GENERAL CHANGE OF VARIABLES FORMULAS FOR SEMIMARTINGALES IN ONE AND FINITE DIMENSIONS

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Abstract. A general one dimensional change of variables formula is established for continuous semimartingales which extends the famous Meyer-Tanaka formula. The inspiration comes from an application arising in stochastic finance theory.

For functions mapping $\mathbf{R}^n$ to $\mathbf{R}$, a general change of variables formula is established for arbitrary semimartingales, where the usual C^2 hypotheses are relaxed.

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1. Introduction.

If $f : \mathbf{R} \to \mathbf{R}$, it was shown in Çinlar-Jacod-Protter-Sharpe [3] that f preserves semimartingales if and only if f is the difference of two convex functions. In this case the famous Meyer-Tanaka formula holds (see Protter [10]):

$$
\begin{aligned}
f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_{-\infty}^{\infty} L_t^x \mu(dx) \\
+ \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - f(X_{s-})\Delta X_s\},
\end{aligned} \tag{1.1}
$$

where X is an arbitrary semimartingale, L_t^x is its local time at the level x, f' is the left continuous version of its derivative, and μ is a signed measure which is the second derivative of f in the generalized function sense.

There is (as yet) no analogous theorem characterizing the functions $f : \mathbf{R}^n \to \mathbf{R}$ which preserve semimartingales, although P. A. Meyer [7] did show in 1976 that if f is convex then it preserves semimartingales.

What we are concerned with here is twofold. In Section Three we extend formula (1.1) to functions of the form $f(x, w, t)$ which are fairly general. We became interested in this extension because of discussions with Ravi Myneni [9], who pointed out that such a formula would be useful in stochastic finance theory. Next, in Section Four we extend the usual Meyer-Itô change of variables formula for functions $f : \mathbf{R}^n \to \mathbf{R}$ that are significantly more general that the usual C^2 hypothesis. Some partial results in very special cases have already been obtained (see Brosamler [1], N. V. Krylov [4], I. Kubo [5], H. H. Kuo and N. R. Shieh [6], P. A. Meyer [8], J. Rosen [11], and M. Yor [12], for example), but we believe ours is the first attempt at a general treatment of the subject for semimartingales (the closest predecessor being that of Krylov [4] for certain diffusions), and our results contain those of Krylov. For a different approach to this problem see E. Carlen and P. Protter [2].

2. A Generalized Local Time.

In Section 3 we establish change of variables formulas where we need to give meaning to terms such as $dL_t^{\theta(t)}$, where L_t^x is the local time at the level x of a continuous semimartingale X, and θ is a continuous, adapted process with paths of finite variation on compacts. In this section we give meaning to $dL_t^{\theta(t)}$, and to avoid confusion with dL_t^x, we denote it $\partial_u L_u^{\theta(u)}$ (see Definitions (2.8) and (2.11)).

Let X and θ be continuous semimartingales, with θ of finite variation on compacts. We make the following hypothesis:

HYPOTHESIS (2.2). $\int_0^t 1_{\{X_u=\theta_u\}} dX_u = \int_0^t 1_{\{X_u=\theta_u\}} d\theta_u = 0.$

In our analysis we have had to give a meaning to terms of the form $dL_u^{\theta(u)}$, where the "d" refers to the variable u, and where $L_u^x = L_u^x(X)$ is the semimartingale local time of X at level x and time u.

THEOREM (2.3). *Let $\pi_n[0,t]$ be a sequence of partitions of $[0,t]$ with $\lim_{n\to\infty} mesh(\pi_n) = 0$. Let H be a continuous, adapted process. Assume Hypothesis (2.2) holds. Then*

$$\lim_{n\to\infty} \sum_{t_i\in\pi_n} H_{t_i}(L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)}) = \int_0^t H_u dL_u^0(X-\theta),$$

where L_t^x is the local time of X at level x, time t.

Proof. By an elementary formula for local time (Protter [10, p.169]),

$$\begin{aligned} L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)} &= |X_{t_{i+1}} - \theta_{t_i}| - |X_{t_i} - \theta_{t_i}| \\ &\quad - \int_{t_i}^{t_{i+1}} \operatorname{sign}(X_u - \theta_{t_i}) dX_u. \end{aligned} \tag{2.4}$$

The first two terms on the right side above equal:

$$\begin{aligned} |X_{t_{i+1}} - \theta_{t_i}| - |X_{t_i} - \theta_{t_i}| &= |X_{t_{i+1}} - \theta_{t_i}| - |X_{t_{i+1}} - \theta_{t_{i+1}}| \\ &\quad + |X_{t_{i+1}} - \theta_{t_{i+1}}| - |X_{t_i} - \theta_{t_i}| \\ &= \int_{t_i}^{t_{i+1}} \operatorname{sign}(X_{t_{i+1}} - \theta_u) d\theta_u + |X_{t_{i+1}} - \theta_{t_{i+1}}| - |X_{t_i} - \theta_{t_i}| \end{aligned} \tag{2.5}$$

because θ is continuous and of finite variation, and hence its local time is identically zero.

Combining (2.4) and (2.5) gives (sums are over t_i in $\pi_n[0,t]$):

$$\begin{aligned}
&\sum H_{t_i}(L^{\theta(t_i)}_{t_{i+1}} - L^{\theta(t_i)}_{t_i}) \\
&\quad= \sum H_{t_i} \int_{t_i}^{t_{i+1}} \operatorname{sign}(X_{t_{i+1}} - \theta(u))d\theta_u \\
&\qquad - \sum H_{t_i} \int_{t_i}^{t_{i+1}} \operatorname{sign}(X_u - \theta_{t_i})dX_u \\
&\qquad + \sum H_{t_i}\{|X_{t_{i+1}} - \theta_{t_{i+1}}| - |X_{t_i} - \theta_{t_i}|\} \\
&\quad= \int_0^t h_n(u)d\theta_u - \int_0^t g_n(u)dX_u \\
&\qquad + \sum H_{t_i}\{|X_{t_{i+1}} - \theta_{t_{i+1}}| - |X_{t_i} - \theta_{t_i}|\}
\end{aligned}$$

where

$$\begin{cases} h_n(u) \equiv \sum\limits_{t_i \in \pi_n} H_{t_i}\operatorname{sign}(X_{t_{i+1}} - \theta_u)1_{(t_i,t_{i+1}]}(u) \\ g_n(u) \equiv \sum\limits_{t_i \in \pi_n} H_{t_i}\operatorname{sign}(X_u - \theta_{t_i})1_{(t_i,t_{i+1}]}(u). \end{cases} \tag{2.6}$$

Since X and θ are both continuous, we know that if $X_{u_0} > \theta_{u_0}(w)$ (resp. $X_{u_0}(w) < \theta_{u_0}(w)$) then there exists an interval $(u_0 - \delta(w), u_0 + \delta(w))$ such that $X_u(w) > \theta_u(w)$ (resp. $X_u(w) < \theta_u(w)$) for all u in the interval. This implies that

$$\begin{aligned}
\lim_{n\to\infty} h_n(u)1_{\{X_u \neq \theta_u\}} &= H_u \operatorname{sign}(X_u - \theta_u)1_{\{X_u \neq \theta_u\}} \\
\lim_{n\to\infty} g_n(u)1_{\{X_u \neq \theta_u\}} &= H_u \operatorname{sign}(X_u - \theta_u)1_{\{X_u \neq \theta_u\}}.
\end{aligned}$$

Since by assumption $\int_0^t 1_{\{X_u = \theta_u\}}dX_u = \int_0^t 1_{\{X_u = \theta_u\}}d\theta_u = 0$, we conclude:

$$\begin{aligned}
\lim_{n\to\infty} \sum H_{t_i}(L^{\theta(t_i)}_{t_{i+1}} - L^{\theta(t_i)}_{t_i}) = &- \int_0^t H_u \operatorname{sign}(X_u - \theta_u)d(X_u - \theta_u) \\
&+ \int_0^t H_u d|X - \theta|_u.
\end{aligned} \tag{2.7}$$

We next use the Meyer-Tanaka formula on $|X - \theta|$, and (2.7) simplifies to:

$$\lim_{n\to\infty} \sum H_{t_i}(L^{\theta(t_i)}_{t_{i+1}} - L^{\theta(t_i)}_{t_i}) = \int_0^t H_u dL^0_u(X - \theta),$$

which was to be proved, where convergence is convergence in probability, uniform in time on compacts (u.c.p.). $\square$

DEFINITION (2.8). *For H continuous, adapted, we define*

$$\int_0^t H_u \partial_u L_u^{\theta(u)}(X) = \lim_{n\to\infty} \sum_{t_i\in\pi_u[0,t]} H_{t_i}(L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)})$$

when X, θ are continuous semimartingales, θ is of finite variation on compacts, and hypothesis (2.2) holds. Convergence is ucp.

With the above definition, Theorem (2.3) gives immediately:

COROLLARY (2.9). *With the hypotheses of Theorem (2.3),*

$$\int_0^t H_u \partial_u L_u^{\theta(u)}(X) = \int_0^t H_u dL_u^0(X-\theta).$$

We can weaken Hypothesis (2.2) in the case where θ is monotone.

THEOREM (2.10). *Let X, θ be continuous semimartingales, with θ increasing, and assume $\int_0^t 1_{\{X_u=\theta_u\}} d\theta_u = 0$. Let H be a continuous, adapted process. Then*

$$\lim_{n\to\infty} \sum_{t_i\in\pi_n} H_{t_i}(L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)}) =$$

$$\int_0^t H_u d\{2L_u^0(X-\theta) - L_u^{0-}(X-\theta)\}.$$

If θ is decreasing, then

$$\lim_{n\to\infty} \sum_{t_i\in\pi_n} H_{t_i}(L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)}) = \int_0^t H_u dL_t^0(X-\theta),$$

where $\pi_n = \pi_n[0,t]$ is a sequence of partitions of $[0,t]$ with $\lim_{n\to\infty} mesh(\pi_n) = 0$. Convergence of the sums is in ucp.

Proof. With the notation (2.6) as used in the proof of Theorem (2.3), note that if $X_u = \theta_u$, then $g_n(u) = H_{t_i}$, because $X_u \geq \theta(t_i)$ for $u \geq t_i$. Hence

$$\lim_{n\to\infty} g_n(u) = H_u = H_u \mathrm{sign}(X_u - \theta_u).$$

Thus as in the proof of Theorem 2.3 (see (2.7)),

$$\begin{aligned}
&\lim_{n\to\infty} \sum H_{t_i}(L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)}) \\
&= \int_0^t H_u \mathrm{sign}(X_u-\theta_u) d\theta_u - \int_0^t H_u \mathrm{sign}(X_u-\theta_u) 1_{\{X_u\neq\theta_u\}} dX_u \\
&\quad + \int_0^t H_u 1_{\{X_u=\theta_u\}} dX_u + \int_0^t H_u d|X-\theta|_u \\
&= -\int_0^t H_u \mathrm{sign}(X_u-\theta_u) d(X_u-\theta_u) \\
&\quad + 2\int_0^t H_u 1_{\{X_u=\theta_u\}} dX_u + \int_0^t H_u d|X-\theta|_u
\end{aligned}$$

and using the Meyer-Tanaka formula on the last term on the right side of the above equality:

$$= 2\int_0^t H_u 1_{\{X_u=\theta_u\}} dX_u + \int_0^t H_u dL_u^0(X-\theta)$$

$$= 2\int_0^t H_u 1_{\{X_u=\theta_u\}} d(X_u-\theta_u) + \int_0^t H_u dL_u^0(X-\theta)$$

(since $\int_0^t 1_{\{X_u=\theta_u\}} d\theta_u = 0$ by hypothesis)

$$= \int_0^t H_u d\{2L_u^0(X-\theta) - L_u^{0-}(X-\theta)\},$$

where we have used Corollary 1, p.177, of Protter [10] ($L_u^{0-} = \lim_{\substack{x\to 0\\ x<0}} L_u^x$). The formula for θ decreasing is established analogously. The lack of symmetry is due to the asymmetric definition of semimartingale local time.

□

DEFINITION (2.11). *For H continuous, adapted; X, θ continuous semimartingales and θ monotone; $\int_0^t 1_{\{X_u=\theta_u\}} d\theta_u = 0$. Define*

$$\int_0^t H_u \partial_u L_u^{\theta(u)}(X) = \lim_{n\to\infty} \sum_{t_u \in \pi_n[0,t]} H_{t_i}(L_{t_{i+1}}^{\theta(t_i)} - L_{t_i}^{\theta(t_i)})$$

where convergence is in ucp.

COROLLARY (2.12). *If θ is increasing in (2.11),*

$$\int_0^t H_u \partial_u L_u^{\theta(u)} = \int_0^t H_u d\{2L_u^0(X-\theta) - L_u^{0-}(X-\theta)\}.$$

If θ is decreasing in (2.11), then

$$\int_0^t H_u \partial_u L_u^{\theta(u)} = \int_0^t H_u dL_u^0(X-\theta),$$

for H continuous, adapted.

Note that by taking $H_u \equiv 1$ we obtain for example, if θ is decreasing,

$$\int_0^t \partial_u L_u^{\theta(u)} = L_t^0(X-\theta)$$

and if θ is increasing,

$$\int_0^t \partial_u L_u^{\theta(u)} = 2L_t^0(X-\theta) - L_t^{0-}(X-\theta).$$

The preceding results raise the question: When is it the case that $\int_0^t 1_{\{X_u=\theta_u\}} dX_u = 0$?

LEMMA (2.13). *Let X, θ be continuous semimartingales, and θ be of finite variation on compacts. Let $X = M + A$ be the unique decomposition of X into a local martingale M and a finite variation process A. Then* $\int_0^t 1_{\{X_u=\theta_u\}} dX_u = \int_0^t 1_{\{X_u=\theta_u\}} dA_u$.

Proof.

$$\int_0^t 1_{\{X_u=\theta_u\}} dX_u = \int_0^t 1_{\{X_u=\theta_u\}} dM_u + \int_0^t 1_{\{X_u=\theta_u\}} dA_u.$$

Let $N_t = \int_0^t 1_{\{X_u=\theta_u\}} dM_u$, a local martingale, with $N_0 = 0$. Then

$$\begin{aligned}
[N,N]_t &= \int_0^t 1_{\{X_u=\theta_u\}} d[M,M]_u \\
&= \int_0^t 1_{\{X_u-\theta_u=0\}} d[X-\theta, X-\theta]_u \\
&= \int_{-\infty}^{\infty} L_t^a(X-\theta) 1_{\{a=0\}} da = 0,
\end{aligned}$$

using the space-time local time formula (see, e.g., Protter [10, p.168]). Since $[N,N]_t = 0$, N must be constant, hence it is identically zero. Therefore

$$\int_0^t 1_{\{X_u=\theta_u\}} dX_u = \int_0^t 1_{\{X_u=\theta_u\}} dA_u.$$

□

THEOREM (2.14). *Let Z be a continuous semimartingale with decomposition $Z = M + A$, and further suppose $dA_u << d[Z,Z]_u$. Let X solve the stochastic differential equation (with σ continuous and never zero),*

$$dX_t = \sigma(X_t, t) dM_t + b(X_t, t) dA_t.$$

Let θ be a continuous semimartingale of finite variation on compacts. Then

$$\int_0^t 1_{\{X_u=\theta_u\}} dX_u = 0.$$

Proof. By Lemma (2.13), we know that

$$\begin{aligned}
\int_0^t 1_{\{X_u=\theta_u\}} dX_u &= \int_0^t 1_{\{X_u=\theta_u\}} b(X_u,u)dA_u \\
&= \int_0^t 1_{\{X_u=\theta_u\}} b(X_u,u)h(u,w)d[Z,Z]_u \\
&= \int_0^t 1_{\{X_u=\theta_u\}} \frac{b(X_u,u)}{\sigma^2(X_u,u)} h(u,w)d[X,X]_u \\
&= \int_0^t 1_{\{X_u-\theta_u=0\}} \frac{b(X_u-\theta_u+\theta_u,u)}{\sigma^2(X_u-\theta_u+\theta_u,u)} h(u,w)d[X-\theta,X-\theta]_u \\
&= \int_{\mathbf{R}} \int_0^t 1_{\{a=0\}} \frac{b(a+\theta_u,u)}{\sigma^2(a+\theta_u,u)} h(u,w)dL_u^a(X-\theta)du \\
&= 0,
\end{aligned}$$

where the penultimate equality uses a generalized Tanaka formula established in San Martin [12] or [13].

□

Note in particular that by letting $M = B$, standard Brownian motion, and $dA_t = dt$, Theorem 2.14 includes Itô diffusions. An analogous result holds when $d\theta_u << d[X,X]_u$, but in general $\int_0^t 1_{\{X_u=\theta_u\}} d\theta_u$ is not zero. For example, take $X_u = B_u$, or standard one dimensional Brownian motion. Take $\theta_u = \sup_{t\le u} B_t$. Then θ is increasing and the support of $d\theta_u$ is carried by $\{u : B_u = \theta_u\}$. Thus $\int_0^t 1_{\{B_u=\theta_u\}} d\theta_u = \theta_t$. Nevertheless by Theorem (2.14), $\int_0^t 1_{\{B_u=\theta_u\}} dB_u = 0$.

3. Change of Variables on $\mathbf{R} \times \Omega \times \mathbf{R}_+$.

In this section we establish several one dimensional change of variables formulas for more general situations than have been previously considered. Our principal result is Theorem (3.2).

We shall consider random functions of the form $f(x,t,w) = \int_0^t h(x,s,w)dA_s = \int_{0+}^t h(x,s,w)dA_s + h(x,0,w)A_0$, where A is a continuous, adapted process of finite variation, and $h : \mathbf{R} \times \mathbf{R}_+ \times \Omega \to \mathbf{R}$ such that

(i) $h(x,\cdot,\cdot)$ is adapted and jointly measurable;

(ii) $h(\cdot,s,w)$ is the difference of two convex functions with generalized second derivative a signed measure $\mu(\cdot,s,w)$;

(iii) $\sup_{\substack{s\le t\\ x\in K}} \left|\frac{\partial h}{\partial x}(x,s,w)\right| < \infty$ for each $t > 0$ and compact $K \subseteq \mathbf{R}$, where $\frac{\partial h}{\partial x}$ is the càdlàg version;

(iv) $h(x,s,w) = 0$ if $x \le 0$, all (s,w).

As in Section 2 let X, θ be continuous semimartingales, with θ of finite variation on compacts. We again make the hypothesis (same as Hypothesis (2.2)):

HYPOTHESIS (3.1). $\int_0^t 1_{\{X_u=\theta_u\}}dX_u = \int_0^t 1_{\{X_u=\theta_u\}}d\theta_u = 0.$

THEOREM (3.2). *Let f, h, and A be as above, and let X, θ be continuous semimartingales with θ of finite variation on compacts, and suppose Hypothesis 3.1 holds. Then*

$$\begin{aligned}
f((X_t-\theta_t)^+,t) &= f((X_0-\theta_0)^+,0) \\
&+ \int_{0+}^t h(X_s-\theta_s,s)dA_s + \int_{0+}^t \frac{\partial f}{\partial x}(X_s-\theta_s,s)1_{(X_s>\theta_s)}d(X-\theta)_s \\
&+ \frac{1}{2}\int_{0+}^t \frac{\partial f}{\partial x}(0,s)\partial_s L_s^{\theta(s)}(X) + \frac{1}{2}\int_0^t\int_{0+}^\infty\int_s^t \partial_u L_u^{y+\theta(u)}(X)\mu(s,\cdot,dy)dA_s \\
&- \int_0^t \nu(\{X_u-\theta_u\},u)d(X-\theta)_u,
\end{aligned}$$

where ν is a measure given by

$$\nu(\Lambda,u) = \int_0^u \mu(\Lambda\cap(0,\infty),s)dA_s.$$

Proof. We begin by using the generalized Itô formula (see San Martin [12] or [13]): for any $\varepsilon > 0$ we have

$$
\begin{aligned}
f((X_t-\theta_t)^+ + \varepsilon, t) &= f((X_0-\theta_0)^+ + \varepsilon, 0) \\
&+ \int_{0+}^t h((X_s-\theta_s)^+ + \varepsilon, s)dA_s \\
&+ \int_{0+}^t \frac{\partial f}{\partial x}([(X_s-\theta_s)^+ + \varepsilon]-, s)d(X-\theta)_s^+ \\
&+ \frac{1}{2}\int_0^t \int_{\mathbf{R}} (L_t^y - L_s^y)\mu(s,\cdot,dy)dA_s
\end{aligned}
$$

where $L_t^y = L_t^y((X-\theta)^+ + \varepsilon) = L_t^{y-\varepsilon}((X-\theta)^+)$ if $y \geq \varepsilon$, and 0 if $y < \varepsilon$.

Next, using the dominated convergence theorem for semimartingales we get:

$$
\begin{aligned}
f((X_t-\theta_t)^+, t) &= f((X_0-\theta_0)^+, 0) \\
&+ \int_{0+}^t h((X_s-\theta_s)^+, s)dA_s \\
&+ \int_{0+}^t \frac{\partial f}{\partial x}((X_s-\theta_s)^+, s)d(X-\theta)_s^+ \\
&+ \frac{1}{2}\int_0^t \int_{0+}^\infty (L_t^{y-} - L_s^{y-})\mu(s,\cdot,dy)dA_s.
\end{aligned}
\tag{3.3}
$$

Let $X = M + C$ be the decomposition of X into a local martingale M and a finite variation process C. Then

$$
L_s^y(X-\theta) - L_s^{y-}(X-\theta) = 2\int_0^s 1_{\{X_u-\theta_u=y\}} d(C-\theta)_u,
$$

since $L_s^y(X-\theta)^+ = L_s^y(X-\theta)$ for $y > 0$. Therefore, letting $\frac{1}{2}I$ denote the last term on the right side of (3.3):

$$
\begin{aligned}
I &= \int_0^t \int_{0+}^\infty (L_t^{y-} - L_s^{y-})\mu(s,\cdot,dy)dA_s \\
&= \int_0^t \int_{0+}^\infty \int_s^t dL_u^y(X-\theta)\mu(s,\cdot,dy)dA_s \\
&- 2\int_0^t \int_{0+}^\infty \int_s^t 1_{\{X_u-\theta_u=y\}} d(C-\theta)_u \mu(s,\cdot,dy)dA_s.
\end{aligned}
$$

By Fubini's theorem the above becomes:

$$
\begin{aligned}
I &= \int_0^t \int_{0+}^\infty \int_s^t dL_u^y(X-\theta)\mu(s,\cdot,dy)dA_s \\
&- 2\int_0^t \int_0^u \int_{0+}^\infty 1_{\{X_u-\theta_u=y\}} \mu(s,\cdot,dy)dA_s d(C-\theta)_u.
\end{aligned}
$$

Letting $\nu(\Lambda, u) \equiv \int_0^u \mu(\Lambda \cap (0,\infty), s) dA_s$, the above becomes:

$$
\begin{aligned}
(3.4) \qquad I = &\int_0^t \int_{0+}^\infty \int_s^t dL_u^y(X-\theta)\mu(s,\cdot,dy)dA_s \\
&-2\int_0^t \nu(\{X_u - \theta_u\}, u) d(C-\theta)_u.
\end{aligned}
$$

We next show that $d(C-\theta)_u$ can be replaced with $d(X-\theta)_u$ in the second term on the right side of (3.4) above. We do this by showing $\int_0^t \nu(\{X_u - \theta_u\}, u) dM_u = 0$, where we recall X has decomposition $X = M + C$. It suffices to show $\int_0^t \nu(\{X_u - \theta_u\}, u) d[M, M]_u = 0$. To this end we have:

$$
\begin{aligned}
|J_t| &\equiv \left| \int_0^t \nu(\{X_u - \theta_u\}, u) d[M, M]_u \right| \\
&= \left| \int_0^t \nu(\{X_u - \theta_u\}, u) d[X-\theta, X-\theta]_u \right| \\
&= \left| \int_{\mathbf{R}} \int_0^t \nu(\{a\}, u) dL_u^a da \right| \\
&= \left| \int_{0+}^\infty \int_0^t \int_0^u \mu(s, \{a\}) dA_s dL_u^a da \right|
\end{aligned}
$$

where $L_u^a = L_u^a(X-\theta)$. The foregoing implies

$$
\begin{aligned}
|J_t| &\le \int_{0+}^\infty \int_0^t \int_0^t |\mu(s,\{a\})| |dA_s| dL_u^a(X-\theta) da \\
&\le \int_{0+}^\infty \left\{ \int_0^t |\mu(s,\{a\})| |dA_s| \right\} L_t^a(X-\theta) da \\
&\le \max_{a \in \mathbf{R}_+} L_t^a(X-\theta) \int_{0+}^\infty \int_0^t |\mu(s,\{a\})| |dA_s| da.
\end{aligned}
$$

However since $\mu(s,\cdot)$ has at most countably many atoms a.s., we deduce $|J_t| = 0$. Thus (3.4) becomes:

$$
\begin{aligned}
(3.5) \qquad I = &\int_0^t \int_{0+}^\infty \int_s^t dL_u^y(X-\theta)\mu(s,\cdot,dy)dA_s \\
&-2\int_0^t \nu(\{X_u - \theta_u\}, u) d(X-\theta)u.
\end{aligned}
$$

Let us return to equation (3.3). We observe that:

$$
d(X-\theta)_s^+ = 1_{(X_s > \theta_s)} d(X-\theta)_s + \frac{1}{2} dL_s^0(X-\theta)
$$

and

$$dL^y_s(X-\theta) = \partial_s L^{y+\theta(s)}_s(X)$$

by Corollary (2.9). Combining this with (3.5), equation (3.3) becomes:

$$\begin{aligned}
f((X-\theta)^+_t, t) =& f((X-\theta)^+_0, 0) + \int_{0+}^t h(X_s-\theta_s, s)dA_s \\
&+ \int_{0+}^t \frac{\partial f}{\partial x}((X_s-\theta_s)^+, s)1_{\{X_s>\theta_s\}}d(X-\theta)_s \\
&+ \int_{0+}^t \frac{\partial f}{\partial x}((X-\theta)^+_s, s)\frac{1}{2}dL^0_s(X-\theta) \\
&+ \frac{1}{2}\int_0^t \int_{0+}^\infty \int_s^t dL^y_u(X-\theta)\mu(s,\cdot,dy)dA_s \\
&- \int_0^t \nu(\{X_u-\theta_u\}, u)d(X-\theta)_u \\
=& f((X-\theta)^+_0, 0) + \int_{0+}^t h(X_s-\theta_s, s)dA_s \\
&+ \int_{0+}^t \frac{\partial f}{\partial x}(X_s-\theta_s)1_{(X_s>\theta_s)}d(X-\theta)_s \\
&+ \frac{1}{2}\int_{0+}^t \frac{\partial f}{\partial x}(0, s)\partial_s L^{\theta(s)}_s(X) \\
&+ \frac{1}{2}\int_0^t \int_{0+}^\infty \int_s^t \partial_u L^{y+\theta(u)}_u(X)\mu(s,\cdot,dy)dA_s \\
&- \int_0^t \nu(\{X_u-\theta_u\}, u)d(X-\theta)_u.
\end{aligned}$$

□

COROLLARY (3.6). *If $x \to h(x,s,w)$ is C^2, we have (under the hypotheses of Theorem (3.2)):*

$$\begin{aligned}
f((X-\theta)^+_t, t) =& f((X_0-\theta_0)^+, 0) + \int_{0+}^t h(X_s-\theta_s, s)dA_s \\
&+ \int_{0+}^t \frac{\partial f}{\partial x}(X_s-\theta_s, s)1_{(X_s>\theta_s)}d(X-\theta)_s \\
&+ \frac{1}{2}\int_{0+}^t \frac{\partial f}{\partial x}(0, s)\partial_s L^{\theta(s)}_s(X) \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_u-\theta_u, u)1_{(X_u>\theta_u)}d[X,X]_u.
\end{aligned}$$

Proof. Note that $\mu(s,\cdot,dy) = \frac{\partial^2 h}{\partial y^2}(y,s,\cdot)dy$, so μ has no atoms, and the formula of Theorem (3.2) reduces to

$$
\begin{aligned}
f((X-\theta)_t^+,t) =& f((X-\theta)_0^+,0) + \int_{0+}^{t} h(X_s-\theta_s,s)dA_s \\
&+ \int_{0+}^{t} \frac{\partial f}{\partial x}(X_s-\theta_s,s)1_{(X_s>\theta_s)}d(X-\theta)_s \\
&+ \frac{1}{2}\int_{0+}^{t} \frac{\partial f}{\partial x}(0,s)\partial_s L_s^{\theta(s)}(X) \\
&+ \frac{1}{2}\int_0^t \int_{0+}^{\infty} \int_s^t \partial_u L_u^{y+\theta(u)}(X)\frac{\partial^2 h}{\partial y^2}(y,s)dydA_s.
\end{aligned}
\tag{3.7}
$$

Consider the last term on the right side of (3.7) above. This term is equal to

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \int_{0+}^{\infty} (L_t^y(X-\theta) - L_s^y(X-\theta))\frac{\partial^2 h}{\partial y^2}(y,s)dydA_s \\
=&\frac{1}{2}\int_{0+}^{\infty} (\int_0^t L_t^y(X-\theta) - L_s^y(X-\theta))\frac{\partial^2 h}{\partial y^2}(y,s)dA_s)dy \\
=&\frac{1}{2}\int_{0+}^{\infty}\int_0^t\int_0^s \frac{\partial^2 h}{\partial y^2}(y,u)dA_u dL_s^y(X-\theta)dy \\
=&\frac{1}{2}\int_{\mathbf{R}} 1_{(y>0)}\int_0^t \frac{\partial^2 f}{\partial y^2}(y,s)dL_s^y(X-\theta)dy \\
=&\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(X_u-\theta_u,u)1_{\{X_u>\theta_u\}}d[X,X]_u,
\end{aligned}
$$

since $[X-\theta, X-\theta] = [X,X]$. □

EXAMPLE (3.8): Let $g = g(x,t)$ be C^2 in x and C^1 in t, and let $\theta(t) = \theta_t(w)$ be a C^1 adapted curve. We wish to consider $f(x,t) = g(x,t)1_{\{x\geq\theta(t)\}}$. In this case it suffices, by taking $g(0,t)\equiv 0$, to consider $f((X_t-\theta_t)^+,t) = g((X_t-\theta_t)^+,t)$ where X is a continuous semimartingale. Since g is C^1, we have $g(x,t) = \int_0^t h(x,s)ds$ where $h(x,s) = \frac{\partial g}{\partial t}(x,s)$. Applying Corollary (3.6) we obtain

$$
\begin{aligned}
f((X_t-\theta_t)^+,t) =& g((X_t-\theta_t)^+,t) \\
=& g((X_0-\theta_0)^+,0) + \int_0^t \frac{\partial g}{\partial t}(X_s-\theta_s,s)ds \\
&+ \int_{0+}^t \frac{\partial g}{\partial x}(X_s-\theta_s,s)1_{(X_s>\theta_s)}d(X-\theta)_s \\
&+ \frac{1}{2}\int_{0+}^t \frac{\partial f}{\partial x}(0,s)\partial_s L_s^{\theta(s)}(X) \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_u-\theta_u,u)1_{(X_u>\theta_u)}d[X,X]_u.
\end{aligned}
$$

□

It is also possible to establish change of variables formulas for functions $f(x,t)$ which are the difference of two convexes in x for each t, without using the generalized process $\partial_u L_u^{\theta(u)}$. The proofs are simpler but the results are less general and less satisfying, since (for example) in Theorem (3.9) below there are terms that have both "t" and "s" in the integrands.

THEOREM (3.9). *Suppose $f : \mathbf{R} \times \mathbf{R}_t \to \mathbf{R}$ and $x \to f(x,t)$ is the difference of two convex functions for each t, and $t \to f(x,t)$ is measurable for each x. Let X be a continuous semimartingale; then*

$$f(X_t,t) = f(X_0,t) + \int_0^t \frac{\partial f}{\partial x}(X_s,t)dX_s + \frac{1}{2}\int_{\mathbf{R}} L_t^x \mu(dx,t), \tag{3.10}$$

where $\dfrac{\partial^2 f}{\partial x^2} = \mu(dx,t)$ in the generalized function sense.

Proof. Since $x \to f(x,t)$ is continuous, $(x,t) \to f(x,t)$ is jointly measurable. For each t there exists a signed measure $\mu(dx,t)$ such that $\dfrac{\partial^2 f}{\partial x^2} = \mu(dx,t)$ in the generalized function sense. Thus f has a representation

$$f(x,t) = \frac{1}{2}\int_{\mathbf{R}} |x-y|\mu(dy,t) + xh(t) + j(t).$$

Moreover

$$\frac{\partial f}{\partial x}(x,t) = \frac{1}{2}\int \operatorname{sign}(x-y)\mu(dy,t) + h(t).$$

By the Meyer-Tanaka formula,

$$|X_t - y| = |X_0 - y| + \int_0^t \operatorname{sign}(X_s - y)dX_s + L_t^y, \tag{3.11}$$

where L_t^y is local time for X at the level y. Thus

$$\begin{aligned} f(X_t,t) &= \frac{1}{2}\int |X_t - y|\mu(dy,t) + X_t h(t) + j(t) \\ &= \frac{1}{2}\int \{|X_0 - y| + \int_0^t \operatorname{sign}(X_s - y)dX_s + L_t^y\}\mu(dy,t) + X_t h(t) + j(t) \\ &= \{f(X_0,t) - X_0 h(t) - j(t)\} + \frac{1}{2}\int_{\mathbf{R}}\int_0^t \operatorname{sign}(X_s - y)dX_s\mu(dy,t) \\ &\quad + \frac{1}{2}\int_{\mathbf{R}} L_t^y \mu(dy,t) + X_t h(t) + j(t), \end{aligned}$$

and using Fubini's theorem for stochastic integrals on the second term on the right

$$
\begin{aligned}
&= f(X_0,t) + \int_0^t \{ \int_{\mathbf{R}} \frac{1}{2} \operatorname{sign}(X_s - y)\mu(dy,t)\} dX_s \\
&\quad + \frac{1}{2}\int_{\mathbf{R}} L_t^y \mu(dy,t) + (X_t - X_0)h(t) \\
&= f(X_0,t) + \int_0^t \frac{\partial f}{\partial x}(X_s,t)dX_s \\
&\quad + \frac{1}{2}\int_{\mathbf{R}} L_t^y \mu(dy,t) + (X_t - X_0)h(t).
\end{aligned}
$$

Note that *a priori* the result holds a.s. for each t. However since (3.11) holds a.s. for all t, and since $f(x,t)$ is deterministic, we have (3.10) holds a.s., all t. $\square$

THEOREM (3.12). *Let $f : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}$ be as in Theorem (3.9), but assume in addition that for each x and each $0 < s < t$, $f_x(x,t) - f_x(x,s) = \int_s^t f_{xt}(x,u)du$. Then*

$$
\begin{aligned}
f(X_t,t) =& f(X_0,t) + (X_t - X_0)h(t) + \int_0^t \frac{\partial f}{\partial x}(X_s,s)dX_s \\
&+ \int_0^t \left\{ \int_0^u \frac{\partial^2 f}{\partial t \partial x}(X_s,u)dX_s \right\} du + \frac{1}{2}\int_{\mathbf{R}} L_t^y \mu(dy,t).
\end{aligned}
$$

Proof. By Theorem (3.9),

$$
\begin{aligned}
f(X_t,t) =& f(X_0,t) \\
&+ (X_t - X_0)h(t) + \int_0^t \frac{\partial f}{\partial x}(X_s,s)dX_s \\
&+ \int_0^t \left\{ \frac{\partial f}{\partial x}(X_s,t) - \frac{\partial f}{\partial x}(X_s,s) \right\} dX_s \\
&+ \frac{1}{2}\int L_t^y \mu(dy,t).
\end{aligned}
$$

Consider the fourth term on the right above. Then

$$
\begin{aligned}
\int_0^t \frac{\partial f}{\partial x}(X_s,t) - \frac{\partial f}{\partial x}(X_s,s)dX_s &= \int_0^t \{ \int_s^t \frac{\partial^2 f}{\partial t \partial x}(X_s,u)du \} dX_s \\
&= \int_0^t \int_0^t 1_{(s \le u)} \frac{\partial^2 f}{\partial t \partial x}(X_s,u)du dX_s,
\end{aligned}
$$

and using the Fubini theorem for stochastic integration, this is:

$$
\int_0^t \{ \int_0^u \frac{\partial^2 f}{\partial t \partial x}(X_s,u)dX_s \} du,
$$

and the result follows. $\square$

COROLLARY (3.13). *With the hypotheses of Theorem (3.12) we also have*

$$
\begin{aligned}
f(X_t,t) =& f(X_0,0) + \int_0^t \frac{\partial f}{\partial t}(X_0,s)ds \\
&+ \int_0^t \frac{\partial f}{\partial x}(X_s,s)dX_s \\
&+ \int_0^t \{ \int_0^u \frac{\partial^2 f}{\partial t \partial x}(X_s,u)dX_s\}du \\
&+ \frac{1}{2}\int_{\mathbf{R}} L_t^y \mu(dy,t).
\end{aligned}
$$

4. Change of variables formulas in n dimensions.

The usual change of variables formulas for functions $f : \mathbf{R}^n \to \mathbf{R}$ assumes that f is C^2. It has been known for some time, however, that a wider class of functions map vectors of semimartingale into semimartingales. For example P. A. Meyer [7] showed in 1976 that if $f : \mathbf{R}^n \to \mathbf{R}$ is convex, then it preserves semimartingales. However to date there are not satisfactory change of variables formulas. Brosamler [1] and Meyer [8] establish a formula for Brownian motion, Krylov [4] has a formula for diffusions, and perhaps the most general formula for general semimartingales is in Protter [10, p.216].

Here we present a much more general formula. We are able to maximize generality by customizing the space of functions under consideration to the vector semimartingale we apply them to.

Let M be a d-dimensional local martingale, and A an m-dimensional adapted, càdlàg, finite variation process. We define the following measures, where Λ is a Borel set in $\mathbf{R}^{m+d}$:

$$
(4.1) \qquad
\begin{aligned}
&\text{(a)} \quad \mu^{ij}(t,w,\Lambda) = \int_0^t 1_\Lambda(A_{s-},M_{s-})d[M^i,M^j]_s^c \\
&\text{(b)} \quad \mu(t,\Lambda) = E\left\{\sum_{i=1}^d \mu^{ii}(t,\cdot,\Lambda)\right\} \\
&\text{(c)} \quad \nu^i(t,w,\Lambda) = \int_0^t 1_\Lambda(A_{s-},M_{s-})|dA_s^i| \\
&\text{(d)} \quad \nu(t,\Lambda) = E\left\{\sum_{i=1}^m \nu^i(t,\cdot,\Lambda)\right\}.
\end{aligned}
$$

(Here $|dA_s^i|$ represents the differential of the total variation process.) Note that by the Kunita-Watanabe inequality $\mu^{ij}(t,w,dz) << \mu^{ii}(t,w,dz)$ for all i,j. Also,

$$
(4.2) \qquad \int_0^t g(A_{s-},M_{s-})d[M^i,M^j]_s^c = \int_{\mathbf{R}^{m+d}} g(z)\mu^{ij}(t,\cdot,dz).
$$

Formula (4.2), which is trivial to establish, is an occupation-time formula for multidimensional semimartingales, and as such $\mu^{ij}(t,w,dz)$ represents an integral version of "local time". We let $\mu(dz)=\mu(\infty,dz)$ in (4.1c), and $\nu(dz)=\nu(\infty,dz)$ in (4.1d). Also, we write

$$Z_s=(A_s,M_s),$$

for the $m+d$ dimensional semimartingale.

Let K_α be a closed, convex set with the property $\{Z_s : s\geq 0\}\subseteq K_\alpha$ a.s. We define the <u>support of Z</u> by

$$\operatorname{supp}(Z)=\bigcap_\alpha K_\alpha. \tag{4.3}$$

Next we define a norm we will use to define our space. Let f, f_i, $1\leq i\leq m+d$, f_{ij}, $1\leq i,j\leq d$ be measurable functions from $\mathbf{R}^m\times\mathbf{R}^d$ to $\mathbf{R}$, and consider the $1+m+d+d^2$ dimensional vector function

$$F=(f,f_1,\ldots,f_{m+d},f_{11},\ldots,f_{dd}). \tag{4.4}$$

We define a norm $||\cdot||_Z$, which depends on the vector-valued semimartingale Z in question, by:

$$\begin{aligned}||F||_Z\equiv ||f||_{L^\infty(Z)}&+||\sum_{i=1}^m|f_i|||_{L^1(d\nu)}\\&+||\left\{\sum_{i=m+1}^{m+d}|f_i|^2\right\}^{1/2}||_{L^2(d\mu)}\\&+\frac{1}{2}||\sum_{i,j}|f_{ij}|||_{L^1(d\mu)},\end{aligned} \tag{4.5}$$

where $||f||_{L^\infty(Z)}\equiv\sup_{z\in\operatorname{supp}(Z)}|f(z)|$. It is convenient to establish some additional notation. For F as in (4.4) we define:

$$D_aF=\sum_{i=1}^m|f_i|;\quad D_xF=\left\{\sum_{i=m+1}^{m+d}|f_i|^2\right\}^{1/2}$$

$$D_{xx}F=\frac{1}{2}\sum_{i,j}|f_{ij}|.$$

Therefore,

$$\begin{aligned}||F||_Z=&||f||_{L^\infty(Z)}+||D_aF||_{L^1(d\nu)}\\&+||D_xF||_{L^2(d\mu)}+||D_{xx}F||_{L^1(d\mu)}.\end{aligned}$$

Our basic space is $C^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R})$, the space of functions from $\mathsf{R}^m \times \mathsf{R}^d$ into R, such that $\frac{\partial f}{\partial u_i}$, $1 \le i \le m$, $\frac{\partial f}{\partial x_i}$, $1 < i \le d$, and $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $l \le i,j \le d$ all exist and are continuous. (The Meyer-Itô change of variables formula applies to the case $C^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R})$.) When we are considering $C^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R})$, in this case we can take:

$$F = (f, \frac{\partial f}{\partial a_1}, \dots, \frac{\partial f}{\partial a_m}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}, \frac{\partial^2 f}{\partial x_1^2}, \dots, \frac{\partial^2 f}{\partial x_d^2})$$

and $D_a F = \sum_{i=1}^{m} |\frac{\partial f}{\partial a_i}|$, $D_x F = \{\sum_{i=1}^{d} |\frac{\partial f}{\partial x_i}|^2\}^{1/2}$, $D_{xx} F = \frac{1}{2} \sum_{i,j} |\frac{\partial^2 f}{\partial x_i \partial x_j}|$. In this case, we write $F \in C^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R})$. If $F(0) = \frac{\partial F}{\partial a_i}(0) = \frac{\partial F}{\partial x_i}(0) = 0$, we write $C_0^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R})$.

DEFINITION (4.6). *Let $\gamma > 0$ be fixed. We define $C_\gamma(Z)$ to be the closure under $\|\cdot\|_Z$ of $\{F \in C_0^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R})$ such that $\|F\|_Z < \infty$ and $\|D_{xx}F\|_{L^\infty(Z)} \le \gamma\}$. We define $C(Z) = \bigcup_{\gamma > 0} C_\gamma(Z)$.*

DEFINITION (4.7). *We say that $F \in C_{\text{loc}}(Z)$ if there exists a sequence of stopping times T^n increasing to ∞ a.s., a sequence of vectors $q_n \in \mathsf{R}^{m+d}$, and a sequence $p_n \in \mathsf{R}$ such that*

$$F(Z) - (q_n \cdot Z + p_n) \in C(Z^{T^n -}).$$

Recall that $Z_t^{T-} = Z_t 1_{\{t<T\}} + Z_{T-} 1_{\{t \ge T\}}$ for a stopping time T, and that Z^{T-} is a semimartingale when Z is one.

If $F \in C_{\text{loc}}(Z)$, then writing $F = (f, f_1, \dots, f_{m+d}, f_{11}, \dots, f_{dd})$, we write $f_i = \frac{\delta F}{\delta a_i}$, $1 \le i \le m$, $f_{m+i} = \frac{\delta F}{\delta x_i}$, $1 \le i \le d$, and $f_{ij} = \frac{\delta^2 F}{\delta x_i \delta x_j}$. We use the symbol "$\delta$" rather than "$\partial$", because not only do the real derivatives need not exist, but in general the functions in question will not be equal to the corresponding derivatives in the generalized function sense.

We remark that we follow the convention that if a function φ is defined only in a Borel subset K of R^{m+d}, we extend it to R^{m+d} by setting $\varphi = 0$ in K^c.

LEMMA (4.8). *For any semimartingale Z:*

(i) $C^{1,2}(\mathsf{R}^m \times \mathsf{R}^d, \mathsf{R}) \subset C_{\text{loc}}(Z)$

(ii) $C_{\text{loc}}(Z)$ *is an algebra.*

Proof. By pre-stopping (that is, replacing Z with Z^{T-} where T is a stopping time) we can assume without loss of generality that all of Z, $\int_0^\infty |dA_s^i|$, $(1 \le i \le m)$ and $[M^i, M^i]_\infty^c$,

$(1 \le i \le d)$ are all bounded a.s. To prove (i), take $F \in C_0^{1,2}(\mathbb{R}^m \times \mathbb{R}^d, \mathbb{R})$, w.l.o.g.. Since: $\mathrm{supp}(Z)$ is compact, μ and ν are finite measures concentrated on $\mathrm{supp}(Z)$, and by the continuity properties of F we have:

$$||F||_Z < \infty, \quad \text{and} \quad ||D_{xx}F||_{L^\infty(Z)} < \infty.$$

This implies $F \in C(Z)$. Removing the pre-stopping localization gives $F \in C_{\mathrm{loc}}(Z)$.

To prove (ii) it suffices to show (once again after pre-stopping as at the beginning of this proof) that if $f \in C_\gamma(Z)$ and $g \in C_{\gamma'}(Z)$, then there exists a $\gamma'' < \infty$ such that $fg \in C_{\gamma''}(Z)$. Further, w.l.o.g. we take $f, g \in C_0^{1,2}(\mathbb{R}^m \times \mathbb{R}^d, \mathbb{R})$. An easy computation gives

$$||fg||_Z \le ||f||_Z||g||_Z, \tag{4.9}$$

and moreover

$$\begin{aligned} ||D_{xx}fg||_{L^\infty(Z)} \le &||D_{xx}f||_{L^\infty(Z)}||g||_{L^\infty(Z)} \\ &+ ||D_{xx}g||_{L^\infty(Z)}||f||_{L^\infty(Z)} \\ &+ ||D_xf||_{L^\infty(Z)}||D_xg||_{L^\infty(Z)}. \end{aligned}$$

Using the fundamental theorem of calculus:

$$\begin{aligned} \frac{\partial f}{\partial x_i}(a,x) &= \frac{\partial f}{\partial x_i}(0,0) + \int_0^1 \sum_{j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(a,tx)x_j dt \\ &= \int_0^1 \sum_{j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(a,tx)x_j dt. \end{aligned}$$

Since the support of Z is assumed compact, we have $||D_xf||_{L^\infty(Z)}$ is bounded, with a bound that depends on γ and $\mathrm{supp}(Z)$ only, and not on f. This holds for g analogously, and the result follows. □

In the following theorem we again use the notation $\frac{\delta F}{\delta a_i}$ instead of $\frac{\partial F}{\partial a_i}$ to underscore that these functions need not be the derivatives of F in the usual nor in the generalized function sense (though if $F \in C^{1,2}(\mathbb{R}^m \times \mathbb{R}^d, \mathbb{R})$ then they agree up to equivalence: $\frac{\partial F}{\partial a_i} = \frac{\delta F}{\delta a_i}$; also in certain cases – e.g., $d[M^i, M^i]_t^c << dt$ and $|dA_t^i| << dt$ – they agree also with the generalized function derivatives. In particular in the case where Z is a diffusion (the solution of an SDE), then the derivatives agree with the usual generalized function derivatives.

THEOREM (4.10). *Let M be a d-dimensional local martingale and let A be an m dimensional adapted, càdlàg process with paths of finite variation on compacts. Let $Z = (A, M)$ and let $F \in C_{\mathrm{loc}}(Z)$. Then*

$$
\begin{aligned}
F(Z_t) =& F(Z_0) + \sum_{i=1}^{m} \int_0^t \frac{\delta F}{\delta a_i}(Z_{s-}) dA_s^i \\
&+ \sum_{i=1}^{d} \int_0^t \frac{\delta F}{\delta x_i}(Z_{s-}) dM_s^i \\
&+ \frac{1}{2} \sum_{i,j} \int_0^t \frac{\delta^2 F}{\delta x_i \delta x_j}(Z_{s-}) d[M^i, M^j]_s^c \\
&+ \sum_{0<s\le t} \left\{ F(Z_s) - F(Z_{s-}) - \sum_{i=1}^{m+d} \frac{\delta F}{\delta z_i}(Z_{s-}) \Delta Z_s^i \right\}.
\end{aligned}
\tag{4.11}
$$

Proof. (In the last term on the right side of (4.11), $\frac{\delta F}{\delta z_i} = \frac{\delta F}{\delta a_i}$ for $1 \le i \le m$, and $\frac{\delta F}{\delta z_i} = \frac{\delta F}{\delta x_i}$ for $m+1 \le i \le m+d$.) By pre-stopping we can assume w.l.o.g. that Z, $\int_0^\infty |dA_s^i|$ $(1 \le i \le m)$, $[M^i, M^i]_\infty^c$ $(1 \le i \le d)$ are all bounded a.s. Further we can and do assume $F \in C_\gamma(Z)$, for some $\gamma > 0$.

With the above assumptions it is simple to check that there exists a sequence $F^n \in C^{1,2} \cap C_\gamma(Z)$ converging to F in $||\cdot||_Z$. Since $F^n \in C^{1,2}$ we have by the standard Meyer-Itô change of variables formula that (4.11) holds for F^n:

$$
\begin{aligned}
F^n(Z_t) =& F^n(Z_0) + \int_0^t \nabla F^n(Z_{s-}) dZ_s \\
&+ \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 F^n}{\partial x_i \partial x_j}(Z_{s-}) d[M^i, M^j]_s^c \\
&+ \sum_{0<s\le t} \left\{ F^n(Z_s) - F^n(Z_{s-}) - \sum_{i=1}^{m+d} \frac{\partial F^n}{\partial z_i}(Z_{s-}) \Delta Z_s^i \right\}.
\end{aligned}
\tag{4.12}
$$

Call the last term on the right side of (4.12) $JF^n(t)$. Then letting n tend to ∞ and using standard techniques we obtain:

$$
\begin{aligned}
F(Z_t) =& F(Z_0) + \int_0^t \nabla_\delta F(Z_{s-}) dZ_s \\
&+ \frac{1}{2} \sum_{i,j} \int_0^t \frac{\delta^2 F}{\delta x_i \delta x_j}(Z_{s-}) d[M^i, M^j]_s^c \\
&+ \lim_{n\to\infty} JF^n(t).
\end{aligned}
\tag{4.13}
$$

It remains to analyze $\lim_{n\to\infty} JF^n(t)$. We write

$$JF^n(t) = \sum_{0<s\leq t} \{F^n(Z_s) - F^n(Z_{s-}) - \nabla_a F^n(Z_{s-})(\Delta A_s + \Delta M_s)\} \tag{4.14}$$

Observe that

$$\begin{aligned}
&E\left\{ \left| \sum_{0<s\leq t} \nabla_{\delta,a} F(Z_{s-})\Delta A_s - \sum_{0<s\leq t} \nabla_a F^n(Z_{s-})\Delta A_s \right| \right\} \\
&\leq E\left\{ \sum_{0<s\leq t} |\nabla_{\delta,a} F(Z_{s-}) - \nabla_a F^n(Z_{s-})||\Delta A_s| \right\} \\
&\leq E\left\{ \int_0^\infty |\nabla_{\delta,a} F(Z_{s-}) - \nabla_a F^n(Z_{s-})||dA_s| \right\} \\
&\leq ||F - F^n||_Z,
\end{aligned}$$

hence

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{0<s\leq t} \{F^n(Z_s) - F^n(Z_{s-}) - \nabla_a F^n(Z_{s-})\Delta A_s\} \\
&= \sum_{0<s\leq t} \{F(Z_s) - F(Z_{s-}) - \nabla_{\delta,a} F(Z_{s-})\Delta A_s\}.
\end{aligned} \tag{4.15}$$

Next consider the ΔM_s terms on the right side of (4.14). We write

$$V_t = \lim_{n\to\infty} \sum_{0<s\leq t} \{F^n(Z_s) - F^n(Z_{s-}) - \nabla_x F^n(Z_{s-})\Delta M_s\},$$

a limit which we now know exists. Moreover by (4.13) and (4.15) we know the jumps of V:

$$\Delta V_t = F(Z_t) - F(Z_{t-}) - \nabla_{\delta,x} F(Z_{t-})\Delta M_t.$$

Therefore it suffices to show that V is a pure jump process. V is the difference of semimartingales, so it too is one. By an examination of (4.13) and (4.15), it is clear that V can jump only if Z jumps. Thus it suffices to show

$$\int_0^t 1_{\{\Delta Z_s \neq 0\}} dV_s = V_t.$$

Since $F^n \in C_\gamma(Z)$, we have:

$$\begin{aligned}
&|F^n(Z_s) - F^n(Z_{s-}) - \nabla_z F^n(Z_{s-})\Delta Z_s| \\
&\leq \gamma k \sum_i (\Delta Z_s^i)^2
\end{aligned}$$

where k is a constant depending on the support of Z only. Since the right side is independent of n, $\sum_{0<s\le t}(\Delta Z_s^i)^2 < \infty$ a.s., we conclude $\lim_{n\to\infty} JF^n(t)$ is a finite variation process. Also, the limit in (4.15) is clearly a finite variation process. Therefore V is a finite variation process as well. Moreover, since

$$\int_0^t 1_{\{\Delta Z_s\neq 0\}}dJF_s^n = JF^n(t),$$

taking limits gives $\int_0^t 1_{\{\Delta Z_s\neq 0\}}dJF_s = JF(t)$. Clearly

$$\begin{aligned}\int_0^t 1_{\{\Delta Z_s\neq 0\}}d&\left\{\sum_{0<r\le s}\{F(Z_r)-F(Z_{r-})-\nabla_{\delta,a}F(Z_{r-})\Delta A_r\}\right\}\\ &=\sum_{0\le s\le t}\{F(Z_s)-F(Z_{s-})-\nabla_{\delta,a}F(Z_{s-})\Delta A_s\},\end{aligned}$$

from which we deduce $\int_0^t 1_{\{\Delta Z_s\neq 0\}}dV_s = V_t$, whence V is pure jump. Finally, we note that the integrals in formula (4.11) of Theorem (4.10) do not depend on the versions of $\frac{\delta F}{\delta a_i}$, $\frac{\delta F}{\delta x_i}$, and $\frac{\delta^2 F}{\delta x_i \delta x_j}$ that are used. □

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