MOSTOW RIGIDITY AND THE BISHOP-STEGER DICHOTOMY FOR SURFACES OF VARIABLE NEGATIVE CURVATURE

by

Steven P. Lalley*
Purdue University

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Department of Statistics
Purdue University

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1. Introduction

Let M be a compact, orientable, C^∞ surface of genus ≥ 2 and with fundamental group $\Gamma = \pi_1(M)$. Let I_1 and I_2 be isomorphisms of Γ into $PSL(2, \mathbf{R})$, the group of linear fractional transformations $z \to \frac{az+b}{cz+d}$ where a, b, c, d $\in \mathbf{R}$ and $ad - bc = 1$, which may also be viewed as the group of isometries of the hyperbolic plane H. When are I_1 and I_2 geometrically conjugate, i.e., when does there exist $\varphi \in PSL(2, \mathbf{R})$ such that $\varphi \circ I_1(\gamma) = I_2(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$? Equivalently, when is there an isometry Φ: $H/I_1(\Gamma) \longrightarrow H/I_2(\Gamma)$ which induces the trivial isomorphism $I_2 \circ I_1^{-1}$ of fundamental groups?

We shall discuss two criteria, the first due to Mostow [M], the second to Bishop and Steger [BS]. Mostow's criterion uses the Dehn-Nielsen boundary correspondence ([De], appendix). This is a homeomorphism ψ: $\hat{\mathbf{R}} \longrightarrow \hat{\mathbf{R}}$ such that $\psi \circ I_1(\gamma) = I_2(\gamma) \circ \psi$ for every $\gamma \in \Gamma$; it is uniquely determined by I_1 and I_2. (NOTE: $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is a circle here.) Mostow's theorem states that I_1 and I_2 are geometrically conjugate iff ψ is absolutely continuous, in which case ψ is the restriction to $\hat{\mathbf{R}}$ of a linear fractional transformation $\varphi \in PSL(2, \mathbf{R})$. The Bishop/Steger criterion is based more directly on the geometric actions of $I_1(\Gamma)$ and $I_2(\Gamma)$ on H, equipped with the Poincaré distance d. It states that I_1 and I_2 are geometrically conjugate iff for any (every) $z \in H$ and $0 < s < 1$,

$$\sum_{\gamma \in \Gamma} \exp\{-sd(I_1(\gamma)z, z) - (1-s)d(I_2(\gamma)z, z)\} = \infty$$

Moreover, if I_1 and I_2 are not geometrically conjugate then there exists $\delta > 0$ (depending on s but not z) such that

$$\sum_{\gamma \in \Gamma} \exp\{(1-\delta)\{-sd(I_1(\gamma)z, z) - (1-s)d(I_2(\gamma)z, z)\}\} < \infty.$$

The purpose of this paper is to prove analogous theorems for surfaces of variable negative curvature and to exhibit their close connection with the ergodic theory of the associated geodesic flows. Let g_1 and g_2 be C^∞ Riemannian metrics on M, each with strictly negative curvature at every point of M. ASSUME that the geodesic flows associated with g_1 and g_2 both have topological entropy 1. (NOTE: Multiplying a metric g by a scalar α has the effect of multiplying the topological entropy of the geodesic flow by $\alpha^{-1/2}$. Also, if g has constant curvature -1 then the geodesic flow has topological entropy 1.) Let π: $\tilde{M} \to M$ be a universal covering space of M, let $\tilde{g}_1$ and $\tilde{g}_2$ be the canonical liftings of g_1 and g_2 to $\tilde{M}$, and let $\tilde{d}_1$ and $\tilde{d}_2$ be the corresponding distance functions on $\tilde{M}$. As before, set $\Gamma = \pi_1(M)$; identify Γ with the group of deck transformations of $\tilde{M}$. Each $\gamma \in \Gamma$ is an isometry of $(\tilde{M}, \tilde{g}_1)$ and also of $(\tilde{M}, \tilde{g}_2)$. Also, Γ is naturally isomorphic to the fundamental group $\pi_1(M)$ of M.

Let g be an arbitrary C^∞ Riemannian metric on M with negative curvature at every point of M. It is known (Prop. 2. below) that there exists in each free homotopy class δ a unique closed geodesic β_δ parametrized by arclength. The function which assigns to each free homotopy class δ the ($g-$) length $\lambda_g(\beta_\delta)$ of β_δ is called the marked length spectrum of (M, g). Observe that the set Δ of free homotopy classes is in one-to-one correspondence with the set Γ^* of conjugacy classes of Γ, so the marked length spectrum may be viewed as a function on this set of conjugacy classes.

THEOREM 1: *For any two negatively curved C^∞ Riemannian metrics g_1 and g_2 on M the following statements are equivalent:*

(1.1) *g_1 and g_2 have the same marked length spectrum;*

(1.2) *for each $0 < s < 1$,*

$$\sum_{\delta \in \Delta} \exp\{-s\lambda_{g_1}(\beta_\delta^{(1)}) - (1-s)\lambda_{g_2}(\beta_\delta^{(2)})\} = \infty; \textit{ and}$$

(1.3) *for each $\tilde{x} \in \tilde{M}$ and $0 < s < 1$,*

$$\sum_{\gamma \in \Gamma} \exp\{-s\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) - (1-s)\tilde{d}_2(\tilde{x}, \gamma\tilde{x})\} = \infty.$$

Furthermore, if (1.1)-(1.3) do not hold then for every $0 < s < 1$ there exists $0 < p < 1$ such that $\forall\, \tilde{x} \in \tilde{M}$

(1.4) $\sum_{\delta \in \Delta} \exp\{-p(s\lambda_{g_1}(\beta_\delta) + (1-s)\lambda_{g_2}(\beta_\delta))\} < \infty$

and

(1.5) $\sum_{\gamma \in \Gamma} \exp\{-p(s\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) + (1-s)\tilde{d}_2(\tilde{x}, \gamma\tilde{x}))\} < \infty.$

NOTE 1: $\beta_\delta^{(i)}$ is the unique closed g_i-geodesic in the free homotopy class δ. The curves $\beta_\delta^{(1)}$ and $\beta_\delta^{(2)}$ will in general be distinct, although they are (of course) homotopic.

NOTE 2: Henceforth, the term "geodesic" will be used for unit speed geodesics, i.e., geodesics parametrized by arclength.

According to a recent theorem discovered independently by Croke [Cr], if g_1 and g_2 have the same marked length spectrum then there is an isometry I: $(M, g_1) \longrightarrow (M, g_2)$ inducing the identity on $\pi_1(M)$.

It is not immediately apparent what the natural generalization of Mostow's theorem should be, because it is not clear what is the "natural" analogue of the Lebesgue measure on the boundary of hyperbolic space. As before, let π: $\tilde{M} \to M$ be a universal cover of M; $\tilde{M}$ has a concrete realization (as a manifold) as the unit disc. There are, of course, many C^∞ covering projections from the unit disc to M. By Koebe's uniformization theorem, there is at least one C^∞ covering projection π: $\tilde{M} \to M$ such that each deck transformation preserves the Poincaré metric on $\tilde{M}$; thus the Poincaré metric projects via π to a C^∞ Riemannian metric on M of constant curvature -1. Henceforth, we shall *assume that π: $\tilde{M} \to M$ is such a covering projection.*

We will see (sec. 2) that if g is any Riemannian metric on M with negative curvature at every point and $\tilde{g}$ is its canonical lift via π to $\tilde{M}$, then every $\tilde{g}$-geodesic ray in $\tilde{M}$

converges (in the Euclidean metric on $\tilde{M}$) to a unique point of $\partial\tilde{M}$ = unit circle. Two natural measure classes on $\partial\tilde{M}$, depending on the Riemannian metric g, may be defined:

(1) Fix $\tilde{x} \in \tilde{M}$ and pick a direction v (at x) at random (the set of directions is the unit circle in the tangent space $T\tilde{M}_{\tilde{x}}$, and "at random" means from the uniform distribution on this circle). There is a unique geodesic ray emanating from $\tilde{x}$ in direction v; it converges to $\xi \in \partial\tilde{M}$. Define $\nu^L_{\tilde{x}}$ to be the distribution of this random point ξ. For any $\tilde{x}$, $\tilde{y}$ the measures $\nu^L_{\tilde{x}}, \nu^L_{\tilde{y}}$ are mutually absolutely continuous (a.c.), so the measure class ν^L is well-defined. We will call it the *Liouville class.*

(2) Fix $\tilde{x} \in \tilde{M}$ and consider the orbit $\Gamma\tilde{x} = \{\gamma\tilde{x}\colon \gamma \in \Gamma\}$ of $\tilde{x}$ under the group Γ of deck transformations. For each $s > 1$ define a probability measure $\nu^s_{\tilde{x}}$ on $\Gamma\tilde{x}$ by placing at each $\gamma\tilde{x}$ a mass $e^{-s\tilde{d}(\tilde{x},\gamma\tilde{x})}/\Sigma_{\gamma'\in\Gamma}e^{-s\tilde{d}(\tilde{x},\gamma'\tilde{x})}$. We will show (sec. 5) that $\nu^s_{\tilde{x}}$ is well-defined for all $s > 1$ and $\tilde{x} \in \tilde{M}$; that as $s \downarrow 1$ the measures $\nu^s_{\tilde{x}}$ converge weakly to a probability measure $\nu_{\tilde{x}}$ supported by $\partial\tilde{M}$; and that for any $\tilde{x}$, $\tilde{y} \in \tilde{M}$ the measures $\nu_{\tilde{x}}$ and $\nu_{\tilde{y}}$ are mutually absolutely continuous. Let ν denote the measure class thus determined. We shall call it the *Patterson class* because its construction is motivated by Patterson's construction [P] of measures on the boundary of the hyperbolic plane for Fuchsian groups.

When the Riemannian metric g has constant curvature -1, the measure classes ν^L and ν coincide. In general, however, they may not.

THEOREM 2: *The Riemannian metrics g_1 and g_2 have the same marked length spectrum iff the Liouville classes $(\nu^L)_{g_1}$ and $(\nu^L)_{g_2}$ are the same; otherwise, $(\nu^L)_{g_1}$ and $(\nu^L)_{g_2}$ are mutually singular.*

THEOREM 3: *The Riemannian metrics g_1 and g_2 have the same marked length spectrum iff the Patterson measure classes $(\nu)_{g_1}$ and $(\nu)_{g_2}$ are the same; otherwise, $(\nu)_{g_1}$ and $(\nu)_{g_2}$ are mutually singular.*

Thus, there are two natural generalizations of Mostow's theorem. Of the two, Th. 2 appears to be the more elementary; but Th. 3 is more closely tied to the Bishop-Steger criterion. Our proof of Th. 2 is very simple and also elementary: it uses only the ergodic theorem for the geodesic flow and some basic facts about negative curvature (sec. 3). Our proofs of Theorems 1 and 3 (secs. 4 and 6, respectively), although not difficult, rely on deeper properties of the geodesic flow, notably the existence of Markov partitions [Ra], [Bo$_1$], together with the machinery of Gibbs states and Ruelle operators [Bo$_2$].

NOTE: While writing this paper I learned from Prof. M. Ramachandran that Theorem 2 was discovered earlier by A. Katok [K]. Katok's paper does not include a proof. Since my proof is relatively short, I have left it in the paper (sec. 3).

2. Background: Negative Curvature and Geodesic Flow

All Riemannian metrics in this paper will be C^∞.

PROPOSITION 2.1: *Let g_1 and g_2 be Riemannian metrics on M, each with negative curvature at every point of M. Then there exists a C^∞ path g_t in the space of Riemannian metrics on M, $1 \leq t \leq 2$, connecting g_1 and g_2 and such that for each $t \in [1,2]$ the metric g_t has negative curvature at every point of M.*

PROOF: For any Riemannian metric g on M with negative curvature there exists a Riemannian metric g^* with constant curvature -1 such that $g = \rho g^*$ for some C^∞, positive, scalar-valued function ρ on M. This is because the Riemannian metric g determines a conformal structure on M, with respect to which M becomes a Riemann surface; its universal cover π: $\tilde{M} \longrightarrow M$ is the unit disc, by the Koebe uniformization theorem, so the Poincaré metric on $\tilde{M}$ projects to a Riemannian metric g^* on M with constant curvature -1. It must be that $g = \rho g^*$ because by construction of g^* the map $(M, g) \xrightarrow{id} (M, g^*)$ is conformal.

There is a C^∞ path connecting g and g^*: just set $g_t = (t + (1-t)\rho)g^*$, $0 \leq t \leq 1$. That each g_t has negative curvature at every point of M follows immediately from Th. 1 of [GR].

Thus it suffices to show that any two Riemannian metrics g_1^* and g_2^* of constant curvature -1 can be connected. But this follows from the connectedness of Teichmüller space, which in turn follows from standard results in the theory of quasiconformal mappings. In brief, the argument runs as follows. Let $\tilde{M}$ be the unit disc and $\tilde{g}$ the Poincaré metric on $\tilde{M}$. By the uniformization theorem there are covering projections π_i: $\tilde{M} \longrightarrow M$ taking $\tilde{g}$ to g_i^*. The mapping id: $M \longrightarrow M$ is (g_1^*, g_2^*)-quasiconformal; it lifts to a quasiconformal mapping Q: $\tilde{M} \longrightarrow \tilde{M}$ satisfying $\pi_2 = \pi_1 \circ Q$. We may assume that Q fixes three points 1, -1, i. By smoothly deforming the dilatation of Q we may obtain a smoothly varying path of quasiconformal mappings Q_t: $\tilde{M} \longrightarrow \tilde{M}$, $1 \leq t \leq 2$, such that $Q_1 = id$ and $Q_2 = Q$ (see [Ah], Th. 5, Ch. V). The deformation of the dilatation μ_t may be performed so that each μ_t is automorphic with respect to the Fuchsian group of deck transformations of $\tilde{M}$ determined by the covering π_1: $\tilde{M} \longrightarrow M$. Consequently, each $\pi_1 \circ Q_t$ projects the Poincaré metric $\tilde{g}$ to a metric g_t^* of constant curvature -1 on M. Since Q_t varies smoothly, so does g_t^*. □

Next we discuss properties of the geodesic flow. Any Riemannian metric g on M determines a Riemannian metric on the unit tangent bundle S^1M. It will be useful to view S^1M as a single manifold for varying g, namely $S^1M = TM^*/\sim$ where $TM^* = \{(x,v)\colon\ x \in M,\ v \in TM_x,\ v \neq 0\}$ and $(x_1, v_2) \sim (x_2, v_2)$ iff $x_1 = x_2$ and $v_1 = \lambda v_2$ for some $\lambda > 0$, $\lambda \in \mathbf{R}$. This allows us to view the geodesic flows for different Riemannian metrics as flows on the same manifold. For any Riemannian metric g on M we will denote the induced metric on S^1M by g also and the corresponding distance function by d.

For a given Riemannian metric g on M the associated geodesic flow $\phi_t\colon S^1M \to S^1M$ is defined as follows: $\phi_t(x,v) = (x_t, v_t)$ where x_t, $t \in \mathbf{R}$, is the unit speed g-geodesic in M with initial point $x_0 = x$ and initial direction $v_0 = v$, and v_t is the direction of $(d/dt)x_t$. Assume that g has negative curvature at every point of M. Then the geodesic flow ϕ_t is an *Anosov flow* (called C-flows or Y-flows in the older Russian literature: [An], [AS]). Several important features of Anosov flows will be fundamental to our arguments; we summarize them here.

PROPOSITION 2.2 [An]: *There exist foliations W^s, W^u, and G of S^1M by C^1 curves, each invariant by ϕ_t, and such that at each $(x,v) \in S^1M$ the leaves of W^s, W^u, G intersect transversally. The leaves of G are orbits of ϕ_t, while the leaves of W^s and W^u are called "stable" and "unstable" manifolds, respectively. There exist constants $0 < C_0 < C_1 < \infty$ and $\lambda_s < 0 < \lambda_u$ such that for any segment L_u of a W^u-leaf and any segment L_2 of a W^s-leaf,*

$$C_o e^{\lambda_s t} \text{ length } (\phi_t(L_s))/\text{length } (L_s) \leq c_1 e^{\lambda_s t},$$
$$C_o e^{\lambda_u t} \text{ length } (\phi_t(L_u))/\text{length } (L_u) \leq c_1 e^{\lambda_u t},$$

for all $t \in \mathbf{R}$.

In particular, if $(x,v), (x',v')$ are in the same stable manifold then $d(\phi_t(x,v), \phi_t(x',v')) \to 0$ as $t \to \infty$. Moreover, for all sufficiently small $\varepsilon > 0$ the set of (x',v') such that $d(\phi_t(x,v), \phi_t(x',v')) < \varepsilon$ for all $t > 0$ is contained in the connected component containing (x,v) of the intersection of the ε-ball centered at (x,v) with the leaf $W^s(x,v)$ of W^s passing through (x,v).

PROPOSITION 2.3 *(Anosov Closing Lemma [An]): For every $\varepsilon > 0$ there exists $\delta > 0$ such that if for any $(x,v) \in S^1M$ and $t > 0$*

$$d((x,v), \phi_t(x,v)) < \delta,$$

then there exist $(x',v') \in S^1M$ and $t' \in (t-\varepsilon, t+\varepsilon)$ such that $\phi_{t'}(x',v') = (x',v')$ and for every $0 \leq s \leq t' \wedge t$

$$d(\phi_s(x,v), \phi_s(x',v')) < \varepsilon.$$

Thus, closed geodesics (periodic orbits of ϕ) are dense in a very strong sense. Note that since S^1M is compact there will exist along any orbit (x_t, v_t) of ϕ many pairs t', t'' such that $d((x_{t'}, v_{t'}), (x_{t''}, v_{t''})) < \delta$.

The final, and most important, property of Anosov flows is their "structural stability".

PROPOSITION 2.4 *(Structural Stability [An]): Assume that g is a Riemannian metric with negative curvature and geodesic flow ϕ_t. Then for any Riemannian metric g' sufficiently close to g in the C^1-topology and with geodesic flow ψ_t there exists a homeomorphism h: $S^1M \longrightarrow S^1M$ C^0-close to the identity which takes orbits of ϕ_t onto orbits of ψ_t. In particular, $\forall$ $(x,v) \in S^1M$ and $t \in \mathbf{R}$*

$$h \circ \phi_t(x,v) = \psi_\tau \circ h(x,v)$$

where $\tau = \tau(t;x,v)$ is jointly continuous and, for each $(x,v) \in S^1M$, $t \longrightarrow \tau$ is a homeomorphism of $\mathbf{R}$ fixing 0. Finally, h: $S^1M \longrightarrow S^1M$ can be made Hõlder continuous with exponent α, for some $\alpha > 0$.

Proofs may be found in [An] and [Ro], with the exception of the statement about Hölder continuity of h. This statement, however, follows from a careful reading of the proofs, and seems to be fairly well known. ([Mo], p. 435, remark (c), mentions that h is Hölder continuous in the corresponding Structural Stability Theorem for Anosov *diffeomorphisms*.)

Let p: $S^1M \longrightarrow M$ be the natural projection.

PROPOSITION 2.5: *Let g_1 and g_2 be Riemannian metrics on M, each with negative curvature at every point of M, and let $\phi_t^{(1)}$ and $\phi_t^{(2)}$ be the corresponding geodesic flows on S^1M. Then there exists a Hölder continuous homeomorphism h: $S^1M \longrightarrow S^1M$ which takes orbits of $\phi^{(1)}$ onto orbits of $\phi^{(2)}$ and (therefore) takes the periodic orbits of $\phi^{(1)}$ onto the periodic orbits of $\phi^{(2)}$. Moreover, h is isotopic to the identity map. Thus, for any $\phi^{(1)}$-periodic orbit α_1 the corresponding $\phi^{(2)}$-periodic orbit $\alpha_2 = h(\alpha_1)$ is homotopic to α_1.*

PROOF: By Prop. 2.1 there is a smooth path g_s in the space of negatively curved Riemannian metrics on M that connects g_1 and g_2. Consequently, Prop. 2.4 implies that there are homeomorphisms h_s: $S^1M \longrightarrow S^1M$ such that (1) $h_1 =$ identity; (2) $s \longrightarrow h_s \in C^0(M, M)$ is continuous; and (3) h_s maps orbits of $\phi^{(1)}$ onto orbits of $\phi^{(s)}$. (Here $\phi^{(s)}$ is the geodesic flow on S^1M for the metric g_s.) Thus, for any periodic orbit α of $\phi^{(1)}$, the mapping $s \longrightarrow p \circ h_s \circ \alpha$ is a homotopy of $p \circ \alpha$. □

PROPOSITION 2.6: *Let g be a Riemannian metric on M with negative curvature at every point. Then in each free homotopy class on M there is a unique closed g-geodesic, which is the shortest closed curve in that homotopy class.*

NOTE: This is reasonably well known, but for completeness we shall prove the uniqueness.

PROOF: The closed geodesics are precisely the p-projections of the periodic orbits of the geodesic flow (counted twice: once forward and once backward). Every homotopy class in M contains at least one closed geodesic, namely any shortest curve in the class ([BC], ch. 11). To show that there is *only* one, it suffices, by Prop. 2.5, to consider a metric g of constant curvature -1. But the closed geodesics for a metric of curvature -1 are precisely those curves whose lifts to the universal covering space $(\tilde{M}, \tilde{g})$, $\tilde{g} =$ Poincaré metric, are geodesics whose endpoints on $\partial\tilde{M}$ are the two fixed points of some $\gamma \in \Gamma$. □

Recall that $\tilde{M}$ is the unit disc and that π: $\tilde{M} \rightarrow M$ is a C^∞ covering projection with the property that each deck transformation is an isometry of $(\tilde{M}, \tilde{g}_p)$ where $\tilde{g}_p$ is the Poincaré metric on $\tilde{M}$. Thus, the Poincaré metric on $\tilde{M}$ projects via π to a Riemannian metric of constant curvature -1 on M. Using this metric in conjunction with Prop. 2.5 and homotopy lifting, we will show that for any other negatively curved metric g on M the Hadamard manifold $(\tilde{M}, \tilde{g})$ has many of the same qualitative features as the hyperbolic plane.

Given $\pi\colon \tilde{M} \to M$, there is an induced covering projection $\bar{\pi}\colon S^1\tilde{M} \to S^1M$ such that

$$\begin{array}{ccc} S^1\tilde{M} & \xrightarrow{\tilde{p}} & \tilde{M} \\ \bar{\pi}\downarrow & & \downarrow\pi \\ S^1M & \xrightarrow{p} & M \end{array}$$

commutes, where p and $\tilde{p}$ are the natural projections.

Let $h\colon S^1M \longrightarrow S^1M$ be a homeomorphism which is isotopic to the identity, such as that produced in Prop. 2.5. There is a natural lifting of h to a homeomorphism $\tilde{h}\colon S^1\tilde{M} \longrightarrow S^1\tilde{M}$ which is also isotopic to the identity (this follows from the homotopy lifting theorem). For any Riemannian metric g on S^1M with lift $\tilde{g}$, there exists a constant $C < \infty$ such that

$$\tilde{d}(\tilde{h}(X), X) \leq C \qquad \forall\, X \in S^1\tilde{M}, \tag{2.1}$$

where $\tilde{d}$ is the distance function determined by $\tilde{g}$.

PROPOSITION 2.7: *Let g_1 and g_2 be negatively curved Riemannian metrics on M, and let $\phi_t^{(1)}$ and $\phi_t^{(2)}$ be the corresponding geodesic flows on S^1M. Let $h\colon S^1M \longrightarrow S^1M$ be a homeomorphism isotopic to the identity taking $\phi^{(1)}$-orbits onto $\phi^{(2)}$-orbits. For each $\phi^{(1)}$-orbit $\psi_1(t)$ define $\psi_2(\tau(t)) = h(\psi_1(t))$ to be the corresponding $\phi^{(2)}$-orbit. Then for any lift $\tilde{\psi}_1$ of ψ_1 to $S^1\tilde{M}$ there is a unique lift $\tilde{\psi}_2$ of ψ_2 to $S^1\tilde{M}$ such that*

$$\max_{i=1,2}\ \sup_{t\in\mathbf{R}}\ \tilde{d}_i(\tilde{\psi}_1(t),\ \tilde{\psi}_2(\tau(t))) \leq C. \tag{2.2}$$

NOTE: g_i induces a Riemannian metric on S^1M which lifts to $S^1\tilde{M}$; $\tilde{d}_i$ is the corresponding distance. The constant $C < \infty$ depends only on the metrics g_1, g_2 not on $\tilde{\psi}_1$.

PROOF: Taking $\tilde{\psi}_2(\tau(t)) = \tilde{h}(\tilde{\psi}_1(t))$ and applying (2.1) proves the existence of a suitable lift $\tilde{\psi}_2$. Uniqueness follows from an easy argument using $\tilde{g}_3$ = Poincaré metric. (NOTE: The uniqueness statement will not be needed.) □

PROPOSITION 2.8: *Let g be a negatively curved Riemannian metric on M, let $\tilde{g}$ be its canonical lifting to $\tilde{M}$, and let $\tilde{d}$ be the corresponding distance function. For any geodesic ray $\tilde{\alpha}(t)$ in $\tilde{M}$ there exists $\xi \in \partial\tilde{M}$ such that*

$$\lim_{t\to\infty} \tilde{\alpha}(t) = \xi \tag{2.3}$$

(in the Euclidean topology on $\tilde{M} \cup \partial\tilde{M}$). For each $\tilde{x} \in \tilde{M}$ and each $\xi \in \partial\tilde{M}$ there exists a unique geodesic ray $\tilde{\alpha}(t)$ such that

$$\tilde{\alpha}(0) = \tilde{x} \text{ and } \lim_{t\to\infty} \tilde{\alpha}(t) = \xi. \tag{2.4}$$

If $\tilde{\alpha}_1(t)$ *and* $\tilde{\alpha}_2(t)$ *are geodesic rays such that* $\lim_{t\to\infty}\alpha_1(t) = \xi_1$ *and* $\lim_{t\to\infty}\tilde{\alpha}_2(t) = \xi_2$ *then*

$$\lim_{t\to\infty} \tilde{d}(\tilde{\alpha}_1(t),\ \tilde{\alpha}_2(t)) = \infty \ \text{if} \ \ \xi_1 \neq \xi_2; \tag{2.5}$$

$$\lim_{t\to\infty} \tilde{d}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t+t_*)) = 0 \ \text{if} \ \ \xi_1 = \xi_2 \tag{2.6}$$

for some $t_* \in \mathbf{R}$.

PROOF: These statements are elementary and well known for the Poincaré metric. For arbitrary g they follow from Props. 2.5 and 2.8 with $\tilde{g}_1 = \tilde{g}$ and $\tilde{g}_2 =$ Poincaré metric. For any $\tilde{g}_1$-geodesic ray there is a unique $\tilde{g}_2$-geodesic ray that stays within a bounded distance (Prop. 2.7); since the $\tilde{g}_2$-geodesic ray tends to a point $\xi \in \partial\tilde{M}$ the same must be true of the $\tilde{g}_1$-geodesic ray. This proves (2.3).

Suppose $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are $\tilde{g}_1$-geodesic rays that tend to ξ_1, $\xi_2 \in \partial\tilde{M}$, respectively. Let $\tilde{\alpha}_1^*$, $\tilde{\alpha}_2^*$ be the unique $\tilde{g}_2$-geodesic rays that track $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ at bounded distances. If $\xi_1 \neq \xi_2$ then $\tilde{\alpha}_1^*$ and $\tilde{\alpha}_2^*$ separate in Poincaré distance; hence $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ must also separate in Poincaré distance, and so also in $\tilde{d}_1$-distance. This proves (2.5). If, on the other hand, $\xi_1 = \xi_2$ then the Poincaré distance between $\tilde{\alpha}_1^*$ and $\tilde{\alpha}_2^*$ goes to zero as they approach ξ_1. Now project to M and pull back to S^1M: the $\phi^{(2)}$-orbits corresponding to $\tilde{\alpha}_1^*$ and $\tilde{\alpha}_2^*$ become increasingly close for large time, and since h: $S^1M \longrightarrow S^1M$ is a homeomorphism the same must be true for the corresponding $\phi^{(1)}$-orbits. (2.6) follows from this.

Now fix $\tilde{x} \in \tilde{M}$ and consider the set of directions at $\tilde{x}$: $S^1\tilde{M}_{\tilde{x}} = \{(\tilde{y}, v) \in S^1M\colon\ \tilde{y} = \tilde{x}\}$. For each $(\tilde{x}, v) \in S^1\tilde{M}_{\tilde{x}}$ there is a unique geodesic ray emanating from $\tilde{x}$ in direction v, and which tends to a point $\xi = \xi(v) \in \partial\tilde{M}$. To prove the existence statement (2.4) it suffices to prove that $v \longrightarrow \xi(v)$ is continuous and $1-1$. But continuity follows easily from the continuity of h: $S^1M \longrightarrow S^1M$ and elementary hyperbolic geometry. As for injectivity, suppose the geodesic rays $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ in directions v_1, v_2 tend to the same $\xi \in \partial\tilde{M}$. Then by the result of the previous paragraph, the $\tilde{d}$-distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ decreases. But this contradicts negative curvature: cf. [BC], sec. 9.5, Cor. 1. □

As in the preceding proof, let $S^1\tilde{M}_{\tilde{x}}$ denote the set of directions at $\tilde{x} \in \tilde{M}$, i.e., $S^1\tilde{M}_{\tilde{x}} = \{(\tilde{y}, v) \in S^1\tilde{M}\colon\ \tilde{y} = \tilde{x}\}$. For each $\tilde{x} \in \tilde{M}$, $S^1\tilde{M}_{\tilde{x}}$ is a circle. Let g be a negatively curved Riemannian metric on M and let $\tilde{g}$ be its lift to $\tilde{M}$. Then for each $\tilde{x} \in \tilde{M}$ there is a homeomorphism $H_{\tilde{x}}$: $S^1\tilde{M}_{\tilde{x}} \to \partial\tilde{M}$ defined by $H_{\tilde{x}}(v) = \xi$ if the geodesic ray starting from $\tilde{x}$ in direction v tends to ξ.

PROPOSITION 2.9: *For any* $\tilde{x}, \tilde{y} \in \tilde{M}$ *the homeomorphism* $H_{\tilde{y}}^{-1}H_{\tilde{x}}:\ S^1\tilde{M}_{\tilde{x}} \longrightarrow S^1\tilde{M}_{\tilde{y}}$ *is* C^1.

NOTE: This will be used in the definition of the Liouville boundary measure class in sec. 3. In general $H_{\tilde{x}}$: $S^1\tilde{M}_{\tilde{x}} \longrightarrow \partial\tilde{M}$ is *not* C^1.

PROOF: It suffices to prove this for $\tilde{y}$ near $\tilde{x}$.

Here is a dynamical description of $H_{\tilde{y}}^{-1}H_{\tilde{x}}$. Project $\tilde{x}$, $\tilde{y}$ to x, $y \in M$. Then S^1M_x and S^1M_y are nearby C^∞ curves in S^1M. Consider the foliation W^s of S^1M by stable manifolds (curves) (see Prop. 2.2). Since (M,g) has negative curvature, each S^1M_z is transverse to W^s. Let N_y be the union of small segments of leaves of W^s passing through points of S^1M_y. For any $(x,v) \in S^1M_x$ the orbit $\phi_t(x,v)$ intersects N_y transversally for some $|t|$ small, in the leaf of W^s passing through some (y,u). It must then be the case that $H_{\tilde{y}}^{-1}H_{\tilde{x}}(v) = u$, because the ϕ_t-orbits through (x,v) and (y,u) are asymptotic.

The fact that $H_{\tilde{y}}^{-1}H_{\tilde{x}}$ is C^1 now follows from the implicit function theorem, because the stable foliation W^s is C^1 (cf. [HP]). □

3. Liouville Measure and Intersection Statistics

Let g be a negatively curved Riemannian metric on M and let ϕ_t be the geodesic flow on S^1M determined by g. The Liouville measure L is the Riemannian volume on S^1M; it has total mass 2π area (M). The Liouville measure is ϕ_t-invariant, ergodic, and mixing [An].

Fix any $\tilde{x} \in \tilde{M}$, and define a probability measure $\nu_{\tilde{x}}^L$ on $\partial\tilde{M}$ as follows. Choose a direction v at $\tilde{x}$ randomly, according to the normalized Lebesgue measure on $\{v \in T\tilde{M}_{\tilde{x}}: \tilde{g}(v,v) = 1\}$. The direction v determines a unique $\tilde{g}$-geodesic ray starting at $\tilde{x}$; this geodesic ray tends to $\xi \in \partial\tilde{M}$. The distribution of the random endpoint ξ is defined to be $\nu_{\tilde{x}}^L$. In the notation of Prop. 2.9, $\xi = H_{\tilde{x}}(\tilde{x}, v)$. Since $H_{\tilde{y}}^{-1}H_{\tilde{x}}$: $S^1\tilde{M}_{\tilde{x}} \longrightarrow S^1\tilde{M}_{\tilde{y}}$ is a C^1-homeomorphism, it follows that for any $\tilde{x}, \tilde{y} \in \tilde{M}$ the measures $\nu_{\tilde{x}}^L$ and $\nu_{\tilde{y}}^L$ are mutually absolutely continuous. Define $(\nu^L)_g$ to be the measure class of $\nu_{\tilde{x}}^L$.

The relation between the Liouville measure on S^1M and the measure class $(\nu^L)_g$ on $\partial\tilde{M}$ is this. Let π: $\tilde{M} \longrightarrow M$ be the covering projection and let $\mathcal{P} \subset \tilde{M}$ be a connected region with compact closure such that π: $\mathcal{P} \longrightarrow M$ is $1-1$ and onto. Choose $(x,v) \in S^1M$ at random according to L, let γ be the geodesic ray in M with initial point x and direction v, and let $\tilde{\gamma}$ be the unique lift to $\tilde{M}$ starting at $\tilde{x} \in \mathcal{P}$. Then $\tilde{\gamma}$ tends to $\xi \in \partial\tilde{M}$. The distribution of ξ is an average of the measures $\nu_{\tilde{x}}^L$, $\tilde{x} \in \mathcal{P}$, hence is a representative of the measure class $(\nu^L)_g$.

Define the intersection number $N_t(\alpha;\beta)$ of the smooth path $\alpha(s)$, $0 \le s \le t$, with the smooth closed curve β to be the number of transversal intersections of α with β. (If β traverses its trace more than once, intersections are counted as multiple.)

PROPOSITION 3.1: *Let (M,g) be a compact Riemannian manifold of (variable) negative curvature, and let L be the corresponding Liouville measure on S^1M. For each $(x,v) \in S^1M$ let $\alpha(s) = \alpha_{x,v}(s)$ be the geodesic on M with initial point x and direction v. Then for every closed geodesic β on M and L-a.e. $(x,v) \in S^1M$,*

$$\lim_{t\to\infty} t^{-1}N_t(\alpha;\beta) = \frac{4 \text{ length } (\beta)}{2\pi \text{ area } (M)}. \tag{3.1}$$

We will refer to these limits as the "intersection statistics" of the geodesic flow. Clearly the intersection statistics determine the marked length spectrum!

PROOF: This is just an instance of Birkhoff's ergodic theorem: the geodesic flow on S^1M is ergodic relative to L, so time averages converge to space averages. Define $F_\varepsilon : SM \longrightarrow \mathbf{R}$ by

$$F_\varepsilon(x,v) = \varepsilon^{-1}\varphi(D(x,v)/\varepsilon)$$

where $\varphi: \mathbf{R} \longrightarrow \mathbf{R}$ is a continuous probability density with support contained in (0, 1) and $D(x,v)$ is the distance to β along the geodesic ray emanating from x in direction v. For any $\varepsilon > 0$ the function F_ε is bounded and Borel measurable, and for any $(x,v) \in S^1M$ such that $\alpha \neq \beta$,

$$|N_t(\alpha;\beta) - \int_0^t F_\varepsilon(\alpha(s))ds| \leq 2.$$

Consequently, it suffices to evaluate $\int F_\varepsilon dL$ (or, failing that, the limit as $\varepsilon \to 0$). Using the coordinate system θ, D, z where z = arc length along β, θ = angle made with β, and D = distance to β, one has for $\varepsilon \to 0$

$$\begin{aligned}\int F_\varepsilon dL &\sim \int_\beta \int_0^{2\pi} \int_0^\varepsilon \varepsilon^{-1}\varphi(D/\varepsilon)dD|\sin\theta|d\theta dz \\ &= 4 \text{ length } (\beta).\end{aligned}$$

□

Suppose now that $\alpha(s)$ and $\alpha'(s)$, $s \geq 0$, are geodesic rays in (M,g) which are asymptotic, i.e., such that $d(\alpha(s), \alpha'(s+s_0)) \to 0$ as $s \to \infty$ for some fixed $s_0 \in \mathbf{R}$. Then α and α' have the same intersection statistics: if for some closed geodesic β the averages $t^{-1}N_t(\alpha,\beta)$ converge to a limit as $t \to \infty$ then so do the averages $t^{-1}N_t(\alpha',\beta)$, and the limit is the same. Thus the existence and values of the limits are functions of the points $\xi \in \partial\tilde{M}$.

The proof of Th. 2 will be based on the fact that intersection statistics are "topologically invariant": if g_1 and g_2 are negatively curved Riemannian metrics and α_1, α_2 are corresponding g_1-, g_2-geodesics (in the sense of Prop. 2.5) then α_1, α_2 have the same intersection statistics. See Prop. 3.4 below. We will prove this in several steps: first we will prove it for *closed* geodesics; then we will use the density of closed geodesics in S^1M to prove it for arbitrary geodesics.

For any two closed C^1 curves α and β on M define $N(\alpha,\beta)$ to be the number of transversal intersections of α with β (if α or β traverses its path more than once then intersections are counted according to multiplicity).

PROPOSITION 3.2: *Let g_1 and g_2 be negatively curved Riemannian metrics on M, and let α_i and β_i be closed g_i-geodesics such that α_1 is homotopic to α_2 and β_1 is homotopic to β_2. Then*

(3.2) $$N(\alpha_1,\beta_1) = N(\alpha_2,\beta_2).$$

NOTE: Recall (Prop. 2.6) that each free homotopy class has exactly one g_i-geodesic.

PROOF: Let g_s, $1 \leq s \leq 2$, be a smooth path in the space of negatively curved Riemannian metrics connecting g_1 and g_2 (Prop. 2.1). Let α_s and β_s denote the closed g_s-geodesics such that α_s is homotopic to α_1 and β_s is homotopic to β_1. Then α_s varies continuously with s, as does β_s, because a closed geodesic is the unique shortest closed curve in its homotopy class, by Prop. 2.6. Consequently, any transversal intersection of α_s with β_s will persist when s is varied slightly. Therefore, the only way $N(\alpha_s, \beta_s)$ could fail to be constant for $1 \leq s \leq 2$ would be for α_s and β_s to have a tangency (nontransversal intersection) for some s. But this is impossible, because distinct geodesics through a point must have different directions. □

LEMMA 3.3: *Let g be a Riemannian metric on M with negative curvature. Then for each closed geodesic β there exists a constant $K = K_\beta < \infty$ with the following properties. For any geodesic segment $\alpha(s)$, $0 \leq s \leq t$,*

$$N_t(\alpha, \beta) \leq K(t+1). \tag{3.3}$$

For any closed geodesic $\gamma(s)$, $0 \leq s \leq t'$, $\gamma \neq \beta$, and any geodesic segment $\alpha(s)$, $0 \leq s \leq t$ with lifts $\tilde{\gamma}$ and $\tilde{\alpha}$ to $\tilde{M}$,

$$|N_t(\alpha, \beta) - N(\gamma, \beta)| \leq K\{2 + \tilde{d}(\tilde{\alpha}(0), \tilde{\gamma}(0)) + \tilde{d}(\tilde{\alpha}(t), \tilde{\gamma}(t'))\}. \tag{3.4}$$

PROOF: The first statement is a consequence of negative curvature. Consider the universal cover $(\tilde{M}, \tilde{g})$; it is tessellated by Γ-images of the "fundamental polygon" $\mathcal{P}_{\tilde{x}} = \{\tilde{y} \in \tilde{M} \colon d(\tilde{y}, \tilde{x}) \leq d(\tilde{y}, \eta\tilde{x}) \ \forall\, \eta \in \Gamma\}$. Let $\tilde{\beta}$ be a lift of β to $\tilde{M}$; since β is a closed geodesic only finitely many Γ-images of $\tilde{\beta}$ will enter $\mathcal{P}_x$. Call this number C. Now let $\tilde{\alpha}$ be any geodesic segment lying entirely in $\mathcal{P}_x$; then $\tilde{\alpha}$ can cross only C Γ-images of $\tilde{\beta}$. Consequently, if α is any geodesic segment with a lift $\tilde{\alpha}$ lying entirely in $\mathcal{P}_x$ then α can only cross β at most C times. Now there is a constant $D < \infty$ such that any geodesic segment $\tilde{\alpha}$ in $\tilde{M}$ of length t enters at most $(Dt+1)$ Γ-images of $\mathcal{P}_x$. Hence, for any geodesic segment α in M of length t, α can only cross β at most $C(Dt+1)$ times.

To prove (3.4) consider the following deformation of $\tilde{\alpha}$ to $\tilde{\gamma}$ in $\tilde{M}$. Move the initial point from $\tilde{\alpha}(0)$ to $\tilde{\gamma}(0)$ along the geodesic segment connecting them, move the final point from $\tilde{\alpha}(t)$ to $\tilde{\gamma}(t')$ along the geodesic segment connecting them, and let the deformation proceed through geodesic segments. Project this deformation back down to M, and consider how the number of intersections with β changes as the deformation proceeds. Transversal intersections persist under small deformations, so the only changes in the intersection number occur when an endpoint passes through β. Thus, we need only estimate the number of intersections of β with the projections of the geodesic segments from $\tilde{\alpha}(0)$ to $\tilde{\gamma}(0)$, and from $\tilde{\alpha}(t)$ to $\tilde{\gamma}(t')$. But upper bounds on these follow immediately from (3.3), yielding (3.4). □

PROPOSITION 3.4: *Let g_1 and g_2 be Riemannian metrics on M, each with negative curvature, and let $\alpha_1(s)$ and $\alpha_2(s)$, $s \geq 0$, be geodesic rays in metrics g_1 and g_2, respectively, with lifts $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ to $\tilde{M}$ that converge to the same point $\xi \in \partial\tilde{M}$. Then there is a constant $C = C_\xi \in (0,\infty)$ such that for each pair β_1, β_2 of homotopic closed geodesics (β_i geodesic in metric g_i)*

$$\lim_{t\to\infty} t^{-1} N_t(\alpha_1,\beta_1) = C \lim_{t\to\infty} t^{-1} N_t(\alpha_2,\beta_2) \tag{3.5}$$

whenever both limits exist.

NOTE: The important point is that C does not depend on the homotopy class of β_1 and β_2.

PROOF: If α_i is asymptotic to a closed geodesic γ_i (note: this is the case for both $i = 1,2$ or neither) then γ_1 and γ_2 are homotopic and the result follows from Prop. 3.2. So we may assume this is not the case.

Recall that the intersection statistics of a g_2-geodesic α_2 depend only on the endpoint $\xi \in \partial\tilde{M}$ of its lift $\tilde{\alpha}_2$. Consequently, it suffices to prove (3.5) for *any* g_2- geodesic α_2 with a lift $\tilde{\alpha}_2$ converging to ξ. Thus, we may assume that the lifts α_1^* and α_2^* of α_1 and α_2 to S^1M are matching $\phi^{(1)}$- and $\phi^{(2)}$-orbits, i.e., that

$$\alpha_2^*(\tau(t)) = h(\alpha_1^*(t)) \quad \forall\, t \in \mathbf{R}$$

for a suitable time change $\tau(t)$, here $h: S^1M \to S^1M$ is the homeomorphism provided by Proposition 2.5.

The plan now is to approximate α_1^* by long periodic orbits of the geodesic flow $\phi^{(1)}$, using the Anosov closing lemma (Prop. 2.3). Since S^1M is compact, $\alpha_1^*(t)$ must visit some δ-ball in S^1M infinitely often as $t \to \infty$; therefore, there exist $0 \leq t_0 < t_1 < t_2 < \dots$ with $t_n \to \infty$ and $\phi^{(1)}$-periodic orbits ${}^{(1)}\gamma_n^*(t)$, $0 \leq t \leq t_n$, such that

$$d_1(\alpha_1^*(t),\ {}^{(1)}\gamma_n^*(t)) < \varepsilon \quad \forall\, 0 \leq t \leq t_n,\ \forall\, n \geq 1$$

(an initial segment of α_1^* may have to be deleted for this). If $\varepsilon > 0$ is sufficiently small then α_1^* and γ_n^* may be lifted to $S^1\tilde{M}$ in such a way that the lifts remain at $\tilde{d}_1$-distance $< \varepsilon$. They may then be projected to $\tilde{M}$, where

$$\tilde{d}_1(\tilde{\alpha}_1(t), {}^{(1)}\tilde{\gamma}_n(t)) < \varepsilon \quad \forall\, 0 \leq t \leq t_n,\ \forall\, n \geq 1.$$

It now follows from Lemma 3.3 that for every $n \geq 1$,

$$|N_{t_n}(\alpha_1,\beta_1) - N({}^{(1)}\gamma_n,\beta_1)| \leq K_{\beta_1}(2 + 2\varepsilon).$$

Each $\phi^{(1)}$-periodic orbit ${}^{(1)}\gamma_n^*$ corresponds via h to a $\phi^{(2)}$-periodic orbit ${}^{(2)}\gamma_n^*(s)$, $0 \leq s \leq s_n$, while α_1^* corresponds to α_2^*. By Proposition 2.7 the orbits ${}^{(2)}\tilde{\gamma}_n^*$ have lifts ${}^{(2)}\tilde{\gamma}_n^*$

to $S^1\tilde{M}$ that stay within a distance C of the corresponding ${}^{(1)}\gamma_n^*$, and similarly for α_2^*. Consequently,

$$\tilde{d}_2(\tilde{\alpha}_2(0), {}^{(2)}\tilde{\gamma}_n(0)) \leq 2C + \varepsilon',$$
$$\tilde{d}_2(\tilde{\alpha}_2(\tau(t_n))), {}^{(2)}\tilde{\gamma}_n(s_n)) \leq 2C + \varepsilon'$$

where $\varepsilon' > 0$ is such that $\tilde{d}_1$-distance $< \varepsilon$ become $\tilde{d}_2$-distances $< \varepsilon'$. Once again, Lemma 3.3 applies, this time to the effect that

$$|N_{\tau(t_n)}(\alpha_2, \beta_2) - N({}^{(2)}\gamma_n, \beta_2)| \leq K_{\beta_2}(2 + 4C + 2\varepsilon').$$

But by Proposition 3.2, for every pair (β_1, β_2) of homotopic closed g_1-, g_2-geodesic and every $n \geq 1$, $N({}^{(1)}\gamma_n, \beta_1) = N^{({}^{(2)}}\gamma_n, \beta_2)$. Thus, the inequalities of the two preceding paragraphs combine to give, for every pair (β_1, β_2),

$$|N_{t_n}(\alpha_1, \beta_1) - N_{\tau(t_n)}(\alpha_2, \beta_2)| = O(1) \text{ as } n \to \infty,$$

$$\Rightarrow$$

$$\lim_{n\to\infty} t_n^{-1} N_{t_n}(\alpha_1, \beta_1) = \lim_{n\to\infty} t_n^{-1} N_{\tau(t_n)}(\alpha_2, \beta_2)$$

whenever at least one of the limits exists. Hence, if both limits in (3.5) exist for some pair (β_1, β_2), then $\lim t_n^{-1}\tau(t_n) = C$ exists. Observe that $0 < C < \infty$ because the Riemannian metrics g_1, g_2 satisfy $C_1 g_1 \leq g_2 \leq C_2 g_1$ for certain constants $0 < C_1 \leq C_2 < \infty$. Finally, note that if both limits in (3.5) exist for more than one pair (β_1, β_2) then the ratio C of the limits is the same for all pairs, because the same sequence $t_n \to \infty$ may be used for each. □

PROOF *of Theorem 2*: If g_1 and g_2 have the same marked length spectrum then Croke's theorem [Cr] implies that g_1 and g_2 are isometric, hence have the same Liouville class.

Suppose g_1 and g_2 are negatively curved Riemannian metrics on M whose Liouville classes $(\nu^L)_{g_1}$ and $(\nu^L)_{g_2}$ coincide. Fix $\tilde{x} \in \tilde{M}$ and choose $\xi \in \partial\tilde{M}$ according to $(\nu^L_{\tilde{x}})_{g_1}$; let $\tilde{\alpha}_1$ be the $\tilde{g}_1$-geodesic ray in $\tilde{M}$ from $\tilde{x}$ to ξ. Then by Prop. 3.1, for every closed g_1-geodesic β_1 in M,

$$\text{(3.6)} \qquad \lim t^{-1} N_t(\alpha_1; \beta_1) = \frac{g_1\text{-length } (\beta_1)}{2\pi(g_1\text{-area } (M))}$$

where $\alpha_1 = \pi \circ \tilde{\alpha}_1$. Similarly, if $\xi \in \partial\tilde{M}$ is chosen according to $(\nu^L_{\tilde{x}})_{g_2}$, $\tilde{\alpha}_2$ is the $\tilde{g}_2$-geodesic ray from $\tilde{x}$ to ξ, and β_2 is any closed g_2-geodesic in M, then

$$\text{(3.7)} \qquad \lim_{t\to\infty} t^{-1} N_t(\alpha_2; \beta_2) = \frac{g_2\text{-length } (\beta_2)}{2\pi(g_2\text{-area } (M))}$$

where $\alpha_2 = \pi \circ \tilde{\alpha}_2$.

Now if $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ are $\tilde{g}_1$-, $\tilde{g}_2$-geodesic rays in $\tilde{M}$ starting at $\tilde{x}$ and ending at ξ and if β_1, β_2 are closed g_1-, g_2-geodesics which are homotopic then by Prop. 3.4

$$\lim_{t\to\infty} t^{-1}N_t(\alpha_1;\beta_1) = C_\xi \lim_{t\to\infty} t^{-1}N_t(\alpha_2;\beta_2) \tag{3.8}$$

provided both limits exist. But, by hypothesis, $(\nu_{\tilde{x}}^L)_{g_1}$ and $(\nu_{\tilde{x}}^L)_{g_2}$ are mutually absolutely continuous, so there must exist $\xi \in \partial\tilde{M}$ such that (3.6) and (3.7) hold for all β_1, β_2. It then follows from (3.8) that there is a constant $0 < C < \infty$ such that for each homotopic pair β_1, β_2 of closed g_1-, g_2-geodesics,

$$g_1\text{-length }(\beta_1) = C(g_2\text{-length }(\beta_2)),$$

i.e., the marked length spectrum of (M, g_1) is a constant multiple of that of (M, g_2). Finally, $C = 1$ because the geodesic flows for g_1 and g_2 both have topological entropy 1 (see [PP]: topological entropy controls the growth of the length spectrum). □

4. Symbolic Dynamics and the Bishop-Steger Dichotomy

Our approach to Theorems 1 and 3 will be via symbolic dynamics for the geodesic flow(s) and the accompanying "thermodynamic formalism": cf. [Ra], [BR], [Bo$_2$], [Ru].

Let A be an aperiodic, irreducible, $k \times k$ matrix of zeros and ones and define

$$\Sigma_A = \{x \in \bigtimes_{n=-\infty}^{\infty} \{1, 2, \ldots, k\}\colon\ A(x_n, x_{n+1}) = 1 \ \ \forall\, n\},$$

The shift σ: $\Sigma_A \to \Sigma_A$ is defined by $(\sigma x)_n = x_{n+1}$. Distances on Σ_A are defined by $d(x, y) = \exp\{-n(x, y)\}$ where $n(x, y)$ is the largest integer n such that $x_m = y_m \ \forall\, |m| < n$.

Now let r: $\Sigma_A \to \mathbf{R}$ be a strictly positive, Hölder continuous function; define the suspension space

$$\Sigma_A^r = \{(x, t) \in \Sigma_A \times \mathbf{R}\colon\ 0 \le t \le r(x)\}.$$

Distances in Σ_A^r are defined by $d((x, t), (y, s)) = d(x, y) + |s - t|$. Now identify $(x, r(x))$ with $(\sigma x, 0)$. The resulting quotient space will still be denoted by Σ_A^r. The symbolic flow ψ_t on Σ_A^r is defined by

$$\begin{aligned} \psi_s(x,t) &= (x, t+2) \quad \text{if} \quad 0 \le t + s \le r(x), \\ \psi_s \circ \psi_t &= \psi_{s+t} \qquad \forall \quad s,\ t \in \mathbf{R}. \end{aligned}$$

In other words, starting at any (x, t), move up the vertical fiber over $(x, 0)$ at unit speed until reaching $(x, r(x))$, then jump instantaneously to $(\sigma x, 0)$ and continue.

The geodesic flow for any negatively curved Riemannian metric has a "representation" as a symbolic flow. Specifically, let g be such a Riemannian metric on the compact surface M and let ϕ_t be the corresponding geodesic flow on $S^1 M$.

PROPOSITION 4.1 ([Ra],[Bo_1]): *There exists a symbolic flow ψ_t on a suspension space Σ_A^r with Hölder continuous height function r and a surjective, Hölder continuous map q: $\Sigma_A^r \to S^1M$ such that*

$$q \circ \psi_t = \phi_t \circ q.$$

Furthermore, q is measure-preserving for the maximum entropy invariant probability measures on Σ_A^r and S^1M, respectively, and is a.e. one-to-one relative to the maximum entropy measure on S^1M.

The map q is not a homeomorphism but is finite-to-one. Thus, it is not necessarily the case that periodic orbits are in one-to-one correspondence. However,

PROPOSITION 4.2: *For all but finitely many periodic orbits γ of ϕ the following is true. There exists a unique periodic orbit β of ψ such that $q \circ \beta$ traverses the same path as γ, and $q \circ \beta = \gamma$ (so β and γ have the same period).*

PROOF (sketch): The existence of q: $\Sigma_A^r \to S^1M$ follows from the existence of a Markov partition — see [Ra]. The only points $\zeta \in S^1M$ for which $q^{-1}(\zeta)$ is *not* a singleton are those points ζ such that the orbit $\phi_t(\zeta)$ passes nontransversally through a wall of one of the "rectangles" of the Markov partition. But these walls consist of sections of stable manifolds or of unstable manifolds (c.f. [Po], sec. 1); consequently at most one periodic orbit can pass through a given wall. Since each rectangle has four walls and the Markov partition has only finitely many rectangles, there are at most finitely many periodic orbits with more than one symbolic representation. □

NOTE 1: It is always the case that if γ is a periodic orbit of ϕ and β is an orbit of ψ such that $q \circ \beta$ follows the path of γ, then β is periodic and the period of β is a multiple of the period of γ.

NOTE 2: Prop. 4.2 could be bypassed in the arguments to follow by using the weaker results of [Bo_1], sec. 5, but this makes the argument slightly more complicated.

To describe the maximum entropy invariant measure for the symbolic flow ψ_t on Σ_A^r we need the concept of a *Gibbs measure* on the sequence space Σ_A. Let f: $\Sigma_A \to \mathbf{R}$ be Hölder continuous, and define S_nf: $\Sigma_A \to \mathbf{R}$ by $S_nf = f + f\circ\sigma + f\circ\sigma^2 + \ldots + f\circ\sigma^{n-1}$. The Gibbs measure μ_f on Σ_A with potential f is the unique shift-invariant Borel probability measure μ on Σ_A with the following property: there exist constants $0 < C_1 < C_2 < \infty$ and $P(f) \in \mathbf{R}$ such that for each $y \in \Sigma_A$ and each $n = 1, 2, \ldots,$

$$C_1 \leq \frac{\mu\{x \in \Sigma_A\colon\ x_j = y_j \ \forall\, 0 \leq j < n\}}{\exp\{S_nf(y) - nP(f)\}} \leq C_2.$$

(See [Bo_2], Ch. 1, for the existence/uniqueness theorem and a catalogue of basic properties of μ_f.) The constant $P(f)$ is called the *pressure* of f.

Each Gibbs measure μ_f on Σ_A, being shift-invariant, corresponds to a unique ψ_t-invariant measure $\bar{\mu}_f$ on Σ_A^r, which is defined as follows: for each continuous G: $\Sigma_A^r \to \mathbf{R}$,

$$\int_{\Sigma_A^r} G d\bar{\mu}_f = \int_{x\in\Sigma_A} \int_{t=0}^{r(x)} G(x,t) dt d\mu_f(x) / \int_{\Sigma_A} r d\mu_f.$$

PROPOSITION 4.3: *Assume that the topological entropy of the symbolic flow ψ_t on Σ_A^r is 1. Then $P(-r) = 0$ and $\bar{\mu}_{-r}$ is the maximum entropy invariant probability measure for ψ_t.*

This is the special case $\varphi = 0$ of [BR], Prop. 3.1.

The main reason for bringing Gibbs measures into the discussion is the existence of necessary and sufficient conditions for two Gibbs measures μ_f and μ_g to be mutually absolutely continuous. Say that f and g are cohomologous if there exists a Hölder continuous function u: $\Sigma_A \to \mathbf{R}$ such that $f - g = u - u \circ \sigma$.

PROPOSITION 4.4 ([Bo$_2$], Th. 1.28 and Prop. 1.14): *Let f, g: $\Sigma_A \to \mathbf{R}$ be Hölder continuous functions satisfying $P(-f) = P(-g) = 0$. Then either $\mu_{-f} = \mu_{-g}$ or μ_{-f} and μ_{-g} are mutually singular. The following are equivalent:*

(4.2) $$\mu_{-f} = \mu_{-g};$$

(4.3) $$f \text{ and } g \text{ are cohomologous;}$$

(4.4) $$S_n f(x) = S_n g(x) \ \forall\, x \in \Sigma_A \text{ satisfying } \sigma^n x = x.$$

Now suppose that r: $\Sigma_A \to \mathbf{R}$ is strictly positive, and consider again the symbolic flow ψ_t on the suspension space Σ_A^r. An orbit of the symbolic flow is periodic iff it passes through a point $(x, 0) \in \Sigma_A^r$ such that $x \in \Sigma_A$ is a periodic sequence; in this case the period of the ψ-orbit is $S_n r(x)$, where n is the least positive integer such that $\sigma^n x = x$. This implies that if f and g are strictly positive on Σ_A and $P(-f) = P(-g) = 0$ then f and g are cohomologous iff corresponding periodic orbits of the symbolic flows ψ_t^f and ψ_t^g have the same periods.

We will need several basic facts about the "presure" functional $f \to P(f)$ (recall that $P(f)$ is the normalizing constant in (4.1)).

PROPOSITION 4.5: *For Hölder continuous f, g: $\Sigma_A \to \mathbf{R}$,*

(4.5) $$P(f) = \lim_{n\to\infty} \frac{1}{n} \log\{ \sum_{x:\ \sigma^n x = x} \exp(S_n f(x))\};$$

$$(4.6) \qquad P(f) = \lim_{n\to\infty} \frac{1}{n} \log\{ \sum_{x:\ \text{per}(x)=n} \exp(S_n f(x))\};$$

$$(4.7) \qquad f > g \Rightarrow P(f) > P(g);$$

$$(4.8) \qquad \|f-g\|_\infty < \varepsilon \Rightarrow |P(f) - P(g)| < \varepsilon;$$

$$(4.9) \qquad P(af + (1-a)g) < aP(f) + (1-a)P(g) \ \forall\ 0 < a < 1$$

unless f and g are cohomologous.

PROOF: (4.5) follows from [Bo$_2$], Lemma 1.20 by a routine argument, and (4.6) follows easily from (4.5). The monotonicity and continuity properties (4.7)–(4.8) follow directly from (4.5). The convexity property (4.9) is known ([Ru], Prop. 4.7) but the proof is somewhat obscure, so we shall give another.

Let $\mu = \mu_{af+(1-a)g}$ be the Gibbs measure for the potential $af + (1-a)g$. By Prop. 4.3 the measures μ, μ_f, and μ_g are all mutually singular. Consequently, from the "Variational Principle" ([Bo$_2$], Prop. 1.21 and Th. 1.22),

$$P(f) > \text{ entropy } (\mu) + \int f d\mu;$$

$$P(g) > \text{ entropy } (\mu) + \int g d\mu;$$

$$P(af + (1-a)g) = \text{ entropy } (\mu) + \int (af + (1-a)g) d\mu,$$

from which (4.9) follows immediately. □

Let M be a compact surface and let S^1M be its circle bundle (as in sec. 2); let g_1 and g_2 be Riemannian metrics on M, each with negative curvature at every point of M; and let $\phi_t^{(1)}$ and $\phi_t^{(2)}$ be the corresponding geodesic flows on S^1M. By Prop. 4.1 there exists a symbolic flow $\psi_t^{(i)}$ "representing" $\phi_t^{(i)}$ for each $i = 1, 2$. These representations do us no good, however, because the suspension spaces may be entirely unrelated. Fortunately, the structural stability theorem (Prop. 2.5) allows us to construct symbolic representations for $\phi^{(1)}$ and $\phi^{(2)}$ which are "compatible" in that the underlying sequence space Σ_A is the same. Let h: $S^1M \to S^1M$ be a Hölder continuous homeomorphism mapping $\phi^{(1)}$-orbits onto $\phi^{(2)}$-orbits and isotopic to the identity (Prop. 2.5).

PROPOSITION 4.6: *There exist a sequence space Σ_A, strictly positive, Hölder continuous functions r_1, r_2: $\Sigma_A \to \mathbf{R}$, and surjective, continuous q_i: $\Sigma_A^{r_i} \to S^1M$ such that $q_i \circ \psi_t^{(i)} = \phi_t^{(i)} \circ q_i$, $i = 1$ and 2. Each q_i is measure-preserving and a.e. one-to-one relative to* *the maximum entropy invariant measures. Furthermore, there exists a homeomorphism H: $\Sigma_A^{r_1} \to \Sigma_A^{r_2}$ such that for each $x \in \Sigma_A$, $H(x,0) = (x,0)$ and H maps $\{(x,t)\colon 0 \le t < r_1(x)\}$ onto $\{(x,t)\colon 0 \le t < r_2(x)\}$, and such that*

$$
\begin{array}{ccc}
\Sigma_A^{r_1} & \xrightarrow{H} & \Sigma_A^{r_2} \\
q_1 \downarrow & & \downarrow q_2 \\
S^1M & \xrightarrow{h} & S^1M
\end{array}
\tag{4.10}
$$

commutes. In addition, $P(-r_1) = P(-r_2) = 0$.

NOTE: The commutativity of the diagram implies that corresponding orbits of $\phi^{(1)}$ and $\phi^{(2)}$ have the same symbolic representation.

PROOF: First apply Prop. 4.1 to obtain Σ_A, r_1, and q_1. Then $h \circ q_1 \colon \Sigma_A^{r_1} \to S^1M$ is a Hölder continuous, surjective map taking $\psi^{(1)}$-orbits onto $\phi^{(2)}$-orbits. For each $x \in \Sigma_A$, define $r_2(x)$ to be the amount of time it takes the $\phi^{(2)}$-orbits through $h \circ q_1(x,0)$ to proceed from $h \circ q_1(x,0)$ to $h \circ q_1(x, r_1(x))$. Then $r_2 \colon \Sigma_A \to \mathbf{R}$ is strictly positive and Hölder continuous (Hölder continuity follows by a routine 3ε-argument from the Hölder continuity of h, q_1, and r, together with smooth dependence on initial conditions for the flows $\phi_t^{(i)}$).

Now consider the suspension space $\Sigma_A^{r_2}$ and the symbolic flow $\psi_t^{(2)}$ on $\Sigma_A^{r_2}$. Define q_2: $\Sigma_A^{r_2} \to S^1M$ by

$$
\begin{aligned}
q_2(x,0) &= h \circ q_1(x,0) \qquad \forall\, x \in \Sigma_A, \\
q_2 \circ \psi_t^{(2)} &= \phi_t^{(2)} \circ q_2 \qquad \forall\, t \in \mathbf{R},
\end{aligned}
$$

and define H: $\Sigma_A^{r_1} \to \Sigma_A^{r_2}$ by $H(x,s) = H(x,t(s))$ where

$$h \circ \phi_s^{(1)} \circ q_1(x,0) = \phi_{t(s)}^{(2)} \circ q_2(x,0).$$

The advertised properties of q_2 and H are easily checked (that $P(-r_1) = P(-r_2) = 0$ follows because $\phi_t^{(1)}$ and $\phi_t^{(2)}$ have topological entropy 1). □

NOTE: The preceding argument is essentially known — see the proof of Prop. 5.4 in [BR]. However, these authors do not mention that Hölder continuity of the structural stability homeomorphism is (apparently) needed for the Hölder continuity of r_2. Hölder continuity of r_2 is essential in order that the results of [Bo_2], Ch. 1 be applicable.

LEMMA 4.7: *Let r_1 and r_2 be as in Proposition 4.6, and fix $\tilde{x} \in M$. For each $\gamma \in \Gamma$ there exist $\omega \in \Sigma_A$ and $n \ge 0$ such that*

$$|\tilde{d}_i(\tilde{x}, \gamma\tilde{x}) - S_n r_i(\omega)| \le C, \qquad i = 1, 2, \tag{4.11}$$

where $C < \infty$ is a constant not depending on γ. Moreover, there exists $k < \infty$ such that the mapping $\gamma \to (n; \omega_0\omega_1 \dots \omega_{n-1})$ is at most k-to-one.

PROOF: Let $\tilde{\alpha}^{(1)}_\gamma(t)$ be the unique $\tilde{g}_1$-geodesic ray in $\tilde{M}$ such that $\tilde{\alpha}^{(1)}_\gamma(0) = \tilde{x}$ and $\tilde{\alpha}^{(1)}_\gamma(t_\gamma) = \gamma\tilde{x}$ where $t_\gamma = \tilde{d}_1(\tilde{x}, \gamma\tilde{x})$. By structural stability (Proposition 2.7) there is a corresponding $\tilde{g}_2$-geodesic ray $\tilde{\alpha}^{(2)}_\gamma$ such that

$$\tilde{d}_i(\tilde{\alpha}^{(1)}_\gamma(t), \tilde{\alpha}^{(2)}_\gamma(\tau(t))) \leq C'$$

for all $t \geq 0$, $i = 1$ and 2, and the constant $C' < \infty$ does not depend on t or γ. The $\tilde{g}_2$-geodesic ray $\tilde{\alpha}^{(2)}_\gamma$ does not necessarily pass through either $\tilde{x}$ or $\gamma\tilde{x}$, but by the triangle inequality

$$|\tilde{d}_2(\tilde{x}, \gamma\tilde{x}) - \tau(t_\gamma)| \leq 2C'.$$

When the geodesic rays $\tilde{\alpha}^{(1)}_\gamma$ and $\tilde{\alpha}^{(2)}_\gamma$ are projected to M and then pulled back to S^1M they become corresponding orbits of the geodesic flows $\phi^{(1)}$ and $\phi^{(2)}$. These orbits then pull back (via q_1, q_2: see Prop. 4.6) to corresponding orbits $\beta^{(1)}_\gamma$ and $\beta^{(2)}_\gamma$ of the symbolic flows $\psi^{(1)}$ and $\psi^{(2)}$. By Proposition 4.6 there exist $\omega \in \Sigma_A$ and $n \geq 0$ such that

$$\begin{aligned}
&\beta^{(1)}_\gamma(0) = (\omega, s) && \text{, some } s \in [0, r_1(\omega)),\\
&\beta^{(2)}_\gamma(0) = (\omega, s') && \text{, some } s' \in [0, r_2(\omega)),\\
&\beta^{(1)}_\gamma(t_\gamma) = (\sigma^{n-1}\omega, s'') && \text{, some } s'' \in [0, r_1(\sigma^{n-1}\omega)),\\
&\beta^{(2)}_\gamma(\tau(t_\gamma)) = (\sigma^{n-1}\omega, s''') && \text{, some } s''' \in [0, r_2(\sigma^{n-1}\omega)).
\end{aligned}$$

Hence

$$\begin{aligned}
&|S_n r_1(\omega) - t_\gamma| \leq 2||r_1||_\infty,\\
&|S_n r_2(\omega) - \tau(t_\gamma)| \leq 2||r_2||_\infty.
\end{aligned}$$

The inequality (4.11) now follows, with $C = 2(C' + ||r_1||_\infty + ||r_2||_\infty)$.

Although the mapping $\gamma \to (n; \omega_0\omega_1 \dots \omega_{n-1})$ is not necessarily one-to-one, there exists $k < \infty$ such that it is at most k-to-one. This is because if γ, γ' map to the same $(n; \omega_0\omega_1 \dots \omega_{n-1})$ then the orbits $\beta^{(1)}_\gamma$ and $\beta^{(1)}_{\gamma'}$ stay close together in $\Sigma^{r_1}_A$, forcing $\tilde{\alpha}^{(1)}_\gamma$ and $\tilde{\alpha}^{(1)}_{\gamma'}$ to stay close together in $\tilde{M}$, and thus forcing an upper bound on $\tilde{d}_1(\gamma\tilde{x}, \gamma'\tilde{x})$. But for any $r > 0$ the number of $\gamma' \in \Gamma$ such that $\tilde{d}_1(\gamma\tilde{x}, \gamma'\tilde{x}) \leq r$ is bounded by a constant k_r independent of γ. □

PROOF *of Theorem 1*: FIRST assume that g_1 and g_2 have the same marked length spectrum. (This is the easy half.) Then $\lambda_{g_1}(\beta^{(1)}_\delta) = \lambda_{g_2}(\beta^{(2)}_\delta) = \ell(\delta)$ for each free homotopy class δ. By a theorem of Margulis [Mr] (see also [PP] and [La], sec. 5), if $N(t)$ is the number of free homotopy classes δ such that $\ell(\delta) \leq t$ then for some constant $0 < C < \infty$,

$$\lim_{t\to\infty} tN(t)e^{-t} = C$$

(here we have also used the fact that the topological entropy of $\phi^{(i)}$ is 1). The conclusion (1.2) follows directly.

To prove (1.3) we will argue that the sum in (1.3) dominates a multiple of the series in (1.2). Fix a free homotopy class δ and consider the closed g_i-geodesic $\beta_\delta^{(i)}$. This closed geodesics has infinitely many lifts to $\tilde{M}$, but at least one lift $\tilde{\beta}_\delta^{(i)}(t)$, $0 \le t \le \ell(\delta)$, such that $\tilde{d}_i(\tilde{x}, \tilde{\beta}_\delta^{(i)}(0)) \le d_i$-diameter (M). For this path $\tilde{\beta}_\delta^{(i)}$ there exists $\gamma \in \Gamma$ such that $\gamma(\tilde{\beta}_\delta^{(i)}(0)) = \tilde{\beta}_\delta^{(i)}(\ell(\delta))$, because $\beta_\delta^{(i)}(0) = \beta_\delta^{(i)}(\ell(\delta))$. Since γ is an isometry, the $\tilde{d}_i$-distance from $\gamma\tilde{x}$ to $\tilde{\beta}_\delta^{(i)}(\ell(\delta))$ is the same as the $\tilde{d}_i$-distance from $\tilde{x}$ to $\tilde{\beta}_\delta^{(i)}(0)$, hence by the triangle inequality,

$$\tilde{d}_i(\tilde{x}, \gamma\tilde{x}) \le \ell(\delta) + 2d_i\text{-diameter}(M).$$

Now let $\tilde{\alpha}(t)$, $0 \le t \le \tilde{d}_i(\tilde{x}, \gamma\tilde{x})$, be the $\tilde{g}_i$-geodesic segment from $\tilde{x}$ to $\gamma\tilde{x}$; since $\tilde{\alpha}(\tilde{d}_i(\tilde{x}, \gamma\tilde{x})) = \gamma\tilde{\alpha}(0)$ and $\tilde{\beta}_\delta^{(i)}(\ell(\delta)) = \gamma\tilde{\beta}_\delta^{(i)}(0)$, there is a homotopy deforming $\tilde{\alpha}$ to $\tilde{\beta}_\delta^{(i)}$ that projects to a homotopy in M deforming α to $\beta_\delta^{(i)}$ through closed curves. Consequently, α is in the free homotopy class δ. Thus, the mapping $\delta \to \gamma$ is one-to-one. It now follows that

$$\begin{aligned}
&\sum_{\delta\in\Delta} \exp\{-s\lambda_{g_1}(\beta_\delta^{(1)}) - (1-s)\lambda_{g_2}(\beta_\delta^{(2)})\} \\
&\le \exp\{2(d_1\text{-diameter } (M) + d_2\text{-diameter } (M))\} \\
&\cdot \sum_{\gamma\in\Gamma} \exp\{-s\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) - (1-s)\tilde{d}_2(\tilde{x}, \gamma\tilde{x})\};
\end{aligned}$$

so (1.3) follows from (1.2).

SECOND, assume that g_1 and g_2 have different marked length spectra. Then by Proposition 4.4 the functions r_1 and r_2 (whose existence is guaranteed by Proposition 4.6) are not cohomologous, and so by (4.9), $P(-sr_1 - (1-s)r_2) < 0$. Consequently, by (4.8), there exists $0 < p < 1$ such that $P(-p(sr_1 + (1-s)r_2)) < 0$, and now by (4.5),

$$\sum_{n=1}^{\infty} \sum_{x:\sigma^n x = x} \exp\{-p(sS_n r_1(x) + (1-s)S_n r_2(x))\} < \infty. \tag{4.12}$$

Recall (Proposition 4.2) that, with finitely many exceptions, periodic orbits of $\phi^{(i)}$ correspond to periodic orbits of $\psi^{(i)}$ with the same periods. Moreover, periodic orbits of $\psi^{(i)}$ are those orbits that pass through points $(x, 0) \in \Sigma_A^{r_i}$ such that $\sigma^n x = x$ for some $n \ge 1$; and the period of such an orbit is $S_n r_i(x)$. By Proposition 4.6, corresponding orbits of $\phi^{(1)}$ and $\phi^{(2)}$ are represented by the same $(x, 0)$. Thus, with finitely many exceptions, each free homotopy class δ is represented in the sum (4.12). The result (1.4) therefore follows from (4.12).

It remains to prove (1.5). According to [Bo$_2$], Lemma 1.20, since $P(-psr_1 - p(1-s)r_2) < 0$,

$$\sum_{n=1}^{\infty} \sum_{a_0 a_1 \ldots a_{n-1}} \exp\{\sup_{a_0 a_1 \ldots a_{n-1}} (-psS_n r_1 - p(1-s)S_n r_2)\} < \infty,$$

where

$$\sup_{a_0 a_1 \dots a_{n-1}} S_n g = \sup\{S_n g(\omega): \omega_i = a_i \ \forall\, 0 \le i < n\}.$$

Consequently, by Lemma 4.7,

$$\sum_{\gamma \in \Gamma} \exp\{-ps\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) - p(1-s)\tilde{d}_2(\tilde{x}, \gamma\tilde{x})\} < \infty. \qquad \square$$

5. Patterson Measures

The definition of the Patterson measure requires an estimate due to Margulis [Mr]. Let g be a negatively curved Riemannian metric on M, let $\tilde{g}$ be its lift to $\tilde{M}$ via the covering projection $\pi: \tilde{M} \to M$, and let $\tilde{d}$ be the associated distance function on $\tilde{M}$. Then, according to [Mr], for each $\tilde{x} \in \tilde{M}$ there exists $0 < C_{\tilde{x}} < \infty$ such that

$$(5.1) \qquad |\{\gamma \in \Gamma: \tilde{d}(\tilde{x}, \gamma\tilde{x}) \le t\}| \sim C_{\tilde{x}} e^{ht}$$

as $t \to \infty$, where h is the topological entropy of the geodesic flow of g. (That the exponential rate in Margulis' formula is the topological entropy follows from [Ma].) Henceforth, we shall assume that $h = 1$.

It follows from (5.1) that for each $\tilde{x} \in \tilde{M}$ and $s > 1$

$$Z_{\tilde{x}}(s) \triangleq \sum_{\gamma \in \Gamma} \exp\{-s\tilde{d}(\tilde{x}, \gamma\tilde{x})\} < \infty,$$

but that $\lim_{s \downarrow 1} Z_{\tilde{x}}(s) = \infty$. Consequently, we may define probability measures $\nu_{\tilde{x}}^s$ on $\tilde{M} \cup \partial\tilde{M}$ by putting mass $\exp\{-s\tilde{d}(\tilde{x}, \gamma\tilde{x})\}/Z_{\tilde{x}}(s)$ at $\gamma\tilde{x}$ for each $\gamma \in \Gamma$, where $s > 1$ and $\tilde{x} \in \tilde{M}$. The main result of this section is the following.

PROPOSITION 5.1: *For each $\tilde{x} \in \tilde{M}$ there exists a Borel probability measure $\nu_{\tilde{x}}$ supported by $\partial\tilde{M}$ such that $\nu_{\tilde{x}}^s \xrightarrow{\omega*} \nu_{\tilde{x}}$ as $s \downarrow 1$. For any two $\tilde{x}$, $\tilde{y} \in \tilde{M}$ the measures $\nu_{\tilde{x}}$ and $\nu_{\tilde{y}}$ are mutually absolutely continuous, and $d\nu_{\tilde{x}}/d\nu_{\tilde{y}}$ is bounded away from 0 and ∞.*

The notation $\xrightarrow{\omega*}$ indicates weak-$*$ convergence: $\nu_n \xrightarrow{\omega*} \nu$ if for every bounded, continuous, real-valued function f on $\tilde{M} \cup \partial\tilde{M}$ it is the case that $\int f d\nu_n \longrightarrow \int f d\nu$. By Helly's selection theorem (called "Alaoglu's theorem" by analysts — see [Ry], Ch. 10, Th. 17), any sequence of Borel probability measures on a compact metric space has a weak-$*$ convergent subsequence. Thus, the existence of $\nu_{\tilde{x}}$ may be proved by showing that for each $\tilde{x} \in \tilde{M}$ there is at most one possible limit of $\nu_{\tilde{x}}^s$ as $s \downarrow 1$.

We will follow the same basic strategy as in [Ni], sec. 4.2, but many of the steps will differ in detail because we cannot use hyperbolic trigonometry. It is worth noting that our standing assumptions concerning the covering projection $\pi: \tilde{M} \to M$, namely that π is C^∞ and that each deck transformation is an isometry of ($\tilde{M}$, Poincaré metric), are crucial:

Prop. 5.1 is easily seen to be *false* for many C^∞ covering maps $\pi: \tilde{M} \to M$. The standing assumptions were used in obtaining certain of the results of sec. 2, which we will call on below. We note also for future reference that, in consequence of the standing assumptions, each element γ of the group Γ of deck transformations acts on $\tilde{M} \cup \partial\tilde{M}$ as a linear fractional transformation preserving $\partial\tilde{M}$. Hence Γ is a Fuchsian group with no parabolic or elliptic elements.

First we shall make some observations about geodesic lines and rays in $(\tilde{M}, \tilde{g})$. Two distinct geodesic lines intersect in at most one point (this follows from negative curvature and the Gauss-Bonnet formula for geodesic triangles). By Prop. 2.8, each geodesic ray in $\tilde{M}$ tends to a point $\xi \in \partial\tilde{M}$ (in the Euclidean topology on $\tilde{M} \cup \partial\tilde{M}$), which we will call its "endpoint"; moreover, for each $\tilde{x} \in \tilde{M}$ and each $\xi \in \partial\tilde{M}$ there is a *unique* geodesic ray with initial point $\tilde{x}$ and endpoint ξ. Any geodesic line $\tilde{\alpha}$ in $\tilde{M}$ has two endpoints ξ, $\xi' \in \partial\tilde{M}$, and $\xi \neq \xi'$ (because if $\xi = \xi'$ then any geodesic line $\tilde{\beta}$ that crosses $\tilde{\alpha}$ transversally would have ξ as an endpoint, since $\tilde{\alpha}, \tilde{\beta}$ cross only once; but then there would be distinct geodesic rays from some $\tilde{x} \in \tilde{M}$ to ξ). If $\tilde{\alpha}$ and $\tilde{\beta}$ are geodesic lines with endpoints ξ_α^+, ξ_α^-, ξ_β^+, $\xi_\beta^- \in \partial\tilde{M}$ and $\tilde{\alpha}$, $\tilde{\beta}$ cross transversally then ξ_α^+, ξ_α^- separate ξ_β^+, ξ_β^- on the circle $\partial\tilde{M}$.

LEMMA 5.2: *Fix $\tilde{x} \in \tilde{M}$. Let $\tilde{z}_n \in \tilde{M}$ be a sequence of points such that $\lim_{n\to\infty} \tilde{z}_n = \xi \in \partial\tilde{M}$ (in the Euclidean topology on $\tilde{M} \cup \partial\tilde{M}$), and let $\xi_n \in \partial\tilde{M}$ be the endpoint of the geodesic ray in $\tilde{M}$ starting at $\tilde{x}$ and passing through z_n. Then*

$$\lim_{n\to\infty} \xi_n = \xi.$$

PROOF: It suffices to show that the sequence ξ_n has no accumulation points on $\partial\tilde{M}$ other than ξ. So choose $\zeta \in \partial\tilde{M}$, $\zeta \neq \xi$.

Consider the geodesic lines through $\tilde{x}$ with endpoints ξ and ζ, respectively. These geodesic lines have second endpoints ξ' and ζ' on $\partial\tilde{M}$, respectively, and the four endpoints are arranged on $\partial\tilde{M}$ in the order ξ, ζ, ξ', ζ'. Choose any $\eta \in \partial\tilde{M}$ on the open arc of $\partial\tilde{M}$ with endpoints ξ, ζ *not* containing ξ', ζ'. The geodesic ray from $\tilde{x}$ to η extends to a geodesic line whose second endpoint on $\partial\tilde{M}$ we call η'; the six endpoints must be arranged on $\partial\tilde{M}$ in the order ξ, η, ζ, ξ', η', ζ'. Let α denote the geodesic line with endpoints η, η'.

Observe that α divides $\tilde{M} \cup \partial\tilde{M}$ into two connected components A_ξ and A_ζ, with $\xi \in A_\xi$ and $\zeta \in A_\zeta$. Any geodesic ray β that starts at $\tilde{x}$ must lie entirely in A_ξ or entirely in A_ζ (otherwise it would cross α twice). If β passes through z_n and z_n is close to ξ then β lies in A_ξ, hence so does its endpoint ξ_n. Consequently, the sequence ξ_n does not accumulate at ζ. □

LEMMA 5.3: *For all $\tilde{x}$, $\tilde{y} \in \tilde{M}$ and $\xi \in \partial\tilde{M}$ there exists a real constant $C(\tilde{x}, \tilde{y}, \xi)$ with the following property. For any sequence $\tilde{z}_n \in \tilde{M}$ such that $\lim_{n\to\infty} \tilde{z}_n = \xi$ in the Euclidean* *topology on $\tilde{M} \cup \partial\tilde{M}$,*

$$(5.2) \qquad \lim_{n\to\infty}\{\tilde{d}(\tilde{x},\tilde{z}_n) - \tilde{d}(\tilde{y},\tilde{z}_n)\} = C(\tilde{x},\tilde{y},\xi).$$

Furthermore, for fixed $\tilde{x},\tilde{y} \in \tilde{M}$ the mapping $\xi \to C(\tilde{x},\tilde{y},\xi)$ is continuous in ξ for $\xi \in \partial\tilde{M}$. Finally, for each $\gamma \in \Gamma$,

$$(5.3) \qquad C(\tilde{x},\tilde{y},\xi) = C(\gamma\tilde{x},\gamma\tilde{y},\gamma\xi).$$

NOTE 1: Compare with [Ni], Lemma 3.2.1.

NOTE 2: It follows from (5.2) and the triangle inequality that

$$(5.4) \qquad |C(\tilde{x},\tilde{y},\xi)| \leq d(\tilde{x},\tilde{y}).$$

NOTE 3: It may be possible to deduce some of this from the existence of the "Busemann function", but the continuity in ξ is a major point.

PROOF: For $\tilde{z} \in \tilde{M}$ and $\xi \in \partial\tilde{M}$ let $\tilde{\alpha}_{\tilde{z},\xi}(t)$ be the geodesic ray (parametrized by arclength) such that $\tilde{\alpha}_{\tilde{z},\xi}(0) = \tilde{z}$ and $\lim_{t\to\infty}\tilde{\alpha}_{\tilde{z},\xi}(t) = \xi$. By Prop. 2.8 there exists, for each pair $\tilde{x}$, $\tilde{y} \in \tilde{M}$ and each $\xi \in \partial\tilde{M}$, a real number $t_\xi = -C(\tilde{x},\tilde{y},\xi)$ such that

$$(5.5) \qquad \lim_{t\to\infty}\tilde{d}(\tilde{\alpha}_{\tilde{x},\xi}(t),\ \tilde{\alpha}_{\tilde{y},\xi}(t+t_\xi)) = 0.$$

We will show that, for fixed $\tilde{x}$, $\tilde{y} \in \tilde{M}$, the mapping $\xi \to t_\xi$ is a continuous function from $\partial\tilde{M}$ to $\mathbf{R}$, and that (5.5) holds uniformly for $\xi \in \partial\tilde{M}$.

Before doing so, however, we will show how this implies the other statements of the lemma. Consider the geodesic ray starting at $\tilde{x}$ that passes through $\tilde{z}_n$; this ray terminates at some point $\xi_n \in \partial\tilde{M}$. Since $\tilde{z}_n \to \xi$, Lemma 5.2 implies that $\xi_n \to \xi$. Also, since $\tilde{z}_n \to \xi \in \partial\tilde{M}$ it must be the case that $\tilde{d}(\tilde{x},\tilde{z}_n) \to \infty$ and $\tilde{d}(\tilde{y},\tilde{z}_n) \to \infty$. A simple argument using (5.5) and the triangle inequality now shows that

$$\lim_{n\to\infty}(d(\tilde{x},\tilde{z}_n) - d(\tilde{y},\tilde{z}_n) + t_{\xi_n}) = 0.$$

The result (5.2) follows from the continuity of $\xi \to t_\xi$, and (5.3) follows because each $\gamma \in \Gamma$ is an isometry of $(\tilde{M},\tilde{g})$ (note that $z_n \to \xi$ implies $\gamma z_n \to \gamma\xi$ because γ acts on $\tilde{M} \cup \partial\tilde{M}$ as a linear fractional transformation).

In proving the continuity of $\xi \to t_\xi$ it suffices to consider $\tilde{x}$, $\tilde{y} \in \tilde{M}$ such that $\tilde{d}(\tilde{x},\tilde{y}) < \varepsilon$ for some small $\varepsilon > 0$, because (5.5) implies that for all $\tilde{x}$, $\tilde{y}$, $\tilde{z} \in \tilde{M}$ and $\xi \in \partial\tilde{M}$,

$$(5.6) \qquad C(\tilde{x},\tilde{y},\xi) + C(\tilde{y},\tilde{z},\xi) = C(\tilde{x},\tilde{z},\xi).$$

Project the geodesic rays $\tilde{\alpha}_{\tilde{x},\xi}(t)$ and $\tilde{\alpha}_{\tilde{y},\xi}(t)$ to M and then pull back to S^1M to obtain orbits $\alpha^*_{x,\xi}(t)$ and $\alpha^*_{y,\xi}(t)$ of the geodesic flow; then $\alpha^*_{x,\xi}(0) = (x, v_\xi) \in S^1M_x$ and $\alpha^*_{y,\xi}(0) = (y, u_\xi) \in S^1M_y$. Recall (Prop. 2.8 and ff.) that $\xi \to (x, v_\xi)$ and $\xi \to (y, u_\xi)$ are homeomorphisms of $\partial\tilde{M}$ onto S^1M_x and S^1M_y, respectively. By (5.5), $\alpha^*_{x,\xi}(t)$ and $\alpha^*_{y,\xi}(t+t_\xi)$ are asymptotic as $t \to \infty$; consequently, if $t_\xi \leq 0$ then $\alpha^*_{x,\xi}$ intersects the stable manifold $W^s(y, u_\xi)$ through (y, u_ξ) at time $-t_\xi$, and this intersection is transversal. (If $t_\xi > 0$ then $\alpha^*_{y,\xi}$ intersects $W^s(x, v_\xi)$ at time t_ξ.) As ξ varies continuously on $\partial\tilde{M}$ the initial point (x, v_ξ) of $\alpha^*_{x,\xi}$ varies continuously along the curve S^1M_x, so by the continuity of the stable foliation W^s and the transversality of the intersections the time t_ξ at which the intersection takes place varies continuously with ξ. The uniformity in ξ of (5.5) also follows from this analysis, because the rate of contraction in the direction of the stable manifold W^s is continuous on S^1M (Proposition 2.2). □

Following [Ni], we define "shadows" $B(\tilde{x}\colon\tilde{y}, r) \subset \partial\tilde{M}$ as follows: for $r > 0$ and $\tilde{x}$, $\tilde{y} \in \tilde{M}$, let $B(\tilde{x}\colon\tilde{y}, r)$ be the set of all $\xi \in \partial\tilde{M}$ such that the geodesic ray from $\tilde{x}$ to ξ intersects the $\tilde{d}$-ball of radius r centered at $\tilde{y}$. Observe that $B(\tilde{x}\colon\tilde{y}, r)$ is an arc of $\partial\tilde{M}$.

LEMMA 5.4: *For any $r > 0$, any $\tilde{x}$, $\tilde{y} \in \tilde{M}$, and any $\xi \in B(\tilde{x}\colon\ \tilde{y}, r)$,*

$$|C(\tilde{x}, \tilde{y}, \xi) - \tilde{d}(\tilde{x}, \tilde{y})| \leq 2r$$

PROOF: Consider the geodesic ray $\tilde{\alpha}_{\tilde{x},\xi}$ from $\tilde{x}$ to ξ. Since $\xi \in B(\tilde{x}\colon\ \tilde{y}, r)$, this geodesic ray intersects the $\tilde{d}$-ball of radius r centered at $\tilde{y}$ in some point $\tilde{z}$. Clearly, $C(\tilde{x}, \tilde{z}, \xi) = \tilde{d}(\tilde{x}, \tilde{z})$ (just take $\tilde{z}_n$ on $\tilde{\alpha}_{\tilde{x},\xi}$ converging to ξ and apply (5.2)). Also, by (5.4), $C(\tilde{y}, \tilde{z}, \xi) \leq r$. Therefore, by (5.6), $|C(\tilde{x}, \tilde{y}, \xi) - \tilde{d}(\tilde{x}, \tilde{z})| \leq r$. The result now follows by another application of the triangle inequality, since $\tilde{d}(\tilde{y}, \tilde{z}) < r$. □

Consider now the probability measures $\nu^s_{\tilde{x}}$ and their weak-$*$ limits as $s \downarrow 1$. By Margulis' formula (5.1) (recall that $h = 1$) $Z_{\tilde{x}}(s) \to \infty$ as $s \downarrow 1$, so as $s \downarrow 1$ all of the mass in $\nu^s_{\tilde{x}}$ floats out to $\partial\tilde{M}$ (specifically, for each $t < \infty$, $\lim_{s\downarrow 1} \nu^s_{\tilde{x}}\{\tilde{y}\colon \tilde{d}(\tilde{x}, \tilde{y}) \leq t\} = 0$). Consequently, any weak-$*$ limit of $\nu^s_{\tilde{x}}$ must be supported by $\partial\tilde{M}$.

LEMMA 5.5: *Fix $\tilde{x}, \tilde{y} \in \tilde{M}$ and suppose $s_n \downarrow 1$ is a sequence such that*

$$\begin{aligned} \mu_{\tilde{x}} &= \textit{weak-}* \lim \nu^{s_n}_{\tilde{x}} \qquad \textit{and} \\ \mu_{\tilde{y}} &= \textit{weak-}* \lim \nu^{s_n}_{\tilde{y}} \end{aligned}$$

both exist. Then $\mu_{\tilde{x}}$ and $\mu_{\tilde{y}}$ are mutually absolutely continuous, and the Radon-Nikodym derivative $d\mu_{\tilde{x}}/d\mu\tilde{y}$ is bounded away from 0 and ∞.

PROOF: For any $\gamma \in \Gamma$ and $s > 1$,

$$\begin{aligned} &\tilde{d}(\tilde{x}, \gamma\tilde{x}) \leq 2\tilde{d}(\tilde{x}, \tilde{y}) + \tilde{d}(\tilde{y}, \gamma\tilde{y}) \leq 4\tilde{d}(\tilde{x}, \tilde{y}) + \tilde{d}(\tilde{x}, \gamma\tilde{x}) \\ \Rightarrow\ &Z_{\tilde{x}}(s) \leq Z_{\tilde{y}}(s)\exp\{2\tilde{d}(\tilde{x}, \tilde{y})\} \leq Z_{\tilde{x}}(s)\exp\{4\tilde{d}(\tilde{x}, \tilde{y})\}. \end{aligned}$$

If $\gamma_n \in \Gamma$ is any sequence such that $\gamma_n \tilde{x} \to \xi \in \partial\tilde{M}$, then it is also the case that $\gamma_n \tilde{y} \to \xi$, so $\gamma_n \tilde{x}$ and $\gamma_n \tilde{y}$ become close in the Euclidean metric on $\tilde{M} \cup \partial\tilde{M}$. Hence, if f: $\partial\tilde{M} \to \mathbf{R}$ is any nonnegative, continuous function then

$$\int f d\mu_{\tilde{x}} \leq \exp\{4\tilde{d}(\tilde{x},\tilde{y})\} \int f d\mu_{\tilde{y}}.$$

It follows that $\mu_{\tilde{x}}$ and $\mu_{\tilde{y}}$ are mutually absolutely continuous and that $d\mu_{\tilde{x}}/d\mu_{\tilde{y}}$ is bounded above and below by $\exp\{\pm 4\tilde{d}(\tilde{x},\tilde{y})\}$. □

LEMMA 5.6: *Let $s_n \downarrow 1$ be a sequence such that $\nu_{\tilde{x}}^{s_n}$ converges weak-$*$ to some probability measure $\mu_{\tilde{x}}$. Then for each $\gamma \in \Gamma$ the sequence $\nu_{\gamma\tilde{x}}^{s_n}$ converges weak-$*$ to a measure $\mu_{\gamma\tilde{x}}$, and*

$$\frac{d\mu_{\gamma\tilde{x}}}{d\mu_{\tilde{x}}}(\xi) = \exp\{C(\tilde{x},\gamma\tilde{x},\xi)\} \qquad \forall\, \xi \in \partial\tilde{M} \tag{5.7}$$

Moreover, for any continuous f: $\partial\tilde{M} \to \mathbf{R}$,

$$\int f d\mu_{\tilde{x}} = \int (f \circ \gamma) d\mu_{\gamma\tilde{x}}. \tag{5.8}$$

PROOF: Note that $Z_{\tilde{x}}(s) = Z_{\gamma\tilde{x}}(s)$ for each $s > 1$, and that $d(\tilde{x},\gamma'\tilde{x}) = d(\gamma\tilde{x},\gamma\gamma'\tilde{x})$ for all γ, $\gamma' \in \Gamma$. Consequently, for any continuous g: $\tilde{M} \cup \partial\tilde{M} \to \mathbf{R}$ and each $s > 1$,

$$\int g d\nu_{\tilde{x}}^{s} = \int (g \circ \gamma) d\nu_{\gamma\tilde{x}}^{s}.$$

Since $g \to g \circ \gamma$ is a linear isometry of $C(\tilde{M} \cup \partial\tilde{M})$, it follows from the Riesz representation theorem that if $\nu_{\tilde{x}}^{s_n} \xrightarrow{\omega*} \mu_{\tilde{x}}$ then $\nu_{\gamma\tilde{x}}^{s_n} \xrightarrow{\omega*} \mu_{\gamma\tilde{x}}$ where $\mu_{\gamma\tilde{x}}$ is defined by (5.8).

For any $\gamma \in \Gamma$ and $s > 1$ the measures $\nu_{\tilde{x}}^{s}$ and $\nu_{\gamma\tilde{x}}^{s}$ are supported by $\Gamma\tilde{x}$, and for each $\gamma' \in \Gamma$

$$\frac{d\nu_{\tilde{x}}^{s}}{d\nu_{\gamma\tilde{x}}^{s}}(\gamma'\tilde{x}) = \exp\{-s\tilde{d}(\tilde{x},\gamma'\tilde{x}) + s\tilde{d}(\gamma\tilde{x},\gamma'\tilde{x})\}.$$

Let f: $\tilde{M} \cup \partial\tilde{M} \to \mathbf{R}$ be continuous, and define f_γ: $\tilde{M} \cup \partial\tilde{M} \to \mathbf{R}$ by

$$f_\gamma(\tilde{y}) = f(\tilde{y}) \exp\{\tilde{d}(\tilde{x},\tilde{y}) - \tilde{d}(\gamma\tilde{x},\tilde{y})\}, \quad \tilde{y} \in \tilde{M};$$
$$f_\gamma(\xi) = f(\xi) \exp\{C(\tilde{x},\gamma\tilde{x},\xi)\}, \qquad \xi \in \partial\tilde{M}.$$

Then by Lemma 5.3, f_γ is continuous. Since $|\tilde{d}(\tilde{x},\tilde{y}) - \tilde{d}(\gamma\tilde{x},\tilde{y})| \leq d(\tilde{x},\gamma\tilde{x})$ for all $\tilde{y} \in \tilde{M}$, we have, as $s \downarrow 1$

$$\int f d\nu_{\gamma\tilde{x}}^{s} = \int f(\tilde{y}) \exp\{s\tilde{d}(\tilde{x},\tilde{y}) - s\tilde{d}(\gamma\tilde{x},\tilde{y})\} d\nu_{\tilde{x}}^{s}(\tilde{y}) \sim \int f_\gamma d\nu_{\tilde{x}}^{s}.$$
$$\Rightarrow \int f d\mu_{\gamma\tilde{x}} = \int f_\gamma d\mu_{\tilde{x}}.$$

Since this holds for any continuous f, it follows that $\mu_{\tilde{x}}$ and $\mu_{\gamma\tilde{x}}$ are mutually absolutely continuous, and that the Radon-Nikodym derivative is given by (5.7). □

LEMMA 5.7: *Let $s_n \downarrow 1$ be a sequence such that $\nu_{\tilde{x}}^{s_n}$ converges weak-* to a probability measure $\mu_{\tilde{x}}$. Then $\mu_{\tilde{x}}$ has no atoms.*

PROOF: By Lemma 5.6, for each $\gamma \in \Gamma$ the measures $\nu_{\gamma\tilde{x}}^{s_n}$ converge weak-* to a probability measure $\mu_{\gamma\tilde{x}}$ which is related to $\mu_{\tilde{x}}$ by (5.7)–(5.8). If $\xi \in \partial\tilde{M}$ were an atom for $\mu_{\tilde{x}}$ of size $p > 0$, then by (5.7) ξ would be an atom for $\mu_{\gamma\tilde{x}}$ of size $p \cdot \exp\{C(\tilde{x}, \gamma\tilde{x}, \xi)\}$. But $C(\tilde{x}, \gamma\tilde{x}, \xi)$ can be made arbitrarily large by a suitable choice of $\gamma \in \Gamma$; this contradicts the fact that each $\mu_{\gamma\tilde{x}}$ is a *probability* measure.

(To see that $C(\tilde{x}, \gamma\tilde{x}, \xi)$ can be made large, consider the geodesic ray $\tilde{\alpha}_{\tilde{x},\xi}$ from $\tilde{x}$ to ξ. For each point $\tilde{y}$ on $\tilde{\alpha}_{\tilde{x},\xi}$ there exists $\gamma \in \Gamma$ such that $\tilde{d}(\tilde{y}, \gamma\tilde{x}) \leq r$ where r is the d-diameter of M. Now choose $\tilde{y}$ on $\tilde{\alpha}_{\tilde{x},\xi}$ so that $\tilde{d}(\tilde{x}, \tilde{y})$ is large and use Lemma 5.4.) □

LEMMA 5.8: *Let $s_n \downarrow 1$ be a sequence such that $\nu_{\tilde{x}}^{s_n}$ converges weak-* to a probability measure $\mu_{\tilde{x}}$. Then for every $\varepsilon > 0$ there exists $r < \infty$ such that for every $\tilde{y} \in \tilde{M}$,*

$$\mu_{\tilde{x}}(B(\tilde{y} : \tilde{x}, r)) > 1 - \varepsilon.$$

PROOF: By Lemma 5.7 the measure $\mu_{\tilde{x}}$ has no atoms; consequently, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every arc J of the circle $\partial\tilde{M}$ of arclength $\leq \delta$, $\mu_{\tilde{x}}(J) < \varepsilon$. Therefore, to prove the lemma it suffices to show that there exists $r > 0$ so large that for every $\tilde{y} \in \tilde{M}$ the arclength of $\partial\tilde{M} \backslash B(\tilde{y}: \tilde{x}, r)$ is $\leq \delta$.

If this were *not* the case then there would be real numbers $r_n \uparrow \infty$, points $\tilde{y}_n \in \tilde{M}$ converging to some $\xi \in \partial\tilde{M}$, and points $\zeta_n \in \partial\tilde{M}$ converging to some $\zeta \in \partial\tilde{M}$, $\zeta \neq \xi$, such that for each $n = 1, 2, \ldots$ the geodesic ray $\tilde{\alpha}_{\tilde{y}_n,\zeta_n}$ does not enter the $\tilde{d}$-ball of radius r_n centered at $\tilde{x}$. We will show that this is impossible.

Choose distinct points ξ_1, ξ_2, ξ_3, $\xi_4 \in \partial\tilde{M}$ arranged in the order ξ_1, ξ_2, ξ_3, ξ_4 on $\partial\tilde{M}$ and in such a way that ξ and ζ are in opposite open intervals $\xi_i\xi_{i+1}$ (e.g., $\xi \in \xi_1\xi_2$ and $\zeta \in \xi_3\xi_4$). For each $i = 1, 2, 3, 4$ let $\tilde{\alpha}_i$ be the geodesic ray from $\tilde{x}$ to ξ_i. Then $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4$ divide $\tilde{M}$ into "quadrants", with ξ, ζ on the boundaries of nonadjacent quadrants. We may arrange it so that $\xi, \zeta, \xi_1, \xi_2, \xi_3, \xi_4$ are distinct points on $\partial\tilde{M}$ and so that they are arranged in the order $\xi_1, \xi, \xi_2, \xi_3, \zeta, \xi_4$.

For sufficiently large n the geodesic ray $\tilde{\alpha}_{\tilde{y}_n,\zeta_n}$ must either cross $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ or cross $\tilde{\alpha}_4$ and $\tilde{\alpha}_1$. Assume the former. By the Gauss-Bonnet formula, the areas of the geodesic triangles formed by $\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_{\tilde{y}_n,\zeta_n}$ are bounded above. But the area of that part of the $\tilde{d}$-ball of radius r_n centered at $\tilde{x}$ lying in the quadrant between $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ converges to ∞ as $n \to \infty$. Therefore, it must be that $\tilde{\alpha}_{\tilde{y}_n,\zeta_n}$ enters the $\tilde{d}$-ball of radius r_n centered at $\tilde{x}$. □

Say that a Borel subset A of $\partial\tilde{M}$ is Γ-invariant if for every $\gamma \in \Gamma$, $A = \gamma A$.

LEMMA 5.9: *Let $s_n \downarrow 1$ be a sequence such that $\nu_{\tilde{x}}^{s_n}$ converges weak-* to a probability measure $\mu_{\tilde{x}}$. Then for any Γ-invariant Borel subset A of $\partial\tilde{M}$,*

$$\mu_{\tilde{x}}(A) = 0 \quad or \quad 1.$$

PROOF: We use essentially the same argument as in [Ni]. Assume that $\mu_{\tilde{x}}(A) > 0$; it suffices to prove that for any $\varepsilon > 0$, $\mu_{\tilde{x}}(A) \geq 1 - \varepsilon$.

Since A has positive $\mu_{\tilde{x}}$-measure, it has a point of density $\zeta \in \partial\tilde{M}$. Fix $r > 0$ large, and take a sequence $\gamma_n \in \Gamma$ such that $\gamma_n \tilde{x} \to \zeta$ (in the Euclidean topology on $\tilde{M} \cup \partial\tilde{M}$) and such that each point $\gamma_n \tilde{x}$ is within $\tilde{d}$-distance r of the geodesic ray from $\tilde{x}$ to ζ. Then for each $n \geq 1$, $\zeta \in B(\tilde{x}\colon \gamma_n \tilde{x}, r)$. Moreover, since $\gamma_n \tilde{x} \to \zeta$, the shadows $B(\tilde{x}\colon \gamma_n \tilde{x}, r)$ shrink to ζ as $n \to \infty$, by Lemma 5.2. Since ζ is a point of density of A, it follows that

$$\lim_{n\to\infty} \frac{\mu_{\tilde{x}}(A \cap B(\tilde{x}\colon \gamma_n \tilde{x}, r))}{\mu_{\tilde{x}}(B(\tilde{x}\colon \gamma_n \tilde{x}, r))} = 1$$

and

$$\mu_{\tilde{x}}(B(\tilde{x}\colon \gamma_n \tilde{x}, r)) > 0 \quad \forall\, n \geq 1.$$

Since A is Γ-invariant, $1_A = 1_A \circ \gamma$ for each $\gamma \in \Gamma$. Thus, by Lemma 5.6, for each $\gamma \in \Gamma$ and $r > 0$

$$\begin{aligned}
\mu_{\tilde{x}}(A) &= \int 1_A d\mu_{\tilde{x}} \\
&= \int 1_A d\mu_{\gamma\tilde{x}} \\
&= \int 1_A(\xi) \exp\{C(\tilde{x}, \gamma\tilde{x}, \xi)\} d\mu_{\tilde{x}}(\xi) \\
&\geq \int_{A \cap B(\tilde{x}:\gamma\tilde{x}, r)} \exp\{C(\tilde{x}, \gamma\tilde{x}, \xi)\} d\mu_{\tilde{x}}(\xi).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{B(\tilde{x}:\ \gamma\tilde{x}, r)} \exp\{C(\tilde{x}, \gamma\tilde{x}, \xi)\} d\mu_{\tilde{x}}(\xi) \\
&\quad = \int_{B(\tilde{x}:\gamma\tilde{x}, r)} d\mu_{\gamma\tilde{x}} \\
&\quad = \int 1_{B(\tilde{x}:\gamma\tilde{x}, r)} \circ \gamma^{-1} d\mu_{\tilde{x}} \\
&\quad = \mu_{\tilde{x}}(B(\gamma^{-1}\tilde{x}\colon \tilde{x}, r))
\end{aligned}$$

By Lemma 5.8, there exists $r > 0$ sufficiently large that $\mu_{\tilde{x}}(B(\tilde{y}:\tilde{x},r)) > 1-\varepsilon$ for every $\tilde{y} \in \tilde{M}$; consequently, for such r,

$$\begin{aligned}
\mu_{\tilde{x}}(A) &\geq \int_{A\cap B(\tilde{x}:\gamma\tilde{x},r)} \exp\{C(\tilde{x},\gamma\tilde{x},\xi)\} d\mu_{\tilde{x}}(\xi) \\
&> 1-\varepsilon - \int_{A^c\cap B(\tilde{x}:\gamma\tilde{x},r)} \exp\{C(\tilde{x},\gamma\tilde{x},\xi)\} d\mu_{\tilde{x}}(\xi) \\
&\geq 1-\varepsilon - \{\frac{\mu_{\tilde{x}}(A^c \cap B(\tilde{x}:\gamma\tilde{x},r))}{\mu_{\tilde{x}}(B(\tilde{x}:\gamma\tilde{x},r))}\} e^{4r} \int_{B(\tilde{x}:\gamma,r)} \exp\{C(\tilde{x},\gamma\tilde{x},\xi)\} d\mu_{\tilde{x}}(\xi) \\
&\geq 1-\varepsilon - \{\frac{\mu_{\tilde{x}}(A^c \cap B(\tilde{x}:\gamma\tilde{x},r))}{\mu_{\tilde{x}}(B(\tilde{x}:\gamma\tilde{x},r))}\} e^{4r}
\end{aligned}$$

because $\exp\{C(\tilde{x},\gamma\tilde{x},\xi)\}/\exp\{C(\tilde{x},\gamma\tilde{x},\xi'\} \leq e^{4r}$ for all $\xi,\xi' \in B(\tilde{x}:\gamma\tilde{x},r)$, by Lemma 5.4. Setting $\gamma = \gamma_n$ and applying the result of the preceding paragraph, we conclude that $\mu_{\tilde{x}}(A) \geq 1-\varepsilon$. □

PROOF *of Proposition 5.1*: It suffices, by Lemma 5.5 and the Helly selection theorem, to prove that for each $\tilde{x} \in \tilde{M}$ the measures $\nu_{\tilde{x}}^s$ have only one weak* limit as $s \downarrow 1$. Suppose there are sequences $s_n \downarrow 1$ and $t_n \downarrow 1$ such that

$$\begin{aligned}
\mu_{\tilde{x}}^{(1)} &= \text{weak-}* \lim \nu_{\tilde{x}}^{s_n}, \\
\mu_{\tilde{x}}^{(2)} &= \text{weak-}* \lim \nu_{\tilde{x}}^{t_n}.
\end{aligned}$$

Then Lemma 5.6 implies that for each $\gamma \in \Gamma$ there exist probability measures $\mu_{\gamma\tilde{x}}^{(i)}$, $i=1,2$, such that $\nu_{\gamma\tilde{x}}^{s_n} \xrightarrow{\omega*} \mu_{\gamma\tilde{x}}^{(1)}$ and $\nu_{\gamma\tilde{x}}^{t_n} \xrightarrow{\omega*} \mu_{\gamma\tilde{x}}^{(2)}$ and

$$\frac{d\mu_{\gamma\tilde{x}}^{(i)}}{d\mu_{\tilde{x}}^{(i)}}(\xi) = \exp\{C(\tilde{x},\gamma\tilde{x},\xi)\}, \quad i=1,2; \tag{5.9}$$

$$\int f d\mu_{\tilde{x}}^{(i)} = \int f\circ\gamma d\mu_{\gamma\tilde{x}}^{(i)} \quad \forall\, \gamma \in \Gamma, \quad i=1,2. \tag{5.10}$$

Set $\bar{\mu}_{\gamma\tilde{x}} = \frac{1}{2}(\mu_{\gamma\tilde{x}}^{(1)} + \mu_{\gamma\tilde{x}}^{(2)})$; then $\mu_{\gamma\tilde{x}}^{(1)}$ is absolutely continuous with respect to $\bar{\mu}_{\gamma\tilde{x}}$. Set $h_{\gamma\tilde{x}}(\xi) = (d\mu_{\gamma\tilde{x}}^{(1)}/d\bar{\mu}_{\gamma\tilde{x}})(\xi)$; then by (5.10)

$$\begin{aligned}
&\int (f\circ\gamma)(h_{\tilde{x}}\circ\gamma) d\bar{\mu}_{\gamma\tilde{x}} \\
&= \int f h_{\tilde{x}} d\bar{\mu}_{\tilde{x}} \\
&= \int f d\mu_{\tilde{x}}^{(1)} \\
&= \int f\circ\gamma d\mu_{\gamma\tilde{x}}^{(1)} \\
&= \int (f\circ\gamma) h_{\gamma\tilde{x}} d\bar{\mu}_{\gamma\tilde{x}}
\end{aligned}$$

for all f: $\partial\tilde{M} \to \mathbf{R}$ continuous and all $\gamma \in \Gamma$. On the other hand, (5.9) implies that $h_{\gamma\tilde{x}} = h_{\tilde{x}}$ for every $\gamma \in \Gamma$. Thus

$$h_{\tilde{x}} \circ \gamma = h_{\tilde{x}} \quad \text{a.e. } (d\bar{\mu}_{\tilde{x}}).$$

It now follows from Lemma 5.9 that $h_{\tilde{x}}$ is a.e. $(\bar{\mu}_{\tilde{x}})$ constant, hence $\mu_{\tilde{x}}^{(1)} = \mu_{\tilde{x}}^{(2)}$.

NOTE: We are justified in applying (5.10) to $fh_{\tilde{x}}$ above despite the fact that $h_{\tilde{x}}$ may not be continuous. This is a standard argument in measure theory: the set $\mathcal{F}$ of Borel sets B such that $\int 1_B d\mu_{\tilde{x}}^{(i)} = \int 1_B \circ \gamma d\mu_{\gamma\tilde{x}}^{(i)}$ is a σ-algebra, and if (5.10) holds for all continuous f then $\mathcal{F}$ contains all closed sets, and consequently $\mathcal{F} = \{\text{Borel sets}\}$. Therefore, (5.10) holds for all simple functions, and, by taking monotone limits, for all nonnegative measurable functions. □

PROPOSITION 5.10: *For each $\tilde{x} \in \tilde{M}$ and all sufficiently large $r > 0$ there exist constants $0 < C_1 \leq C_2 < \infty$ such that for every $\gamma \in \Gamma$*

$$C_1 \leq \frac{\nu_{\tilde{x}}(B(\tilde{x}\colon \gamma\tilde{x}, r))}{\exp\{-\tilde{d}(\tilde{x}, \gamma\tilde{x})\}} \leq C_2.$$

PROOF: By Lemma 5.6,

$$\nu_{\tilde{x}}(B(\tilde{x}\colon \gamma\tilde{x}, r)) = \int_{B(\tilde{x}:\gamma\tilde{x},r)} \exp\{-C(\tilde{x}, \gamma\tilde{x}, \xi)\} d\nu_{\gamma\tilde{x}}(\xi)$$

and

$$\nu_{\gamma\tilde{x}}(B(\tilde{x}\colon \gamma\tilde{x}, r)) = \nu_{\tilde{x}}(B(\gamma^{-1}\tilde{x}\colon \tilde{x}, r)).$$

By Lemma 5.8, $\nu_{\tilde{x}}(B(\tilde{y}\colon \tilde{x}, r)) \geq 1/2$ for all $\tilde{y} \in \tilde{M}$ and all sufficiently large $r > 0$. Therefore, the result follows from Lemma 5.4. □

6. Patterson Measures and Mostow Rigidity

In this section we shall use the Bishop-Steger criterion (Theorem 1) and some of the machinery of sec. 4 to prove Theorem 3. Assume that g_1 and g_2 are Riemannian metrics on M (normalized so that the associated geodesic flows have topological entropy 1), and let $\nu_{\tilde{x}}^{(1)}$, $\nu_{\tilde{x}}^{(2)}$ be the Patterson measure (cf. Prop. 5.1) for the distances $\tilde{d}_1$, $\tilde{d}_2$, respectively. Let $C_1(\tilde{x}, \tilde{y}, \xi)$ and $C_2(\tilde{x}, \tilde{y}, \xi)$ be the functions defined by (5.2) for $\tilde{d}_1$ and $\tilde{d}_2$.

PROPOSITION 6.1: *If for some $\tilde{x} \in \tilde{M}$*

$$\sum_{\gamma\in\Gamma} \exp\{-\frac{1}{2}(\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) + \tilde{d}_2(\tilde{x}, \gamma\tilde{x}))\} = \infty \tag{6.1}$$

then $\nu_{\tilde{x}}^{(1)}$ *and* $\nu_{\tilde{x}}^{(2)}$ *are mutually absolutely continuous.*

PROOF: If the sum is infinite then by Theorem 1 the metrics g_1 and g_2 have the same marked length spectrum. Hence, by Prop. 4.4, the height functions r_1 and r_2 of Prop. 4.6 are cohomologous, and therefore by [Bo$_2$], Th. 1.28, there exists a constant $K < \infty$ such that $\|S_n r_1 - S_n r_2\|_\infty \leq K$ for all $n \geq 1$. It now follows from (4.11) that for every $\gamma \in \Gamma$

$$|\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) - \tilde{d}_2(\tilde{x}, \gamma\tilde{x})| \leq K + 2(\|r_1\|_\infty + \|r_2\|_\infty + C).$$

Consequently, there are constants $0 < C_1 < C_2 < \infty$ such that for all $s > 1$

$$C_1(\nu_{\tilde{x}}^s)_{g_1} \leq (\nu_{\tilde{x}}^s)_{g_2} \leq C_2(\nu_{\tilde{x}}^s)_{g_1},$$

so $\nu_{\tilde{x}}^{(1)}$ and $\nu_{\tilde{x}}^{(2)}$ are mutually absolutely continuous, with $d\nu_{\tilde{x}}^{(2)}/d\nu_{\tilde{x}}^{(1)}$ bounded above and below by C_1 and C_2. □

PROPOSITION 6.2: *If for some* $\tilde{x} \in \tilde{M}$

$$\sum_{\gamma\in\Gamma} \exp\{-\frac{1}{2}\tilde{d}_1(\tilde{x}, \gamma\tilde{x}) - \frac{1}{2}\tilde{d}_2(\tilde{x}, \gamma\tilde{x})\} < \infty \tag{6.2}$$

then $\nu_{\tilde{x}}^{(1)}$ *and* $\nu_{\tilde{x}}^{(2)}$ *are mutually singular.*

PROOF: Set $\Gamma_1 = \{\gamma \in \Gamma: \tilde{d}_1(\tilde{x}, \gamma\tilde{x}) \geq \tilde{d}_2(\tilde{x}, \gamma\tilde{x})\}$ and $\Gamma_2 = \Gamma \backslash \Gamma_1$. If (6.1) holds then

$$\sum_{\gamma\in\Gamma_1} \exp\{-\tilde{d}_1(\tilde{x}, \gamma\tilde{x})\} < \infty \tag{6.3}$$

and

$$\sum_{\gamma\in\Gamma_2} \exp\{-\tilde{d}_2(\tilde{x}, \gamma\tilde{x})\} < \infty. \tag{6.4}$$

Consider the "shadows" $B_i(\tilde{x}: \tilde{y}, R)$ for the metrics $\tilde{g}_i$; thus, $B_i(\tilde{x}: \tilde{y}, R)$ is the set of all $\xi \in \partial\tilde{M}$ such that the $\tilde{g}_i$-geodesic ray starting at $\tilde{x}$ and terminating at ξ enters the $\tilde{d}_i$-ball of radius R centered at $\tilde{y}$. Since the metrics $\tilde{g}_1$ and $\tilde{g}_2$ are comparable (i.e., $C_1\tilde{g}_1 \leq \tilde{g}_2 \leq C_2\tilde{g}_1$), for each $R_1 > 0$ there exists $R_2 > 0$ such that $\forall\, \tilde{x}, \tilde{y}$, $B_1(\tilde{x}: \tilde{y}, R_1) \subset B_2(\tilde{x}: \tilde{y}, R_2)$; conversely, for each $R_2 > 0$ there exists $R_1 > 0$ such that $B_2(\tilde{x}: \tilde{y}, R_2) \subset B_1(\tilde{x}: \tilde{y}, R_1)$.

Fix $R > d_1$-diameter (M). For each $t \geq R$ define

$$\begin{aligned}
\Gamma(t) &= \{\gamma \in \Gamma: t - R \leq \tilde{d}_1(\tilde{x}, \gamma\tilde{x}) \leq t + R\}, \\
\Gamma(t)^* &= \{\gamma \in \Gamma(t): \tilde{d}_1(\tilde{x}, \gamma\tilde{x}) < \tilde{d}_2(\tilde{x}, \gamma\tilde{x})\}, \\
A(t) &= \cup_{\gamma\in\Gamma(t)} B_1(\tilde{x}: \gamma\tilde{x}, R), \\
A(t)^* &= \cup_{\gamma\in\Gamma(t)^*} B_1(\tilde{x}: \gamma\tilde{x}, R).
\end{aligned}$$

Then for each $t \geq R$, $A(t) = \partial \tilde{M}$, because every $\tilde{g}_1$-geodesic ray that starts at $\tilde{x}$ must pass within distance R of some $\gamma\tilde{x}$ at distance $t \pm R$ of $\tilde{x}$. Hence, $\nu_{\tilde{x}}^{(1)}(A(t)) = 1$ for each $t \geq R$. It now follows from Proposition 5.10 and (6.3) that $\lim_{t\to\infty} \nu_{\tilde{x}}^{(1)}(A(t)\backslash A(t)^*) = 0$, so

$$\lim_{t\to\infty} \nu_{\tilde{x}}^{(1)}(A(t)^*) = 1.$$

But by the preceding paragraph, (6.4), and Proposition 5.10,

$$\lim_{t\to\infty} \nu_{\tilde{x}}^{(2)}(A(t)^*) = 0. \qquad \square$$

PROOF *of Theorem 3*: By Theorem 1, g_1 and g_2 have the same marked length spectrum iff (6.1) holds. Therefore, by Propositions 6.1–6.2, (6.1) holds iff $\nu_{\tilde{x}}^{(1)}$ and $\nu_{\tilde{x}}^{(2)}$ are mutually absolutely continuous; otherwise $\nu_{\tilde{x}}^{(1)}$ and $\nu_{\tilde{x}}^{(2)}$ are mutually singular. $\square$

REFERENCES

[Ah] Ahlfors, L. *Lectures on QuasiConformal Mappings.* Van Nostrand Press, Princeton (1966).

[An] Anosov, D. Geodesic flows on closed Riemannian manifolds of negative curvature. *Proc. Steklov Inst. Math.* **90** (1967).

[AS] Ansov, D. and Sinai, Ja. Some smooth ergodic systems. *Russian Math. Surveys* **22** (1967) 103–168.

[BC] Bishop, R. and Crittenden, R. *Geometry of Manifolds.* Academic Press, New York (1964).

[BS] Bishop, C. and Steger, T. To appear in *Acta. Math.*

[Bo$_1$] Bowen, R. Symbolic Dynamics for hyperbolic flows, *Amer. J. Math.* **95** (1973), 429–450.

[Bo$_2$] Bowen, R. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphism*, Springer Lecture Notes in Math. **470** (1975).

[BR] Bowen, R. and Ruelle, D. Ergodic theory of Axiom A flows. *Inv. Math.* **29** (1975), 181–202.

[Cr] Croke, C. Rigidity for surfaces of non-positive curvature. *Comm. Math. Helv.* **65** (1990), 150–169.

[De] Dehn, M. *Papers on Group Theory and Topology.* Springer (1987).

[GR] Grauert, H. and Reckziegel, H. Hermitesche Metriken und normale Familien holomorpher Abbildungen. *Math. Zeit.* **89** (1963), 108–125.

[HP] Hirsch, M. and Pugh, C. Smoothness of horocycle foliations. *J. Diff. Geom.* **10** (1975), 255–238.

[K] Katok, A. Four applications of conformal equivalence to geometry and dynamics. *Ergodic Th. Dynamical Systems* **8*** (1988), 139–152.

[La] Lalley, S. Renewal theorems in symbolic dynamics, with applications to geodesic flows, noneuclidean tessellations, and their fractal limits. *Acta. Math.* **163** (1989), 1–55.

[Ma] Manning, A. Topological entropy for geodesic flows. *Ann. Math.* **110** (1979), 567–573.

[M] Margulis, G. Applications of ergodic theory to the investigation of manifolds of negative curvature. *Funktsional. An. i Prilozhen* **3** (1969), 89–90.

[Mo] Moser, J. On a theorem of Anosov. *J. Diff. Eq.* **5** (1969), 411–440.

[M] Mostow, G. *Strong Rigidity of Locally Symmetric Spaces.* Princeton U. Press, Princeton (1973).

[Ni] Nicholls, P. The Ergodic Theory of Discrete Groups. Cambridge U. Pess,Cambridge (1989).

[PP] Parry, W. and Pollicott, M. An analogue of the prime number theorem for closed orbits of Axiom A flows. *Ann. Math.* **118** (1983) 573–591.

[P] Patterson, S. J. The limit set of a Fuchsiun group. *Acta Math.* **136** (1976), 241–273.

[Po] Pollicott, M. Analytic extensions of the zeta function for surfaces of variable negative curvature. *J. Diff. Geom* **29**, 699–706.

[Ra] Ratner, M. Markov partitions for Anosov flows on n-dimensional manifolds. *Israel J. Math.* **15** (1973), 92–114.

[Ro] Robinson, C. Structural stability of vector fields. *Ann. Math.* **99** (1974), 154–175.

[Ry] Royden, H. *Real Analysis*. MacMillan, New York (1988).

[Ru] Ruelle, D. *Thermodynamic Formalism.* Addison-Wesley, Reading MA (1978).