FROM DISCRETE TO CONTINUOUS TIME FINANCE: WEAK CONVERGENCE OF THE FINANCIAL GAIN PROCESS

by

Darrell Duffie and Philip Protter
Stanford University Purdue University

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Department of Statistics
Purdue University

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From Discrete to Continuous Time Finance: Weak Convergence of the Financial Gain Process

Darrell Duffie and Philip Protter

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Abstract: Conditions, suitable for applications in finance, are given for the weak convergence (or convergence in probability) of stochastic integrals. For example, consider a sequence S^n of security price processes converging in distribution to S and a sequence θ^n of trading strategies converging in distribution to θ. We survey conditions under which the financial gain process $\int \theta^n \, dS^n$ converges in distribution to $\int \theta \, dS$. Examples include convergence from discrete to continuous time settings, and in particular, generalizations of the convergence of binomial option replication models to the Black-Scholes model. Counterexamples are also provided.

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1. Introduction

Although a large part of financial economic theory is based on models with continuous-time security trading, it is widely felt that these models are relevant insofar as they characterize the behavior of models in which trades occur discretely in time. It seems natural to check that the limit of discrete-time security market models, as the lengths of periods between trades shrink to zero, produces the effect of continuous-time trading. That is one of the principal aims of this paper. If $\{(S^n, \theta^n)\}$ is a sequence of security price processes and trading strategies converging in distribution to some such pair (S, θ), we are concerned with additional conditions under which the sequence $\{\int \theta_t^n \, dS_t^n\}$ of stochastic integrals defining the gains from trade converges in distribution to the stochastic integral $\int \theta_t \, dS_t$. Conditions recently developed by Jakubowski, Mémin, and Pagès (1989) and Kurtz and Protter (1991a, 1991b) are re-stated here in a manner suitable for easy applications in finance, and several such examples are worked out in this paper. The paper gives parallel conditions for convergence of gains in probability. In short, this paper is more of a "user's guide" than a set of new convergence results.

A good motivating example is Cox, Ross, and Rubinstein's (1979) proof that the Black-Scholes (1973) Option Pricing Formula is the limit of a discrete-time binomial option pricing formula (due to William Sharpe) as the number of time periods per unit of real time goes to infinity. Aside from providing a simple interpretation of the Black-Scholes formula, this connection between discrete and continuous time financial models led to a standard technique for estimating continuous-time derivative asset prices by using numerical methods based on discrete-time reasoning. One of the examples of this paper is of the following related form. Suppose that $\{S^n\}$ is a sequence of security price processes converging in distribution to the geometric Brownian motion price process S of the Black-Scholes model, with $\{S^n\}$ satisfying a basic technical condition. (For this, we show that it is enough that the cumulative return process R^n for S^n converges in distribution to the Brownian motion cumulative return process R underlying S, plus the same technical condition on the sequence $\{R^n\}$. For example, S^n could be a price process that is adjusted only at discrete-time intervals of length $1/n$, with *i.i.d.* or α-mixing returns satisfying a regularity condition.) We show that if an investor, in ignorance of the distinction between S^n and S, or perhaps at a loss for what else to do, attempts to replicate a call option payoff by following the associated Black-Scholes stock hedging strategy $C_x(S_t^n, T-t)$, then the investor will be successful in the limit (in the sense that the final payoff of the hedging strategy converges in distribution to the option payoff as $n \to \infty$), and the required initial investment converges to the Black-Scholes call option value. (This can be compared with the non-standard proof of the Black-Scholes formula given by Cutland, Kopp, and Willinger (1991), which draws a different sort of connection between the discrete and continuous models.) While this kind of stability result is to be expected, we feel that it is important to have precise and easily verifiable mathematical conditions that are sufficient for this kind of convergence result. As we show in counterexamples, there are conditions that are not obviously pathological under which convergence fails. Our general goal is to provide a useful set of tools for exploring the boundaries between discrete and continuous time financial models, as well as the stability of the financial gain process $\int \theta \, dS$ with respect to simultaneous perturbations of the price process S and trading strategy θ.

2. Preliminaries

This section sets out some of the basic definitions and notation. We let $\mathbb{D}^d$ denote the space of $\mathbb{R}^d$-valued càdlàg[1] sample paths on a fixed time interval $\mathcal{T} = [0, T]$. There are natural extensions of our results in each case to $\mathcal{T} = [0, \infty)$. The Skorohod topology[2] on $\mathbb{D}^d$ is used throughout, unless otherwise noted. A càdlàg process is a random variable S on some probability space valued in $\mathbb{D}^d$. A sequence $\{S^n\}$ of càdlàg processes (which may be defined on different probability spaces) converges in distribution to a càdlàg process S, denoted $S^n \Longrightarrow S$, if $E[h(S^n)] \to E[h(S)]$ for any bounded continuous real-valued function h on $\mathbb{D}^d$.

A famous example is Donsker's Theorem, whereby a normalized "coin toss" random walk converges in distribution to Brownian Motion. That is, let $\{Y_k\}$ be a sequence of independent random variables with equally likely outcomes $+1$ and -1, and let $R_t^n =$

[1] That is, $f \in \mathbb{D}$ means that $f : \mathcal{T} \to \mathbb{R}$ has a limit $f(t-) = \lim_{s \uparrow t} f(s)$ from the left for all t, and that the limit from the right $f(t+)$ exists and is equal to $f(t)$ for all t. By convention, $f(0-) = f(0)$. The expression "RCLL" (right continuous with left limits) is also used in place of càdlàg (continu à droite, limites à gauche).

[2] The Skorohod topology is defined by the convergence of x_n to x in $\mathbb{D}^d$ if and only if there is a sequence $\lambda_n : \mathcal{T} \to \mathcal{T}$ of strictly increasing continuous functions ("time changes") such that, for each $t_0 \in \mathcal{T}$, $\sup_{t \le t_0} |\lambda_n(t) - t| \to 0$ and $\sup_{t \le t_0} |x_n[\lambda_n(t)] - x(t)| \to 0$.

$(Y_1 + \cdots + Y_{[nt]})/\sqrt{n}$ for any time t, where $[t]$ denotes the smallest integer less than or equal to t. Then $R^n \Longrightarrow B$, where B is a Standard Brownian Motion. Donsker's Theorem applies to more general forms of random walk and to a class of martingales; Billingsley (1968), Ethier and Kurtz (1986), or Jacod and Shiryayev (1988) are good general references.

In financial models, we are more likely to think of $\{Y_k\}$ as a discrete-time return process, so that R^n is the normalized cumulative return process. The corresponding price process S^n is defined by $S_t^n = S_0 \mathcal{E}(R^n)_t$, for some initial price $S_0 > 0$, where the stochastic exponential $\mathcal{E}(R^n)$ of R^n is given in this case by

$$\mathcal{E}(R^n)_t = \prod_{k=1}^{[nt]} \left(1 + \frac{Y_k}{\sqrt{n}}\right).$$

The general definition of the stochastic exponential, introduced into this financial context by Harrison and Pliska (1981), is given in Section 3. It is well known that $S^n \Longrightarrow S$, where $S_t = S_0 e^{B_t - t/2}$. That is, with returns generated by a coin toss random walk, the asset price process converges in distribution to the solution of the stochastic differential equation $dS_t = S_t \, dB_t$. This is the classical Black-Scholes example (leaving out, for simplicity, constants for the interest rate and the mean and variance of stock returns). We return later in the paper to extend this example, showing that the Black-Scholes formula can be found as the limit of discrete-time models with a general class of cumulative return processes R^n converging in distribution to Brownian Motion.

A process X is a *semimartingale* if there exists a decomposition $X = M + A$ where M is a local martingale and A is an adapted càdlàg process with paths of finite variation on compact time intervals. Semimartingales are the most general processes having "stochastic differentials." Protter (1990) is an introductory treatment of stochastic integration and stochastic differential equations; Dellacherie-Meyer (1982) is a comprehensive treatment of semimartingales and stochastic integration.

3. Two Counterexamples

This section presents two counterexamples. In each case, and obviously for different reasons, even though a trading strategy θ^n converges in distribution to a trading strategy θ and a price process S^n converges in distribution to a price process S, it is not true that the financial gain process $\int \theta^n \, dS^n$ converges in distribution to the financial gain process $\int \theta \, dS$.

EXAMPLE 1. Our first example is deterministic and well known. It is essentially the same as Example 1.1 of Kurtz and Protter (1991a). Let there be $d = 1$ security and consider the trading strategies $\theta^n = \theta = 1_{(T/2,T]}$, all of which hold one unit of the security after time $T/2$. Let $S^n = 1_{[T/2+1/n,T]}$ for $n > 2/T$ and let $S = 1_{[T/2,T]}$. Although $\theta^n \Longrightarrow \theta$ and $S^n \Longrightarrow S$, it is not the case that $(\theta^n, S^n) \Longrightarrow (\theta, S)$ in the sense explained in Section 2. On the other hand, $\int_0^t \theta^n \, dS^n = 1$ for all $n > 2/T$ and all $t > T/2 + 1/n$, while $\int_0^t \theta \, dS = 0$ for all t. Failure of weak convergence occurs for a rather obvious reason that will be excluded by our main convergence conditions.

EXAMPLE 2. Our second example is more subtle. Let B be a standard Brownian motion and let $R = \sigma B$ describe the "ideal" cumulative return on a particular investment, for some constant σ. Suppose, however, that returns are only credited with a lag, on a moving average basis, with $R_t^n = n \int_{t-1/n}^t R(s) \, ds$, so that we are dealing instead with the "stale" returns. Suppose an investor chooses to invest total wealth, X_t at time t, by placing a fraction $g(X_t)$ in this risky investment, with the remainder invested risklessly (and for simplicity, at a zero interest rate). We assume for regularity that g is bounded with a bounded derivative. In the ideal case, the wealth process is given by

$$X_t = x + \int_0^t g(X_s) X_s \, dR_s,$$

where x is initial wealth. With stale returns, likewise, the wealth process X^n is given by

$$X_t^n = x + \int_0^t X_s^n g(X_s^n) \, dR_s^n.$$

It can be shown that the "stale" cumulative return process R^n converges in distribution to R. Is it true, as one might hope, that the corresponding wealth process X^n converges in distribution to X? The answer is typically "No." In fact, we show in an appendix that $X^n \Longrightarrow Y$, where

$$Y_t = X_t + \frac{1}{2} \int_0^t g(X_s) X_s^2 [1 + g'(X_s)] \, ds. \tag{1}$$

For instance, with $g(x) = k$, a constant investment strategy, for all x, we have $Y_t = e^{kt} X_t$, which can represent a substantial discrepancy between the limit of the gains and the gain of the limit strategy and returns. In particular, the price process $S = S_0 \mathcal{E}(R)$ corresponding to the limit return process is not the same as the limit of the price processes $S^n = S_0 \mathcal{E}(R^n)$.

Again, the sufficient conditions in our convergence results to follow would preclude this example. At the least, however, the example shows that care must be taken.

4. Weak Convergence Results for Stochastic Integrals

This section presents recently demonstrated conditions for weak convergence of stochastic integrals, in a form simplified for applications in financial economic models.

The following setup is fixed for this section. For each n, there is a probability space $(\Omega^n, \mathcal{F}^n, P^n)$ and a filtration $\{\mathcal{F}_t^n : t \in \mathcal{T}\}$ of sub-σ-fields of $\mathcal{F}^n$ (satisfying the usual conditions) on which X^n and H^n are càdlàg adapted processes valued respectively in $I\!R^m$ and $I\!M^{km}$ (the space of $k \times m$ matrices). (We can, and do, always fix a càdlàg version of any semimartingale.) We let E_n denote expectation with respect to $(\Omega^n, \mathcal{F}^n, P^n)$.

There is also a probability space and filtration on which the corresponding properties hold for X and H, respectively. Moreover, $(H^n, X^n) \Longrightarrow (H, X)$. For "$(H^n, X^n) \Longrightarrow (H, X)$," we emphasize that the definition requires that there exists one (and not two) sequence λ_n of time changes such that $\lambda_n(s)$ converges to s uniformly, and $(H^n_{\lambda^n(s)}, X^n_{\lambda^n(s)})$ converges in law uniformly in s to (H, X). We assume throughout that X^n is a semimartingale for each n, which implies the existence of $\int H^n_{s-}\, dX^n_s$.

4.1. Good Sequences of Semimartingales

The following property of $\{X^n\}$ is obviously the key to our goal.

DEFINITION. *A sequence $\{X^n\}$ of semimartingales is good if, for any $\{H^n\}$, the convergence of (H^n, X^n) to (H, X) in distribution (respectively, in probability) implies that X is a semimartingale and also implies the convergence of $(H^n, X^n, \int H^n_-\, dX^n)$ to $(H, X, \int H_- dX)$ in distribution (respectively, in probability).*

The following result, showing that "goodness" is closed under appropriate stochastic integration, is due to Kurtz and Protter (1991b), who also provide necessary and sufficient conditions for goodness.

PROPOSITION 1. *If (X^n) is good and (H^n, X^n) converges in distribution, then $(\int H^n_-\, dX^n)$ is also good.*

For our purposes, it remains to establish some simple conditions for a sequence $\{X^n\}$ of semimartingales to be good. We will start with a relatively simple condition for "goodness," and then extend in generality. Before stating the condition, recall that each semimartingale X is defined by the fact that it can be written as the sum $M + A$ of a local martingale M with $M_0 = 0$ and an adapted process A of finite variation. The total variation of A at time t is denoted $|A|_t$, and the quadratic variation of M is denoted $[M, M]$. (If M is vector-valued, $[M, M]$ is the matrix-valued process whose (i, j)-element is $[M^i, M^j]$.)

CONDITION A. *A sequence $\{X^n = M^n + A^n\}$ of semimartingales satisfies Condition A if both $\{E_n([M^n, M^n]_T)\}$ and $\{E_n(|A^n|_T)\}$ are bounded.*

If M is a local martingale with $E([M, M]_T) < \infty$, then $E([M, M]_T) = \text{var}\ (M_T)$ [for example, see Protter (1990), p. 66]. Thus the following condition is sufficient for Condition A, and may be easier to check in practice.

CONDITION A′. *A sequence $\{X^n = M^n + A^n\}$ of semimartingales satisfies Condition A′ if M^n is a martingale for all n, $\{\text{var}_n\ (M_T^n)\}$ is bounded, and $\{E_n(|A^n|_T)\}$ is bounded.*

THEOREM 1. *If $\{X^n\}$ has uniformly bounded jumps and satisfies Condition A or A′, then $\{X^n\}$ is good.*

This result can be shown as an easy corollary of results in Kurtz and Protter (1991a). The assumption of uniformly bounded jumps for $\{X^n\}$ is strong, and not often satisfied in practice, but we also obtain convergence if, with jumps appropriately truncated, $\{X^n\}$ satisfies Condition A. Since the obvious method of truncating jumps is not continuous in the Skorohod topology, we proceed as follows. For each $\delta \in [0, \infty)$, let $h_\delta : [0, \infty) \to [0, \infty)$ be defined by $h_\delta(r) = (1 - \delta/r)^+$, and let $J_\delta(X)$ be the process defined by

$$J_\delta(X)_t = X_t - \sum_{s \leq t} h_\delta(|\Delta X_s|)\Delta X_s,$$

where $\Delta X_s = X_s - X_{s-}$.

THEOREM 2. *If, for some δ, the sequence $\{J_\delta(X^n)\}$ satisfies Condition A or A′, then $\{X^n\}$ is good.*

A proof is given in Kurtz and Protter (1991a), and also in Jakubowski, Mémin, and Pagès (1989). The conditions here are designed to be easy to verify in practice.

In some cases, the restriction on the quadratic variation in Condition A may be more difficult to verify than the following condition.

CONDITION B. *A sequence $\{X^n = M^n + A^n\}$ of semimartingales satisfies Condition B if $\{E_n\left(\sup_{t\leq T}|\Delta M_t^n|\right)\}$ and $\{E_n\left(|A^n|_T\right)\}$ are bounded.*

THEOREM 3. *If $\{X^n\}$ satisfies Condition B, then $\{X^n\}$ is good.*

This result is proved in an earlier version of this paper, and follows from a result by Jakubowski, Mémin, and Pagès (1989), based on an application of Davis' inequality (Dellacherie and Meyer (1982), VII.90).

Comment: If the semimartingales X^n have uniformly bounded jumps, then they are *special*: that is, there exists a unique decomposition $X^n = M^n + A^n$, for which the finite variation process A^n is predictable with $A_0^n = 0$. Such a decomposition is called *canonical.* For the canonical decomposition, it can be shown that the jumps of M^n (and hence of A^n) are also bounded, and therefore for the canonical decomposition in the case of bounded jumps, $\{X^n\}$ is good if $\{E_n(|A^n|_T)\}$ is bounded. In fact, in Theorem 3, it is enough to replace the restriction that $\{E_n(|A^n|_T)\}$ is bounded with a weaker restriction: that the measures induced by $\{A^n\}$ on $[0, T]$ are tight. (See, for example, Billingsley (1968) for a definition of tightness of measures.) We also offer the following help in verifying this tightness condition for special semimartingales.

LEMMA 1. *Suppose $\{Z^n\}$, with $Z^n = M^n + A^n$, is a sequence of special semimartingales for which the measures induced on $[0, T]$ by (A^n) are tight. Then, for the canonical decomposition $Z^n = \widetilde{M}^n + \widetilde{A}^n$, the measures induced by $\{\widetilde{A}^n\}$ are also tight.*

PROOF: Since Z^n is special, A^n is locally of integrable variation [Dellacherie and Meyer (1982), page 214]. Since $\widetilde{A}^n$ is the predictable compensator of A^n, the result follows from the Corollary of Appendix B, Lemma B1. ▮

4.2. Stochastic Differential Equations

We now address the case of stochastic differential equations of the form

$$Z_t^n = H_t^n + \int_0^t f_n(s, Z_{s-}^n)\, dX_s^n,$$
$$Z_t = H_t + \int_0^t f(s, Z_{s-})\, dX_s,$$

where f_n and f are continuous real-valued functions on $\mathbb{R}_+ \times \mathbb{R}^k$ into $\mathbb{M}^{km}$ such that:

(i) $x \mapsto f_n(t,x)$ is Lipschitz (uniformly in t), each n,

(ii) $t \mapsto f_n(t,x)$ is LCRL (left continuous with right limits, or "càglàd") for each x, each n, and

(iii) for any sequence (x_n) of càdlàg functions with $x_n \to x$ in the Skorohod topology, (y_n, x_n) converges to (y, x) (Skorohod), where $y_n(s) = f_n(s+, x_n(s))$, $y(s) = f(s+, x(s))$.

(If $f_n = f$ for all n, then condition (iii) is automatically true.) The following theorem is proved in more generality in Kurtz and Protter (1991a). See also Słomiński (1989).

THEOREM 4. *Suppose $\{X^n\}$ is good and let $(f_n)_{n\geq 1}$ and f satisfy (i)-(iii) above. Suppose (H^n, X^n) converges to (H, X) in distribution (respectively, in probability). Let Z^n, Z be solutions[3] of*

$$Z_t^n = H_t^n + \int_0^t f_n(s, Z_{s-}^n)\, dX_s^n$$
$$Z_t = H_t + \int_0^t f(s, Z_{s-})\, dX_s,$$

respectively. Then (Z^n, H^n, X^n) converges to (Z, H, X) in distribution (respectively, in probability). Moreover, if $H^n = Z_0^n$ and $H = Z_0$, then $\{Z^n\}$ is good.

An important special case is the stochastic differential equation

$$Z_t = 1 + \int_0^t Z_{s-}\, dX_s,$$

which defines the stochastic exponential[4] $Z = \mathcal{E}(X)$ of X. The solution, extending the special case of Section 2, is

$$Z_t = \exp\left(X_t - \frac{1}{2}[X,X]_t^c\right) \prod_{0<s\leq t} (1+\Delta X_s)e^{-\Delta X_s},$$

where $[X,X]^c$ denotes the continuous part of the quadratic variation $[X,X]$ of X. With a Standard Brownian Motion B, for example, $[B,B]_t^c = [B,B]_t = t$ and $\mathcal{E}(B)_t = e^{B_t - t/2}$.

[3] Unique solutions exist. See Protter (1990).

[4] This is also known as the Doléans-Dade exponential.

5. Convergence of Discrete-Time Strategies

In order to apply our results to "discrete-time" trading strategies θ^n and corresponding price processes S^n, we need conditions under which $(\theta^n, S^n) \Longrightarrow (\theta, S)$. We will consider strategies that are discrete-time with respect to a *grid*, defined by times $\{t_0, \ldots, t_k\}$ with $0 = t_0 < t_1 < \cdots < t_k = T$. The *mesh size* of the grid is $\sup_k \mid t_k - t_{k-1} \mid$.

The following convergence result is sufficient for many purposes. This result is trivial if f is uniformly continuous. The content of the lemma is to reduce it to that case.

LEMMA 2. *Let (S^n) and S be $\mathbb{D}^d$-valued on the same probability space, S be continuous, and $S^n \Longrightarrow S$. For each n, let the random times $\{T_k^n\}$ define a grid on $[0,T]$ with mesh size converging with n to 0 almost surely. For some continuous $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, let $H_t^n = f\left[S^n(T_k^n), T_k^n\right]$, $t \in [T_k^n, T_{k+1}^n)$, and $H_t = f(S_t, t)$. Then $(H^n, S^n) \Longrightarrow (H, S)$.*

Since the limit process S is continuous, convergence in the Skorohod topology is equivalent here to convergence in the uniform metric topology, so the proof is straightforward and omitted.

COROLLARY 1. *Suppose, moreover, that $\{S^n\}$ is good. Then $\int H_{t-}^n \, dS_t^n \Longrightarrow \int H_{t-} \, dS_t$.*

The following corollary allows the function f defining the trading strategies to depend on n. The proof involves only a slight adjustment.

COROLLARY 2. *Suppose $f_n : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ is continuous for each n such that: For any $\epsilon > 0$, there is some N large enough that, for any (x,t) and $n \geq N$, $|f_n(x,t) - f(x,t)| < \epsilon$. Then, with $H_t^n = f_n[S(T_k^n), T_k^n]$, $t \in [T_k^n, T_{k+1}^n)$, the conclusions of Lemma 2 and Corollary 1 follow.*

6. Example: Convergence to the Black-Scholes Model

The objective of this section is to show that the weak convergence methods presented in this paper are easy to apply to a standard situation: the Black-Scholes (1973) option pricing formula. Under standard regularity conditions, the unique arbitrage-free price of a call option with time τ to expiration and exercise price K, when the current stock price is x, and the continuously compounding interest rate is $r \geq 0$, is

$$C(x, \tau) = \Phi(h)x - Ke^{-r\tau}\Phi(h - \sigma\sqrt{\tau}),$$

where Φ is the standard normal cumulative distribution function and

$$h = \frac{\log(x/K) + r\tau + \sigma^2\tau/2}{\sigma\sqrt{\tau}},$$

provided the stock price process S satisfies the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t; \quad S_0 = x > 0, \tag{2}$$

for constants μ, r, and $\sigma > 0$.[5] We will show convergence to the Black-Scholes formula in two cases:

(*a*) A fixed stock-price process S satisfying (2) and a sequence of stock trading strategies $\{\theta^n\}$ corresponding to discrete-time trading with trading frequency increasing in n, with limit equal to the Black-Scholes stock trading strategy $\theta_t = C_x(S_t, T-t)$, where T is the expiration date of the option and $C_x(x,\tau) = \frac{\partial}{\partial x}C(x,\tau)$.

(*b*) A sequence of stock price processes $\{S^n\}$ constructed as the stochastic exponentials of cumulative return processes $\{X^n\}$ converging in distribution to a Brownian Motion X, and trading strategies $\{\theta^n\}$ defined by $\theta^n(t) = C_x(S_t^n, T-t)$ for discretely chosen t.

Case (a) handles applications such as those of Leland (1985); Case (b) handles extensions of the Cox-Ross-Rubinstein (1979) results.

CASE (*a*) INCREASING TRADING FREQUENCY.

Let $T > 0$ be fixed, and let the set of stopping times $\mathcal{T}_n = \{T_k^n\}$ define a sequence of grids (as in Lemma 2) with mesh size shrinking to zero almost surely. In the n-th environment, the investor is able to trade only at stopping times in $\mathcal{T}_n$. That is, the trading strategy θ^n must be chosen from the set Θ^n of square-integrable predictable processes with $\theta^n(t) = \theta^n(T_{k-1}^n)$ for $t \in (T_{k-1}^n, T_k^n]$. For a simple case, let $T_k^n = k/n$, or n trades per unit of time, deterministically.

We take the case $r = 0$ for simplicity, since this allows us to consider stock gains alone, bond trading gains being zero. For $r > 0$, a standard trick of Harrison and Kreps (1979)

[5] Note that S is the stochastic exponential of the semimartingale $X_t = \mu t + \sigma B_t$.

allows one to normalize to this case without loss of generality. We consider the stock trading strategy $\theta^n \in \Theta^n$ defined by $\theta^n(0)$ arbitrary and

$$\theta^n(t) = C_x[S(T_k^n),\ T - T_k^n], \qquad t \in (T_k^n, T_{k+1}^n].$$

For riskless discount bonds maturing after T, with a face value of one dollar (the unit of account) and bearing zero interest, we define the bond trading strategy $\alpha^n \in \Theta^n$ by the self-financing restriction

$$\alpha^n(t) = \alpha^n(0) + \int_0^{T_k^n} \theta_t^n \, dS_t - \theta^n(T_k^n)S(T_k^n) + \theta^n(0)S(0), \qquad t \in (T_k^n, T_{k+1}^n], \tag{3}$$

where

$$\alpha^n(0) = C(S_0, T) - \theta^n(0)S_0.$$

The total initial investment $\alpha_0^n + \theta_0^n S_0$ is the Black-Scholes option price $C(S_0, T)$. (Note that, $\alpha^n \in \Theta^n$.) The total payoff of this self-financing strategy (α^n, θ^n) at time T is $C(S_0, T) + \int_0^T \theta_t^n \, dS_t$. For our purposes, it is therefore enough to show that

$$C(S_0, T) + \int_0^T \theta_t^n \, dS_t \Longrightarrow (S_T - K)^+,$$

the payoff of the option. This can be done by direct (tedious) calculation (as in, say, Leland (1985)), but our general weak convergence results are quite simple to apply here. It should be conceded, of course, that in simple cases such as that considered by Leland (1985), one could likely obtain[6] almost sure convergence.

PROPOSITION 1. *In the limit, the discrete-time self-financing strategy θ^n pays off the option. That is,* $C(S_0, T) + \int_0^T \theta_t^n \, dS_t \Longrightarrow (S_T - K)^+$.

PROOF: For $X^n = X = S$, it is clear that $\{X^n\}$ is good. With $H_t^n = C_x(S(T_{k-1}^n), T - T_{k-1}^n)$, $t \in [T_{k-1}^n, T_k^n)$, the conditions of Corollary 1 of Lemma 2 are satisfied since C_x is continuous. Since $\theta_t^n = H_{t-}^n$, it follows that $C(S_0, T) + \int_0^T \theta_t^n \, dS_t \Longrightarrow C(S_0, T) + \int_0^T \theta_t \, dS_t$. By Black and Scholes (1973), $C(S_0, T) + \int_0^T \theta_t \, dS_t = (S_T - K)^+$ a.s. [For the details, see, for example, Duffie (1988), Section 22.] Thus $C(S_0, T) + \int_0^T \theta_t^n dS_t \Longrightarrow (S_T - K)^+$. ∎

[6] Leland allows for transactions costs that converge to zero, and (despite appearances) actually makes an argument for convergence in mean, which does not necessarily imply almost sure convergence.

We can generalize the result as follows. We can allow S to be any diffusion process of the form $dS_t = \mu(S_t, t)\, dt + \sigma(S_t, t)\, dB_t$. Then, subject to technical restrictions, for any terminal payoff $g(S_T)$, there is a sequence of discrete-time trading strategies whose terminal payoff converges in distribution to $g(S_T)$. The following technical regularity conditions are far in excess of the minimum known sufficient conditions. For weaker conditions, see, for example, the references cited in Section 21 of Duffie (1988).

CONDITION C. *The functions $\sigma : I\!R \times [0, T] \to I\!R$ and $g : I\!R \to I\!R$ together satisfy Condition C if they are Lipschitz and have Lipschitz first and second derivatives.*

PROPOSITION 2. *Let (σ, g) satisfy Condition C. Suppose $dS_t = \mu(S_t, t)\, dt + \sigma(S_t, t)\, dB_t$, and that $\{S^n\}$ is good with $S^n \Longrightarrow S$. Then there exist (discrete-time self-financing) strategies (θ^n) in Θ^n such that*

$$E\left[g(X_T)\right] + \int_0^T \theta_t^n \, dS_t^n \Longrightarrow g(S_T)\,,$$

where $X_t = S_0 + \int_0^t \sigma(X_s, s) dB_s\,,\ t \in [0, T]$.

The result implies that one obtains the usual "risk-neutral" valuation and exact replication of the derivative payoff $g(S_T)$ in the limit, as $S^n \Longrightarrow S$.

PROOF: Let $F(x, t) = E\left[g(X_T^{x,t})\right]$, where $X_\tau^{x,t} = x + \int_t^\tau \sigma(X_s^{x,t}, s)\, dB_s\,,\quad \tau \geq t$. Then, as in Duffie (1988) Section 22, the partial F_x is a well-defined continuous function and $\theta_t = F_x(S_t, t)$ satisfies $E\left[g(X_T)\right] + \int_0^T \theta_t \, dS_t = g(S_T)$ a.s. For the trading strategies $\theta_t^n = f[S(T_k^n), T_k^n]$, $t \in (T_k^n, T_{k+1}^n]$, the result then follows as in the proof of Proposition 1. ∎

Related results have been obtained independently by He (1990).

CASE (b) (CUMULATIVE RETURNS THAT ARE APPROXIMATELY BROWNIAN MOTION).

The cumulative return process X corresponding to the price process S of (2) is the Brownian Motion X defined by

$$X_t = \mu t + \sigma B_t. \tag{4}$$

That is, $S = S_0 \mathcal{E}(X)$, where $\mathcal{E}(X)$ is the stochastic exponential of X as defined in Section 2. We now consider a sequence of cumulative return processes $\{X^n\}$ with $X^n \Longrightarrow X$.

Example 1. (Binomial Returns)

A classical example is the coin-toss walk "with drift" used by Cox, Ross, and Rubinstein (1979). That is, let

$$X_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} Y_k^n, \tag{5}$$

where, for each n, $\{Y_k^n\}$ is a sequence of independent and identically distributed binomial trials with $\sqrt{n}E(Y_1^n) \to \mu$ and $\mathrm{var}(Y_1^n) \to \sigma^2$. It is easy to show that $X^n \Longrightarrow X$. (See, for example, Duffie (1988), Section 22.)

Let us show that the assumptions of Theorem 3 (for example) are satisfied in this case. For any number t, recall that $[t]$ denotes the largest integer less than or equal to t. Since the $(Y_k^n)_{k\geq 1}$ are independent and have finite means, we know that

$$M_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} [Y_k^n - E\,(Y_k^n)]$$

is a martingale, and thus a decomposition of X^n is:

$$X_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} [Y_k^n - E(Y_1^n)] + \frac{1}{\sqrt{n}}[nt]E\,(Y_1^n) \equiv M^n + A^n.$$

The jumps of M^n are uniformly bounded. In order to verify Condition B for goodness, it is therefore enough to show that $E(|A^n|_T)$ is bounded. This follows from the fact that A^n is deterministic and $A_t^n \to \mu t$. Thus X^n is good. With $S^n = S_0^n \mathcal{E}(X^n)$ and $S_0^n \to S_0$, Theorem 4 implies that $\{S^n\}$ is good and that $S^n \Longrightarrow S$.

We consider the discrete-time stock-trading strategy $\theta^n \in \Theta^n$ defined by

$$\theta_t^n = C_x\,[S^n(T_k^n), T_k^n]\,, \qquad t \in (T_k^n, T_{k+1}^n],$$

where $S^n = S_0^n \mathcal{E}(X^n)$. In order to show that Black-Scholes applies in the limit, we must show that $C(S_0^n, 0) + \int_0^T \theta_t^n \, dS_t^n \Longrightarrow (S_T - K)^+$. [The self-financing bond trading strategy α^n is defined by the obvious analogue to (3), and the initial investment is the Black-Scholes value of the option, $C(S_0^n, T)$.] It is implicit in the following statement that all processes are defined on the same probability space unless the stopping times $\{T_k^n\}$ are deterministic.

Proposition 3. *Suppose $S_0^n \to S_0 > 0$, $\{X^n\}$ is good, and $X^n \Longrightarrow X$, where X is the Black-Scholes cumulative return process (4). Then $S^n = \mathcal{E}(X^n)S_0^n \Longrightarrow \mathcal{E}(X)S_0 = S$ and $C(S_0^n, 0) + \int_0^T \theta_t^n \, dS_t^n \Longrightarrow (S_T - K)^+$.*

Proof: To apply Corollary 1 of Lemma 2, we need only show that $S^n \Longrightarrow S$ and that S^n is good. This is true by Theorem 4. Since $C(S_0, 0) + \int_0^T \theta_t \, dS_t = (S_T - K)^+$ a.s., we are done. ∎

What examples, in addition to the coin-toss random walks $\{X^n\}$ satisfy the hypotheses of Proposition 3?

Example 2. (*iid Returns*). Suppose $\{X^n\}$ is a sequence of stock return processes defined by (5), where:

(i) $\{Y_k^n\}$ are uniformly bounded,

(ii) for each n, $\{Y_k^n\}$ is i.i.d.,

(iii) $\sqrt{n}\, E(Y_k^n) \to_n \mu$, and

(iv) $\sqrt{n}\, \mathrm{var}(Y_k^n) \to_n \sigma^2$.

Then, using Lindeberg's Central Limit in the proof of Donsker's Theorem, we have $X^n \Longrightarrow X$, where X is given by (4). Furthermore, $\{X^n\}$ satisfies the hypotheses of Theorem 3. Thus, the hypotheses of Proposition 3 are satisfied. This ends Example 2.

Example 3. (*Mixing returns*). Let the sequence $\{X^n\}$ of cumulative return processes be defined by (5), where the following conditions apply:

(i) $\{Y_k^n\}$ are uniformly bounded, $\mathbb{R}$-valued, and stationary in k (for each n).

(ii) For $\mathcal{F}_m^n = \sigma\{Y_k^n; k < m\}$, $\mathcal{G}_n^m = \sigma\{Y_k^n; k \leq m\}$, and $\varphi_p^n(m) = \varphi_p^n(\mathcal{G}_{m+\ell}^n \mid \mathcal{F}_\ell^n)$, where

$$\varphi_p^n(\mathcal{A} \mid \mathcal{B}) = \sup_{A \in \mathcal{A}} \|P_n(A \mid \mathcal{B}) - P_n(A)\|_{L^p},$$

$C_n{'} = \sum_{m=1}^\infty [\varphi_p^n(m)]^\alpha < \infty$, each n, where $p = \frac{2+\delta}{1+\delta}$, $\alpha = \frac{\delta}{1+\delta}$, for some $\delta > 0$.

(iii) $\sqrt{n} E_n(Y_k^n) \to \mu$ and, for $U_k^n = Y_k^n - E_n[Y_k^n]$, $\quad \sup_n \sqrt{n} C_n \|U_1^n\|_{L^{2+\delta}} < \infty$.

(iv) $\sigma_n^2 = E_n[(U_1^n)^2] + 2\sum_{k=2}^\infty E_n(U_1^n U_k^n)$ is well-defined and $\sigma_n^2 \to_n \sigma^2$.

Under (i)-(iv), for X^n defined by (5), we have $X^n \Longrightarrow X$. [See Ethier-Kurtz (1986), pp. 350-353, for calculations not given here.]

In order to invoke goodness, we need to find suitable semimartingale decompositions of X^n. To this end, following Ethier-Kurtz (1986) (p. 350 ff), define:

$$M_\ell^n = \sum_{k=1}^{\ell} U_k^n + \sum_{m=1}^{\infty} E_n\left(U_{\ell+m}^n \mid \mathcal{F}_\ell^n\right).$$

The series on the right is convergent as a consequence of the mixing hypotheses (see Ethier-Kurtz (1986), p. 351), and M_ℓ^n is a martingale (with jumps bounded by twice the bound on $\{Y_k^n\}$) with respect to the filtration $(\mathcal{F}_\ell^n)_{\ell\geq 1}$. We have

$$X_t^n = \frac{1}{\sqrt{n}} M_{[nt]}^n + A_t^n,$$

where

$$A_t^n = \frac{1}{\sqrt{n}} V_{[nt]}^n + \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} E_n(Y_k^n),$$

with

$$V_\ell^n = - \sum_{m=1}^{\infty} E_n(U_{\ell+m}^n \mid \mathcal{F}_\ell^n).$$

Note that the total variation of the paths of the process V^n are majorized in that

$$E_n(\mid V^n \mid_\ell) \leq 2E_n\left(\sum_{k=1}^{\ell} \left| \sum_{m=1}^{\infty} E_n(U_{k+m}^n \mid \mathcal{F}_k^n)\right|\right),$$

and using a standard estimate [Ethier-Kurtz (1986), p. 351], it follows that

$$E_n(\mid V^n \mid_\ell) \leq \sum_{k=1}^{\ell} \left(\sum_{m=1}^{\infty} 8\varphi_p^{\delta/1+\delta}(m)\right) \|U_1^n\|_{L^{2+\delta}} \leq 8\ell C_n \|U_1^n\|_{L^{2+\delta}}.$$

Then

$$\sup_n \frac{1}{\sqrt{n}} E_n\left(\mid V^n \mid_{[nt]}\right) \leq \sup_n \delta\sqrt{n}\, t\, C_n \,\|U_1^n\|_{L^{2+\delta}} < \infty$$

by hypothesis (iii). Since $\sqrt{n}E_n(Y_k^n) \to_n \mu$, it follows that $\sup_n E_n(|A^n|_T) < \infty$. Since M^n has bounded jumps, Condition B is satisfied, so X^n is good, and Proposition 3 applies once again. This ends Example 3.

Appendix A: "Non-Convergence with Stale Returns,"

This appendix explains the failure of weak convergence for Example 2 of Section 3, in which returns in model n are given by the "stale return" process R^n, which converges to the Brownian motion R. We need to show that the wealth process X^n defined by the investment policy g converges to the process Y given by (1). This is really an extension of the "Wong-Zakai pathology" that was pursued by Kurtz and Protter (1991a). To this end, consider the following calculations. (Without loss of generality, let $\sigma = 1$.) We have

$$\begin{aligned} R^n(t) &= R(t) + [R^n(t) - R(t)] \\ &= V_n(t) + Z_n(t), \end{aligned}$$

where $V_n(t) = R(t)$ for all n and where $Z_n(t) = R^n(t) - R(t)$. Then, clearly, $V_n \Longrightarrow R$ and $Z_n \Longrightarrow 0$. The key is to look at

$$\begin{aligned} H_n(t) &= \int_0^t Z_n(s)\, dZ_n(s) \\ &= \frac{1}{2} Z_n(t)^2 - \frac{1}{2}[Z_n, Z_n]_t \\ &= \frac{1}{2} Z_n(t)^2 - \frac{1}{2} t, \end{aligned}$$

since $[Z_n, Z_n]_t = [R^n - R, R^n - R]_t = [R, R]_t = t$. Since $Z_n \Longrightarrow 0$, so does Z_n^2 by the continuous mapping theorem, and hence $H_n(t) \Longrightarrow -t/2$. Similarly, letting $K_n = [V_n, Z_n]$, we have $K_n(t) = -t$. Then $U_n = H_n - K_n \Longrightarrow U$, where $U_t = t/2$. Theorem 5.10 of Kurtz and Protter (1991a) then implies that $X^n \Longrightarrow Y$ defined by (1).

What is "really" going on in this example is that the Brownian motion R, which is a continuous martingale with paths of infinite variation on compacts, is being approximated by continuous processes R^n of finite variation; moreover the processes have no martingale properties. Thus the calculus of the R^n processes is the classical path by path Riemann-Stieltjes "first order" calculus; while the calculus of the Brownian motion R is the Itô "second order" calculus. This leads to a discontinuity (or lack of robustness) when we approximate R by R^n. This discontinuity is precisely computable in the above calculations, applied to Theorem (5.10) of Kurtz and Protter (1991a).

Appendix B: An Aid to Checking Goodness of Special Semimartingales

LEMMA B1. *Suppose $\{Z^n\}$, with $Z^n = M^n + A^n$, is a sequence of special semimartingales for which the measures on $[0,T]$ induced by (A^n) are tight. Suppose further that $\sup_n E_n(\sup_{t\leq T} |\Delta A_t^n|) < \infty$. Then, for the canonical decomposition $Z^n = \tilde{M}^n + \tilde{A}^n$, the measures induced by $\tilde{A}_n$ are also tight.*

PROOF: Let

$$H_t^n = \frac{d\widetilde{A}_t^n}{d|\widetilde{A}^n|_t},$$

where $|\widetilde{A}^n|_t = \int_0^n |d\widetilde{A}_s^n|$ denotes the total variation of the paths of the process $\widetilde{A}^n$. Then H^n is predictable and, for any stopping time τ,

$$E\left[\int_0^\tau H_t^n\, dA_t^n\right] = E\left[\int_0^\tau H_t^n\, d\widetilde{A}_t^n\right] = E\left[|\widetilde{A}^n|_\tau\right].$$

Since $|H_t^n| = 1$,

$$E\left[|A^n|_\tau\right] \geq E\left[\int_0^\tau H_t^n\, dA_t^n\right] = E\left[|\widetilde{A}^n|_\tau\right].$$

By the Lenglart Domination Theorem [Jacod and Shiryaev (1987), Lemma 3.30 (b), page 35, with $\epsilon = b$ and $\eta = \sqrt{b}$],

$$\lim_{b\to\infty} \sup_n P_n\left(|\widetilde{A}^n|_\tau \geq b\right) \leq \lim_{b\to\infty} \sup_n \left\{\frac{1}{b}\left[\sqrt{b} + E_n\left(\sup_{t\leq\tau} |\Delta A^n|_t\right)\right] + P_n\left(|A^n|_\tau \geq \sqrt{b}\right)\right\} = 0.$$

Since the measures induced by (A^n) are tight, it follows that the measures induced by $\widetilde{A}^n$ are tight. ∎

COROLLARY. *Suppose the measures induced by (A^n) are tight and A^n is locally of integrable variation for all n. If $\widetilde{A}^n$ is the predictable compensator for A^n, then the measures induced by $(\widetilde{A}^n)$ are tight.*

PROOF: For given b, let $T^n = \inf\{t \geq 0 : |A^n|_t \geq b\}$. The stopped process $(A^n)^{T^n_-}$ is of bounded total variation. For given $t_0 > 0$ and $\epsilon > 0$, there exists b large enough that

$$\sup_n P_n(T^n \leq t_0) \leq \sup_n P_n\left(|A^n|_{t_0} \geq b\right) < \epsilon, \tag{6}$$

since the measures induced by (A^n) are tight. It follows that

$$\begin{aligned}\lim_{b\to\infty} \sup_n P_n\left(|\widetilde{A}^n|_{t_0} \geq b\right) &\leq \lim_{b\to\infty} \sup_n \left\{P_n\left(\left\{|\widetilde{A}^n|_{t_0} \geq b\right\} \bigcap \{T^n > t_0\}\right) + P_n(T^n \leq t_0)\right\} \\ &\leq \lim_{b\to\infty} \sup_n \left\{P_n\left(\left| (\widetilde{A}^n)^{T^n_-} \right|_{t_0} \geq b\right) + P_n(T^n \leq t_0)\right\} < \epsilon,\end{aligned}$$

by the lemma and (6). Since ϵ is arbitrary, the lim sup is 0, so the measures induced by $(\widetilde{A}^n)$ are tight. ∎

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