ESTIMATION OF THE LOCATION AND SCALE PARAMETERS OF THE EXTREME VALUE DISTRIBUTION BASED ON MULTIPLY TYPE-II CENSORED SAMPLES*

by

N. Balakrishnan
McMaster University
Hamilton, Ontario CANADA

Shanti S. Gupta
Purdue University
West Lafayette, IN 47907 USA

and

S. Panchapakesan
Southern Illinois University
Carbondale, IL, USA

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Estimation of the location and scale parameters of the extreme value distribution based on multiply Type–II censored samples

N. Balakrishnan*, Shanti S. Gupta** and S. Panchapakesan+

*Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada

**Department of Statistics, Purdue University, West Lafayette, Indiana, U.S.A.

+Department of Mathematics, Southern Illinois University, Carbondale, Illinois, U.S.A.

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Abstract

In this paper, we consider the problem of estimating the location and scale parameters of an extreme value distribution based on multiply Type–II censored samples. We first describe the best linear unbiased estimators and the maximum likelihood estimators of these parameters. After observing that the best linear unbiased estimators need the construction of some tables for its coefficients and that the maximum likelihood estimators do not exist in an explicit algebraic form and hence need to be found by numerical methods, we develop approximate maximum likelihood estimators by appropriately approximating the likelihood equations. In addition to being simple explicit estimators, these estimators turn out to be nearly as efficient as the best linear unbiased estimators and the maximum likelihood estimators. Next, we derive the asymptotic

variances and covariance of these estimators in terms of the first two single moments and the product moments of order statistics from the standard extreme value distribution. Finally, we present an example in order to illustrate all the methods of estimation of parameters discussed in this paper.

1. Introduction

Consider the extreme value distribution with probability density function

$$g(y;\mu,\sigma) = \frac{1}{\sigma}e^{(y-\mu)/\sigma}e^{-e^{(y-\mu)/\sigma}}, -\infty < y < \infty, -\infty < \mu < \infty, \sigma > 0, \qquad (1.1)$$

and cumulative distribution function

$$G(y;\mu,\sigma) = 1 - e^{-e^{(y-\mu)/\sigma}}, -\infty < y < \infty, -\infty < \mu < \infty, \sigma > 0. \qquad (1.2)$$

Order statistics from the above given extreme value distribution and their moments have been studied by various authors including Lieblein and Zelen (1956), Lieblein and Salzer (1957), White (1969), and Balakrishnan and Chan (1991). Of these, Lieblein and Salzer (1957) and White (1969) give the means and variances of order statistics, respectively, for sample sizes up to 100. While Lieblein and Zelen (1956) give the covariances for sample sizes up to 6, Balakrishnan and Chan (1991) have recently computed the means, variances and covariances of all order statistics for sample sizes up to 30.

By using Lloyd's (1952) least–squares theory on the linear estimation of the location and scale parameters (for details, see David (1981) or Balakrishnan and Cohen (1990)), Lieblein and Zelen (1956) tabulated the best linear unbiased estimators of μ and σ for very small sample sizes. Recently, Balakrishnan and Chan (1991) have tabulated the best linear unbiased estimators of μ and σ by considering complete as well as Type–II censored samples for sample sizes up to 30. Mann (1967a) derived the best linear invariant estimators of μ and σ and the necessary tables for the calculation of these estimators for sample sizes up to 25 (based on complete and Type–II right–censored samples only) have been prepared by Mann (1967a,b, 1968) and Mann, Schafer and Singpurwalla (1974, pp. 194 – 207). D'Agostino (1971) considered some approximations to the best linear unbiased estimators as well as the best linear invariant estimators. Mann and Fertig (1977) and Engelhardt and Bain (1977) have made surveys of several such linear estimators that have been proposed in the literature and require fewer tables than the best linear unbiased estimators or the best linear invariant estimators. Linear estimation of the parameters μ and σ based on optimally selected order statistics and some associated inference problems have been studied in great detail by Chan and Kabir (1969), Chan and Mead (1971), Hassanein (1968, 1969, 1972), and Hassanein, Saleh and Brown (1984, 1986).

Harter and Moore (1968) proposed an iterative procedure for obtaining the maximum likelihood estimates of μ and σ based on complete and doubly Type–II censored samples. They examined the bias, variances, covariance and conditional bias of these estimates by using Monte Carlo simulations. These results are reproduced by Harter (1970). Recently, by appropriately approximating the likelihood equations for μ and σ based on a general doubly Type–II censored sample, Balakrishnan and Varadan (1991) derived the approximate maximum likelihood estimators of μ and σ.

Let us assume that the following multiply Type–II censored sample from a sample of size n

$$Y_{r_1+1:n} \le \cdots \le Y_{r_1+s_1:n} \le Y_{r_2+1:n} \le \cdots \le Y_{r_2+s_2:n} \le \cdots \le Y_{r_k+1:n} \le \cdots \le Y_{r_k+s_k:n} \tag{1.3}$$

is available from the extreme value distribution in (1.2). In Section 2, we describe the best linear unbiased estimation of the parameters μ and σ based on the multiply Type–II censored sample in (1.3). In Section 3, we present the maximum likelihood estimation of the parameters μ and σ based on the multiply Type–II censored sample in (1.3) and note that these estimators do not exist in an explicit algebraic form and that they need to be determined by numerically solving the two likelihood equations simultaneously. By appropriately approximating these two likelihood equations, we derive in Section 4 the approximate maximum likelihood estimators of μ and σ based on the multiply Type–II censored sample in (1.3). These estimators are simple explicit estimators which turn out to be nearly as efficient as the best linear unbiased estimators and the maximum likelihood estimators. In Section 5, we derive the asymptotic variances and covariance of these estimators in terms of the first two single moments and the product moments of order statistics from the standardized extreme value distribution. In Section 6, we take an example from a life–testing experiment considered earlier by Mann and Fertig (1973) and Lawless (1982) and illustrate all the methods of estimation of parameters μ and σ discussed in this paper. Similar work for the normal and the logistic populations have been recently carried out by Balakrishnan, Gupta and Panchapakesan (1991 a,b).

The Weibull distribution with probability density function

$$e^{-x^a/b^a}\, a\,\frac{x^{a-1}}{b^a}, \qquad x \geq 0,\ a > 0,\ b > 0$$

and cumulative distribution function

$$1 - e^{-x^a/b^a}, \qquad x \geq 0,\ a > 0,\ b > 0$$

is used extensively as a failure–time model. It is very easy to see that the random variable $Y = \ln X$ has its density function to be

$$e^{-e^{a(y-\ln b)}}\, a e^{a(y-\ln b)}, \quad -\infty < y < \infty,\ a > 0,\ b > 0$$

and its cumulative distribution function to be

$$1 - e^{-e^{a(y-\ln b)}}, \qquad -\infty < y < \infty,\ a > 0,\ b > 0.$$

Upon comparing these expressions with Eqs. (1.1) and (1.2), we simply observe that the variable Y defined above has the extreme value distribution in (1.2) with the location parameter $\mu = \ln b$ and the scale parameter $\sigma = 1/a$. Thus, the methods of estimation discussed in this paper may very well be applied to estimate the parameters a and b of the above given Weibull distribution.

2. Best Linear Unbiased Estimation

Let $X_{i:n} = (Y_{i:n} - \mu)/\sigma$, $i = r_1 + 1, \ldots, r_1 + s_1, r_2 + 1, \ldots, r_2 + s_2, \ldots, r_k + 1, \ldots, r_k + s_k$. Then, $X_{i:n}$ are simply order statistics from a sample of size n from a standard extreme value population with probability density function

$$f(x) = e^x \, e^{-e^x}, \; -\infty < x < \infty, \tag{2.1}$$

and cumulative distribution function

$$F(x) = 1 - e^{-e^x}, \; -\infty < x < \infty. \tag{2.2}$$

Let us denote $E(X_{i:n})$ by $\alpha^*_{i:n}$, $E(X^2_{i:n})$ by $\alpha^{*(2)}_{i:n}$, $Var(X_{i:n})$ by $\beta^*_{i,i:n}$, $E(X_{i:n}X_{j:n})$ by $\alpha^*_{i,j:n}$, and $Cov(X_{i:n}, X_{j:n})$ by $\beta^*_{i,j:n}$. Then, we immediately have $E(Y_{i:n}) = \mu + \sigma \, \alpha^*_{i:n}$, $Var(Y_{i:n}) = \sigma^2 \, \beta^*_{i,i:n}$, and $Cov(Y_{i:n}, Y_{j:n}) = \sigma^2 \, \beta^*_{i,j:n}$. Further, let us denote

$$\underset{\sim}{Y} = \left[Y_{r_1+1:n} \cdots Y_{r_1+s_1:n} \; Y_{r_2+1:n} \cdots Y_{r_2+s_2:n} \cdots Y_{r_k+1:n} \cdots Y_{r_k+s_k:n}\right]^T,$$

$$\underset{\sim}{\alpha} = \left[\alpha^*_{r_1+1:n} \cdots \alpha^*_{r_1+s_1:n} \; \alpha^*_{r_2+1:n} \cdots \alpha^*_{r_2+s_2:n} \cdots \alpha^*_{r_k+1:n} \cdots \alpha^*_{r_k+s_k:n}\right]^T,$$

$$\underset{\sim}{1} = \left[1 \;\; 1 \; \ldots . \; 1\right]^T_{\sum_1^k s_i \times 1},$$

$$\underset{\sim}{\beta} = \left[\left[\beta^*_{i,j:n}\right]\right] \text{ for } i, j \in I \text{ where } I = \left\{r_1 + 1, \ldots, r_1 + s_1, r_2 + 1, \ldots, r_2 + s_2, \ldots, r_k + 1, \ldots, r_k + s_k\right\},$$

and

$$\underset{\sim}{\Omega} = \underset{\sim}{\beta}^{-1}.$$

Then, the Best Linear Unbiased Estimators of μ and σ based on the multiply Type–II censored sample in (1.3) derived by minimizing the generalized variance

$$\left[\underset{\sim}{Y} - \mu\underset{\sim}{1} - \sigma\underset{\sim}{\alpha}\right]^T \underset{\sim}{\Omega} \left[\underset{\sim}{Y} - \mu\,\underset{\sim}{1} - \sigma\,\underset{\sim}{\alpha}\right] \tag{2.3}$$

are given by (see David, 1981; Balakrishnan and Cohen, 1990)

$$\mu^* = \left\{\frac{\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha}\ \underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ -\ \underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}\ \underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1})^2}\right\} \underset{\sim}{Y}$$

$$= - \underset{\sim}{\alpha}^T \underset{\sim}{\Delta}\, \underset{\sim}{Y}$$

$$= \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} a_j\, Y_{j:n} \tag{2.4}$$

and

$$\sigma^* = \left\{\frac{\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}\ \underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ -\ \underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha}\ \underset{\sim}{1}^T\ \underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1})^2}\right\} \underset{\sim}{Y}$$

$$= \underset{\sim}{1}^T \underset{\sim}{\Delta}\, \underset{\sim}{Y}$$

$$= \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} b_j\, Y_{j:n}, \tag{2.5}$$

where $\underset{\sim}{\Delta}$ is a skew–symmetric matrix of order $\sum_{i=1}^{k} s_i$ given by

$$\underset{\sim}{\Delta} = \frac{\underset{\sim}{\Omega}(\underset{\sim}{1}\ \underset{\sim}{\alpha}^T - \underset{\sim}{\alpha}\ \underset{\sim}{1}^T)\ \underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1})^2}. \tag{2.6}$$

The variances and covariance of the estimators μ^* and σ^* in (2.4) and (2.5), respectively, are given by

$$\mathrm{Var}(\mu^*) = \sigma^2\left\{\frac{\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha}}{(\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1})^2}\right\}, \tag{2.7}$$

$$\mathrm{Var}(\sigma^*) = \sigma^2\left\{\frac{\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}}{(\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1})^2}\right\}, \tag{2.8}$$

and

$$\mathrm{Cov}(\mu^*, \sigma^*) = -\sigma^2\left\{\frac{\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}}{(\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{\alpha})\ (\underset{\sim}{1}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T\ \underset{\sim}{\Omega}\ \underset{\sim}{1})^2}\right\}. \tag{2.9}$$

By using the values of means, variances and covariances of order statistics from the standard extreme value distribution tabulated by Lieblein and Salzer (1957), White (1969), and Balakrishnan and Chan (1991), we may compute the coefficients a_j and b_j in Eqs. (2.4) and (2.5) and also the variances and covariance of the best linear unbiased estimators μ^* and σ^* from Eqs. (2.7) – (2.9). For large sample sizes, we may determine these quantities approximately by making use of the approximate expressions of means, variances and covariances of order statistics from the standard extreme value distribution derived by David and Johnson's (1954) method; for example, refer to David (1981) or Arnold and Balakrishnan (1989).

3. Maximum Likelihood Estimation

The likelihood function based on the multiply Type–II censored sample in (1.3) can be written as

$$L = \frac{n!}{\prod_{i=1}^{k+1} (r_i - r_{i-1} - s_{i-1})! \, \sigma^{\sum_1^k s_i}} \left\{F\left[X_{r_1+1:n}\right]\right\}^{r_1} \left\{1 - F\left[X_{r_k+s_k:n}\right]\right\}^{n-r_k-s_k}$$

$$\times \prod_{i=2}^{k} \left\{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]\right\}^{r_i-r_{i-1}-s_{i-1}} \prod_{i=1}^{k} \prod_{j=r_i+1}^{r_i+s_i} f(X_{j:n}), \tag{3.1}$$

where as before, $X_{i:n} = (Y_{i:n} - \mu)/\sigma$, f(x) and F(x) are the density function and the distribution function of the standard extreme value population as given in Eqs. (2.1) and (2.2), respectively, and $r_o = s_o = 0$ and $r_{k+1} = n$. From Eq. (3.1), we have the log–likelihood function to be

$$\ln L = \text{Const} - A \ln \sigma + r_1 \ln\left\{F\left[X_{r_1+1:n}\right]\right\} + \left[n - r_k - s_k\right] \ln\left\{1 - F\left[X_{r_k+s_k:n}\right]\right\}$$

$$+ \sum_{i=2}^{k} t_i \ln \left\{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]\right\} + \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \ln f(X_{j:n}), \tag{3.2}$$

where $t_i = r_i - r_{i-1} - s_{i-1}$ for $i = 2,3,\ldots,k$, and $A = \sum_{i=1}^{k} s_i$ is the size of the available multiply Type–II censored sample. From Eq. (3.2), we obtain the likelihood equations for μ and σ to be

$$\frac{\partial \ell n L}{\partial \mu} = -\frac{1}{\sigma}\Bigg[r_1 \frac{f(X_{r_1+1:n})}{F(X_{r_1+1:n})} - (n - r_k - s_k)\frac{f(X_{r_k+s_k:n})}{1-F(X_{r_k+s_k:n})}$$
$$+ \sum_{i=2}^{k} t_i \frac{f(X_{r_i+1:n}) - f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}$$
$$+ \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \frac{f'(X_{j:n})}{f(X_{j:n})}\Bigg]$$
$$= 0, \qquad (3.3)$$

and

$$\frac{\partial \ell n L}{\partial \sigma} = -\frac{1}{\sigma}\Bigg[A + r_1 X_{r_1+1:n}\frac{f(X_{r_1+1:n})}{F(X_{r_1+1:n})} - \left[n-r_k-s_k\right] X_{r_k+s_k:n}\frac{f(X_{r_k+s_k:n})}{1-F(X_{r_k+s_k:n})}$$
$$+ \sum_{i=2}^{k} t_i \frac{X_{r_i+1:n} f(X_{r_i+1:n}) - X_{r_{i-1}+s_{i-1}:n} f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}$$
$$+ \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} X_{j:n}\frac{f'(X_{j:n})}{f(X_{j:n})}\Bigg]$$
$$= 0, \qquad (3.4)$$

where $f'(x)$ denotes the derivative of the function $f(x)$ given in (2.1). Eqs. (3.3) and (3.4) cannot be solved explicitly. But, they may be solved by using numerical methods in order to determine the maximum likelihood estimates of μ and σ.

4. Approximate Maximum Likelihood Estimation

Following the notations of Balakrishnan, Gupta and Panchapakesan (1991a,b), let us set $p_i = i/(n+1)$, $q_i = 1 - p_i$, and $\xi_i = F^{-1}(p_i) = \ell n(-\ell n\, q_i)$. Let

$$h_1(X_{r_1+1:n}) = \frac{f(X_{r_1+1:n})}{F(X_{r_1+1:n})} = \frac{e^{X_{r_1+1:n}} e^{-e^{X_{r_1+1:n}}}}{1-e^{-e^{X_{r_1+1:n}}}}, \tag{4.1}$$

$$h_2(X_{j:n}) = \frac{f'(X_{j:n})}{f(X_{j:n})} = 1 - e^{X_{j:n}}, \tag{4.2}$$

and

$$h_3\left[X_{r_k+s_k:n}\right] = \frac{f(X_{r_k+s_k:n})}{1-F(X_{r_k+s_k:n})} = e^{X_{r_k+s_k:n}}. \tag{4.3}$$

By expanding the functions $h_1(X_{r_1+1:n})$, $h_2(X_{j:n})$ and $h_3(X_{r_k+s_k:n})$ in (4.1), (4.2) and (4.3) around the points ξ_{r_1+1}, ξ_j and $\xi_{r_k+s_k}$, respectively, in Taylor series (see David (1981) or Arnold and Balakrishnan (1989) for reasoning) we may approximate them by

$$h_1\left[X_{r_1+1:n}\right] = \frac{f(X_{r_1+1:n})}{F(X_{r_1+1:n})} \simeq \gamma - \delta X_{r_1+1:n}, \tag{4.4}$$

$$h_2\left[X_{j:n}\right] = \frac{f'(X_{j:n})}{f(X_{j:n})} \simeq \alpha_j - \beta_j X_{j:n}, \tag{4.5}$$

and

$$h_3(X_{r_k+s_k:n}) = \frac{f(X_{r_k+s_k:n})}{1-F(X_{r_k+s_k:n})} \simeq 1 - \alpha_{r_k+s_k} + \beta_{r_k+s_k} X_{r_k+s_k:n}, \tag{4.6}$$

where

$$\gamma = -\frac{q_{r_1+1}}{p_{r_1+1}} \ell n\, q_{r_1+1} \left\{1 - \ell n(-\ell n\, q_{r_1+1})\right\} + \frac{q_{r_1+1}}{p^2_{r_1+1}} \left[\ell n\, q_{r_1+1}\right]^2 \ell n \left[-\ell n\, q_{r_1+1}\right], \tag{4.7}$$

$$\delta = \frac{q_{r_1+1}}{p_{r_1+1}} \ell n\, q_{r_1+1} \left\{1 + \frac{1}{p_{r_1+1}} \ell n\, q_{r_1+1}\right\}, \tag{4.8}$$

$$\alpha_j = 1 + \ell n\, q_j \left\{1 - \ell n\left[-\ell n\, q_j\right]\right\}, \tag{4.9}$$

and

$$\beta_j = -\ell n\, q_j. \tag{4.10}$$

From Eq. (4.10), it is clear that $\beta_j > 0$. Also, from Eq. (4.8) upon using the fact that $\ell n(1-z) < -z$ for $0 < z < 1$, we easily observe that $\delta > 0$.

Now let

$$k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] = \frac{f(X_{r_i+1:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})} \tag{4.11}$$

and

$$k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] = \frac{f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}. \tag{4.12}$$

Upon expanding the functions $k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$ and $k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$ in (4.11) and (4.12) around the point $\left[\xi_{r_{i-1}+s_{i-1}}, \xi_{r_i+1}\right]$ in bivariate Taylor series, we may approximate these functions by

$$k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \simeq \eta_{oi} + \eta_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \eta_{2i}\, X_{r_i+1:n} \tag{4.13}$$

and

$$k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \simeq \eta^*_{oi} + \eta^*_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \eta^*_{2i}\, X_{r_i+1:n}, \tag{4.14}$$

where

$$\eta_{1i} = \frac{q_{r_{i-1}+s_{i-1}}\, q_{r_i+1}\, \ell n\, q_{r_{i-1}+s_{i-1}}\, \ell n\, q_{r_i+1}}{(p_{r_i+1} - p_{r_{i-1}+s_{i-1}})^2}, \tag{4.15}$$

$$\eta_{2i} = \frac{q_{r_i+1}\, \ell n\, q_{r_i+1}}{(p_{r_i+1} - p_{r_{i-1}+s_{i-1}})^2} \left\{\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] + \left[1 - p_{r_{i-1}+s_{i-1}}\right] \ell n\, q_{r_i+1}\right\}, \tag{4.16}$$

$$\eta_{oi} = -\frac{q_{r_i+1}\, \ell n\, q_{r_{i+1}}}{p_{r_i+1} - p_{r_{i-1}+s_{i-1}}} - \eta_{1i}\, \ell n\left[-\ell n\, q_{r_{i-1}+s_{i-1}}\right] + \eta_{2i}\, \ell n\left[-\ell n\, q_{r_i+1}\right], \tag{4.17}$$

$$\overset{*}{\eta}_{1i} = -\frac{q_{r_{i-1}+s_{i-1}} \ \ln q_{r_{i-1}+s_{i-1}}}{(p_{r_i+1} - p_{r_{i-1}+s_{i-1}})^2} \left\{ \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] - \left[1 - p_{r_i+1}\right] \ln q_{r_{i-1}+s_{i-1}} \right\}, \tag{4.18}$$

$$\overset{*}{\eta}_{2i} = \eta_{1i} = \frac{q_{r_{i-1}+s_{i-1}} \ q_{r_i+1} \ \ln q_{r_{i-1}+s_{i-1}} \ \ln q_{r_i+1}}{(p_{r_i+1} - p_{r_{i-1}+s_{i-1}})^2}, \tag{4.19}$$

and

$$\overset{*}{\eta}_{oi} = -\frac{q_{r_{i-1}+s_{i-1}} \ln q_{r_{i-1}+s_{i-1}}}{p_{r_i+1} - p_{r_{i-1}+s_{i-1}}} - \overset{*}{\eta}_{1i} \ln\left[-\ln q_{r_{i-1}+s_{i-1}}\right] + \overset{*}{\eta}_{2i} \ln(-\ln q_{r_i+1}). \tag{4.20}$$

It is readily seen from Eqs. (4.15) and (4.19) that $\eta_{1i} = \overset{*}{\eta}_{2i}$ is positive. In order to see from Eq. (4.16) that η_{2i} is positive, we simply have to note that $\ln q_{r_i+1} < -p_{r_i+1}$ and consequently

$$\begin{aligned} &\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] + \left[1 - p_{r_{i-1}+s_{i-1}}\right] \ln q_{r_i+1} \\ &\quad < p_{r_i+1} - p_{r_{i-1}+s_{i-1}} - p_{r_i+1} + p_{r_{i-1}+s_{i-1}} \, p_{r_i+1} \\ &\quad = -p_{r_{i-1}+s_{i-1}} \, q_{r_i+1} \\ &\quad < 0. \end{aligned}$$

Similarly, in order to note from Eq. (4.18) that $\overset{*}{\eta}_{1i}$ is positive, we simply have to use the fact that $\ln q_{r_{i-1}+s_{i-1}} < -p_{r_{i-1}+s_{i-1}}$ and consequently

$$\begin{aligned} &\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] - \left[1 - p_{r_i+1}\right] \ln q_{r_{i-1}+s_{i-1}} \\ &\quad > p_{r_i+1} - p_{r_{i-1}+s_{i-1}} + p_{r_{i-1}+s_{i-1}} - p_{r_{i-1}+s_{i-1}} \, p_{r_i+1} \\ &\quad = p_{r_i+1} \, q_{r_{i-1}+s_{i-1}} \\ &\quad > 0. \end{aligned}$$

By making use of the approximations in Eqs. (4.13) and (4.14), we obtain

$$k\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] = k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] - k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$$

$$= \frac{f(X_{r_i+1:n}) - f(X_{r_{i-1}+s_{i-1}:n})}{F(X_{r_i+1:n}) - F(X_{r_{i-1}+s_{i-1}:n})}$$

$$\simeq \zeta_{oi} - \zeta_{1i} X_{r_{i-1}+s_{i-1}:n} - \zeta_{2i} X_{r_i+1:n}, \tag{4.21}$$

where

$$\zeta_{1i} = \zeta_{1i}^{*} - \zeta_{1i}$$

$$= - \frac{q_{r_{i-1}+s_{i-1}} \ln q_{r_{i-1}+s_{i-1}}}{(p_{r_i+1} - p_{r_{i-1}+s_{i-1}})^2} \left\{ \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}} \right] + q_{r_i+1} \left[\ln q_{r_i+1} - \ln q_{r_{i-1}+s_{i-1}} \right] \right\}, \tag{4.22}$$

$$\zeta_{2i} = \zeta_{2i} - \zeta_{2i}^{*}$$

$$= - \frac{q_{r_i+1} \ln q_{r_i+1}}{(p_{r_i+1} - p_{r_{i-1}+s_{i-1}})^2} \left\{ q_{r_{i-1}+s_{i-1}} \left[\ln q_{r_{i-1}+s_{i-1}} - \ln q_{r_i+1} \right] - \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}} \right] \right\}, \tag{4.23}$$

and

$$\zeta_{oi} = \zeta_{oi} - \zeta_{oi}^{*}$$

$$= \frac{q_{r_{i-1}+s_{i-1}} \ln q_{r_{i-1}+s_{i-1}} - q_{r_i+1} \ln q_{r_i+1}}{p_{r_i+1} - p_{r_{i-1}+s_{i-1}}} + \zeta_{1i} \ln(-\ln q_{r_{i-1}+s_{i-1}}) + \zeta_{2i} \ln(-\ln q_{r_i+1}). \tag{4.24}$$

Now, by first defining the function

$$\ell_1(x,y) = (y - x) - (1 - y) \ln(1 - x) + (1 - y) \ln(1 - y), \; y \geq x,$$

and noting that $\ell_1(x,x) \equiv 0$ and that $\ell_1(x,y)$ is monotonically increasing in y since $\frac{\partial}{\partial y} \ell_1(x,y) = \ln \left[\frac{1-x}{1-y}\right] > 0$, we have

$$\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}} \right] + q_{r_i+1} \left[\ln q_{r_i+1} - \ln q_{r_{i-1}+s_{i-1}} \right]$$
$$= \ell_1 \left[p_{r_{i-1}+s_{i-1}}, p_{r_i+1} \right] > 0$$

which immediately implies from Eq. (4.22) that $\zeta_{1i} > 0$. Similarly, by defining the function

$$\ell_2(x,y) = (1-x)\,\ell n(1-x) - (1-x)\,\ell n(1-y) - y + x, \quad y \geq x,$$

and noting that $\ell_2(x,x) \equiv 0$ and that $\ell_2(x,y)$ is monotonically increasing in y since $\frac{\partial}{\partial y}$ $\ell_2(x,y) = (y-x)/(1-y) > 0$, we have

$$q_{r_{i-1}+s_{i-1}} \left[\ell n\, q_{r_{i-1}+s_{i-1}} - \ell n\, q_{r_i+1}\right] - \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]$$
$$= \ell_2(p_{r_{i-1}+s_{i-1}},\, p_{r_i+1}) > 0$$

which readily implies from Eq. (4.23) that $\zeta_{2i} > 0$.

Now, upon using the approximations in Eqs. (4.4), (4.5), (4.6) and (4.21) into the likelihood equation for μ in (3.3), we get the approximate likelihood equation for μ to be

$$r_1\left[\gamma - \delta X_{r_1+1:n}\right] - \left[n - r_k - s_k\right]\left[1 - \alpha_{r_k+s_k} + \beta_{r_k+s_k} X_{r_k+s_k:n}\right]$$
$$+ \sum_{i=2}^{k} t_i\left[\zeta_{oi} - \zeta_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \zeta_{2i}\, X_{r_i+1:n}\right]$$
$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \left[\alpha_j - \beta_j\, X_{j:n}\right] = 0. \tag{4.25}$$

Eq. (4.25), when solved for μ, yields the approximate maximum likelihood estimator of μ to be

$$\hat{\mu} = B - \sigma\, C, \tag{4.26}$$

where

$$t_i = r_i - r_{i-1} - s_{i-1},\ i = 2,3,\ldots,k,$$

$$m = r_1\delta + \left[n - r_k - s_k\right]\beta_{r_k+s_k} + \sum_{i=2}^{k} t_i\left[\zeta_{1i} + \zeta_{2i}\right] + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \beta_j,$$

$$B = \frac{1}{m}\Big\{r_1\delta\, Y_{r_1+1:n} + \left[n - r_k - s_k\right]\beta_{r_k+s_k}\, Y_{r_k+s_k:n}$$
$$+ \sum_{i=2}^{k} t_i\,\zeta_{1i}\, Y_{r_{i-1}+s_{i-1}:n} + \sum_{i=2}^{k} t_i\,\zeta_{2i}\, Y_{r_i+1:n} + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \beta_j Y_{j:n}\Big\},$$

and

$$C = \frac{1}{m}\left\{r_1\gamma - \left[n - r_k - s_k\right]\left[1 - \alpha_{r_k+s_k}\right] + \sum_{i=2}^{k} t_i\,\zeta_{oi} + \sum_{i=1}^{k}\ \sum_{j=r_i+1}^{r_i+s_i} \alpha_j\right\}. \tag{4.27}$$

Next, upon using the approximations in Eqs. (4.4), (4.5), (4.6), (4.13) and (4.14) into the likelihood equation for σ in (3.4), we get the approximate likelihood equation for σ to be

$$\begin{aligned} A &+ r_1 X_{r_1+1:n}\left[\gamma - \delta\, X_{r_1+1:n}\right] - \left[n - r_k - s_k\right] X_{r_k+s_k:n}\left[1 - \alpha_{r_k+s_k} + \beta_{r_k+s_k} X_{r_k+s_k:n}\right] \\ &+ \sum_{i=2}^{k} t_i\ X_{r_i+1:n}\left[\eta_{oi} + \eta_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \eta_{2i}\, X_{r_i+1:n}\right] \\ &- \sum_{i=2}^{k} t_i\, X_{r_{i-1}+s_{i-1}:n}\left[\overset{*}{\eta}_{oi} + \overset{*}{\eta}_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \eta_{1i}\, X_{r_i+1:n}\right] \\ &+ \sum_{i=1}^{k}\ \sum_{j=r_i+1}^{r_i+s_i} X_{j:n}\left(\alpha_j - \beta_j\, X_{j:n}\right) = 0. \end{aligned} \tag{4.28}$$

Eq. (4.28), when solved for σ simultaneously by using the solution for μ in (4.26), yields the approximate maximum likelihood estimator of σ to be

$$\hat{\sigma} = \left\{- D + \left(D^2 + 4AE\right)^{1/2}\right\} / 2A, \tag{4.29}$$

where

$A = \sum_{i=1}^{k} s_i$ is the size of the available sample,

$$\begin{aligned} D = {} & r_1\,\gamma\, Y_{r_1+1:n} - \left[n - r_k - s_k\right]\left[1 - \alpha_{r_k+s_k}\right] Y_{r_k+s_k:n} + \sum_{i=2}^{k} t_i\,\eta_{oi}\, Y_{r_i+1:n} \\ & - \sum_{i=2}^{k} t_i\,\overset{*}{\eta}_{oi}\, Y_{r_{i-1}+s_{i-1}:n} + \sum_{i=1}^{k}\ \sum_{j=r_i+1}^{r_i+s_i} \alpha_j\, Y_{j:n} - mBC, \end{aligned}$$

and

$$\begin{aligned} E = {} & r_1\delta\, Y^2_{r_1+1:n} + \left[n - r_k - s_k\right]\beta_{r_k+s_k}\, Y^2_{r_k+s_k:n} + \sum_{i=2}^{k} t_i\,\zeta_{1i}\, Y^2_{r_{i-1}+s_{i-1}:n} \\ & + \sum_{i=2}^{k} t_i\,\zeta_{2i}\, Y^2_{r_i+1:n} + \sum_{i=1}^{k}\ \sum_{j=r_i+1}^{r_i+s_i} \beta_j\, Y^2_{j:n} \\ & + \sum_{i=2}^{k} t_i\,\eta_{1i}\left[Y_{r_i+1:n} - Y_{r_{i-1}+s_{i-1}:n}\right]^2 - mB^2 \end{aligned}$$

$$= r_1\delta\left[Y_{r_1+1:n} - B\right]^2 + \left[n - r_k - s_k\right]\beta_{r_k+s_k}\left[Y_{r_k+s_k:n} - B\right]^2$$
$$+ \sum_{i=2}^{k} t_i\, \zeta_{1i}\left[Y_{r_{i-1}+s_{i-1}:n} - B\right]^2 + \sum_{i=2}^{k} t_i\, \zeta_{2i}\left[Y_{r_i+1:n} - B\right]^2$$
$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j\left[Y_{j:n} - B\right]^2 + \sum_{i=2}^{k} t_i\, \eta_{1i}\left[Y_{r_i+1:n} - Y_{r_{i-1}+s_{i-1}:n}\right]^2. \tag{4.30}$$

It needs to be mentioned here that upon solving Eq. (4.28) we get a quadratic equation in σ which has two roots; however, one of them becomes negative and hence inadmissible since δ, β_i, ζ_{1i}, ζ_{2i} and η_{1i} are all positive and consequently $E > 0$.

__Remark__: For the special case when the available sample is a doubly Type–II censored sample $Y_{r+1:n}, Y_{r+2:n}, \ldots, Y_{n-s:n}$, that is, when $r_1 = r$, $r_2 = r+1, \ldots, r_k = r + k - 1$, $s_1 = s_2 = \ldots = s_{k-1} = 1$ and $s_k = n - r - s - k + 1$, the estimators $\hat{\mu}$ and $\hat{\sigma}$ in Eqs. (4.26) and (4.29), respectively, simply reduce to the approximate maximum likelihood estimators of μ and σ derived by Balakrishnan and Varadan (1991).

5. Approximate Variances and Covariance of the Estimators

By using the linear approximations in (4.4), (4.5), (4.6), (4.13) and (4.14), we also derive from the likelihood equations for μ and σ in Eqs. (3.3) and (3.4) that

$$E\left[-\frac{\partial^2 \ell n\ L}{\partial \mu^2}\right] \simeq \frac{m}{\sigma^2}, \tag{5.1}$$

$$E\left[-\frac{\partial^2 \ell n\ L}{\partial \mu\, \partial \sigma}\right] \simeq \frac{m}{\sigma^2} V_1, \tag{5.2}$$

and

$$E\left[-\frac{\partial^2 \ell n\ L}{\partial \sigma^2}\right] \simeq \frac{m}{\sigma^2} V_2, \tag{5.3}$$

where, as earlier,

$$m = r_1\delta + \left[n - r_k - s_k\right] \beta_{r_k+s_k} + \sum_{i=2}^{k} t_i\left[\zeta_{1i} + \zeta_{2i}\right] + \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \beta_j, \tag{5.4}$$

$$V_1 = \frac{2}{m}\Big\{r_1\,\delta\,\alpha^*_{r_1+1:n} + (n - r_k - s_k)\,\beta_{r_k+s_k}\,\alpha^*_{r_k+s_k:n} + \sum_{i=2}^{k} t_i\,\zeta_{1i}\,\alpha^*_{r_{i-1}+s_{i-1}:n}$$
$$+ \sum_{i=2}^{k} t_i\,\zeta_{2i}\,\alpha^*_{r_i+1:n} + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \beta_j\,\alpha^*_{j:n}\Big\} - C, \tag{5.5}$$

and

$$V_2 = \frac{3}{m}\Big\{r_1\,\delta\,\alpha^{*(2)}_{r_1+1:n} + \Big[n - r_k - s_k\Big]\,\beta_{r_k+s_k}\,\alpha^{*(2)}_{r_k+s_k:n} + \sum_{i=2}^{k} t_i\,\eta^*_{1i}\,\alpha^{*(2)}_{r_{i-1}+s_{i-1}:n}$$
$$+ \sum_{i=2}^{k} t_i\,\eta_{2i}\,\alpha^{*(2)}_{r_i+1:n} + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \beta_j\,\alpha^{*(2)}_{j:n} - 2\sum_{i=2}^{k} t_i\,\eta_{1i}\,\alpha^*_{r_{i-1}+s_{i-1},r_i+1:n}\Big\}$$
$$-\frac{2}{m}\Big\{r_1\,\gamma\,\alpha^*_{r_1+1:n} - \Big[n - r_k - s_k\Big]\Big[1 - \alpha_{r_k+s_k}\Big]\,\alpha^*_{r_k+s_k:n} + \sum_{i=2}^{k} t_i\,\eta_{0i}\,\alpha^*_{r_i+1:n}$$
$$- \sum_{i=2}^{k} t_i\,\eta^*_{0i}\,\alpha^*_{r_{i-1}+s_{i-1}:n} + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} \alpha_j\,\alpha^*_{j:n}\Big\} - \frac{A}{m}. \tag{5.6}$$

From these expressions, we may compute

$$\mathrm{Var}(\hat{\mu}) \simeq \frac{\sigma^2}{m}\left\{\frac{V_2}{V_2 - V_1^2}\right\}, \tag{5.7}$$

$$\mathrm{Var}(\hat{\sigma}) \simeq \frac{\sigma^2}{m}\left\{\frac{1}{V_2 - V_1^2}\right\}, \tag{5.8}$$

and

$$\mathrm{Cov}(\hat{\mu}, \hat{\sigma}) \simeq -\frac{\sigma^2}{m}\left\{\frac{V_1}{V_2 - V_1^2}\right\}. \tag{5.9}$$

Approximate variances and covariance of the estimators $\hat{\mu}$ and $\hat{\sigma}$ may be determined from Eqs. (5.7) – (5.9) either by directly using the tabulated values of means, variances and covariances of order statistics from the standard extreme value distribution given by Lieblein and Salzer (1957), White (1969) and Balakrishnan and Chan (1991), or by using approximate expressions of these quantities presented by David (1981) or Arnold and Balakrishnan (1989).

Theorem: Asymptotically, $\hat{\mu}$ and $\hat{\sigma}$ jointly have a bivariate normal distribution with mean vector $\binom{\mu}{\sigma}$ and variance–covariance matrix

$$\frac{\sigma^2}{m(V_2-V_1^2)}\begin{bmatrix} V_2 & -V_1 \\ -V_1 & 1 \end{bmatrix},$$

where m, V_1 and V_2 are as given in Eqs. (5.4), (5.5) and (5.6), respectively.

For a proof of this theorem, one may refer to Kendall and Stuart (1973) or Rao (1975).

6. Illustrative Example

Let us consider the example of Mann and Fertig (1973) who give failure times of airplane components for a life test in which 13 components were placed on a test. Failure times in hours observed are as follows:

0.22, 0.50, 0.88, 1.00, 1.32, –, 1.54, 1.76, 2.50, 3.00, –, –, –

The failure time of the sixth component to fail was not observed due to experimental difficulties and, in addition the test terminated at the time of the tenth failure resulting in the censoring of the last three observations.

By assuming that the above data arose from a Weibull distribution, in order to obtain estimates of the parameters we transform the above data to the extreme value form by taking the logs of the observations, which are as follows:

–1.541, –0.693, –0.128, 0, 0.278, –, 0.432, 0.565, 0.916, 1.099, –, –, –

Now by assuming that the above given multiply Type–II censored sample has come from an extreme value population, we shall use the results developed in this paper to estimate the parameters μ and σ and also to obtain approximate confidence intervals for these parameters.

For the approximate maximum likelihood estimation of μ and σ, we have:

$n = 13,$

$r_1 = 0,\ s_1 = 5,\ r_2 = 6,\ s_2 = 4,$

$t_2 = 1,$

$A = s_1 + s_2 = 9,$

j	p_j	q_j	α_j	β_j
1	0.0714	0.9286	0.7331	0.0741
2	0.1429	0.8571	0.5575	0.1542
3	0.2143	0.7857	0.4158	0.2412
4	0.2857	0.7143	0.2971	0.3365
5	0.3571	0.6429	0.1973	0.4418
6	0.4286	0.5714	0.1155	0.5597
7	0.5000	0.5000	0.0528	0.6931
8	0.5714	0.4286	0.0123	0.8472
9	0.6429	0.3571	0.0004	1.0297
10	0.7143	0.2857	0.0296	1.2528

$$\eta_{12} = \frac{(0.6429)\,(0.5)\,\ln(0.6429)\,\ln(0.5)}{(0.5-0.3571)^2} = 4.8202,$$

$$\eta_{22} = \frac{0.5\,\ln(0.5)}{(0.5-0.3571)^2}\left\{\left[0.5-0.3571\right] + 0.6429\,\ln\left[0.5\right]\right\} = 5.1378,$$

$$\eta_{02} = -\frac{0.5\,\ln(0.5)}{(0.5-0.3571)} - \eta_{12}\,\ln\left[-\ln 0.6429\right] + \eta_{22}\,\ln\left[-\ln 0.5\right] = 4.4802,$$

$$\overset{*}{\eta}_{12} = -\frac{0.6429\,\ln(0.6429)}{(0.5-0.3571)^2}\left\{\left[0.5-0.3571\right] - 0.5\,\ln\left[0.6429\right]\right\} = 5.0596,$$

$$\overset{*}{\eta}_{22} = \eta_{12} = 4.8202,$$

$$\overset{*}{\eta}_{02} = -\frac{0.6429\,\ln(0.6429)}{0.5-0.3571} - \overset{*}{\eta}_{12}\,\ln(-\ln 0.6429) + \overset{*}{\eta}_{22}\,\ln\left[-\ln 0.5\right] = 4.3544,$$

$$\zeta_{12} = \overset{*}{\eta}_{12} - \eta_{12} = 5.0596 - 4.8202 = 0.2394,$$

$$\zeta_{22} = \eta_{22} - \overset{*}{\eta}_{22} = 5.1378 - 4.8202 = 0.3176,$$

$$\zeta_{02} = \eta_{02} - \overset{*}{\eta}_{02} = 4.4802 - 4.3544 = 0.1258,$$

$m = 9.3860,$

$B = 7.3033/9.3860 = 0.7781,$

$C = -0.4895/9.3860 = -0.0522,$

$D = -3.2568,$

$E = 2.1139,$

and hence

$$\hat{\sigma} = \frac{-D+(D^2+4AE)^{1/2}}{2A} = 0.6982$$

and

$$\hat{\mu} = B - \hat{\sigma}C = 0.8145.$$

Also, from Eqs. (5.5) and (5.6) we have

$$V_1 = -0.2112 \text{ and } V_2 = 1.8753$$

using which we obtain the standard errors of the estimates $\hat{\mu}$ and $\hat{\sigma}$ to be

$$SE(\hat{\mu}) = \frac{\hat{\sigma}}{\sqrt{m}}\left\{\frac{V_2}{V_2-V_1^2}\right\}^{1/2} = 0.6982(0.1091)^{1/2} = 0.2306$$

and

$$SE(\hat{\sigma}) = \frac{\hat{\sigma}}{\sqrt{m}}\left\{\frac{1}{V_2-V_1^2}\right\}^{1/2} = 0.6982(0.0582)^{1/2} = 0.1684.$$

Now, upon applying the asymptotic normality of the estimators $\hat{\mu}$ and $\hat{\sigma}$ (see the Theorem in Section 5), we obtain approximate 95% confidence intervals for μ and σ to be

$$[0.8145 - 1.96(0.2306), 0.8145 + 1.96(0.2306)] = [0.3625, 1.2665]$$

and

$$[0.6982 - 1.96(0.1684), 0.6982 + 1.96(0.1684)] = [0.3681, 1.0283],$$ respectively.

By using the results presented in Section 2 and the tables of means, variances and covariances of order statistics from the extreme value distribution prepared recently by Balakrishnan and Chan (1991), we find the best linear unbiased estimates of μ and σ to be

$$\begin{aligned}\mu^* = &-0.0034(-1.541) + 0.0046(-0.693) + 0.0139(-0.128)\\ &+ 0.0244(0.000) + 0.0584(0.278) + 0.0932(0.432)\\ &+ 0.0841(0.565) + 0.1061(0.916) + 0.6187(1.099)\\ = &\ 0.8814\end{aligned}$$

and

$$\begin{aligned}\sigma^* = &-0.0904(-1.541) - 0.0946(-0.693) - 0.0932(-0.128)\\ &- 0.0878(0.000) - 0.1072(0.278) - 0.0836(0.432)\end{aligned}$$

$- 0.0240(0.565) + 0.0073(0.916) + 0.5735(1.099)$

$= 0.7743$

and the standard errors of the estimates μ^* and σ^* to be

$$SE(\mu^*) = \sigma^*(0.1062)^{1/2} = 0.7743(0.1062)^{1/2} = 0.2523$$

and

$$SE(\sigma^*) = \sigma^*(0.0849)^{1/2} = 0.7743(0.0849)^{1/2} = 0.2256.$$

Now, upon applying the asymptotic normality of μ^* and σ^* (since they are linear functions of order statistics), we obtain approximate 95% confidence intervals for μ and σ to be

$[0.8814 - 1.96(0.2523), 0.8814 + 1.96(0.2523)] = [0.3869, 1.3759]$

and

$[0.7743 - 1.96(0.2256), 0.7743 + 1.96(0.2256)] = [0.3321, 1.2165]$,

respectively.

It should be mentioned here that with the sixth observation as 1.33 (or the transformed observation as 0.285), by starting with the graphical estimate of 0.69 as an initial guess for σ Lawless (1982) used Newton's method to solve iteratively the likelihood equations based on the Type–II right censored sample in order to obtain the maximum likelihood estimates of μ and σ to be 0.821 and 0.706, respectively. In this case of right censoring only, Balakrishnan and Varadan (1991) computed the approximate maximum likelihood estimates of μ and σ to be 0.81098 and 0.71010, respectively. It is pleasing to note here that the results obtained for the multiple censoring case are quite close to the results based on the right censoring only.

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References

Arnold, B.C. and Balakrishnan, N. (1989). **Relations, Bounds and Approximations for Order Statistics**, Lecture Notes in Statistics No. 53, Springer–Verlag, New York.

Balakrishnan, N. and Chan, P.S. (1991). Means, variances and covariances of order statistics, and BLUEs of the location and scale parameters based on complete and censored samples from extreme value distribution, **Submitted for publication**.

Balakrishnan, N. and Cohen, A.C. (1990). **Order Statistics and Inference: Estimation Methods**, Academic Press, Boston.

Balakrishnan, N., Gupta, S.S., and Panchapakesan, S. (1991a). Estimation of the mean and standard deviation of the normal distribution based on multiply Type–II censored samples, **Submitted for publication**.

Balakrishnan, N. Gupta, S.S., and Panchapakesan, S. (1991b). Estimation of the mean and standard deviation of the logistic distribution based on multiply Type–II censored samples, **Submitted for publication**.

Balakrishnan, N. and Varadan, J. (1991). Approximate MLEs for the location and scale parameters of the extreme value distribution with censoring, To appear in **IEEE Trans. on Reliab.**

Chan, L.K. and Kabir, A.B.M.L. (1969). Optimum quantiles for the linear estimation of the parameters of the extreme–value distribution in complete and censored samples, **Naval Res. Logist. Quart. 16**, 381 – 404.

Chan, L. K. and Mead, E. R. (1971). Linear estimation of the parameters of the extreme–value distribution based on suitably chosen order statistics, **IEEE Trans. on Reliab. R–20**, 74 – 83.

D'Agostino, R.B. (1971). Linear estimation of the Weibull parameters, **Technometrics 13**, 171 – 182.

David, F.N. and Johnson, N.L. (1954). Statistical treatment of censored data. I. Fundamental formulae, **Biometrika 41**, 228 – 240.

David, H.A. (1981). **Order Statistics**, Second edition, John Wiley & Sons, New York.

Engelhardt, M. and Bain, L.J. (1977). Simplified statistical procedures for the Weibull or extreme value distribution, **Technometrics 19**, 323 – 331.

Harter, H.L. (1970). **Order Statistics and their Uses in Testing and Estimation**, Vol. 2, U.S. Govt. Printing Office, Washington, D. C.

Harter, H.L. and Moore, A.H. (1968). Maximum–likelihood estimation, from doubly censored samples, of the parameters of the first asymptotic distribution of extreme values, **J. Amer. Statist. Assoc. 63**, 889 – 901.

Hassanein, K.M. (1968). Analysis of extreme value data by sample quantiles for very large samples, **J. Amer. Statist. Assoc. 61**, 852– 855.

Hassanein, K.M. (1969). Estimation of the parameters of the extreme value distribution by use of two or three order statistics, **Biometrika 56**, 429 – 436.

Hassanein, K.M. (1972). Simultaneous estimation of the parameters of the extreme value distribution by sample quantiles, **Technometrics 14**, 63 – 70.

Hassanein, K.M., Saleh, A.K. Md. E., and Brown, E.F. (1984). Quantile estimates in complete and censored samples from extreme–value and Weibull distributions, **IEEE Trans. on Reliab. R–33**, 370 – 373.

Hassanein, K.M., Saleh, A.K. Md. E., and Brown, E. F. (1986). Estimation and testing of quantiles of the extreme–value distribution, **J. Statist. Plann. Inf. 14**, 389 – 400.

Kendall, M.G. and Stuart, A. (1973). **The Advanced Theory of Statistics**, Vol. 2, Charles Griffin and Co., London.

Lawless, J. F. (1982). **Statistical Models & Methods For Lifetime Data**, John Wiley & Sons, New York.

Lieblein, J. and Salzer, H.E. (1957). Table of the first moment of ranked extremes, **J. Res. Nat. Bur. Stand. 59**, 203 – 206.

Lieblein, J. and Zelen, M. (1956). Statistical investigation of the fatigue life of deep–grove ball bearings, **J. Res. Nat. Bur. Stand. 57**, 273 – 316.

Lloyd, E. H. (1952). Least squares estimation of location and scale parameters using order statistics, **Biometrika 39**, 88 – 95.

Mann, N. R. (1967a). Tables for obtaining the best linear invariant estimates of parameters of the Weibull distribution, **Technometrics 9**, 629 – 645.

Mann, N. R. (1967b). Results on location and scale parameter estimation with application to the extreme–value distribution, **Technical Report No. 23**, Aerospace Research Laboratories, Wright–Patterson AFB, Ohio.

Mann, N. R. (1968). Point and interval estimation procedures for the two–parameter Weibull and extreme–value distributions, **Technometrics 10**, 231 – 256.

Mann, N. R. and Fertig, K. W. (1973). Tables for obtaining Weibull confidence bounds and tolerance bounds based on best linear invariant estimates of parameters of the extreme value distribution, **Technometrics 15**, 87 – 102.

Mann, N. R. and Fertig, K. W. (1977). Efficient unbiased quantile estimators for moderate–size complete samples from extreme–value and Weibull distributions: confidence bounds and tolerance and prediction intervals, **Technometrics 19**, 87 – 94.

Mann, N. R., Schafer, R. E., and Singpurwalla, N.D. (1974). **Methods for Statistical Analysis of Reliability and Life Data**, John Wiley & Sons, New York.

Rao, C. R. (1975). **Linear Statistical Inference and Its Applications**, Third edition, John Wiley & Sons, New York.

White, J. S. (1969). The moments of log–Weibull order statistics, **Technometrics 11**, 373 – 386.