SOME RESULTS ON BAYES AND EMPIRICAL BAYES RULES FOR RANKING PAIRWISE COMPARED TREATMENTS *

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Abstract

Scores of 'k' players have been observed. The problem of selecting the player with the highest expected wins, more generally the problem of partitioning k players according to the expected wins is considered. We assume that the players are playing pairwise. Bayes rules are obtained for general ranking problem under general class of loss functions. Sequence of parametric empirical Bayes rules is proposed and proved to be optimal of the order $O(e^{-cn})$ for some positive constant c.

Key Words: Selection and ranking, Pairwise comparison, Bayes selection rules, Empirical Bayes rules.
AMS 1985 subject classification: 62C20.

1 Introduction

Selection and ranking problems arise in many practical situations where the tests of homogeneity do not provide the answer the experimenter wants. In this paper we study the problem of ranking treatments, when only pairwise comparisons of the treatments are available.

In the method of paired comparison several "treatments" under investigation are presented in all possible pairwise combinations to a judge who states which member of each pair he prefers. We do not allow expression of no preference. This experiment may be repeated several times independently. Score for the "treatment" i is defined as a number of times the "treatment" i is preferred over other treatments. We define the treatment with the highest expected score as the best "treatment." Here the term "treatment" may stand for objects, machines, tennis players and the like. The method of paired comparison is widely used when no meaningful absolute measurement can be made on the "treatments."

The method of paired comparison has great practical simplicity. It has been used extensively in experimental situations where subjective judgment of individuals lead to quantitative responses, and situations where measurements are difficult or costly to obtain. Trawinski and David (1963) proposed a procedure for selecting the best treatment. Contributions have been made by David (1963), Bradley (1984), Bradley (1976), among others. A Brief review of subset selection and indifference zone approaches to the selection of the

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best treatment in a paired comparison experiment is in David (1989). We will assume that the Bradley-Terry preference model holds.

In this paper we consider a general problem of partitioning k treatments into r non-empty, mutually exclusive subsets, $S_1, S_2, \ldots, S_r$, where S_1 is of size t_1, S_2 is of size t_2 and so on. The goal is to partition the k treatments into these r subsets, so that S_1 contains the t_1 worst treatments, S_2 contains next t_2 worst treatments, ... and S_r contains the t_r best treatments. When $r = 2$ and $\#S_2 = 1$, this is a problem of selecting the best treatment.

In many situations, an experimenter may have some prior information about the parameter of interest, and he would like to use that information for making the decision. In section 2, the problem is described and the Bayes rule is derived, when the unknown parameter p has a conjugate prior. In section 3, it is assumed that the prior is partially known. In this case we consider the situation where one is repeatedly dealing with the same selection problem independently. In such instances, at each stage, one would like to use the past information and use it to derive a rule which is close to a Bayes rule. This approach is known as the empirical Bayes and is due to Robbins (1956). Empirical Bayes rules have been derived for subset selection goals by Deely (1965), for selecting good populations by Gupta and Hsiao (1983). Gupta and Liang (1986, 1988) have considered the problem of selecting good and best binomial populations. Recently, Gupta nad Liang (1989) delt with the problem of selecting the best multinomial cell and Gupta and Hande (1991) considered the problem of partitioning k multinomial cells and the problem of selecting the cell associated with the largest probability and estimating the probability associated with the selected cell.

2 Bayes Rules

Suppose that there are k treatments, $T_1, T_2, \ldots, T_k$. These treatments are to be compared in pairs N times, independently. Let for $i \neq j$ and for $1 \leq \gamma \leq N$,

$$X_{ij\gamma} = \begin{cases} 1 & \text{if } i\text{th treatment is preferred in } \gamma\text{th} \\ & \quad \text{comparison of the } i\text{th and } j\text{th treatments,} \\ 0 & \text{otherwise.} \end{cases}$$

We notice that $X_{ij\gamma} + X_{ji\gamma} = 1 \ \forall \ i \neq j$ and $\forall \ \gamma$. We assume that $X_{ij\gamma}$ for all $i < j$ and for all γ are independent and $P(X_{ij\gamma} = 1) = \pi_{ij}$. The expected "score" of the ith treatment is given by

$$N \ \theta_i = \sum_{j \neq i, \gamma} EX_{ij\gamma} = N \sum_{j \neq i} \pi_{ij}.$$

Our goal is to select the treatment with the highest expected score. We assume the Bradley-Terry model, that is, we assume that there exists a probability vector $\underset{\sim}{p} = (p_1, p_2, \ldots, p_k)$ such that for each i $p_i \geq 0$, $\sum_{i=1}^{k} p_i = 1$ and $\pi_{ij} = p_i/(p_i + p_j)$. Thus the treatment associated with the largest p_i has the highest expected score. The distribution of

$$Y_{ij} = \sum_{\gamma=1}^{N} X_{ij\gamma} \quad \text{for } i < j \ ,$$

is given by

$$P(y_{11}, y_{12}, \ldots, y_{k-1k}) = \prod_{i<j} \binom{N}{y_{ij}} \pi_{ij}^{y_{ij}} (1-\pi_{ij})^{y_{ji}}.$$

Notice that

$$\begin{aligned}
\prod_{i<j} \binom{N}{y_{ij}} \pi_{ij}^{y_{ij}} (1-\pi_{ij})^{y_{ji}} &= \prod_{i<j} \binom{N}{y_{ij}} \prod_{i<j} p_i^{x_{ij}} p_j^{x_{ji}} (p_i+p_j)^{-N} \\
&= \prod_{i<j} \binom{N}{y_{ij}} \prod_{i<j} (p_i+p_j)^{-N} \prod_{i<j} p_i^{y_{ij}} \prod_{i<j} p_j^{y_{ji}} \\
&= \prod_{i<j} \binom{N}{y_{ij}} \prod_{i<j} (p_i+p_j)^{-N} \prod_{i<j} p_i^{y_{ij}} \prod_{j<i} p_i^{y_{ij}} \\
&= \prod_{i<j} \binom{N}{y_{ij}} \prod_{i<j} (p_i+p_j)^{-N} \prod_{i=1}^{k} \prod_{j\neq i} p_i^{y_{ij}} \\
&= \prod_{i<j} \binom{N}{y_{ij}} \prod_{i<j} (p_i+p_j)^{-N} \prod_{i=1}^{k} p_i^{\sum_{j\neq i} y_{ij}}.
\end{aligned}$$

Hence if for each i, $x_i = \sum_{j\neq i} y_{ij}$, then $x_1, x_2, \ldots, x_k$ forms a sufficient statistic.

Let $p_{[1]} \leq p_{[2]} \cdots \leq p_{[k]}$ be the ordered values of the parameters $p_1, p_2, \ldots, p_k$. We assume there is no prior knowledge about the exact pairing between the ordered and the unordered parameters. Our goal is to partition k treatments into r mutually exclusive subsets, $S_1, S_2 \ldots, S_r$ such that S_1 the set of t_1 treatments associated with the probabilities $p_{[1]}, p_{[2]}, \ldots, p_{[t_1]}$, S_2 the set of treatments associated with the probabilities $p_{[t_1+1]}, p_{[t_1+2]}, \ldots, p_{[t_1+t_2]}$ and S_r the set of treatments associated with the probabilities $p_{[k-t_r+1]}, p_{[k-t_r+2]}, \ldots, p_{[k]}$. Here $t_1, t_2, \ldots, t_r$ are fixed integers in advance, such that $\sum_{i=1}^{r} t_i = k$.

Let

$$\mathcal{A} = \left\{ \ (S_1, S_2, \ldots, S_r): \quad S_i \cap S_j = \emptyset, |S_i| = t_i \ \ \forall i \neq j; \text{ and } S_i \subset \{1, 2, \ldots, k\} \ \right\}$$

be the action space and let

$$\Omega = \left\{ \ (p_1, p_2, \ldots, p_k) : \textstyle\sum_{i=1}^{k} p_i = 1; p_i \geq 0 \ \ \forall \ \ i \ \right\}$$

be the parameter space.

We will assume that the prior distribution of the parameter $\underset{\sim}{p}$ follows a Dirichlet distribution, G, with the hyperparameters $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive constants. Let $\alpha_0 = \sum_{i=1}^{k} \alpha_i$.

The density of $\underset{\sim}{p}$, $g(.)$ is given by,

$$g(\underset{\sim}{p}) = \begin{cases} \frac{\Gamma(\alpha_0)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} p_i^{\alpha_i - 1} & \text{if } p_i \geq 0 \ \text{ for } i = 1, 2, \ldots, k \ \text{ and } \sum_{i=1}^{k} p_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $L : \mathcal{A} \times \Omega \longrightarrow \mathbf{R}$ be the loss function. We will assume that the loss function is non negative, permutation invariant and "increasing". Mathematically we write these

conditions as
(1) $L(p,a) \geq 0 \;\; \forall \;\; \underset{\sim}{p} \in \Omega$ and $a \in \mathcal{A}$
(2) $L(\pi \underset{\sim}{p}, \pi a) = L(\underset{\sim}{p}, a) \;\; \forall \;\; \underset{\sim}{p} \in \Omega; \quad a \in \mathcal{A}$ and for each permutation π.
Here $\pi(\tilde{a}) = (\pi S_1, \pi S_2, \ldots, \pi S_k)$ and for any $S \subset \{1, 2, \ldots, k\}$, $S = \{i_1, i_2, \ldots, i_q\}$ then $\pi S = \{\pi(i_1), \pi(i_2), \ldots, \pi(i_q)\}$
(3) Let $\underset{\sim}{p} = (p_1, p_2, \ldots, p_k)$ and $p_i \leq p_j$; $a \in \mathcal{A}$, $a = (S_1, S_2, \ldots, S_r)$ and $i \in S_{r_1}, j \in S_{r_2}$, $r_1 \leq r_2$.
Let Π_{ij} be the permutation which inter changes only ith and jth co-ordinates then, $L(\Pi_{ij}\underset{\sim}{p}, a) \leq L(\underset{\sim}{p}, a)$.
(4) There exists a positive constant δ, such that for every $a \in \mathcal{A}$
$\int L^{1+\delta}(\underset{\sim}{p}, a) g(\underset{\sim}{p}) dp \; < \infty$.

The condition (1) says that the loss function is non-negative, (2) says it is permutation invariant, (3) says it is "monotone".

Now we prove that any decision rule that "ranks" the treatment T_i according to the rank of $x_i + \alpha_i$ in $x_1 + \alpha_1, x_2 + \alpha_2, \ldots, x_k + \alpha_k$ is a Bayes rule.

Theorem 2.1 *:*
If the parameter $\underset{\sim}{p}$ has the prior distribution $g = g(. \,|\underset{\sim}{\alpha})$ with $\underset{\sim}{\alpha}$ as a hyperparameter and if $\underset{\sim}{X} = (X_1, X_2, \ldots, X_k)$ represent the score of the $i-th$ treatment under the Baradley-Terry model, then, for the loss function as described earlier, any decision rule $\delta = \delta(x_1, x_2, \ldots, x_k) = (S_1(\underset{\sim}{x}), S_2(\underset{\sim}{x}), \ldots, S_r(\underset{\sim}{x}))$ such that for every $r_2 > r_1$, $i \in S_{r_1}$ $j \in S_{r_2}$ then $x_i + \alpha_i \leq x_j + \alpha_j$ is a Bayes rule.

Proof: The posterior density of $\underset{\sim}{p}$, $g(\underset{\sim}{p}|\underset{\sim}{x})$ is given by

$$g(\underset{\sim}{p}|\underset{\sim}{x}) = h_1(\underset{\sim}{x}) h_2(\underset{\sim}{p}) \prod_{t=1}^{k} p_t^{x_t + \alpha_t - 1}$$

where $h_1(\underset{\sim}{x})$ is some function of $\underset{\sim}{x}$ and is free of $\underset{\sim}{p}$, and $h_2(\underset{\sim}{p})$ is a function of $\underset{\sim}{p}$ and is permutation invariant.

Let B be the class of non randomized decision rules which ranks the cell Π_i according to the rank $x_i + \alpha_i$ in $x_1 + \alpha_1, x_2 + \alpha_2, \ldots, x_k + \alpha_k$.

First we will prove that if a non-randomized rule δ is not in B, then there is a rule in B which has smaller Bays risk than δ. Then we will prove that all decision rules in the class B have the same Bays risk, and this would prove the theorem.

Let $\delta = \delta(\underset{\sim}{x}) = (S_1(\underset{\sim}{x}), S_2(\underset{\sim}{x}), \ldots, S_r(\underset{\sim}{x}))$ be any decision rule. Let,

$$R(\underset{\sim}{p}, \delta) = E_{\underset{\sim}{x}|\underset{\sim}{p}} L(\underset{\sim}{p}, \delta(\underset{\sim}{x}))$$

be the risk function and

$$r(\delta) = E_{\underset{\sim}{p}} R(\underset{\sim}{p}, \delta)$$

be the Bayes risk of the decision rule δ.

The Bayes rule is a rule which minimizes the posterior Bayes risk. To find out the Bayes rules, it is enough to consider the non-randomized rules. Let $r(\delta|\underset{\sim}{x})$ denote the posterior Bayes risk of the decision rule δ, thus

$$r(\delta) = \int R(\underset{\sim}{p}, \delta) g(\underset{\sim}{p}) d\underset{\sim}{p}$$

and

$$r(\delta|\underset{\sim}{x}) = \int L(\underset{\sim}{p}, \delta(\underset{\sim}{x})) g(\underset{\sim}{p}|\underset{\sim}{x}) d\underset{\sim}{p}$$

where $g(\underset{\sim}{p}|\underset{\sim}{x})$ is the density of $\underset{\sim}{p}$ for given $\underset{\sim}{X} = \underset{\sim}{x}$.

We assume that the decision rule δ is not in the class B. Thus, we assume that there exist $\underset{\sim}{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in \mathcal{X}$ and $r_1, r_2,\quad 1 \le r_1 < r_2 \le r$ such that $i \in S_{r_1}(\underset{\sim}{x}^0);\quad j \in S_{r_2}(\underset{\sim}{x}^0)$ and $\quad x_i^0 + \alpha_i > x_j^0 + \alpha_j$.

Now we will construct a new decision rule δ' whose Bayes risk will be smaller than that of δ.

Let $\delta'(\underset{\sim}{x}) = (S_1'(\underset{\sim}{x}), S_2'(\underset{\sim}{x}), \ldots, S_r'(\underset{\sim}{x}))$

$$\begin{aligned}
S_i'(\underset{\sim}{x}^0) &= S_i(\underset{\sim}{x}^0) \quad \forall i \ne r_1 \quad \text{and} \quad i \ne r_2 \\
S_{r_1}(\underset{\sim}{x}^0) &= (S_{r_1}(\underset{\sim}{x}^0) - \{i\}) \bigcup \{j\} \\
S_{r_2}(\underset{\sim}{x}^0) &= (S_{r_2}(\underset{\sim}{x}^0) - \{j\}) \bigcup \{i\};
\end{aligned}$$

and let

$$\delta'(\underset{\sim}{x}) = \delta(\underset{\sim}{x}) \ \forall \ \underset{\sim}{x} \ne \underset{\sim}{x}^0.$$

To prove that the new decision rule δ' has smaller Bayes risk than the decision rule δ, it is enough to prove that, $r(\delta|\underset{\sim}{x}) - r(\delta'|\underset{\sim}{x}) \ge 0$ for all $\underset{\sim}{x}$.
For $\underset{\sim}{x} \ne \underset{\sim}{x}^0, \delta(\underset{\sim}{x}) = \delta'(\underset{\sim}{x})$ thus, $r(\delta|\underset{\sim}{x}) = r(\delta'|\underset{\sim}{x})$. Let $g(\underset{\sim}{p}|\underset{\sim}{x})$ be the posterior density of $\underset{\sim}{p}$.

$$\begin{aligned}
g(\underset{\sim}{p}|\underset{\sim}{x}) &= h_1(\underset{\sim}{x}) h_2(\underset{\sim}{p}) \prod_{t=1}^{k} p_t^{x_t+\alpha_t-1} \\
&= h_1(\underset{\sim}{x}) h_2(\underset{\sim}{p}) \prod_{t=1}^{k} p_t^{x_t+\alpha_t-1}.
\end{aligned}$$

Then the posterior Bayes risk of the rule δ is given by,

$$\begin{aligned}
r(\delta|\underset{\sim}{x}) &= \int L(\underset{\sim}{p}, \delta(\underset{\sim}{x}))\, g(\underset{\sim}{p}|\underset{\sim}{x})\, d\underset{\sim}{p} \\
&= h_1(\underset{\sim}{x}_0) \int h_2(\underset{\sim}{p}) L(\underset{\sim}{p}, \delta(\underset{\sim}{x})) \prod_{t=1}^{k} p_t^{x_t+\alpha_t-1} d\underset{\sim}{p}.
\end{aligned}$$

Hence

$$\begin{aligned} r(\delta|\underset{\sim}{x}^0) - r(\delta'|\underset{\sim}{x}^0) &= h_1(\underset{\sim}{x}^0) \int h_2(\underset{\sim}{p})[L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0)) - L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0))] \prod_{t=1}^{k} p_t^{x_t^0+\alpha_t-1} d\underset{\sim}{p} \\ &= h_1(\underset{\sim}{x}^0) \int_{p_i > p_j} h_2(\underset{\sim}{p})[L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0)) - L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0)] \prod_{t=1}^{k} p_t^{x_t^0+\alpha_t-1} d\underset{\sim}{p} \\ &\quad + h_1(\underset{\sim}{x}^0) \int_{p_j > p_i} h_2(\underset{\sim}{p})[L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0) - L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0))] \prod_{t=1}^{k} p_t^{x_t^0+\alpha_t-1} d\underset{\sim}{p}. \end{aligned}$$

Since L is invariant (condition 2) , interchanging the variables p_i and p_j inside the integral, we get,

$$\begin{aligned} & h_1(\underset{\sim}{x}^0) \int_{p_j \geq p_i} h_2(\underset{\sim}{p})[L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0)) \ - \ L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0)] \prod_{t=1}^{k} p_t^{x_t^0+\alpha_t-1} \ d\underset{\sim}{p} \\ = & \ -h_1(\underset{\sim}{x}^0) \int_{p_i \geq p_j} h_2(\underset{\sim}{p})[L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0)) - L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0))] \\ & \ [\prod_{t \neq i \neq j}^{k} p_t^{x_t^0+\alpha_t-1}] p_j^{x_i^0+\alpha_i-1} p_i^{x_j^0+\alpha_j-1} d\underset{\sim}{p}. \end{aligned}$$

This implies

$$\begin{aligned} & r(\delta|\underset{\sim}{x}^0) - r(\delta'|\underset{\sim}{x}^0) \\ = & \ h_1(\underset{\sim}{x}^0) \int_{p_i \geq p_j} h_2(\underset{\sim}{p})[L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0) - L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0))][p_i^{x_i^0+\alpha_i-1} p_j^{x_j^0+\alpha_j-1} - p_i^{x_j^0+\alpha_j-1} p_j^{x_i^0+\alpha_i-1}] \\ & \prod_{t \neq i \neq j} p_t^{x_t^0+\alpha_t-1} d\underset{\sim}{p} \ . \end{aligned}$$

But from condition (3) on the loss function and from $p_i \geq p_j$, we have

$$L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0)) \geq L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0))$$

that is

$$L(\underset{\sim}{p}, \delta(\underset{\sim}{x}^0)) - L(\underset{\sim}{p}, \delta'(\underset{\sim}{x}^0)) \geq 0 \ .$$

Also for $x_i^0 + \alpha_i \geq x_j^0 + \alpha_j$, and $p_i \geq p_j$ we have

$$p_i^{x_i^0+\alpha_i-1} p_j^{x_j^0+\alpha_j-1} - p_i^{x_j^0+\alpha_j-1} p_j^{x_i^0+\alpha_i-1} \geq 0.$$

Hence from the above equations we get,

$$r(\delta|\underset{\sim}{x}^0) - r(\delta'|\underset{\sim}{x}^0) \geq 0$$

and since $\delta(\underset{\sim}{x}) = \delta'(\underset{\sim}{x}) \ \forall \ \underset{\sim}{x} \neq \underset{\sim}{x}^0$, we have

$$r(\delta|\underset{\sim}{x}) = r(\delta'|\underset{\sim}{x}) \ \ \forall \underset{\sim}{x} \neq \underset{\sim}{x}^0.$$

Hence $r(\delta) \geq r(\delta')$. Since there are finite number of elements in $\mathcal{X}$, by the same method we can obtain a decision rule in B which has smaller Bayes risk than δ. It is straightforward to see that all rules in B have the same Bayes risk. And this completes the proof. □

Remark 2.1 :
It should be noted that the Bayes rule is not necessarily unique.

3 Empirical Bayes rules

Here we consider a situation in which one repeatedly deals with the same ranking problem independently, and we assume that the prior is partially known.

We assume that the form of the prior is known and is Dirichelet prior, but the hyperparameters $\alpha_1, \alpha_2, \ldots, \alpha_k$ are unknown. We consider following two cases, Case 1: $\alpha_0 = \sum_{i=1}^k \alpha_i$ known, Case 2: $\alpha_0 = \sum_{i=1}^k \alpha_i$ unknown.

In this section we will derive a sequence of empirical Bayes rules for selecting the best treatment, or for partitioning the k treatment in r subsets. For each $m = 1, 2, 3, \ldots\ldots$, $(\underset{\sim}{Y}^m, p_m)$ are independent random vectors where

$$\underset{\sim}{Y}^m = (Y_{12m}, Y_{13m}, \ldots Y_{k-1km})$$

denote the observable random vector, Y_{ijm} denote the score the of ith treatment over jth treatment at mth stage.

Let $Y^1, Y^2, Y^3, \ldots, Y^n$ be the past available observations, and $Y^{n+1} = Y = (Y_{12}, Y_{13}, \ldots, Y_{k-1k})$ be the present observation. Here we assume that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are unknown but $\alpha_0 = \sum_{i=1}^k \alpha_i$ is known. To derive a sequence of a empirical Bayes rules we need to get estimates of the hyperparameters $\alpha_1, \alpha_2, \ldots, \alpha_k$. Note that

$$\begin{aligned}
N^{-1}EY_{ij} &= E\pi_{ij} \\
&= Ep_i(p_i + p_j)^{-1} \\
&= \frac{\alpha_i}{\alpha_i + \alpha_j} \\
&= \frac{1}{1 + \frac{\alpha_j}{\alpha_i}}.
\end{aligned}$$

Hence

$$z_{ijn} = (nN)^{-1} \sum_{m=1}^{n} y_{ij}^m$$

is a moment estimator of $(1 + \frac{\alpha_j}{\alpha_i})^{-1}$.

$$\begin{aligned}
EZ_{ijn} &= \frac{1}{1 + \frac{\alpha_j}{\alpha_i}} \\
\frac{\alpha_j}{\alpha_i} &= \frac{1 - EZ_{ijn}}{EZ_{ijn}}
\end{aligned}$$

$$\begin{aligned} \alpha_j &= \alpha_i \frac{1 - EZ_{ijn}}{EZ_{ijn}} \\ \text{thus} \quad \alpha_0 - \alpha_i, &= \alpha_i \sum_{j \neq i} \frac{1 - EZ_{ijn}}{EZ_{ijn}} \\ \text{hence} \quad \alpha_i &= \alpha_0 (1 + \sum_{j \neq i} \frac{1 - EZ_{ijn}}{EZ_{ijn}})^{-1}. \end{aligned}$$

Now we are ready to define the moment estimators of the hyperparameters $\alpha_1, \alpha_2, \ldots \alpha_k$. For $1 \leq i \leq k$ define

$$\hat{\alpha}_i = \begin{cases} \alpha_0 (1 + \sum_{j \neq i} \frac{1 - Z_{ijn}}{Z_{ijn}})^{-1} & \text{if } Z_{ijn} > 0 \; \forall \; i \neq j \\ = \frac{\alpha_0}{k} & \text{otherwise.} \end{cases}$$

Now we will propose the sequence of empirical Bayes rules $\{\delta_n\}_1^\infty$, where δ_n is a rule which ranks the ith treatment according to the rank of $x_i + \hat{\alpha}_i$ in $x_1 + \hat{\alpha}_1, x_2 + \hat{\alpha}_2, \ldots, x_k + \hat{\alpha}_k$. In the case of ties use randomization.

The optimality of a sequence of a empirical Bayes rules can be judged by considering how small its risk is as compared to the minimum Bayes risk at n th stage. And how fast it goes to 0. We need the following definition,

Definition 3.1 :- *A sequence of the empirical Bay's rules $\{\delta_n\}$ is said to be asymptotically optimal at least of order β_n to the prior distribution G if ,*

$$r(G, \delta_n) - r(G) \leq O(\beta_n) \quad as \quad n \to \infty,$$

where $r(G)$ is a Bayes risk of a Bayes rule and $r(G, \delta)$ is a Bayes risk of rule δ.

To prove that the sequence of the empirical Bayes rules is asymptotically optimal of order $O(e^{-cn})$ for some positive constant c, as we need to prove that each i, $P(\hat{\alpha}_i - \alpha_i > \epsilon)$ and $P(\hat{\alpha}_i - \alpha_i < -\epsilon)$ are of order $O(e^{-cn})$. In order to prove these results we need the following lemma due to Hoeffding (1963). For sake of completeness we state it.

Lemma 3.1 *(Hoeffding):*
If $Y_1, Y_2, \ldots, Y_n$ are the independent random variables such that for each i there exists real numbers a_i and b_i such that $P(a_i \leq Y_i \leq b_i) = 1$ then for any $t > 0$

$$P(\bar{Y} - \mu \geq t) \leq e^{-2n^2 t^2 (\sum_{i=1}^n (b_i - a_i)^2)^{-1}},$$

where $\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$ and $E\bar{Y} = \mu$.

Now we prove that for each i, $P(\hat{\alpha}_i - \alpha_i > \epsilon)$ and $P(\hat{\alpha}_i - \alpha_i < -\epsilon)$ are of order $O(e^{-cn})$.

Lemma 3.2 :
For $0 \leq i \leq k$ there exist a positive constant c such that

$$\begin{aligned} P(\hat{\alpha}_i - \alpha_i > \epsilon) &\leq O(e^{-cn}) \\ and \quad P(\hat{\alpha}_i - \alpha_i < -\epsilon) &\leq O(e^{-cn}), \end{aligned}$$

where ϵ is a fixed positive number.

Proof: We prove the first part of the lemma. Second part can be proven similarly. Without loss of generality assume $i = 1$, then

$$\begin{aligned} P(\hat{\alpha}_1 - \alpha_1 > \epsilon) &= P(\alpha_0(1 + \sum_{j \neq 1} \frac{1 - Z_{1jn}}{Z_{1jn}})^{-1} - \alpha_1 > \epsilon \text{ and } Z_{1jn} > 0 \, \forall \, j \neq 1 \,) \\ &+ P(Z_{1jn} \leq 0 \text{ for some } j \neq 1 \,). \end{aligned}$$

Also

$$P(Z_{1jn} \leq 0 \text{ for some } j \neq 1 \,) \leq \sum_{j \neq 1} P(Z_{1jn} \leq 0).$$

Notice that Z_{1jn} is average of *i.i.d* bounded random variables with positive expectation, hence each term on the right is of order $O(e^{-cn})$ for some positive constant c.

$$\begin{aligned} & P(\alpha_0(1 + \sum_{j \neq 1} \frac{1 - Z_{1jn}}{Z_{1jn}})^{-1} - \alpha_1 > \epsilon \text{ and } Z_{1jn} > 0 \, \forall \, j \neq 1 \,) \\ & \leq P(\alpha_0 - (1 + \sum_{j \neq 1} \frac{1 - Z_{ijn}}{Z_{ijn}})\alpha_1 > (1 + \sum_{j \neq 1} \frac{1 - Z_{ijn}}{Z_{ijn}})\epsilon) \\ & \leq P((\alpha_0 - \alpha_1) - \alpha_1 \sum_{j \neq 1} \frac{1 - Z_{1jn}}{Z_{1jn}} > \epsilon + \epsilon \sum_{j \neq 1} \frac{1 - Z_{1jn}}{Z_{1jn}}) \\ & = P(\sum_{j \neq 1} (\alpha_j - [\alpha_1 + \epsilon] \frac{1 - Z_{1jn}}{Z_{1jn}}) > \epsilon) \\ & \leq \sum_{j \neq 1} P((\alpha_j - [\alpha_1 + \epsilon] \frac{1 - Z_{1jn}}{Z_{1jn}}) > \frac{\epsilon}{k}) \end{aligned}$$

Let $\frac{\epsilon}{k} = \epsilon'$ then

$$P((\alpha_j - [\alpha_1 + \epsilon] \frac{1 - Z_{1jn}}{Z_{1jn}}) > \epsilon') = P([\alpha_j - \epsilon'] Z_{1jn} - [\alpha_1 + \epsilon](1 - Z_{1jn}) > 0).$$

Note that $[\alpha_j - \epsilon'] Z_{1jn} - [\alpha_1 + \epsilon](1 - Z_{1jn})$ is average of n bounded *i.i.d.* variables with negative expectation, hence by Hoeffdings lemma, there exists some positive constant c such that right side of above equation is of order $O(e^{-cn})$. And this completes the proof of the lemma. □

This lemma leads to:

Theorem 3.1 *:*
The sequence of the empirical Bayes rules $\{\delta_n\}_1^\infty$ defined above is asymptotically optimal of order e^{-cn} for some positive constant c.

Proof:

For $1 \leq l \neq t \leq k$, define

$$A_{lt}^n = \left\{ \underset{\sim}{x} : \begin{array}{c} x_l + \hat{\alpha_{nl}} \geq x_t + \hat{\alpha_{nt}} \\ x_l + \alpha_l < x_t + \alpha_t \end{array} \right\}$$

Let δ_G be the Bayes rule and let $r(G)$ be Bayes risk of the Bayes rule.

$$
\begin{aligned}
& r(G,\delta_n) - r(G) \\
&= \int\int L(\underset{\sim}{p}_{n+1}, \delta_n(\underset{\sim}{x})) - L(\underset{\sim}{p}_{n+1}, \delta_G(\underset{\sim}{x}_{n+1}))] \prod_{j=1}^{n+1} f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) g_{\underset{\sim}{\alpha}}(\underset{\sim}{p}_j) d\underset{\sim}{x} d\underset{\sim}{p} \\
&\leq \int\int |L(\underset{\sim}{p}_{n+1}, \delta_n(\underset{\sim}{x}) - L(\underset{\sim}{p}_{n+1}, \delta_G(\underset{\sim}{x}_{n+1}))| \\
&\quad I_{(\cup_{l\neq t} A_{lt}^n)} \prod_{j=1}^{n+1} \ f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) g_{\underset{\sim}{\alpha}}(\underset{\sim}{p}_j) \ d\underset{\sim}{x} d\underset{\sim}{p}.
\end{aligned}
$$

$I_D(.\,)$ denote an indicator function of the set D, *i.e.*

$$
\begin{aligned}
I_D(\omega) &= 1 \text{ if } \ \omega \in D \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

Now using the Holder's inequality we have

$$
\begin{aligned}
& r(G,\delta_n) - r(G) \\
&\leq \left(\int\int |L(\underset{\sim}{p}_{n+1}, \delta_n(\underset{\sim}{x})) - L(\underset{\sim}{p}_{n+1}, \delta_G(\underset{\sim}{x}_{n+1}))|^{1+\delta} \prod_{j=1}^{n+1} f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) g_{\underset{\sim}{\alpha}}(\underset{\sim}{p}_j) \ d\underset{\sim}{x} d\underset{\sim}{p} \right)^{\frac{1}{1+\delta}} \\
&\quad \times \left(\int\int I_{\cup_{l\neq t} A_{lt}^n}(\underset{\sim}{x}) \prod_{j=1}^{n+1} f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) g_{\underset{\sim}{\alpha}}(\underset{\sim}{p}_j) \ d\underset{\sim}{x} d\underset{\sim}{p} \right)^{\frac{\delta}{1+\delta}}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int\int |L(\underset{\sim}{p}_{n+1}, \delta_n(\underset{\sim}{x}_{n+1})) - L(\underset{\sim}{p}_{n+1}, \delta_G(\underset{\sim}{x}_{n+1}))|^{1+\delta} \ \ \textstyle\prod_{j=1}^{n+1} \ \ & f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) g_\alpha(\underset{\sim}{p}_j) \, d\underset{\sim}{x} \ d\underset{\sim}{p} \\
\leq \quad & 2 \sup_a \int L^{1+\delta}(\underset{\sim}{p}_{n+1}, a) g(\underset{\sim}{p}_{n+1}) d\underset{\sim}{p}.
\end{aligned}
$$

The supremum is taken over all $a \in \mathcal{A}$. Since $\mathcal{A}$ is finite and since we assume(4) on the loss function the right hand side of the above equation is finite, say m.

$$
\begin{aligned}
\int\int I_{\cup_{l\neq t} A_{lt}^n} a(\underset{\sim}{x}) \ \prod_{j=1}^{n+1} f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) \ g_{\underset{\sim}{\alpha}}(\underset{\sim}{p}_j) \, d\underset{\sim}{x} d\underset{\sim}{p} \ &\leq \ \sum_{l\neq t} \int\int I_{A_{lt}^n}(\underset{\sim}{x}) \prod_{j=1}^{n+1} f(\underset{\sim}{x}_j|\underset{\sim}{p}_j) g_{\underset{\sim}{\alpha}}(\underset{\sim}{x}_j) \ dx \ \ d\underset{\sim}{p} \\
&= \ \sum_{l\neq t} P(A_{lt}^n).
\end{aligned}
$$

Let $\epsilon' = \min\{|x_l + \alpha_l - x_t + \alpha_t| : |x_l + \alpha_l - x_t - \alpha_t| \neq 0,\}$. To prove the theorem, it is enough to prove that each term on the right is $O(e^{-cn})$. Without loss of generality let us assume that $l = 1$ and $t = 2$.

$$\begin{aligned}
& P(A_{12}^n) \\
= \;& P(X_{n+1,1} + \hat{\alpha_{1n}} \geq X_{n+1,2} + \hat{\alpha_{2n}} \text{ and } X_{n+1,1} + \alpha_1 \leq X_{n+1,2} + \alpha_2) \\
\leq \;& P(\hat{\alpha_2} - \hat{\alpha_1} - \alpha_2 + \alpha_1 < -\epsilon) \; (\epsilon = \epsilon'/2) \\
= \;& P(\frac{1}{n}\sum_{j=1}^{n}[\frac{(x_{2j} - x_{1j})}{N} - \frac{1}{\alpha_0}[\alpha_2 - \alpha_1]] < -\epsilon) \\
\leq \;& O(e^{-cn}),.
\end{aligned}$$

The last inequality follows from the Lemma 3.2. This completes the proof of the theorem. □

Now consider the Case 2: α_0 unknown. As before, to derive a sequence of a sequence of empirical Bayes rules, based on the past observations, we need to get estimates of the parameters, $\alpha_1, \alpha_2, \ldots, \alpha_k$. To do this, note that

$$\begin{aligned}
E[Y_{ij}] &= N\frac{\alpha_i}{\alpha_i + \alpha_j} \\
E[Y_{ij}^2] &= NE[\frac{P_i P_j}{(P_i + P_j)^2}] + N^2 E[\frac{P_i}{P_i + P_j}]^2 \\
&= NE[\frac{P_i}{P_i + P_j}] - NE[\frac{P_i}{P_i + P_j}]^2 + N^2 E[\frac{P_i}{P_i + P_j}]^2 \\
&= NE[\frac{P_i}{P_i + P_j}] + N(N-1)E[\frac{P_i}{P_i + P_j}]^2 \\
&= N\frac{\alpha_i}{\alpha_i + \alpha_j} + N(N-1)\frac{\alpha_i(\alpha_i + 1)}{(\alpha_i + \alpha_j)(\alpha_i + \alpha_j + 1)}.
\end{aligned}$$

Hence,

$$\frac{EY_{ij}^2 - EY_{ij}}{N(N-1)} = \frac{\alpha_i(\alpha_i + 1)}{(\alpha_i + \alpha_j)(\alpha_i + \alpha_j + 1)}.$$

Let

$$z'_{ijn} = \frac{1}{nN(N-1)}\Sigma_{m=1}^n {y_{ijm}}^2.$$

then

$$\frac{EZ'_{ijn} - [EZ_{ijn}]^2}{EZ'_{ijn}} = \frac{\alpha_j}{(\alpha_i + \alpha_j)(\alpha_i + 1)}.$$

After some straight forward calculations it follows that,

$$\alpha_i = \frac{[EZ'_{ijn}][1 - EZ_{ijn}]}{[EZ'_{ijn}] - [EZ_{ijn}]^2} - 1.$$

For the case when $\alpha_o, \alpha_1, \ldots, \alpha_k$ are unknown, define

$$\begin{aligned} \hat{\alpha}'_{in} &= \frac{Z'_{ijn}[1 - Z_{ijn}]}{Z'_{ijn} - [Z_{ijn}]^2} - 1 \qquad \text{if } \frac{Z'_{ijn}[1-Z_{ijn}]}{Z'_{ijn}-[Z_{ijn}]^2} \geq 1 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Now we propose a sequence of empirical Bayes rules $\{\delta'_n\}_1^\infty$, where δ'_n is a rule which ranks the ith treatment according to the rank of $x_i + \hat{\alpha}'_i$ in $x_1 + \hat{\alpha}'_1, x_2 + \hat{\alpha}'_2, \ldots, x_k + \hat{\alpha}'_k$. In the case of ties use randomization. As for Case 1, following is the theorem about the asymptotic optimality of the sequence of empirical Bayes rules $\{\delta'_n\}_{n=1}^\infty$. The proof of this theorem is similar to that of Theorem 3.1 and hence omited.

Theorem 3.2 *:*
The sequence of the empirical Bayes rules $\{\delta'_n\}_1^\infty$ defined above is asymptotically optimal of order e^{-cn} for some positive constant c.

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References

[1] Bechhofer, R. E. (1954) *"A single sample multiple decision procedure for ranking means of normal populations with known variances"* Annals of Mathematical Statistics 25, 16-39.

[2] Bradley, R. A. (1976) *"Science, statistics and paired comparison"* Biometrics 32, 213-232.

[3] Bradley, R. A. (1984) *"Paired comparisons: some basic procedures and examples"* Handbook of Statistics, vol. 4, ed. P. R. Krishnaiah and P. K. Sen, North Holland, Amsterdam, 229-326.

[4] David, H. A. (1963) *"The method of paired comparison"* Griffin, London.

[5] David, H. A. (1989) *"Ranking and selection from paired-comparison data"* to appear in the proceedings of the second international advanced seminar/workshop on inference procedure associated with statistical ranking and selection. (Ed. Eve Bofinger).

[6] Deely, J. J. (1965) *"Multiple decision procedures from an empirical Bayes approach"* Ph. D. thesis (Mimeo. Ser. No. 45), Department of Statistics, Purdue University, West Lafayette, Indiana.

[7] Gupta, S. S. (1956) *"On a decision rule for a problem of ranking means"* Memeograph series No. 150 , Institute of statistics, University of North Carolina, Chapel Hill.

[8] Gupta, S. S. and Hande, S. N. (1991) *"Single-Stage Bayes and empirical Bayes rule for ranking multinomial events"* Technical Report # 91-14, Department of Statistics, Purdue University, West Lafayette, Indiana.

[9] Gupta, S. S. and Liang, T. C. (1986) *"Empirical Bayes rules for selecting good binomial populations"* Adaptive Statistical Procedures and related topics (Ed. J. Van Ryzin), IMS Lecture Notes-Monograph Series, Vol. 8, 110-128.

[10] Gupta, S. S. and Liang, T. C. (1988) *"Empirical Bayes rules for selecting best binomial population"* Statistical Decision Theory and Related topics Reliability and Quality control; A. P. Basu (Ed.); NHollnd/AElsvr.

[11] Gupta, S. S, and Liang, T. C. (1989) *"Parametric empirical Bayes rules for selecting the most probable multinomial event"* Contributions to Probability and Statistics, Essays in Honor of Ingram Olkin, (Ed. Gleser et. al.), Springer-Verlag, 318-328.

[12] Hoeffding, W. (1963) *"Probability inequalities for sums of bounded random variables"* Amer. Statist. Assoc. 58, 13-30.

[13] Robbins, H. (1956) *"An empirical Bayes approach to statistical decision problems"* Proc. Third Berkeley Symp. Math. Stat. Probab. 1, 157-163, University of California Press.

[14] Trawinski, B. J. and David, H. A. (1963) *"Selection of the best treatment in a paired-comparison experiment"* Annals of Mathematical Statistics 34, 75-91.