ESTIMATION OF THE MEAN AND STANDARD DEVIATION OF THE LOGISTIC DISTRIBUTION BASED ON MULTIPLY TYPE-II CENSORED SAMPLES*

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Estimation of the Mean and Standard Deviation of the Logistic Distribution Based On Multiply Type–II Censored Samples

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Abstract

In this paper, we discuss the problem of estimating the mean and standard deviation of a logistic population based on multiply Type–II censored samples. First, we discuss the best linear unbiased estimation and the maximum likelihood estimation methods. Next, by appropriately approximating the likelihood equations we derive approximate maximum likelihood estimators for the two parameters and show that these estimators are quite useful as they do not need the construction of any special tables (as required for the best linear unbiased estimators) and are explicit estimators (unlike the maximum likelihood estimators which need to be determined by numerical methods). We show that these

estimators are also quite efficient, and derive the asymptotic variances and covariance of the estimators. Finally, we present an example to illustrate the methods of estimation discussed in this paper.

1. Introduction

Order statistics from the logistic distribution and their moments were first examined by Birnbaum and Dudman (1963). They tabulated the means and standard deviations of logistic order statistics for some selected sample sizes. Gupta and Shah (1965) and Tarter and Clark (1965) studied the distribution of order statistics and the sample range and derived explicit exact expressions for the first four moments of order statistics in terms of gamma function and its derivatives. Variances and covariances of order statistics were tabulated by Gupta, Qureishi and Shah (1967). Shah (1966, 1970) established some recurrence relations satisfied by the single and the product moments of order statistics which would enable one to compute these moments for all sample sizes in a simple recursive manner. After Kjelsberg (1962) pointed out various applications of the truncated logistic distribution, Tarter (1966) studied order statistics from the truncated logistic distribution and derived exact explicit expressions for the means, variances and covariances of order statistics in terms of a finite series involving logarithms and dilogarithms of the constants of truncation. By proceeding on the lines of Shah (1966, 1970), Balakrishnan and Joshi (1983) and Balakrishnan and Kocherlakota (1986) derived several recurrence relations satisfied by the single and the product moments of order statistics from a general truncated logistic distribution. All these developments have been reviewed in a recent article by Gupta and Balakrishnan (1991).

The best linear unbiased estimation of the mean μ and standard deviation σ based on doubly Type–II censored samples was discussed by Gupta, Qureishi and Shah (1967) who have also set up the necessary tables for some selected sample sizes up to 25 and for some selected choices of censoring. Raghunandanan and Srinivasan (1970) proposed some simple estimators of μ and σ based on a search made on some specific linear functions of order statistics and set up the necessary tables for sample sizes up to 20. The maximum likelihood estimation of μ and σ based on doubly Type–II censored samples was studied by Harter and Moore (1967). They examined the bias and mean square error of the estimators through Monte Carlo simulations for sample sizes 10 and 20 and over various choices of censoring; see also Harter (1970). Tiku (1968) derived the modified maximum likelihood estimators of μ and σ based on doubly Type–II censored samples. Furthermore, the estimation of the parameters μ and σ based on selected order statistics was considered by several authors including Gupta and Gnanadesikan (1966), Chan (1969), Hassanein (1969), Chan, Chan and Mead (1971), Chan and Cheng (1972, 1974), and Beyer, Moore and Harter (1976). Many of these developments have been presented in the recent books by Balakrishnan and Cohen (1990) and Balakrishnan (1991b). In this paper, we consider the problem of estimating the mean μ and standard deviation σ of a logistic population based on multiply Type–II censored samples.

Consider the logistic distribution with probability density function

$$g(y;\, \mu,\, \sigma) = \frac{\pi}{\sigma\sqrt{3}}\, \frac{\exp\{-\pi(y-\mu)/\sigma\sqrt{3}\}}{[1+\exp\{-\pi(y-\mu)/\sigma\sqrt{3}\}]^2},\ -\infty < y < \infty, \tag{1.1}$$

and cumulative distribution function

$$G(y;\, \mu,\, \sigma) = [1 + \exp\{-\pi(y-\mu)/\sigma\sqrt{3}\}]^{-1},\ -\infty < y < \infty. \tag{1.2}$$

Let us assume that the following multiply Type–II censored sample from a sample of size n

$$Y_{r_1+1:n} \le \cdots \le Y_{r_1+s_1:n} \le Y_{r_2+1:n} \le \cdots \le Y_{r_2+s_2:n} \le \cdots \le Y_{r_k+1:n} \le \cdots \le Y_{r_k+s_k:n} \tag{1.3}$$

is available from the logistic population in (1.1). Here, it is assumed that among the n items placed on a life–testing experiment, the smallest r_1, the largest $n - r_k - s_k$, and in addition some middle life–times are not observed. In Section 2, we first present the best linear unbiased estimators of μ and σ based on the multiply censored sample in (1.3). The maximum likelihood estimation of μ and σ based on the multiply censored sample in (1.3) is discussed in Section 3. We note that these maximum likelihood estimators do not exist in an explicit algebraic form and need to be determined by numerically solving the two likelihood equations. In Section 4, by appropriately approximating the likelihood equations by making use of some linear approximations, we derive the approximate maximum likelihood estimators of μ and σ based on the multiply Type–II censored sample in (1.3). These estimators, in addition to being simple and explicit, are nearly as efficient as the best linear unbiased estimators and the maximum likelihood estimators. We derive the asymptotic variances and covariance of these approximate maximum likelihood estimators in Section 5. Finally, in Section 6 we present an example from a life–testing experiment and illustrate the methods of estimation of μ and σ discussed in this paper. It should be mentioned here that similar work for the normal distribution has been carried out recently by Balakrishnan, Gupta and Panchapakesan (1991).

2. Best Linear Unbiased Estimation

Let $X_{i:n} = \pi\left[Y_{i:n} - \mu\right]/\sigma\sqrt{3}$, i = 1,2,...,n. Then, $X_{i:n}$ are order statistics from the standard logistic population with probability density function

$$f(x) = e^{-x}/(1 + e^{-x})^2, \qquad -\infty < x < \infty, \tag{2.1}$$

and cumulative distribution function

$$F(x) = 1/(1 + e^{-x}), \qquad -\infty < x < \infty. \tag{2.2}$$

Let us denote $E(X_{i:n})$ by $\alpha^*_{i:n}$, $E(X^2_{i:n})$ by $\alpha^{*(2)}_{i:n}$, $Var(X_{i:n})$ by $\beta^*_{i,i:n}$, $E(X_{i:n}\,X_{j:n})$ by $\alpha^*_{i,j:n}$ and $Cov(X_{i:n}, X_{j:n})$ by $\beta^*_{i,j:n}$. We then immediately have

$E(Y_{i:n}) = \mu + \sigma\frac{\sqrt{3}}{\pi}\,\alpha^*_{i:n}$, $Var(Y_{i:n}) = \sigma^2\,\frac{3}{\pi^2}\,\beta^*_{i,i:n}$, and

$Cov(Y_{i:n},Y_{j:n}) = \sigma^2\,\frac{3}{\pi^2}\,\beta^*_{i,j:n}$. Further, let us denote

$$\underset{\sim}{Y} = \left[Y_{r_1+1:n} \cdots Y_{r_1+s_1:n}\; Y_{r_2+1:n} \cdots Y_{r_2+s_2:n} \cdots Y_{r_k+1:n} \cdots Y_{r_k+s_k:n}\right]^T,$$

$$\underset{\sim}{\alpha} = \left[\alpha^*_{r_1+1:n} \cdots \alpha^*_{r_1+s_1:n}\ \alpha^*_{r_2+1:n} \cdots \alpha^*_{r_2+s_2:n} \cdots \alpha^*_{r_k+1:n} \cdots \alpha^*_{r_k+s_k:n}\right]^T,$$

$$\underset{\sim}{1} = \left[1\ \ 1 \ldots\ldots 1\right]^T_{\sum_1^k s_i \times 1},$$

$$\underset{\sim}{\beta} = \left[\left[\beta^*_{i,j:n}\right]\right] \text{ for } i,j \in I \text{ where } I = \left\{r_1 + 1,\ldots,r_1 + s_1,\ r_2 + 1,\ldots,r_2 + s_2,\ \ldots,r_k + 1,\ldots,r_k + s_k\right\}, \text{ and } \underset{\sim}{\Omega} = \underset{\sim}{\beta}^{-1}.$$

Then, the Best Linear Unbiased Estimators of μ and σ based on the multiply Type–II censored sample in (1.3) may be derived by minimizing the generalized variance (see David, 1981; Balakrishnan and Cohen, 1990) given by

$$\left[\underset{\sim}{Y} - \mu\,\underset{\sim}{1} - \sigma\,\frac{\sqrt{3}}{\pi}\,\underset{\sim}{\alpha}\right]^T \Omega \left[\underset{\sim}{Y} - \mu\,\underset{\sim}{1} - \sigma\,\frac{\sqrt{3}}{\pi}\,\underset{\sim}{\alpha}\right]. \tag{2.3}$$

The best linear unbiased estimators of μ nd σ obtained by minimizing the generalized variance in (2.3) are given by

$$\begin{aligned}\mu^* &= \left\{\frac{\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{\alpha}\,\underset{\sim}{1}^T\,\underset{\sim}{\Omega} - \underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1}\,\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{\alpha})(\underset{\sim}{1}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1})^2}\right\}\underset{\sim}{Y}\\ &= -\underset{\sim}{\alpha}^T\,\underset{\sim}{\Delta}\,\underset{\sim}{Y}\\ &= \sum_{i=1}^{k}\ \sum_{j=r_i+1}^{r_i+s_i} a_j\,Y_{j:n}\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\sigma^* &= \frac{\pi}{\sqrt{3}}\left\{\frac{\underset{\sim}{1}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1}\,\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega} - \underset{\sim}{1}^T\,\Omega\,\underset{\sim}{\alpha}\,\underset{\sim}{1}^T\,\underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1})^2}\right\}\underset{\sim}{Y}\\ &= \frac{\pi}{\sqrt{3}}\underset{\sim}{1}^T\,\underset{\sim}{\Delta}\,\underset{\sim}{Y}\\ &= \sum_{i=1}^{k}\ \sum_{j=r_i+1}^{r_i+s_i} b_j\,Y_{j:n},\end{aligned} \tag{2.5}$$

where $\underset{\sim}{\Delta}$ is a skew–symmetric matrix of order $\sum_{i=1}^{k} s_i$ given by

$$\underset{\sim}{\Delta} = \frac{\underset{\sim}{\Omega}(\underset{\sim}{1}\,\underset{\sim}{\alpha}^T - \underset{\sim}{\alpha}\,\underset{\sim}{1}^T)\,\underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\,\underset{\sim}{\Omega}\,\underset{\sim}{1})^2}. \tag{2.6}$$

The variances and covariance of the estimators μ^* and σ^* (David, 1981; Balakrishnan and Cohen, 1990) are given by

$$\mathrm{Var}(\mu^*) = \sigma^2 \left\{ \frac{\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{\alpha}}{(\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{\alpha})(\underset{\sim}{1}^T \underset{\sim}{\Omega} \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{1})^2} \right\}, \tag{2.7}$$

$$\mathrm{Var}(\sigma^*) = \frac{\pi^2}{3} \sigma^2 \left\{ \frac{\underset{\sim}{1}^T \underset{\sim}{\Omega} \underset{\sim}{1}}{(\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{\alpha})(\underset{\sim}{1}^T \underset{\sim}{\Omega} \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{1})^2} \right\}, \tag{2.8}$$

and

$$\mathrm{Cov}(\mu^*, \sigma^*) = -\frac{\pi}{\sqrt{3}} \sigma^2 \left\{ \frac{\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{1}}{(\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{\alpha})(\underset{\sim}{1}^T \underset{\sim}{\Omega} \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \underset{\sim}{\Omega} \underset{\sim}{1})^2} \right\}. \tag{2.9}$$

By using the values of means, variances and covariances of logistic order statistics tabulated by Gupta, Qureishi and Shah (1967) for sample sizes up to twenty five and more recently by Balakrishnan and Malik (1991) for sample sizes up to fifty, we may determine the coefficients a_j and b_j in Eqs. (2.4) and (2.5) and also the variances and covariance of the best linear unbiased estimators from Eqs. (2.7), (2.8) and (2.9), respectively. For sample sizes larger than fifty, we may determine these coefficients and the variances and the covariance of the estimators approximately upon using approximate expressions of means, variances and covariances of logistic order statistics obtained by David and Johnson's (1954) method; for details, refer to David (1981) and Arnold and Balakrishnan (1989).

3. Maximum Likelihood Estimation

With $X_{i:n} = \pi(Y_{i:n} - \mu)\sigma/\sqrt{3}$, we have the likelihood function based on the multiply Type–II censored sample in (1.3) to be

$$L = \frac{n!}{\prod_{i=1}^{k+1} (r_i - r_{i-1} - s_{i-1})!} \left[\frac{\pi}{\sigma\sqrt{3}}\right]^{\sum_1^k s_i} \left\{F\left[Z_{r_1+1:n}\right]\right\}^{r_1}$$

$$\times \prod_{i=2}^{k} \left\{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]\right\}^{r_i - r_{i-1} - s_{i-1}}$$

$$\times \left\{1 - F\left[X_{r_k+s_k:n}\right]\right\}^{n-r_k-s_k} \prod_{i=1}^{k} \prod_{j=r_i+1}^{r_i+s_i} f\left[X_{j:n}\right], \tag{3.1}$$

where f(x) and F(x) are the density function and the cumulative distribution function of the standard logistic population as given in Eqs. (2.1) and (2.2), respectively, and $r_0 = s_0 = 0$ and $r_{k+1} = n$. From Eq. (3.1), we have the log-likelihood function to be

$$\ell n\ L = \text{Const} - A\ \ell n\ \sigma + r_1\ \ell n\left\{F(X_{r_1+1:n})\right\}$$

$$+ \sum_{i=2}^{k} t_i\ \ell n\left\{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]\right\}$$

$$+ \left[n - r_k - s_k\right] \ell n\left\{1 - F\left[X_{r_k+s_k:n}\right]\right\}$$

$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \ell n\left\{F\left[X_{j:n}\right]\right\} + \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \ell n\left\{1 - F\left[X_{j:n}\right]\right\} \tag{3.2}$$

upon using the fact that f(x) = F(x) (1 − F(x)), where $A = \sum_{i=1}^{k} s_i$ is the size of the available multiply Type–II censored sample and $t_i = r_i - r_{i-1} - s_{i-1}$ for i = 2,...,k. From Eq. (3.2), we obtain the likelihood equations for μ and σ to be

$$\frac{\partial\ \ell n\ L}{\partial \mu} = - \frac{\pi}{\sigma\sqrt{3}} \left[r_1\left\{1 - F\left[X_{r_1+1:n}\right]\right\} - \left[n - r_k - s_k\right] F\left[X_{r_k+s_k:n}\right]\right.$$

$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \left\{1 - 2F\left[X_{j:n}\right]\right\}$$

$$\left. + \sum_{i=2}^{k} t_i \left\{\frac{f(X_{r_i+1:n}) - f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}\right\}\right]$$

$$= 0, \tag{3.3}$$

and

$$\frac{\partial \ \ell n \ L}{\partial \sigma} = -\frac{1}{\sigma}\Big[A + r_1 \ X_{r_1+1:n}\Big\{1 - F\Big[X_{r_1+1:n}\Big]\Big\}$$

$$- \Big[n - r_k - s_k\Big] X_{r_k+s_k:n} \ F\Big[X_{r_k+s_k:n}\Big]$$

$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} X_{j:n}\Big\{1 - 2 \ F\Big[X_{j:n}\Big]\Big\}$$

$$+ \sum_{i=2}^{k} t_i \left\{\frac{X_{r_i+1:n} \ f(X_{r_i+1:n}) - X_{r_{i-1}+s_{i-1}:n} \ f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}\right\}\Big]$$

$$= 0. \tag{3.4}$$

Eqs. (3.3) and (3.4) do not admit explicit solutions even though f(.) and F(.) are simple explicit functions. But, the maximum likelihood estimates of μ and σ may be determined by numerically solving Eqs. (3.3) and (3.4).

4. Approximate Maximum Likelihood Estimation

Let $p_i = i/(n+1)$, $q_i = 1 - p_i$, and $\xi_i = F^{-1}(p_i) = \ell n \ (p_i/q_i)$. Then, by expanding the function $F(X_{i:n})$ around the point ξ_i in Taylor series (see David (1981) or Arnold and Balakrishnan (1989) for reasoning), we may approximate it by

$$F(X_{i:n}) \simeq \alpha_i + \beta_i \ X_{i:n}, \tag{4.1}$$

where

$$\alpha_i = F(\xi_i) - \xi_i f(\xi_i) = p_i\Big\{1 - q_i \ \ell n\Big[p_i/q_i\Big]\Big\} \tag{4.2}$$

and

$$\beta_i = f(\xi_i) = p_i \ q_i > 0. \tag{4.3}$$

Now, let

$$h_1\Big[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\Big] = \frac{f(X_{r_i+1:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})} \tag{4.4}$$

and

$$h_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] = \frac{f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}. \tag{4.5}$$

By expanding the functions $h_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$ and $h_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$ in (4.4) and (4.5) around the point $\left[\xi_{r_{i-1}+s_{i-1}}, \xi_{r_i+1}\right]$ in bivariate Taylor series, respectively, we may then approximate them by

$$h_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \simeq \delta_i^* + \gamma_i X_{r_{i-1}+s_{i-1}:n} - \left[\gamma_i + \beta_{r_i+1}\right] X_{r_i+1:n} \tag{4.6}$$

and

$$h_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \simeq \delta_i^{**} + \left[\gamma_i + \beta_{r_{i-1}+s_{i-1}}\right] X_{r_{i-1}+s_{i-1}:n} - \gamma_i X_{r_i+1:n}, \tag{4.7}$$

where

$$\gamma_i = p_{r_{i-1}+s_{i-1}}\, p_{r_i+1}\, q_{r_{i-1}+s_{i-1}}\, q_{r_i+1} / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2 > 0, \tag{4.8}$$

$$\begin{aligned}\delta_i^* &= \beta_{r_i+1}\left\{\ell n\left[\frac{p_{r_i+1}}{q_{r_i+1}}\right] + 1/\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]\right\} \\ &\quad + \gamma_i\left\{\ell n\left[\frac{p_{r_i+1}}{q_{r_i+1}}\right] - \ell n\left[\frac{p_{r_{i-1}+s_{i-1}}}{q_{r_{i-1}+s_{i-1}}}\right]\right\},\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\delta_i^{**} &= \beta_{r_{i-1}+s_{i-1}}\left\{1/\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] - \ell n\left[\frac{p_{r_{i-1}+s_{i-1}}}{q_{r_{i-1}+s_{i-1}}}\right]\right\} \\ &\quad + \gamma_i\left\{\ell n\left[\frac{p_{r_i+1}}{q_{r_i+1}}\right] - \ell n\left[\frac{p_{r_{i-1}+s_{i-1}}}{q_{r_{i-1}+s_{i-1}}}\right]\right\}.\end{aligned} \tag{4.10}$$

By making use of the approximations in (4.6) and (4.7), we obtain

$$\begin{aligned}h\left[X_{r_i+s_{i-1}:n}, X_{r_i+1:n}\right] &= h_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] - h_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \\ &= \frac{f(X_{r_i+1:n}) - f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}\end{aligned}$$

$$\simeq \delta_i - \beta_{r_{i-1}+s_{i-1}} X_{r_{i-1}+s_{i-1}:n} - \beta_{r_i+1} X_{r_i+1:n}, \tag{4.11}$$

where β_i is as given in Eq. (4.3) and

$$\delta_i = \delta_i^* - \delta_i^{**}$$

$$= \beta_{r_i+1}\left\{1/\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] + \ln\left[\frac{p_{r_i+1}}{q_{r_i+1}}\right]\right\}$$

$$- \beta_{r_{i-1}+s_{i-1}}\left\{1/\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] - \ln\left[\frac{p_{r_{i-1}+s_{i-1}}}{q_{r_{i-1}+s_{i-1}}}\right]\right\}. \tag{4.12}$$

Now, upon using the approximations in Eqs. (4.1) and (4.11) into the likelihood equation for μ in (3.3), we obtain the approximate likelihood equation for μ to be

$$r_1\left\{1 - \alpha_{r_1+1} - \beta_{r_1+1} X_{r_1+1:n}\right\} - \left[n - r_k - s_k\right]\left\{\alpha_{r_k+s_k} + \beta_{r_k+s_k} X_{r_k+s_k:n}\right\}$$

$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \left\{1 - 2\alpha_j - 2\beta_j X_{j:n}\right\}$$

$$+ \sum_{i=2}^{k} t_i\left\{\delta_i - \beta_{r_{i-1}+s_{i-1}} X_{r_{i-1}+s_{i-1}:n} - \beta_{r_i+1}X_{r_i+1:n}\right\} = 0, \tag{4.13}$$

which when solved for μ yields the approximate maximum likelihood estimator of μ to be

$$\hat{\mu} = B - \frac{\sqrt{3}}{\pi}\sigma C, \tag{4.14}$$

where

$$t_i = r_i - r_{i-1} - s_{i-1}, \qquad i = 2,3,\ldots,k,$$

$$A = \sum_{i=1}^{k} s_i,$$

$$m = r_1\beta_{r_1+1} + \left[n - r_k - s_k\right]\beta_{r_k+s_k} + 2\sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \beta_j$$

$$+ \sum_{i=2}^{k} t_i\,\beta_{r_{i-1}+s_{i-1}} + \sum_{i=2}^{k} t_i\,\beta_{r_i+1},$$

$$B = \frac{1}{m}\Big\{r_1\beta_{r_1+1}\, Y_{r_1+1:n} + \Big[n - r_k - s_k\Big]\, \beta_{r_k+s_k}\, Y_{r_k+s_k:n}$$
$$+ 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j\, Y_{j:n} + \sum_{i=2}^{k} t_i\, \beta_{r_{i-1}+s_{i-1}}\, Y_{r_{i-1}+s_{i-1}:n}$$
$$+ \sum_{i=2}^{k} t_i\, \beta_{r_i+1}\, Y_{r_i+1:n}\Big\},$$

and

$$C = \frac{1}{m}\Big\{A + r_1 - r_1\, \alpha_{r_1+1} - (n - r_k - s_k)\, \alpha_{r_k+s_k}$$
$$- 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \alpha_j + \sum_{i=2}^{k} t_i\, \delta_i\Big\}. \tag{4.15}$$

Next, upon using the approximations in Eqs. (4.1), (4.6) and (4.7) into the likelihood equation for σ in (3.4), we obtain the approximate likelihood equation for σ to be

$$A + r_1\, X_{r_1+1:n}\Big\{1 - \alpha_{r_1+1} - \beta_{r_1+1}\, X_{r_1+1:n}\Big\}$$
$$- \Big[n - r_k - s_k\Big]\, X_{r_k+s_k:n}\Big\{\alpha_{r_k+s_k} + \beta_{r_k+s_k} X_{r_k+s_k:n}\Big\}$$
$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} X_{j:n}\Big\{1 - 2\alpha_j - 2\beta_j\, X_{j:n}\Big\}$$
$$+ \sum_{i=2}^{k} t_i\, X_{r_i+1:n}\Big\{\delta_i^{*} + \gamma_i\, X_{r_{i-1}+s_{i-1}:n} - \Big[\gamma_i + \beta_{r_i+1}\Big] X_{r_i+1:n}\Big\}$$
$$- \sum_{i=2}^{k} t_i\, X_{r_{i-1}+s_{i-1}:n}\Big\{\delta_i^{**} + \Big[\gamma_i + \beta_{r_{i-1}+s_{i-1}}\Big] X_{r_{i-1}+s_{i-1}:n} - \gamma_i\, X_{r_i+1:n}\Big\}$$
$$= 0, \tag{4.16}$$

which when solved for σ (simultaneously, by using the solution for μ in (4.14)) yields the approximate maximum likelihood estimator of σ to be

$$\hat{\sigma} = \frac{\pi}{\sqrt{3}}\left\{\frac{-D + (D^2 + 4AE)^{1/2}}{2A}\right\}, \tag{4.17}$$

where

$$A = \sum_{i=1}^{k} s_i \text{ as before,}$$
$$D = r_1\Big[1 - \alpha_{r_1+1}\Big] Y_{r_1+1:n} - \Big[n - r_k - s_k\Big]\alpha_{r_k+s_k}\, Y_{r_k+s_k:n}$$

$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \left[1 - 2\alpha_j\right] Y_{j:n} + \sum_{i=2}^{k} t_i\, \delta_i^{*}\, Y_{r_i+1:n}$$
$$- \sum_{i=2}^{k} t_i\, \delta_i^{**}\, Y_{r_i+s_{i-1}:n} - mBC,$$

and

$$E = r_1 \beta_{r_1+1} Y^2_{r_1+1:n} + \left[n - r_k - s_k\right] \beta_{r_k+s_k} Y^2_{r_k+s_k:n} + 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j Y^2_{j:n}$$
$$+ \sum_{i=2}^{k} t_i\, \beta_{r_i+1} Y^2_{r_i+1:n} + \sum_{i=2}^{k} t_i\, \beta_{r_{i-1}+s_{i-1}} Y^2_{r_{i-1}+s_{i-1}:n}$$
$$+ \sum_{i=2}^{k} t_i\, \gamma_i \left[Y_{r_i+1:n} - Y_{r_{i-1}+s_{i-1}:n}\right]^2 - mB^2$$
$$= r_1 \beta_{r_1+1} \left[Y_{r_1+1:n} - B\right]^2 + \left[n - r_k - s_k\right] \beta_{r_k+s_k} \left[Y_{r_k+s_k:n} - B\right]^2$$
$$+ 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j \left[Y_{j:n} - B\right]^2 + \sum_{i=2}^{k} t_i\, \beta_{r_i+1} \left[Y_{r_i+1:n} - B\right]^2$$
$$+ \sum_{i=2}^{k} t_i\, \beta_{r_{i-1}+s_{i-1}} \left[Y_{r_{i-1}+s_{i-1}:n} - B\right]^2 + \sum_{i=2}^{k} t_i\, \gamma_i \left[Y_{r_i+1:n} - Y_{r_{i-1}+s_{i-1}:n}\right]^2. \tag{4.18}$$

It should be mentioned here that upon solving Eq. (4.16) we obtain a quadratic equation in σ which has two roots; however, one of them is negative and hence inadmissible since both β_i and γ_i are positive and consequently $E > 0$.

<u>Remark 1:</u> For the special case when $s_1 = s_2 = \ldots = s_{k-1} = 1$, $r_1 = r$, $r_2 = r + 1, \ldots$, $r_k = r + k - 1$ and $s_k = n - r - s - k + 1$, then the available sample in (1.3) simply becomes a Type–II censored sample $Y_{r+1:n} \le Y_{r+2:n} \le \ldots \le Y_{n-s:n}$, where the smallest r and the largest s observations have been censored. In this case, the estimators $\hat{\mu}$ and $\hat{\sigma}$ in Eqs. (4.14) and (4.17), respectively, simply reduce to the approximate maximum likelihood estimators of μ and σ derived by Balakrishnan (1991a).

<u>Remark 2</u>: For the special case when the available multiply Type–II censored sample in (1.3) is symmetric (that is, if $Y_{i:n}$ is available in the sample then so also is $Y_{n-i+1:n}$), it

can be shown from Eq. (4.15) that C = 0. As a result, the estimator $\hat{\mu}$ in (4.14) simply becomes

$$\hat{\mu} = B$$

which is a linear function of the available order statistics with equal weights for the symmetric order statistics. Due to the symmetry of the logistic density function in (2.1) and hence the relation $E(X_{i:n}) = -E\left[X_{n-i+1:n}\right]$ (see David (1981) or Arnold and Balakrishnan (1989)), it may be easily shown that the above estimator $\hat{\mu}$ is an unbiased estimator of μ.

5. Approximate Variances and Covariance of the Estimators

By using the linear approximations in (4.1), (4.6), (4.7) and (4.11), we also obtain from the likelihood equations for μ and σ in Eqs. (3.3) and (3.4) that

$$E\left[-\frac{\partial^2 \ln L}{\partial \mu^2}\right] \simeq \frac{m\,\pi^2}{3\sigma^2}, \tag{5.1}$$

$$E\left[-\frac{\partial^2 \ln L}{\partial \mu\, \partial \sigma}\right] \simeq \frac{m\,\pi}{\sqrt{3}\,\sigma^2} V_1, \tag{5.2}$$

and

$$E\left[-\frac{\partial^2 \ln L}{\partial \sigma^2}\right] \simeq \frac{m}{\sigma^2} V_2, \tag{5.3}$$

where, as before,

$$\begin{aligned} m = {} & r_1\beta_{r_1+1} + \left[n - r_k - s_k\right] \beta_{r_k+s_k} + 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j \\ & + \sum_{i=2}^{k} t_i\, \beta_{r_{i-1}+s_{i-1}} + \sum_{i=2}^{k} t_i\, \beta_{r_i+1}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} V_1 = {} & \frac{2}{m}\Big\{r_1\, \beta_{r_1+1}\, \alpha^*_{r_1+1:n} + \left[n - r_k - s_k\right] \beta_{r_k+s_k}\, \alpha^*_{r_k+s_k:n} \\ & + 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j\, \alpha^*_{j:n} + \sum_{i=2}^{k} t_i\, \beta_{r_{i-1}+s_{i-1}}\, \alpha^*_{r_{i-1}+s_{i-1}:n} \\ & + \sum_{i=2}^{k} t_i\, \beta_{r_i+1}\, \alpha^*_{r_i+1:n}\Big\} - C, \end{aligned} \tag{5.5}$$

and

$$V_2 = \frac{3}{m}\Big\{r_1\,\beta_{r_1+1}\,\alpha^{*(2)}_{r_1+1:n} + \Big[n - r_k - s_k\Big]\,\beta_{r_k+s_k}\,\alpha^{*(2)}_{r_k+s_k:n} + 2\sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i}\beta_j\,\alpha^{*(2)}_{j:n}$$
$$+ \sum_{i=2}^{k} t_i\,\beta_{r_i+1}\,\alpha^{*(2)}_{r_i+1:n} + \sum_{i=2}^{k} t_i\,\beta_{r_{i-1}+s_{i-1}}\,\alpha^{*(2)}_{r_{i-1}+s_{i-1}:n}$$
$$+ \sum_{i=2}^{k} t_i\,\gamma_i\Big[\alpha^{*(2)}_{r_i+1:n} + \alpha^{*(2)}_{r_{i-1}+s_{i-1}:n} - 2\,\alpha^{*}_{r_{i-1}+s_{i-1},\,r_i+1:n}\Big]\Big\}$$
$$-\frac{2}{m}\Big\{r_1(1-\alpha_{r_1+1})\,\alpha^{*}_{r_1+1:n} - (n - r_k - s_k)\,\alpha_{r_k+s_k}\,\alpha^{*}_{r_k+s_k:n}$$
$$+ \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i}(1-2\alpha_j)\alpha^{*}_{j:n} + \sum_{i=2}^{k} t_i\,\delta^{*}_i\,\alpha^{*}_{r_i+1:n} - \sum_{i=2}^{k} t_i\,\delta^{**}_i\,\alpha^{*}_{r_{i-1}+s_{i-1}:n}\Big\}$$
$$-\frac{A}{m}. \tag{5.6}$$

In the above equations, like in Section 2 $\alpha^{*}_{i:n}$, $\alpha^{*(2)}_{i:n}$ and $\alpha^{*}_{i,j:n}$ denote the first and second single moments and the product moments, respectively, of order statistics from the standard logistic distribution in (2.2). From these formulae, we may compute

$$\mathrm{Var}(\hat{\mu}) \simeq \frac{3}{\pi^2}\,\frac{\sigma^2}{m}\left\{\frac{V_2}{V_2 - V_1^2}\right\}, \tag{5.7}$$

$$\mathrm{Var}(\hat{\sigma}) \simeq \frac{\sigma^2}{m}\left\{\frac{1}{V_2 - V_1^2}\right\}, \tag{5.8}$$

and

$$\mathrm{Cov}\Big[\hat{\mu},\,\hat{\sigma}\Big] \simeq -\frac{\sqrt{3}}{\pi}\,\frac{\sigma^2}{m}\left\{\frac{V_1}{V_2 - V_1^2}\right\}; \tag{5.9}$$

refer to Kendall and Stuart (1973) or Rao (1975).

The approximate variances and covariance of the estimators $\hat{\mu}$ and $\hat{\sigma}$ can be determined from Eqs. (5.7) – (5.9) either by directly using the tables of means, variances and covariances of logistic order statistics prepared by Balakrishnan and Malik (1991) for sample sizes up to fifty or by using approximate expressions of means, variances and covariances of logistic order statistics presented by Arnold and Balakrishnan (1989).

The asymptotic distribution of the estimators $\hat{\mu}$ and $\hat{\sigma}$ is given in the following theorem; for a proof, refer to Kendall and Stuart (1973) or Rao (1975).

Theorem 1: Asymptotically, the approximate maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ jointly have a bivariate normal distribution with mean vector $\begin{bmatrix}\mu \\ \sigma\end{bmatrix}$ and variance–covariance matrix

$$\frac{3\sigma^2}{m\pi^2\left[V_2 - V_1^2\right]} \begin{bmatrix} V_2 & -\frac{\pi}{\sqrt{3}} V_1 \\ -\frac{\pi}{\sqrt{3}} V_1 & \frac{\pi^2}{3} \end{bmatrix},$$

where m, V_1 and V_2 are as given in Eqs. (5.4), (5.5) and (5.6), respectively.

Remark 3: For the special case when the available multiply Type–II censored sample in (1.3) is symmetric, by using the facts that $\alpha^*_{i:n} = -\alpha^*_{n-i+1:n}$ and C = 0 as noted earlier in Remark 2, it can be very easily shown from Eq. (5.5) that $V_1 = 0$. Hence, we have the estimators $\hat{\mu}$ and $\hat{\sigma}$ to be uncorrelated in this case. Also, we have from Eqs. (5.7) and (5.8) that

$$\text{Var}\left[\hat{\mu}\right] \simeq \frac{3\sigma^2}{m\pi^2} \quad \text{and} \quad \text{Var}\left[\hat{\sigma}\right] \simeq \frac{\sigma^2}{mV_2}.$$

6. Illustrative Example

Let us consider the following lifetime data of 20 electronic units given by Balakrishnan, Gupta and Panchapakesan (1991):

—, —, 128.887, 132.585, 133.196, 140.734, 141.816, 146.864, 148.350, —, —, 154.671, 159.188, 163.117, 166.252, 166.770, 172.017, 174.744, —, —

Out of the 20 units placed on test, the first two units failed before the measurement started resulting in the censoring of the first two observations, two central observations are censored as the failure times of those two units were not recorded due to experimental difficulties, and the experiment was stopped immediately after the failure of the eighteenth unit resulting in the censoring of the last two observations.

We shall now assume that the above given multiply Type–II censored sample has come from a logistic distribution in (1.2) and estimate the unknown mean μ and the unknown standard deviation σ and also construct approximate confidence intervals for both these parameters.

For the approximate maximum likelihood estimation developed in this paper, we have:

i	p_i	q_i	α_i	β_i	$1-2\alpha_i$
3	0.1429	0.8571	0.3623	0.1225	0.2754
4	0.1905	0.8095	0.4136	0.1542	0.1728
5	0.2381	0.7619	0.4491	0.1814	0.1018
6	0.2857	0.7143	0.4727	0.2041	0.0546
7	0.3333	0.6667	0.4874	0.2222	0.0252
8	0.3810	0.6190	0.4955	0.2358	0.0090
9	0.4286	0.5714	0.4990	0.2449	0.0020
10	0.4762	0.5238	0.5000	0.2494	0
11	0.5238	0.4762	0.5000	0.2494	0
12	0.5714	0.4286	0.5010	0.2449	–0.0020
13	0.6190	0.3810	0.5045	0.2358	–0.0090
14	0.6667	0.3333	0.5126	0.2222	–0.0252
15	0.7143	0.2857	0.5273	0.2041	–0.0546
16	0.7619	0.2381	0.5509	0.1814	–0.1018
17	0.8095	0.1905	0.5864	0.1542	–0.1728
18	0.8571	0.1429	0.6377	0.1225	–0.2754

$n = 20,$

$r_1 = 2,\ s_1 = 7,\ r_2 = 11,\ s_2 = 7,$

$t_2 = 2,$

$A = s_1 + s_2 = 14,$

$$\gamma_2 = \left[\frac{0.4286 \times 0.5714}{0.5714 - 0.4286}\right]^2 = 2.9412,$$

$$\delta_2^* = 0.2449 \left\{\ln\left[\frac{0.5714}{0.4286}\right] + \frac{1}{0.5714-0.4286}\right\} + 2.9412 \left\{\ln\left[\frac{0.5714}{0.4286}\right] - \ln\left[\frac{0.4286}{0.5714}\right]\right\}$$

$$= 3.4770,$$

$$\delta_2^{**} = 0.2449 \left\{\frac{1}{0.5714-0.4286} - \ln\left[\frac{0.5714}{0.4286}\right]\right\} + 2.9412\left\{\ln\left[\frac{0.5714}{0.4286}\right] - \ln\left[\frac{0.4286}{0.5714}\right]\right\}$$

$$= 3.4770,$$

$\delta_2 = \overset{*}{\delta_2} - \overset{**}{\delta_2} = 0$,

m = 6.93,

B = 1053.5435/6.93 = 152.0265,

C = 0,

D = –39.4441,

E = 1490.0910,

and hence

$$\hat{\sigma} = \frac{\pi}{\sqrt{3}} \left\{ \frac{-D + (D^2+4AE)^{1/2}}{2A} \right\} = 21.4413$$

and

$$\hat{\mu} = B - \frac{\sqrt{3}}{\pi} \hat{\sigma} C = B = 152.0265.$$

Also, from Eqs. (5.5) and (5.6) we have

$$V_1 = 0 \text{ and } V_2 = 4.21153$$

so that we have the approximate standard errors of the estimates $\hat{\mu}$ and $\hat{\sigma}$ to be

$$SE(\hat{\mu}) = \frac{\sqrt{3}}{\pi} \frac{\hat{\sigma}}{\sqrt{m}} = 4.4905$$

and

$$SE(\hat{\sigma}) = \frac{\hat{\sigma}}{(mV_2)^{1/2}} = 3.9688.$$

By applying the asymptotic normality of the estimators $\hat{\mu}$ and $\hat{\sigma}$ (see Theorem 1), we get approximate 95% confidence intervals for μ and σ to be

[152.0265 – 1.96(4.4905), 152.0265 + 1.96(4.4905)] = [143.2251, 160.8279] and

[21.4413 – 1.96(3.9688), 21.4413 + 1.96(3.9688)] = [13.6625, 29.2201], respectively.

By using the results presented in Section 2 and the tables of means, variances and covariances of logistic order statistics prepared recently by Balakrishnan and Malik (1991), we find the best linear unbiased estimates of μ and σ to be

$$
\begin{aligned}
\overset{*}{\mu} &= 0.0581(128.887) + 0.0431(132.585) + 0.0524(133.196) + 0.0602(140.734) \\
&\quad + 0.0664(141.816) + 0.0711(146.864) + 0.1488(148.350) + 0.1488(154.671) \\
&\quad + 0.0711(159.188) + 0.0664(163.117) + 0.0602(166.252) + 0.0524(166.770) \\
&\quad + 0.0431(172.017) + 0.0581(174.744) \\
&= 152.0655
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{*}{\sigma} &= -\,0.2741(128.887) - 0.0899(132.585) - 0.0808(133.196) - 0.0691(140.734) \\
&\quad - 0.0555(141.816) - 0.0406(146.864) - 0.0274(148.350) + 0.0274(154.671) \\
&\quad + 0.0406(159.188) + 0.0555(163.117) + 0.0691(166.252) + 0.0808(166.770) \\
&\quad + 0.0899(172.017) + 0.2741(174.744) \\
&= 22.4462
\end{aligned}
$$

and the standard errors of the estimates $\overset{*}{\mu}$ and $\overset{*}{\sigma}$ to be

$$\mathrm{SE}(\overset{*}{\mu}) = \overset{*}{\sigma}(0.0465)^{1/2} = 22.4462(0.0465)^{1/2} = 4.8403$$

and

$$\mathrm{SE}(\overset{*}{\sigma}) = \overset{*}{\sigma}(0.0457)^{1/2} = 22.4462(0.0457)^{1/2} = 4.7984.$$

Now by using the asymptotic normality of the estimators $\overset{*}{\mu}$ and $\overset{*}{\sigma}$ (since they are linear functions of order statistics), we obtain approximate 95% confidence intervals for μ and σ to be

$[152.0655 - 1.96(4.8403),\ 152.0655 + 1.96(4.8403)] = [142.5785,\ 161.5525]$ and

$[22.4462 - 1.96(4.7984),\ 22.4462 + 1.96(4.7984)] = [13.0413,\ 31.8511]$, respectively.

It is of interest to mention here that by assuming this multiply Type–II censored sample to have come from a normal population, Balakrishnan, Gupta and Panchapakesan (1991) computed the approximate maximum likelihood estimates of the mean μ and the standard deviation σ to be 151.9806 and 19.4392, respectively, and the best linear unbiased estimates of μ and σ to be 151.9804 and 20.7525, respectively. Balakrishnan, Gupta and Panchapakesan (1991) also worked out approximate 95% confidence intervals for μ and σ based on the approximate maximum likelihood estimators to be [143.2868, 160.6744] and [12.5543, 26.3241], respectively, and the approximate 95% confidence intervals for μ and σ based on the best linear unbiased estimators to be [142.7051, 161.2557] and [12.8235, 28.6815], respectively. The results obtained in this paper by assuming the multiply Type–II censored sample to have come from a logistic population are seen to be close to the results obtained by Balakrishnan, Gupta & Panchapakesan (1991) under the assumption of a normal distribution for the given multiply Type–II censored sample.

Furthermore, upon comparing the results obtained by the two methods, we observe that the best linear unbiased estimates of μ and σ are numerically close to the approximate maximum likelihood estimates of μ and σ. However, the best linear unbiased estimates have slightly larger standard errors than the corresponding approximate maximum likelihood estimates and, therefore, the confidence intervals based on the best linear unbiased estimates turn out to be slightly wider than the corresponding confidence intervals based on the approximate maximum likelihood estimates.

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