int_1^{1+b_n}\frac{dx}{x^2}\right] \quad (\text{since } \alpha \geq 1) \\
&= 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&P\left\{M_n(1)-f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)\geq 0\right\}\\
\geq &P\left\{M_n(1)-f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)-E\left[M_n(1)-f_n(1)\left(\frac{1}{\alpha+2}+C_n\right)\right]\geq 0\right\}.
\end{aligned} \tag{4.5}
$$

By central limit theorem, the R.H.S. of (4.5) will tend to $\frac{1}{2}$ as $n\to\infty$. Therefore, we have

$$E[(\varphi_n(1)-\varphi_G(1))^2]\geq O(C_n^2)=O(b_n^2) \text{ since } (\alpha+2)C_n=b_n. \qquad \square$$

Proof of Theorem 4.1.: Theorem 4.1 is a direct result of (3.1), (3.2) and Lemmas 4.2 and 4.3. $\square$

From Theorems 3.1 and 4.1, there is a prior distribution G such that Conditions A1 and A2 hold, and $R(G,\varphi_n)-R(G,\varphi_G)=O\left(\frac{1}{nb_n}\right)+O(b_n^2)$. In order to obtain the best convergence rate, we let $b_n=n^{-1/3}$, and the convergence rate is of order $O(n^{-2/3})$.

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