NONLINEAR MARKOV RENEWAL THEORY

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Abstract

Let $\{S_n\}$ be a Markov random walk satisfying the conditions of Keston's Markov renewal theorem. It is shown that if $\{Z_n\}$ is a stochastic process whose finite-dimensional, conditional distributions are asymptotically close to those of $\{S_n\}$ (in the sense of weak convergence), then $\{Z_n\}$ also satisfies a renewal theorem. In the traditional setting of nonlinear renewal theory, where $\{Z_n\}$ is represented as a perturbed Markov random walk, this allows substantial weakening of the slow change condition on the perturbation process; more importantly, no such representation is required.

Keywords and Phrases: Markov random walk, nonlinear renewal theory, Prokhorov metric, Markov renewal theory.

1. Introduction

Let $S_0, S_1, \ldots$ be a stochastic process for which a renewal theorem is known, *i.e.*, for which it is known that the overshoot $\{S_{\tau_a} - a : a \geq 0\}$ converges in distribution (to a known limiting distribution) as $a \to \infty$. Here $\tau_a = \inf\{n \geq 1 : S_n > a\}$.

In many applications, especially in statistics, what is needed is a renewal theorem for a process $Z_0, Z_1, \ldots$ which is asymptotically close to $\{S_n\}$ in some sense. This has spurred the development of such renewal theorems, usually called nonlinear renewal theorems, during the past fifteen years. (The adjective "nonlinear" is used because such theorems may be used to obtain the limiting distribution of the overshoot of the original process $\{S_n\}$ over a nonlinear boundary.)

When S_n is the n^{th} partial sum of iid random variables (i.e. a random walk) satisfying certain conditions (e.g. the distribution of the summands is nonarithmetic and has finite positive mean), it is well–known (Blackwell's Renewal Theorem) that $\{S_{\tau_a} - a\}$ converges in distribution as $a \to \infty$. In this setting, nonlinear renewal theory has been explored by Lai and Siegmund (1977), Lalley (1982), Zhang (1988), and Woodroofe (1990), among others. In all but the last reference, it is assumed that $\{Z_n\}$ is a perturbed random walk, *i.e.*, $Z_n = S_n + \xi_n$, where the sequence $\{\xi_n\}$ of perturbations satisfies a slow change condition such as

(1) $$\tfrac{1}{n}\max\{|\xi_1|, \ldots, |\xi_n|\} \to 0 \text{ in probability as } n \to \infty; \text{ and}$$

(2) $$\lim_{\delta \to 0} \sup_{n \geq 1} \mathbb{P}\{\max_{0 \leq k \leq n\delta} |\xi_{n+k} - \xi_n| \geq \epsilon\} = 0 \quad \forall \epsilon > 0.$$

(Actually, somewhat more general conditions of the same type are allowed in the above references, but this is not relevant here.) The paper of Woodroofe (1990) presents a more general formulation which does not require that $\{Z_n\}$ is a perturbed random walk; rather, it requires closeness of conditional distributions.

Each of the papers above contains applications of the theory to probability and statistics. For surveys of the theory, as well as many more applications, see Woodroofe (1982) and Siegmund (1984, 1985).

Recently, Su (1990) and Melfi (1992) have developed nonlinear renewal theory when $\{S_n\}$ is a Markov random walk. (Recall the definition of a Markov random walk. It begins with a Markov chain $Y_0, Y_1, \ldots$ with state space E. The summands X_i have the property that the conditional distribution of X_n given $\{Y_i : i \geq 0\}$ and $\{X_j : j \neq n\}$ depends only on Y_{n-1} and Y_n. The process $\{Y_n, X_n\}$ is called a *Markov renewal process*, and $\{S_n = X_1 + \ldots + X_n : n \geq 1\}$ is called a *Markov random walk.* A more precise definition is given in the next section.) In Su (1990) the state space E of the Markov chain is required to be finite; in Melfi (1992) it can be any complete separable metric space.

Both of the above references require that $Z_n = S_n + \xi_n$, with $\{\xi_n\}$ satisfying a slow change condition analogous to that given above. The purpose of this paper is to present a renewal theory for processes $\{Z_n\}$ which satisfy an alternative (much weaker) condition of asymptotic closeness to $\{S_n\}$. This condition does not require any representation of the form $Z_n = S_n + \xi_n$; rather, it only requires that finite dimensional, conditional distributions coalesce (in the sense of the Prokhorov metric). In one direction, this allows weakening of the slowly changing conditions in the case where $Z_n = S_n + \xi_n$. More interestingly, it allows processes $\{Z_n\}$ which have no such representation; for example, $\{Z_n\}$ may have a dependence structure like that of a Markov random walk, but where the driving process is *not* Markovian. One motivation for developing renewal theory for such processes is the study of clinical trials where allocation is adaptive and randomized. Details of such applications will be worked out in a future paper.

In the next section Kesten's Markov renewal theorem is described, along with a result which shows that the convergence in this theorem holds uniformly (in the starting point of the Markov chain) on compacts. In Section 3, nonlinear extensions of Kesten's Markov renewal theorem are presented. These theorems are proved in Section 4. Section 5 contains some comments on the applicability of these theorems, as well as some techniques for verifying some of the conditions. In section 6, a machine breakdown example is presented.

2. Kesten's Markov Renewal Theorem

Markov renewal theorems have been proved by a variety of authors; notably, Orey

(1961), Jacod (1971), Kesten (1974), and Athreya, McDonald, and Ney (1978). At present, the most general result available seems to be that of Kesten, so this theorem will be the embarkation point for the nonlinear renewal theory below.

The statement of Kesten's Markov renewal theorem follows. Let (E, d) be a separable metric space with Borel σ-algebra $\mathcal{E}$. Begin with a stochastic transition kernel Q on E, i.e. a function $Q: E \times \mathcal{E} \to [0,1]$ which is a measurable function for fixed $A \in \mathcal{E}$ and is a probability measure (on $\mathcal{E}$) for fixed $y \in E$, and let $Y_0, Y_1, \ldots$ be a Markov chain with transition kernel Q, so that for $A \in \mathcal{E}$ and $n, k \geq 1$,

$$\mathsf{P}\{Y_{n+k} \in A | Y_0, Y_1, \ldots, Y_n\} = Q^k(Y_n; A). \tag{3}$$

The summands $\{X_n\}$ are required to satisfy

$$\mathcal{D}(X_n | \{Y_i : i \geq 0\}, \{X_j : j \neq n\}) = F(\cdot | Y_{n-1}, Y_n), \tag{4}$$

where $\{F(\cdot|x, y): x \in E, y \in E\}$ is a family of distributions on $\mathcal{B}(\mathsf{R})$. Define $S_0 = 0$ and

$$S_n = X_1 + \ldots + X_n, n \geq 1. \tag{5}$$

Then $\{S_n\}$ is a Markov random walk.

It is convenient to assume that $\{Y_n, X_n\}$ is the coordinate process on the canonical probability space $(\Omega, \mathcal{F}) = ((E \times \mathsf{R})^{\mathsf{N}}, (\mathcal{E} \times \mathcal{B})^{\mathsf{N}})$, where N denotes the nonnegative integers and $\mathcal{B}$ is the Borel σ-algebra on R. Also, for $y \in E$, P_y represents the probability measure pertaining to paths with $Y_0 = y$. (For details on the construction of these processes, see Melfi (1992) and Revuz (1984).)

2.1. Kesten's Conditions

Two of the conditions for Kesten's Theorem, (K2) and (K3) below, are rather mundane; (K2) is analogous to the condition that the mean is positive and finite in ordinary renewal theory, while (K3) is an aperiodicity condition. Condition (K1) is a recurrence condition on the Markov chain which is satisfied for some non–Harris recurrent Markov chains (since it only requires that $\{Y_n\}$ return to ϕ–positive *open* sets w.p.1), but requires the existence of a *finite* invariant measure. Condition (K4) is a continuity condition which is crucial for proving uniform convergence in Theorem 1 below. Some explanation of (K4) will be given after its statement.

From now on, the notation "a.s." means "a.e. $[P_y]$ for each y." For $f: (E \times \mathbf{R})^{\mathbf{N}} \to \mathbf{R}$ and $\delta > 0$, define

$$
\begin{aligned}
& f^{\delta}(y_0, s_0, y_1, s_1, \ldots) \\
(6) \qquad & = \lim_{m\to\infty} \sup\{f(y_0', s_0', y_1', s_1', \ldots): d(y_i, y_i') + |s_i - s_i'| < \delta \quad \forall \quad i \le m\} \\
& = \lim_{m\to\infty} f_m^{\delta}, \text{ say.}
\end{aligned}
$$

(This will be used in (K4) below). Note that f^{δ} is $(\mathcal{E} \times \mathcal{B})^{\mathbf{N}}$–measurable for every (not necessarily measurable) f. In fact, it may be shown that for every f, f_m^{δ} is lower semicontinuous for each m. The conditions follow.

(K1) There exists a probability measure ϕ on $\mathcal{E}$ which is invariant for Q, i.e., for every $A \in \mathcal{E}$,

$$\phi(A) = \int \phi(dy) Q(y; A). \tag{7}$$

Also, for all open A with $\phi(A) > 0$,

$$P_y\{Y_n \in A \ \exists\ n \ge 1\} = 1 \text{ for all } y \in E. \tag{8}$$

(K2)

$$
\begin{aligned}
& \int E_y|X_1|\phi(dy) < \infty, \\
\mu := & \int E_y(X_1)\phi(dy) > 0, \text{ and} \\
& \lim_{n\to\infty} \tfrac{1}{n} S_n = \mu \text{ a.s.}
\end{aligned}
$$

(K3) There exists a sequence $\{\zeta_\nu\} \subset \mathbf{R}$ such that the group generated by $\{\zeta_\nu\}$ is dense in $\mathbf{R}$ and such that for each ν and each $\delta > 0$ there is a $z = z(\nu, \delta) \in E$ with the following property: For each $\epsilon > 0$ there is an $A \in \mathcal{E}$ with $\phi(A) > 0$, integers m_1, m_2, and an $\eta \in \mathbf{R}$ such that for each $y \in A$,

$$P_y\{d(Y_{m_1}, z) < \epsilon, |S_{m_1} - \eta| \le \delta\} > 0$$

and

$$P_y\{d(Y_{m_2}, z) < \epsilon, |S_{m_2} - \eta - \zeta_\nu| \le \delta\} > 0.$$

(K4) For each $y \in E$ and $\delta > 0$ there is a $q = q(y, \delta)$ such that for all product measurable functions $f : (E \times \mathbb{R})^{\mathbb{N}} \to \mathbb{R}$,

$$E_y f(Y_0, S_0, Y_1, S_1, \ldots) \le E_z f^\delta(Y_0, S_0, Y_1, S_1, \ldots) + \delta \sup |f|$$

and

$$E_z f(Y_0, S_0, Y_1, S_1, \ldots) \le E_y f^\delta(Y_0, S_0, Y_1, S_1, \ldots) + \delta \sup |f|$$

whenever $d(y, z) < q$.

Some insight into the above condition is provided by the observation that (K4) implies that the transition operators for the whole process $\{Y_n, S_n\}_{n \ge 0}$ are weakly continuous under the product topology on $(E \times \mathbb{R})^{\mathbb{N}}$. In other words, for $y \in E$ and $A \in (\mathcal{E} \times \mathcal{B})^{\mathbb{N}}$, let $N(y; A) = P_y\{(Y_0, S_0, Y_1, S_1, \ldots) \in A\}$. Then (K4) implies that $N(y'; \cdot) \Rightarrow N(y; \cdot)$ as $y' \to y$, where $\Rightarrow$ denotes convergence in distribution.

To see this, let a metric e on $(E \times \mathbb{R})^{\mathbb{N}}$ be defined by

$$e((\mathbf{y}, \mathbf{s}), (\mathbf{y}', \mathbf{s}')) = \sum_{i=0}^{\infty} \frac{1}{2^i} \min\{1, d(y_i, y_i') + |s_i - s_i'|\}.$$

Then e metrizes the product topology (Kelley (1955), p. 122). Now fix $\delta > 0$ and $y \in E$, let $q = q(y, \frac{\delta}{2})$ be given by (K4), and let m be so large that $\frac{1}{2^m} < \frac{\delta}{2}$. For any $(\mathbf{y}, \mathbf{s}) \in (E \times \mathbb{R})^{\mathbb{N}}$ with $d(y_i, y_i') + |s_i - s_i'| < \frac{\delta}{2} \quad \forall \quad i \le m$, $e((\mathbf{y}, \mathbf{s}), (\mathbf{y}', \mathbf{s}')) < \delta$. Thus for any $A \in (\mathcal{E} \times \mathcal{B})^{\mathbb{N}}$,

$$(1_A)^{\delta/2} \le 1_{A^\delta},$$

where $A^\delta = \{\mathbf{z} \in (E \times \mathbb{R})^{\mathbb{N}}; e(\mathbf{z}, A) < \delta\}$.

So, for y' with $d(y, y') < q$ and $A \in (\mathcal{E} \times \mathcal{B})^{\mathbb{N}}$,

$$\begin{aligned} N(y'; A) &= E_{y'} 1_A(Y_0, S_0, Y_1, S_1, \ldots) \\ &\le E_y (1_A)^{\delta/2}(Y_0, S_0, Y_1, S_1, \ldots) + \delta/2 \\ &\le E_y 1_{A^\delta}(Y_0, S_0, Y_1, S_1, \ldots) + \delta/2 \\ &\le N(y; A^\delta) + \delta/2. \end{aligned}$$

Thus the Prokhorov distance between $N(y';\cdot)$ and $N(y;\cdot)$ is less than δ. Since $((E\times\mathbb{R})^{\mathbb{N}}, e)$ is separable, and since the Prokhorov distance metrizes weak convergence on separable metric spaces, $N(y';\cdot) \Rightarrow N(y;\cdot)$ as $y' \to y$.

2.2. Kesten's Markov Renewal Theorem

Let $\{Y_n', X_n'; n \in \mathbb{Z}\}$ be the coordinate process on the space $(\Omega', \mathcal{F}') = ((E\times\mathbb{R})^{\mathbb{Z}}, (\mathcal{E}\times\mathcal{B})^{\mathbb{Z}})$, and let P' be the probability measure on $(\Omega', \mathcal{F}')$ under which $\{Y_n', X_n'\}$ is the two–sided stationary process associated with the original Markov renewal process $\{Y_n, X_n : n \in \mathbb{N}\}$. (Details of this construction may be found in Melfi (1992); see also Doob (1953), p. 456).

Define

$$S_n' = \begin{cases} \sum_{i=1}^{n} X_n' & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum_{i=n+1}^{0} X_n' & \text{if } n < 0, \end{cases}$$

and define a measure ψ on $\mathcal{E}$ by

$$\psi(A) = P'\{\sup_{n<0} S_n' < 0,\ Y_0' \in A\}.$$

Also, let the distribution K be given by

$$K(A\times(r,\infty)) = \frac{1}{\mu}\int_E \psi(dz)\int_{E\times(0,\infty)} P_z\{Y_{\tau_0}\in dw,\ S_{\tau_0}\in d\lambda\}\int_0^{\lambda} 1_{(A\times(r,\infty))}(w,s)ds,$$

where

$$\tau_a = \inf\{n \geq 1 : S_n > a\} \quad a \geq 0. \tag{9}$$

Kesten's Markov Renewal Theorem (Kesten, 1974): Assume that conditions (K1)–(K4) are satisfied. Then for any starting point $y \in E$, $(Y_{\tau_a}, S_{\tau_a} - a)$ has joint limiting distribution K, as $a \to \infty$. In particular, for any $y \in E$ and $r > 0$,

$$\lim_{a\to\infty} P_y\{S_{\tau_a} - a > r\} = \frac{1}{\mu}\int_E \psi(dz)\int_r^{\infty} (\lambda - r)P_z\{S_{\tau_0}\in d\lambda\}.$$

2.3. Uniform Convergence

It is interesting that, with no further conditions, the convergence in Kesten's Theorem holds uniformly (in the starting point) on compacts. That is the content of Theorem 1 below. The proof of Theorem 1 is omitted; a similar result is proved in Melfi (1992). The key to the proof is a clever use of condition (K4), although there are some technical details that need to be addressed. (For those who are interested, the proof of Theorem 1 may be found in Melfi (1991).)

Theorem 1: For each $\epsilon > 0$ and compact set $C \in \mathcal{E}$, there is an $a_0 = a_0(\epsilon, C) < \infty$ such that for all $a \geq a_0$,

$$\rho[K_a^*(y;\cdot), K(\cdot)] < \epsilon \quad \forall \quad y \in C.$$

3. Nonlinear Markov Renewal Theorems

Let $\{W_n : n \geq 0\}$ be a stochastic process taking values in $(E, \mathcal{E})$, and let $\{Z_n : n \geq 0\}$ be a real–valued stochastic process. Both are defined on the same probability space $(\Gamma, \mathcal{A}, \mathbb{P})$. For $a \geq 0$ define

$$t_a = \inf\{n \geq 1 : Z_n > a\}; \tag{10}$$

$$\begin{aligned} R_a &= Z_{t_a} - a; \text{ and} \\ W_a &= W_{t_a}. \end{aligned} \tag{11}$$

Also, let $\{\mathcal{G}_k : k \geq 1\}$ be a filtration for which $\mathcal{G}_k \supseteq \sigma(W_0, \ldots, W_k, Z_1, \ldots, Z_k)$ for all k. Of primary interest below are limiting distributions for R_a and (W_a, R_a).

The Prokhorov distance plays an important role in what follows, so its definition and a few relevant properties are recalled. For probability measures P, Q on a metric space $(\mathcal{X}, d)$ equipped with its Borel σ-algebra $\mathcal{A}$, define the Prokhorov distance ρ between P and Q by

$$\rho(P, Q) = \inf\{\epsilon > 0 : P(A) \leq Q(A^\epsilon) + \epsilon \;\; \forall\, A \in \mathcal{A}\},$$

where $A^\epsilon = \{z \in \mathcal{X} : d(z, A) < \epsilon\}$.

Then ρ is a metric on the set of all probability measures on $\mathcal{A}$ (Dudley, 1989, p. 309). An important property is that if $(\mathcal{X}, d)$ is separable, then the Prokhorov distance metrizes the topology of weak convergence, *i.e.*, if $P, P_1, P_2, \ldots$ are probability measures on $\mathcal{A}$, then

$$\rho(P_n, P) \to 0 \qquad \text{if and only if} \qquad P_n \Rightarrow P. \tag{12}$$

For this and other properties of ρ, see Dudley (1989).

3.1. Limiting Distribution of the Overshoot

Let $\mathcal{G}_a$ and $\{Z_{a,k}: k \geq 1\}$ be the prior σ–algebra and post–t_a delayed process, respectively, i.e., for $a \geq 0$ and $k \geq 1$,

$$\mathcal{G}_a = \{A \in \mathcal{A}: A \cap \{t_a = n\} \in \mathcal{G}_n \text{ for all } n \geq 1\}; \text{ and}$$
$$Z_{a,k} = Z_{t_a+k} - Z_{t_a}.$$

Additionally, for $a \geq 0,\ m \geq 1,\ y \in E,\ B \in \mathcal{B}^m$, and $\gamma \in \Gamma$, let

$$L_{a,m}(\gamma; B) = \mathsf{P}\{(Z_{a,1}, \ldots, Z_{a,m}) \in B | \mathcal{G}_a\}(\gamma); \text{ and}$$
$$L^*_m(y; B) = P_y\{(S_1, \ldots, S_m) \in B\},$$

where $\{S_n\}$ is a Markov random walk. Let ρ_m represent the Prokhorov metric for distributions on $\mathcal{B}^m$. The conditions for a renewal theorem follow.

(I) There exists a Markov random walk satisfying (K1)–(K4) for which

$$\rho_m[L_{a,m}, L^*_m(W_a; \cdot)] \to 0 \text{ in probability}$$

for each $m \geq 1$.

(II) $\{R_a: a \geq 0\}$ is tight.

(III) $\{W_a: a \geq 0\}$ is tight.

Theorem 2: Assume conditions (I)–(III). Then

$$\lim_{a\to\infty} \mathsf{P}\{Z_{t_a} - a > r\} = \frac{1}{\mu} \int \psi(dz) \int_r^\infty (\lambda - r) P_z\{S_{\tau_0} \in d\lambda\},$$

i.e., $\{Z_{t_a} - a\}$ has the same limiting distribution as $\{S_{\tau_a} - a\}$.

3.2. Joint Limiting Distribution

Much of the notation in this section remains the same as in Section 3.1. It will be necessary, however, to define joint analogues of $L_{a,m}$ and L_m^*. So, for $a \geq 0,\ m \geq 1,\ y \in E,\ B \in (\mathcal{E} \times \mathcal{B})^m$, and $\gamma \in \Gamma$, let

$$W_{a,m} = W_{t_a+m};$$

$$\mathbf{L}_{a,m}(\gamma; B) = \mathsf{P}\{(W_{a,1}, Z_{a,1}, \ldots, W_{a,m}, Z_{a,m}) \in B|\mathcal{G}_a\}(\gamma); \text{ and}$$

$$\mathbf{L}_m^*(y; B) = P_y\{(Y_1, S_1, \ldots, Y_m, S_m) \in B\}.$$

Also, let ρ_m be the Prokhorov metric for distributions on $(\mathcal{E} \times \mathcal{B})^m$. The analogue of condition (I) is

(I′) There exists a Markov random walk satisfying (K1)–(K4) for which

$$\rho_m[\mathbf{L}_{a,m}, \mathbf{L}_m^*(W_a; \cdot)] \to 0 \text{ in probability as } a \to \infty$$

for each $m \geq 1$.

Theorem 3: Assume conditions (I′), (II), and (III). Then (W_a, R_a) has joint limiting distribution K, *i.e.*, $(W_{t_a}, Z_{t_a} - a)$ has the same limiting distribution as $(Y_{\tau_a}, S_{\tau_a} - a)$.

4. Proofs

4.1. Proof of Theorem 2.

In this section, all the conditions of Theorem 2 are in force. A bit more notation will facilitate the proof. For $0 < b < \infty$, let K_b^* represent the distribution of the overshoot of the Markov random walk over b, (i.e.),

$$K_b^*(y; A) = P_y\{S_{\tau_b} - b \in A\}$$

for $A \in \mathcal{B}$ and $y \in E$. Also, for $\epsilon > 0$ and $0 < b < \infty$, let $0 = v_0 < v_1 < \ldots < v_p = b$ represent a partition of $[0, b]$ satisfying $\max_{1 \leq i \leq p}(v_i - v_{i-1}) < \epsilon$ and $p < (b+1)/\epsilon$.

It is useful to focus attention on uniform behavior on compact sets. The following lemma records two useful facts in this regard.

Lemma 2: Fix $\epsilon > 0, b_0 < \infty$, and a compact set $C \subseteq E$. Then there exist $b \in (b_0, \infty)$ and $m \in \mathbb{N}$ such that

(a) $$\rho_1[K_b^*(y;\cdot),\ K_{2b-v_i}^*(y;\cdot)] < \epsilon \quad \forall \quad i \leq p \text{ and } y \in C; \text{ and}$$

(b) $$P_y\{S_i \leq 3b \quad \forall \quad i \leq m\} < \epsilon \quad \forall \quad y \in C.$$

Proof: Fix ϵ, b_0, and C. Existence of a number b such that (a) holds is an immediate consequence of the uniform convergence on compacts established in Theorem 1. For (b), define, for each $y \in E$,

$$J_y = J_y(\epsilon, b) = \inf\{j\colon P_y\{S_i \leq 3b + 1 \quad \forall \quad i \leq j\} < \epsilon/2\}.$$

Then $J_y < \infty$ a.e. $[P_y]$ since (by (K2)) $P_y\{S_n \to \infty\} = 1$. Let $0 < \delta < \epsilon/2$. By (K4), if $y' \in B(y; q(y, \delta))$, then

$$\begin{aligned} P_{y'}\{S_i \leq 3b \quad \forall \quad i \leq J_y\} &\leq P_y\{S_i \leq 3b + \delta \quad \forall \quad i \leq J_y\} + \delta \\ &\leq P_y\{S_i \leq 3b + 1 \quad \forall \quad i \leq J_y\} + \delta \\ &< \epsilon/2 + \delta < \epsilon. \end{aligned}$$

Now $\{B(y; q(y;\delta))\colon y \in C\}$ is an open cover of C, so there are $y_1, \ldots, y_\ell$ such that $\cup_{i=1}^{\ell} B(y_i; q(y_i, \delta)) \supset C$. Let $m = \max\{J_{y_1}, \ldots, J_{y_\ell}\}$, and fix $y \in C$.
Then $y \in B(y_i; q(y_i, \delta))$ for some $i \leq \ell$. By the above,

$$\begin{aligned} &P_y\{S_i \leq 3b \quad \forall \quad i \leq m\} \\ \leq &P_y\{S_i \leq 3b + 1 \quad \forall \quad i \leq m\} + \delta < \epsilon. \end{aligned}$$

■

A useful decomposition is presented next. For fixed ϵ, b, partition $0 = v_0 < \ldots < v_p = b$, $r > 4\epsilon, m, a, C \subset E$ and $i = 1, \ldots, p$, define

$$C_i = C_i(a) = \{W_a \in C\} \cap \{v_{i-1} < R_a \leq v_i\};$$

$$A_i = \{\mathbf{z} \in \mathbb{R}^m : z_j \leq 2b - v_i - \epsilon \quad \forall \quad j < k \text{ and } z_k > 2b - v_{i-1} + r + \epsilon \quad \exists \quad k \leq m\};$$

$$B_i = \{\mathbf{z} \in \mathbb{R}^m : z_j \leq 2b - v_{i-1} \quad \forall \quad j < k \text{ and } z_k > 2b - v_i + r \quad \exists \quad k \leq m\}$$
$$\cup \{\mathbf{z} \in \mathbb{R}^m\colon z_j \leq 2b - v_{i-1} \quad \forall \quad j \leq m\}.$$

Note that $C_i \in \mathcal{G}_a$ and that the sets A_i, B_i, and C_i satisfy

$$C_i \cap \{(Z_{a,1}, \ldots, Z_{a,m}) \in A_i^\epsilon\} \subseteq C_i \cap \{R_{a+2b} > r\} \subseteq C_i \cap \{(Z_{a,1}, \ldots, Z_{a,m}) \in B_i\}.$$

(These inclusions are easy to verify if the time is taken to sort out the notation.)

The next result is the key to the proof of Theorem 2.

Proposition 1: Fix $\epsilon > 0$, $b_0 < \infty$, and a compact set $C \subseteq E$. Let $0 < b < \infty$ be as in Lemma 2. Then there is an $a_0 < \infty$ such that for all $a \geq a_0$ and $r > 4\epsilon$,

$$\int_{\{W_a \in C, R_a \leq b\}} K_b^*(W_a; (r + 4\epsilon, \infty)) d\mathbb{P} - 4\epsilon \leq \mathbb{P}\{W_a \in C, R_a \leq b, R_{a+2b} > r\} \leq \int_{\{W_a \in C, R_a \leq b\}} K_b^*(W_a; (r - 4\epsilon, \infty)) d\mathbb{P} + 4\epsilon.$$

Proof: Fix $\epsilon > 0, r > 4\epsilon, 0 < b_0 < \infty$, and a compact set C. Let b and m be as in Lemma 2. Let a_0 be so large that

$$\mathbb{P}\{\rho_m[L_{a,m}, L_m^*(W_a; \cdot)] \geq \epsilon\} \leq \frac{\epsilon^2}{b+1} \quad \forall \quad a \geq a_0.$$

Then for $a \geq a_0$ and $i = 1, \ldots, p$,

$$\mathbb{P}[C_i \cap \{(Z_{a,1}, \ldots, Z_{a,m}) \in B_i\}] = \int_{C_i} L_{a,m}(\gamma; B_i) d\mathbb{P}(\gamma) \leq \int_{C_i} [L_m^*(W_a; B_i^\epsilon) + \epsilon] d\mathbb{P} + \frac{\epsilon^2}{b+1}.$$

Also, for $y \in C$ and $i = 1, \ldots, p$,

$$\begin{aligned} L_m^*(y; B_i^\epsilon) \leq\ & L_m^*(y; \{\mathbf{z} \in \mathbb{R}^m : z_j \leq 2b - v_{i-1} + \epsilon \quad \forall \quad j < k \text{ and} \\ & \qquad z_k > 2b - v_i - \epsilon + r \quad \exists \quad k \leq m\}) \\ & + L_m^*(y; (-\infty, 3b]^m) \\ & \leq K_{2b - v_{i-1} + \epsilon}^*(y; (r - 3\epsilon, \infty)) + \epsilon \\ & \leq K_b^*(y; (r - 4\epsilon, \infty)) + 2\epsilon, \end{aligned}$$

where the second and third inequalities follow from Lemma 2.

So for $a \geq a_0$,

$$\begin{aligned}
\mathbb{P}\{W_a \in C, R_a \leq b, R_{a+2b} > r\} &= \mathbb{P}\left\{\bigcup_{i=1}^{P}(C_i \cap \{R_{a+2b} > r\})\right\} \\
&\leq \mathbb{P}\left\{\bigcup_{i=1}^{P}(C_i \cap \{(Z_{a,1},\ldots,Z_{a,m}) \in B_i\})\right\} \\
&\leq \sum_{i=1}^{p}\left\{\int_{C_i}[K_b^*(W_a;(r-4\epsilon,\infty)) + 2\epsilon]d\mathbb{P} + \frac{\epsilon^2}{b+1}\right\} \\
&\leq \int_{\{W_a \in C, R_a \leq b\}} K_b^*(W_a;(r-4\epsilon,\infty))d\mathbb{P} + 3\epsilon.
\end{aligned}$$

This proves the second inequality in the proposition; the first may be established by a similar argument. ■

The proof of Theorem 2 now may be completed. Fix $\delta > 0$. It is enough to show that there is a $b < \infty$ such that for all $r > \delta$,

$$\limsup_{a\to\infty} \mathbb{P}\{R_{a+2b} > r\} \leq K(E \times (r-\delta,\infty)) + \delta; \text{ and}$$
$$\liminf_{a\to\infty} \mathbb{P}\{R_{a+2b} > r\} \geq K(E \times (r+\delta,\infty)) - \delta.$$

To do this, let $\epsilon < \delta/7$. Then there is a $b_0 < \infty$ and a compact set $C \subseteq E$ such that

$$\mathbb{P}\{R_a \leq b_0\} \geq 1 - \epsilon;$$

$$\mathbb{P}\{W_a \in C\} \geq 1 - \epsilon; \text{ and}$$

$$|K_b^*(y;(r-4\epsilon,\infty)) - K(E \times (r-4\epsilon,\infty))| \leq \epsilon,$$

for all $b \geq b_0$, $a \geq 0$, and $y \in C$.

For these values of ϵ, b_0, and C, let b and m be the constants guaranteed by Lemma 2. Then, by Proposition 1, there is an $a_0 < \infty$ such that for all $a \geq a_0$,

$$\mathbb{P}\{R_a \leq b, R_{a+2b} > r, W_a \in C\} \leq \int_{\{W_a \in C, R_a \leq b\}} K_b^*(W_a;(r-4\epsilon,\infty))d\mathbb{P} + 4\epsilon.$$

So for $a \geq a_0$,

$$\begin{aligned}
\mathbb{P}\{R_{a+2b} > r\} &\leq \mathbb{P}\{R_{a+2b} > r, R_a \leq b\} + \epsilon \\
&\leq \mathbb{P}\{R_{a+2b} > r, R_a \leq b, W_a \in C\} + 2\epsilon \\
&\leq \int_{\{W_a \in C, R_a \leq b\}} K_b^*(W_a;(r-4\epsilon,\infty))d\mathbb{P} + 6\epsilon \\
&\leq K(E \times (r-4\epsilon,\infty)) + 7\epsilon < K(E \times (r-\delta,\infty)) + \delta.
\end{aligned}$$

This establishes the inequality for lim sup. The companion inequality for lim inf may be proved similarly. ■

4.2. Proof of Theorem 3

The proof of Theorem 3 parallels that of Theorem 2; whatever intuition there was in the proof of Theorem 2 is, however, disguised by the complication of working with joint distributions.

The partitions $0 = v_0 < \ldots < v_p = b$ and the sets C_i are defined precisely as in the last section. The sets B_i, however, must be redefined. For $i = 1, \ldots, p,\ A \subseteq E \times [0, \infty)$, and $m \geq 1$, define

$$B_i = \{(\mathbf{w}, \mathbf{z}) \colon z_j \leq 2b - v_{i-1} \quad \forall \quad j < k,\ (w_k, z_k + v_i - 2b) \in A^{\epsilon}, \exists k \leq m\}$$
$$\cup \{(\mathbf{w}, \mathbf{z}) \colon z_j \leq 2b - v_{i-1} \quad \forall \quad j \leq m\}.$$

It is easy to verify that (as long as $A \cap (E \times [0, \epsilon]) = \phi$),

$$C_i \cap \{(W_{a+2b}, R_{a+2b}) \in A\} \subseteq C_i \cap \{(W_{a,1}, Z_{a,1}, \ldots, W_{a,m}, Z_{a,m}) \in B_i\}.$$

Let $\mathbf{K}_b^*$ represent the joint distribution of Y_{τ_b} and $S_{\tau_b} - b$, *i.e.*, for $y \in E$ and $A \in \mathcal{E} \times \mathcal{B}[0, \infty)$,

$$\mathbf{K}_b^*(y; A) = P_y\{(Y_{\tau_a}, S_{\tau_a} - a) \in A\}.$$

Lemma 2 is restated in this context. The proof is entirely analogous.

Lemma 3: Fix $\epsilon > 0, b_0 < \infty$, and a compact set $C \subseteq E$. Then there exist $b \in (b_0, \infty)$ and $m \in \mathbb{N}$ such that

$$\rho[\mathbf{K}_b^*(y; \cdot), \mathbf{K}_{2b-v_i}^*(y; \cdot)] < \epsilon \quad \forall \quad i \leq p \text{ and } y \in C;$$

(a) $$\rho[\mathbf{K}_b^*(y; \cdot), K(\cdot)] < \epsilon \quad \forall \quad y \in C; \text{ and}$$

(b) $$P_y\{S_i \leq 3b \quad \forall \quad i \leq m\} < \epsilon \quad \forall \quad y \in C.$$

Proposition 2: Fix $\epsilon > 0, 0 < b_0 < \infty$, and a compact set C. Let $0 < b < \infty$ be as in Lemma 3. Then there is an $a_0 < \infty$ such that for all $a \geq a_0$ and $A \in \mathcal{E} \times \mathcal{B}[0, \infty)$ satisfying

$A \cap (E \times [0, 4\epsilon]) = \phi,$

$$\mathbb{P}\{W_a \in C, R_a \le b, (W_{a+2b}, R_{a+2b}) \in A\} \le K(A^{6\epsilon}) + 6\epsilon.$$

Again the proof is analogous to the proof of Proposition 1 and will be omitted.

To prove Theorem 3, fix $\delta > 0$ and let $\epsilon < \delta/8$. Find $0 < b_0 < \infty$ and a compact set C satisfying

$$\mathbb{P}\{R_a \le b_0\} \ge 1 - \epsilon \le \mathbb{P}\{W_a \in C\}. \quad \forall \quad a \ge 0.$$

Let b and m be given by Lemma 3. For $A \subseteq E \times [0, \infty)$, define $A(\epsilon) = A \cap (E \times (4\epsilon, \infty))$. By Proposition 2, there is an $a_0 < \infty$ such that for all $a \ge a_0$ and $A \in \mathcal{E} \times \mathcal{B}[0, \infty)$,

$$\begin{aligned} &\mathbb{P}\{(W_{a+2b}, R_{a+2b}) \in A\} \\ &\quad = \mathbb{P}\{(W_{a+2b}, R_{a+2b}) \in A \backslash A(\epsilon)\} + \mathbb{P}\{(W_{a+2b}, R_{a+2b}) \in A(\epsilon)\} \\ &\quad \le \mathbb{P}\{(R_{a+2b} \le 4\epsilon\} + \mathbb{P}\{W_a \in C, R_a \le b, (W_{a+2b}, R_{a+2b}) \in A(\epsilon)\} + 2\epsilon \\ &\quad \le \mathbb{P}\{(R_{a+2b} \le 4\epsilon\} + K(A^{6\epsilon}) + 6\epsilon + 2\epsilon \\ &\quad \le \mathbb{P}\{(R_{a+2b} \le \delta\} + K(A^{\delta}) + \delta. \end{aligned}$$

So for every $\delta > 0$ and $A \in \mathcal{E} \times \mathcal{B}[0, \infty)$,

$$\limsup_{c \to \infty} \mathbb{P}\{(W_c, R_c) \in A\} \le \limsup_{c \to \infty} \mathbb{P}\{R_c \le \delta\} + K(A^{\delta}) + \delta. \tag{13}$$

Now specialize to closed sets. If F is closed, then $F^{\delta} \downarrow F$ as $\delta \downarrow 0$. Also, by Theorem 2, $\limsup_{c \to \infty} \mathbb{P}\{R_c \le \delta\} \Rightarrow K(E \times [0, \delta]) \to 0$ as $\delta \downarrow 0$. So, let $\delta \downarrow 0$ in (13) to obtain

$$\limsup_{c \to \infty} \mathbb{P}\{(W_c, R_c) \in F\} \le K(F) \text{ for all closed } F,$$

which, by the Portmanteau Theorem (Billingsley, 1968, pp. 11–12) proves Theorem 3. ■

5. Comments on the Conditions

In this section some comments will be made about the conditions of Theorems 2 and 3. The following material (in particular Section 5.1) will make clear the connection between those results and the more traditional nonlinear renewal theory for perturbed processes.

Throughout, it is assumed that $\{S_n\}$ is a Markov random walk satisfying conditions (K1)–(K4).

Section 5.1. Condition (I)

It is shown that the following conditions on $\{\xi_n\}$ are sufficient for Condition (I) to hold in the setting of a perturbed Markov random walk: there exists a $\beta \in (\frac{1}{2}, 1]$ for which

(14) $$a^{-\beta}\{t_a - \frac{a}{\mu}\} \to 0 \text{ in } P_y\text{–probability for each } y, \text{ as } a \to \infty;$$

(15) $$\lim_{\delta \to 0} \sup_{n \geq 1} P_y\{\max_{K \leq \delta n^{\beta}} |\xi_{n+k} - \xi_n| \geq \epsilon\} = 0 \quad \forall \quad \epsilon > 0, \forall y; \quad \text{and}$$

(16) For each $n \geq 1, \xi_n$ is $\sigma(Y_0, \ldots, Y_n, S_0, \ldots, S_n)$–measurable.

Notice that these conditions are much weaker than those assumed in Melfi (1992) and Su (1990).

Proposition 3: Assume that $Z_n = S_n + \xi_n$, with $\{\xi_n\}$ satisfying (14)–(16). Then $\{Z_n\}$ satisfies condition (I).

Proof: For $a \geq 0$ and $k \geq 1$, define

$$S_{a,k} = S_{t_a+k} - S_{t_a};$$

$$\xi_{a,k} = \xi_{t_a+k} - \xi_{t_a}; \text{ and}$$

$$\mathcal{G}_k = \sigma(Y_0, \ldots, Y_n, S_1, \ldots, S_n, \xi_1, \ldots, \xi_n).$$

Then $Z_{a,k} = S_{a,k} + \xi_{a,k}$, and (by the strong Markov property and (16))

$$\mathcal{D}((S_{a,1}, \ldots, S_{a,m})|\mathcal{G}_a) = L_m^*(Y_a; \cdot)$$

for all $m \geq 1$ and $a \geq 0$.

Thus for $B \in \mathcal{B}^m$ and $\delta > 0$,

$$\begin{aligned} \mathsf{P}\{(Z_{a,1}, \ldots, Z_{a,m}) \in B|\mathcal{G}_a\} &\leq \mathsf{P}\{(S_{a,1}, \ldots, S_{a,m}) \in B^{\delta}|\mathcal{G}_a\} + \\ &\quad + \mathsf{P}\{\max_{k \leq m} |\xi_{a,k}| \geq \delta|\mathcal{G}_a\} \\ &= L_m^*(Y_a; B^{\delta}) + \mathsf{P}\{\max_{k \leq m} |\xi_{a,k}| \geq \delta|\mathcal{G}_a\}, \end{aligned}$$

i.e. $$\rho_m[L_{a,m}, L_m^*(Y_a; \cdot)] \leq \delta + \mathsf{P}\{\max_{k \leq m} |\xi_{a,k}| \geq \delta|\mathcal{G}_a\}.$$

But (14) and (15) imply that $\xi_{a,k} \to 0$ in P_y–probability as $a \to \infty$, so letting $a \to \infty$ and then $\delta \to 0$ in the above proves Proposition 3. ■

Section 5.2. Tightness of $\{R_a\}$

Let $\{W_n, Z_n\}$ and $\{S_n\}$ be as in the set–up for Theorems 2 and 3. It is shown below that if $\xi_n = Z_n - S_n$ satisfies

$$\{\xi_{t_a} : a \geq 0\} \text{ is tight; and} \tag{17}$$

$$\{\xi_{\tau_a} : a \geq 0\} \text{ is tight,} \tag{18}$$

then $\{R_a\}$ is tight.

Lemma 4: Assume (17) and (18). Then for every $\epsilon > 0$, there is a $b < \infty$ such that for all $a \geq 0$,

(i) $$\mathrm{P}\{\tau_{a-b} \leq t_a\} \geq 1 - \epsilon; \text{ and}$$

(ii) $$\mathrm{P}\{t_a \leq \tau_{a+b}\} \geq 1 - \epsilon.$$

Proof: Fix $\epsilon > 0$. The assumptions guarantee $b < \infty$ for which $\mathrm{P}\{\xi_{t_a} > b\} < \epsilon > \mathrm{P}\{\xi_{t_a} < -b\}$ for all $a \geq 0$. To prove (i), note that since $Z_{t_a} > a$,

$$\begin{aligned} \mathrm{P}\{S_{t_a} \leq a - b\} &= \mathrm{P}\{Z_{t_a} \leq a - b + \xi_{t_a}\} \\ &\leq \mathrm{P}\{\xi_{t_a} > b\} + \mathrm{P}\{Z_{t_a} \leq a - b + \xi_{t_a}, \xi_{t_a} \leq b\} \\ &< \epsilon + 0 = \epsilon. \end{aligned}$$

$$\begin{aligned} \mathrm{P}\{S_{t_a} \leq a - b\} &= \mathrm{P}\{Z_{t_a} \leq a - b + \xi_{t_a}\} \\ &\leq \mathrm{P}\{\xi_{t_a} > b\} + \mathrm{P}\{Z_{t_a} \leq a - b + \xi_{t_a}, \xi_{t_a} \leq b\} \\ &< \epsilon + 0 = \epsilon. \end{aligned}$$

Now use the relation $\{S_{t_a} > a - b\} \subseteq \{\tau_{a-b} \leq t_a\}$ to get $\mathrm{P}\{\tau_{a-b} \leq t_a\} \geq \mathrm{P}\{S_{t_a} > a - b\} \geq 1 - \epsilon$, proving (i).

For (ii), since $S_{\tau_{a+b}} > a + b$,

$$\begin{aligned}\mathbb{P}\{Z_{\tau_{a+b}} \le a\} &= \mathbb{P}\{S_{\tau_{a+b}} \le a - \xi_{\tau_{a+b}}\} \\ &\le \mathbb{P}\{\xi_{\tau_{a+b}} \le -b\} + \mathbb{P}\{S_{\tau_{a+b}} \le a - \xi_{\tau_{a+b}}, \xi_{\tau_{a+b}} > -b\} \\ &< \epsilon + 0 = \epsilon.\end{aligned}$$

Now use the relation $\{Z_{\tau_{a+b}} > a\} \subseteq \{t_a \le \tau_{a+b}\}$ to get

$$\mathbb{P}\{\tau_a \le t_{a+b}\} \ge \mathbb{P}\{Z_{\tau_{a+b}} > a\} \ge 1 - \epsilon. \blacksquare$$

The preceding lemma shows that, with high probability, t_a is between τ_{a-b} and τ_{a+b} for every a. The next result shows that, with high probability, τ_{a+b} and τ_{a-b} are not too far apart.

Proposition 4: For any fixed $b < \infty$, $\{(\tau_{a+b} - \tau_{a-b}) : a \ge 0\}$ is tight.

Proof: For $a > 0$ and $k \ge 1$, define

$$S_{a,k} = S_{\tau_{a+k}} - S_{\tau_a}.$$

Let $b < \infty$. Then for $R > 0, \{(\tau_{a+b} - \tau_{a-b}) > R\} \subseteq \{S_{a-b,k} \le 2b \quad \forall \quad K \le R\}$. Fix $\epsilon > 0$, and let C be a compact set for which $\mathbb{P}\{Y_{\tau_a} \in C\} < \epsilon/2$ for all $a \ge 0$. By Lemma 2, there is an $R \in \mathbb{N}$ for which

$$L_R^*(y; (-\infty, 2b]^R) < \epsilon/2 \quad \forall \quad y \in C.$$

So for any $a \ge 0$,

$$\begin{aligned}\mathbb{P}\{(\tau_{a+b} - \tau_{a-b}) > R\} &\le \mathbb{P}\{S_{a-b,k} \le 2b \quad \forall \quad k \le R\} \\ &= \int \mathbb{P}\{S_{a-b,k} \le 2b \quad \forall \quad k \le R | \mathcal{F}_{\tau_{a-b}}\} d\mathbb{P} \\ &= \int L_R^*(Y_{\tau_{a-b}}; (-\infty, 2b]^R) d\mathbb{P} \\ &\le \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

where the final inequality follows by integrating separately over the sets $\{Y_{\tau_{a-b}} \in C\}$ and $\{Y_{\tau_{a-b}} \notin C\}$. $\blacksquare$

Proposition 5: If (17) and (18) are true, then $\{R_a: a \geq 0\}$ is tight.

Proof: For $a \geq 0$,

$$\begin{aligned} R_a = Z_{t_a} - a &= S_{t_a} + \xi_{t_a} - a \\ &= (S_{\tau_a} - a) + (S_{t_a} - S_{\tau_a}) + \xi_{t_a}. \end{aligned}$$

The first and third terms on the right are tight (by the Markov renewal theorem and (16)), so it is only necessary to consider the second term.

For this term, note that

$$|S_{t_a} - S_{\tau_a}| \leq |S_{t_a} - S_{\tau_{a-b}}| + |S_{\tau_a} - S_{\tau_{a-b}}|.$$

The last term is bounded above by $|S_{\tau_a} - a| + |S_{\tau_{a-b}} - (a-b)| + b$, which is tight.

Next, for $R > 0$,

$$\begin{aligned} \mathbb{P}\{|S_{t_a} - S_{\tau_a}| > R\} \leq \mathbb{P}\{|S_{t_a} - S_{\tau_a}| > R, \tau_{a-b} \leq t_a \leq \tau_{a+b}, (\tau_{a+b} - \tau_{a-b}) \leq m, Y_{\tau_{a-b}} \in C\} \\ + \mathbb{P}\{t_a < \tau_{a-b}\} + \mathbb{P}\{t_a > \tau_{a+b}\} + \mathbb{P}\{(\tau_{a+b} - \tau_{a-b}) > m\} \\ + \mathbb{P}\{Y_{\tau_{a-b}} \notin C\}. \end{aligned}$$

For fixed $\epsilon > 0$, use Lemma 4 to find a $b < \infty$ for which the second and third terms on the right are less than $\epsilon/5$ for all a, then use the Markov renewal theorem to find a compact set C such that the last term is less than $\epsilon/5$ for all a, then use Proposition 4 to find an m for which the fourth term on the right is less than $\epsilon/5$ for all a.

It only remains to show that for this b, C, and m, there is an $R < \infty$ for which the first term on the right is less than $\epsilon/5$ for all a. This follows from condition (K4) by an argument like that in Lemma 2(b). The details are omitted. ■

Section 5.3. Tightness of $\{W_a\}$

Condition (III) of Theorems 2 and 3 requires that $\{W_a\} = \{W_{t_a}: a \geq 0\}$ is tight. Some simple conditions which imply that $\{W_a\}$ is tight are

(a) E is compact;

(b) $E = \mathbb{R}$ and $\{W_a\}$ is L'–bounded;

(c) $W_{t_a} - Y_{\tau_a} \xrightarrow{P} 0$.

Tightness of $\{W_a\}$ follows immediately from (a); from (b) by Chebyshev's Inequality, and from (c) by the Markov renewal theorem and Slutsky's Theorem.

In general, tightness of $\{W_a\}$ may be difficult to verify. The following gives some conditions under which $\{W_a\}$ may be shown to be tight, and also serves to indicate the sort of argument which may be used to prove tightness.

It is natural to assume that $\{W_n\}$ and $\{Y_n\}$ are asymptotically close in some sense, since often $\{W_n\}$ will be a perturbed version of $\{Y_n\}$. The following result shows how to exploit such closeness to prove tightness of $\{W_a\}$.

Lemma 5: If $W_n - Y_n \xrightarrow{\text{a.s.}} 0$ and if $\{Y_{t_a}: a \geq 0\}$ is tight, then $\{W_{t_a}: a \geq 0\}$ is tight.

Proof: Since the convergence of $W_n - Y_n$ is a.s., n may be replaced by t_a to get $W_{t_a} - Y_{t_a} \xrightarrow{\text{a.s.}} 0$. This, along with tightness of $\{Y_{t_a}\}$, proves the result. ∎

Thus the key in this setting is tightness of $\{Y_{t_a}\}$. It may be shown that this follows from (17) and (18) above. The proof is almost the same as for tightness of $\{R_a\}$ given above.

Proposition 6: Assume that $W_n - Y_n \xrightarrow{\text{a.s.}} 0$, and that (17) and (18) hold. Then $\{W_a\}$ is tight.

Proof: All that remains is to show that (17) and (18) imply that $\{Y_{t_a}: a \geq 0\}$ is tight. For $K \in \mathcal{E}, a \geq 0, b < \infty, C \in \epsilon$, and $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}\{Y_{t_a} \notin K\} \leq \mathbb{P}\{Y_{t_a} \notin K,\ Y_{\tau_{a-b}} \in C,\ \tau_{a-b} \leq t_a \leq \tau_{a+b},\ (\tau_{a+b} - \tau_{a-b}) \leq m\} \\ + \mathbb{P}\{t_a > \tau_{a+b}\} + \mathbb{P}\{t_a < \tau_{a-b}\} + \mathbb{P}\{Y_{\tau_{a-b}} \notin C\} \\ + \mathbb{P}\{(\tau_{a+b} - \tau_{a-b}) > m\}. \end{aligned}$$

Lemma 4 and Proposition 4 serve to bound the latter 4 terms, as in the proof of Proposition 5. Then (K4) is used to bound the first term on the right. The details are omitted. ∎

6. A Machine Breakdown Example

One application of Theorems 2 and 3 is the study of breakdown times of machines which have some control or adjustment made at the time of repair. This section is devoted to the investigation of such a process, where the distribution of the time between breakdowns depends on an underlying Markov chain and the machine setting.

Specifically, let $\{\epsilon_n : n \in \mathbb{Z}\}$ be iid Uniform $[-\frac{1}{2}, \frac{1}{2}]$, and let $Y_n = (\frac{1}{2})Y_{n-1} + \epsilon_n,\ n \geq 1$. Let V_n be a stochastic process with values in some measurable space $(M, \mathcal{M})$, and let $\{T_n : n \geq 1\}$ be a real-valued stochastic process satisfying

$$\mathcal{D}(T_n | \{Y_i : i \geq 0\},\ \{T_j : j \neq n\},\ \{V_j : j < n\}\} = \begin{cases} F_{\theta_n} & \text{if } Y_n - Y_{n-1} > 0 \\ G_{\theta_n} & \text{if } Y_n - Y_{n-1} \leq 0; \end{cases}$$

for $A \in \mathcal{B}$ and $n \geq 1$. Here V_n represents some measurement (for example, quantity of output) at time n; $\theta_n = \theta_n(\{Y_i, T_i, V_i\} : i < n)$ represents the machine setting at time n; and $\{T_n : n \geq 1\}$ is the sequence of times between breakdown.

It is assumed that all of the above random elements are defined on the same probability space $(\Gamma, \mathcal{A}, \mathbb{P})$, and that $\theta = \lim_{n\to\infty} \theta_n$ exists a.e. $[\mathbb{P}]$. It is also assumed that $\{F_z, G_z : z \in \mathbb{R}\}$ is a family of distributions with common compact support, that $F_z \to F_\theta$ and $G_z \to G_\theta$ in total variation as $z \to \theta$, and that F_θ and G_θ are absolutely continuous with respect to Lebesgue measure. Let $\lambda = \lambda_\theta$ and $\nu = \nu_\theta$ be the means of F_θ and G_θ respectively, and assume that $\lambda + \nu > 0$.

Let $\{X_n : n \geq 1\}$ satisfy

$$\mathcal{D}(X_n | \{Y_i : i \geq 0\},\ \{X_j : j \neq n\}) = \begin{cases} F_\theta & \text{if } Y_n - Y_{n-1} > 0 \\ G_\theta & \text{if } Y_n - Y_{n-1} \leq 0, \end{cases}$$

and define

$$S_n = X_1 + \ldots + X_n; \text{ and}$$

$$Z_n = T_1 + \ldots + T_n.$$

Then $\{S_n\}$ is a Markov random walk, and it is shown next that $\{Y_n, S_n, Z_n\}$ satisfies the conditions of Theorem 3, so that $(Y_{t_a}, Z_{t_a} - a)$ converges in distribution as $a \to \infty$.

First, conditions (K1)–(K4) must be verified for the process $\{Y_n, X_n\}$. It is well–known that $\{Y_n\}$ satisfies condition (K1) with ϕ given by

$$\phi(A) = \mathbb{P}\left[\left(\sum_{n=1}^{\infty}(\tfrac{1}{2})^n \varepsilon_n\right) \in A\right], \; A \in \mathcal{B}.$$

(See Nummelin, 1984, e.g.).

A short excursion into the theory of Markov chains facilitates the verification of (K2). Sources of further information include Breiman (1968) and Revuz (1984). First a few definitions are needed. Let $\{\zeta_n : n \geq 0\}$ be a Markov chain with state space $(D, \mathcal{D})$ and transition function $N(\cdot,\cdot)$. Also, for $x \in D$, let $\mathbf{P}_x$ represent the unique probability measure pertaining to paths of $\{\zeta_n\}$ starting at x.

Definition: $\{\zeta_n : n \geq 0\}$ is said to be indecomposable if and only if there do not exist two disjoint, nonempty subsets A and B in $\mathcal{D}$ for which

$$N(x, A) = 1 \quad \forall \quad x \in A \text{ and } N(x, \mathcal{B}) = 1 \quad \forall \quad x \in \mathcal{B}.$$

Definition: $\{\zeta_n : n \geq 0\}$ is said to be Harris recurrent (with respect to the measure m) if and only if for every $A \in \mathcal{D}$ with $m(A) > 0$,

$$\mathbf{P}_x\left\{\sum_{n=1}^{\infty} 1_A(\zeta_n) = \infty\right\} = 1 \quad \forall \quad x \in D.$$

The Markov Chain of interest is defined next. Define $D \subseteq [-1,1]^2$ by

$$D = \left\{(x,y) \in [-1,1]^2 : y \in \left[\frac{x-1}{2}, \frac{x+1}{2}\right]\right\}.$$

Let N be the transition function on $(D, \mathcal{D})$ given by $N((x,y),C) = Q(y; C_y)$, where $C_y = \{x : (x,y) \in C\}$, and let $\zeta_0, \zeta_1, \ldots$ be a Markov chain with transition function N. Then for $y \in [-1,1]$ and $A \in \mathcal{D}^{\mathbb{N}}$,

$$P_y\{[(Y_0,Y_1),\ (Y_1,Y_2),\ldots] \in A\} = \mathbf{P}_{\eta_y}\{(\zeta_0,\zeta_1,\ldots) \in A\},$$

where η_y is the measure on $\mathcal{D}$ defined by

$$\eta_y(A \times \mathcal{B}) = 1_A(y) \int_B Q(y; dz).$$

Lemma 6: The measure π, defined by

$$\pi(C) = \int_C \phi(dx)Q(x; dy), \ C \in \mathcal{D},$$

is stationary for $\{\zeta_n\}$.

Proof: The Lemma follows from the fact that ϕ is stationary for $\{Y_n\}$ and Proposition 6.6 of Breiman (1968).

The Markov chain $\{\zeta_n\}$ is clearly indecomposable. This, along with Lemma 6, implies that π is the *unique* stationary distribution and that if ζ_0 has distribution π, then $\{\zeta_n\}$ is ergodic. (See Breiman, 1968, Theorem 7.16).

Let m denote Lebesgue measure on $\mathcal{D}$.

Proposition 7: The Markov chain $\{\zeta_n\}$ is Harris recurrent with respect to m.

Proof: (This proof is similar to an argument in Bélisle, Romeijn and Smith (1990).) First note that π and m are mutually absolutely continuous. Thus π has a strictly positive density, say g, with respect to m. Also note that for any $x \in D$, $N^2(x; \cdot)$ is absolutely continuous with respect to m. Fix $A \in \mathcal{D}$. By the ergodicity of $\{\zeta_n\}$,

$$\begin{aligned} 1 = \mathbf{P}_\pi \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} 1\{\zeta_i \in A\} = \pi(A) \right\} \\ = \int \mathbf{P}_x \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} \pi(dx) \\ = \int \mathbf{P}_x \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} g(x)dm(x). \end{aligned}$$

Thus

$$\mathbf{P}_x \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} = 1 \text{ for a.e. } x[m],$$

i.e., $m(D_0) = 1$, where

$$D_0 = \left\{ x \in D : \mathbf{P}_x \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=1}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} = 1 \right\}.$$

To prove Harris recurrence, it is necessary to know that $D_0 = D$. So fix $x \in D$. Then

$$\begin{aligned}
&\mathbf{P}_x \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} \\
&= \int_D \mathbf{P}_x \left\{ \lim_{n\to\infty} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) | \zeta_2 = y \right\} N^2(x; dy) \\
&= \int_D \mathbf{P}_y \left\{ \lim_{n\to\infty} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} N^2(x, dy) \\
&\geq \int_{D_0} \mathbf{P}_y \left\{ \lim_{n\to\infty} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} N^2(x, dy) \\
&= N^2(x; D_0) = N^2(x; D) = 1,
\end{aligned}$$

where the penultimate equality follows from the fact that $N^2(x; \cdot)$ is absolutely continuous with respect to m.

Thus

$$\mathbf{P}_x \left\{ \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} 1_A(\zeta_i) = \pi(A) \right\} = 1 \quad \forall \quad x \in D.$$

Now let A satisfy $m(A) > 0$. Then $\pi(A) > 0$, so that the above relation implies that

$$\mathbf{P}_x \left\{ \sum_{i=1}^{\infty} 1_A(\zeta_i) = \infty \right\} = 1 \quad \forall \quad x \in D,$$

i.e., $\{\zeta_n\}$ is Harris recurrent with respect to m. ■

Condition (K2) is now easy to verify. By the law of large numbers for Harris recurrent Markov chains (Revuz, 1984, p. 140), for each $A \in \mathcal{D}$,

$$\tfrac{1}{n} \sum_{i=1}^{n} 1_A(\zeta_n) \longrightarrow \pi(A) \text{ a.e. } [\mathbf{P}_x] \quad \forall x. \tag{19}$$

Proposition 8:

$$\tfrac{1}{n} \sum_{i=1}^{n} 1\{Y_i - Y_{i-1} > 0\} \to \tfrac{1}{2} \text{ a.s.}$$

Proof: Let $A = \{(x, y) \in D : y > x\}$. Then

$$\begin{aligned}
\pi(A) = P_\phi\{Y_1 > Y_0\} &= \int P_z\{Y_1 > z\}\phi(dz) \\
&= \int \left(\tfrac{1}{2} - \tfrac{z}{2}\right) \phi(dZ) = \tfrac{1}{2},
\end{aligned}$$

since the mean of ϕ is zero. Thus

$$\begin{aligned} P_z &\left[\frac{1}{n}\sum_{i=1}^{n} 1\{Y_i - Y_{i-1} > 0\} \to \frac{1}{2}\right] \\ &= \mathbf{P}_{\eta z}\left[\frac{1}{n}\sum_{i=1}^{n} 1_A(\zeta_i) \to \frac{1}{2}\right] \\ &= \int \mathbf{P}_x\left[\frac{1}{n}\sum_{i=1}^{n} 1_A(\zeta_i) \to \frac{1}{2}\right]\eta_z(dx) = 1 \end{aligned}$$

by (19). ■

For condition (K2), first note that

$$\begin{aligned} E_y X_1 &= \lambda P_y\{Y_1 > y\} + \nu P_y\{Y_1 \le y\} \\ &= \tfrac{1}{2}\lambda(1-y) + \tfrac{1}{2}\nu(1+y). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu &\equiv \int E_y X_1 \phi(dy) \\ &= \tfrac{1}{2}\int(\lambda + \nu - \lambda y + \nu y)\phi(dy) = \tfrac{1}{2}(\lambda+\nu), \end{aligned}$$

since the mean of ϕ is zero.

So, letting $X_1^F, X_2^F, \ldots$ be i.i.d. F_θ and $X_1^G, X_2^G, \ldots$ be iid G_θ,

$$\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i^F 1\{Y_i - Y_{i-1} > 0\} + \frac{1}{n}\sum_{i=1}^{n-1} X_i^G 1\{Y_i - Y_{i-1} \le 0\} \overset{\text{a.s.}}{\to} \mu$$

by Proposition 4 and the Strong Law of Large Numbers.

Condition (K3) is clearly true. For (K4), more notation is needed. For $y \in [-1,1]$ and $n \ge 1$, let

$$Y_n(y) = (\tfrac{1}{2})^n y + \sum_{k=1}^{n} (\tfrac{1}{2})^{n-k}\epsilon_k,$$

and let $S_n(y)$ be defined as S_n is above, but with $Y_n(y)$ in place of Y_n. Then the P_0–distribution of $\{Y_n(y), S_n(y)\}$ is equal to the P_y–distribution of $\{Y_n, S_n\}$.

Now

$$
\begin{aligned}
Y_n(y) - Y_{n-1}(y) &= -(\tfrac{1}{2})^n y + \epsilon_n + \sum_{k=1}^{n-1} \left[(\tfrac{1}{2})^{n-k} - (\tfrac{1}{2})^{n-1-k}\right] \epsilon_k \\
&= -(\tfrac{1}{2})^n y + \epsilon_n - \tfrac{1}{2}\sum_{k=1}^{n-1} (\tfrac{1}{2})^{n-1-k} \epsilon_k \\
&= -(\tfrac{1}{2})^n y + \epsilon_n - \tfrac{1}{2}\sum_{k=2}^{n} (\tfrac{1}{2})^{n-k} \epsilon_{k-1} \\
&= -(\tfrac{1}{2})^n y + J_n, \text{ say.}
\end{aligned}
$$

Lemma 7: For each $\delta > 0$ and $y \in [-1,1]$, there is a $q > 0$ such that if $|y - z| < q$, then

(20) $$P_0\{1_{(0,\infty)}[Y_k(y) - Y_{k-1}(y)] \neq 1_{(0,\infty)}[Y_k(z) - Y_{k-1}(z)] \ \exists \ k \geq 1\} < \delta.$$

Proof: Fix $\delta > 0$ and $y > 0$. (The argument for $y \leq 0$ is analogous). Then for $k \geq 1$ and $z > 0$,

$$
\begin{aligned}
&P_0\{1_{(0,\infty)}[Y_k(y) - Y_{k-1}(y)] \neq 1_{(0,\infty)}[Y_k(z) - Y_{k-1}(z)]\} \\
&\qquad = P_0\{Y_k(y) - Y_{k-1}(y) > 0, \ Y_k(z) - Y_{k-1}(z) \leq 0\} \\
&\qquad + P_0\{Y_k(y) - Y_{k-1}(y) \leq 0, \ Y_k(z) - Y_{k-1}(z) > 0\} \\
&\qquad = P_0\{\tfrac{1}{2^k} y < J_k \leq \tfrac{1}{2^k} z\} + P_0\{\tfrac{1}{2^k} z < J_k \leq \tfrac{1}{2^k} y\}.
\end{aligned}
$$

One of the quantities on the right will be zero, depending on whether $y > z$ or $y < z$. For simplicity, assume that $y < z$.

It may be shown that for each k, the density of J_k is bounded by 1. Thus

$$
\begin{aligned}
\text{LHS}(20) &\leq \sum_{k=1}^{\infty} P_0\{\tfrac{1}{2^k} y < J_k \leq \tfrac{1}{2^k} z\} \\
&\leq \sum_{k=1}^{\infty} \tfrac{1}{2^k}(z - y) < \delta,
\end{aligned}
$$

whenever $(z - y) < \delta$. For $y > z$, a similar argument shows that LHS(20) $< \delta$ whenever $(y - z) < \min\{y, \delta\}$. (Taking the minimum of δ and y avoids problems with $z < 0$). ■

To verify (K4), let $y \in [-1,1]$ and $\delta > 0$. Let $q = q(y,\delta)$ be as in Lemma 7, and for $z \epsilon [-1,1]$, define $E(y,z) = \{1_{(0,\infty)}[Y_k(y) - Y_{k-1}(y)] = 1_{(0,\infty)}[Y_k(z) - Y_{k-1}(z)] \ \forall \ k \geq 1\}$.

Now fix $z \in B(y;q)$ and note that by Lemma 7, $P_0\{(E(y,z))^c\} < \delta$. So for any

$$
\begin{aligned}
&f{:}([-1,1] \times [0,\infty))^{\mathbb{N}} \to \mathbb{R},\\
&E_y f(Y_0, S_0, Y_1, S_1, \ldots) - E_z f^\delta(Y_0, S_0, Y_1, S_1, \ldots)\\
&= \int [f(Y_0(y), S_0(y), Y_1(y), S_1(y), \ldots) - f^\delta(Y_0(z), S_0(z), Y_1(z), S_1(z), \ldots)] dP_0\\
&\leq 0 + \delta \sup |f|,
\end{aligned}
$$

where the inequality follows by integrating separately over $E(y,z)$ and $E(y,z)^c$. This establishes one inequality in (K4); the other follows by the same argument.

Conditions (II) and (III) are clearly true.

Lemma 8: The process $\{Y_n, z_n\}$ satisfies (I′).

Proof: For $(x,y) \in [-1,1]^2$ and $\kappa \in \Theta$, let

$$
H_\kappa(\cdot|x,y) = \begin{cases} F_\kappa(\cdot) & \text{if } y > x; \\ G_\kappa(\cdot) & \text{if } y \leq x. \end{cases}
$$

Fix $\epsilon > 0$ and $m \geq 1$. Let $\delta > 0$ be so small that if $|\kappa - \theta| < \delta$, then the total variation distance between $H_\kappa(\cdot|x,y)$ and $H_\theta(\cdot|x,y)$ is less than ϵ/m for all $(x,y)\epsilon[-1,1]^2$. Fix a $\gamma \in \Gamma$ for which $\theta_n(\gamma) \to \theta$, and let n be so large that $|\theta_n(\gamma) - \theta| < \delta$. Then for $A \in ([-1,1]^2 \times [0,\infty))^m$,

$$
\begin{aligned}
\mathbb{P}\{(Y_{n+1}, &Z_{n,1}, \ldots, Y_{n+m}, Z_{n,m}) \in A|\mathcal{F}_n\}(\gamma)\\
&= \int_A Q(Y_n(\gamma); dy_1) \ldots Q(y_{m-1}; dy_m) \times\\
&\qquad F_{\theta_{n+1}(\gamma)}(dz_1|Y_n(\gamma), y_1) \ldots F_{\theta_{n+m}(\gamma)}(dz_m - z_{m-1}|y_{m-1}, y_m)\\
&\leq \int_A Q(Y_n(\gamma|, dy_1) \ldots Q(y_{m-1}; dy_m) \times\\
&\qquad F_{\theta_{n+1}(\gamma)}(dz_1|Y_n(\gamma), y_1) \ldots [F_\theta(dz_m - z_{m-1}|y_{m-1}, y_m) + \epsilon/m].
\end{aligned}
$$

Iterating this relation shows that

$$
\begin{aligned}
\mathbb{P}\{(Y_{n+1}, &Z_{n,1}, \ldots, Y_{n+m}, Z_{n,m}) \in A | \mathcal{F}_n\}(\gamma) \\
&\leq \int_A Q(Y_n(\gamma); dy_1) \ldots Q(y_{m-1}; dy_m) \times \\
&\qquad F_\theta(dz_1 | Y_n(\gamma), y_1) \ldots F_\theta(dz_m - z_{m-1} | y_{m-1}, y_m) + m\epsilon/m \\
&= L_m^*(Y_n(\gamma); A) + \epsilon.
\end{aligned}
$$

Thus

$$
\rho[\mathbb{P}\{(Y_{n+1}, Z_{n,1}, \ldots, Y_{n+m}, Z_{n,m}) \in \cdot | \mathcal{F}_n\}, L_m^*(Y_n; \cdot)] \stackrel{\text{a.s.}}{\to} 0,
$$

from which the lemma follows. ■

Theorem 4: The pair $(Y_{t_a}, z_{t_a} - a)$ has joint limiting distribution k. In particular,

$$
\lim_{a \to \infty} \mathbb{P}\{Z_{t_a} - a > r\} = \frac{2}{\lambda + \nu} \int \psi(dz) \int_r^\infty (s - r) P_z\{S_{\tau_0} \in ds\}.
$$

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