AN EDGEWORTH EXPANSION FOR U-STATISTICS WITH WEAKLY DEPENDENT OBSERVATIONS

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An Edgeworth expansion for U-statistics with weakly dependent observations

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Summary. Under mild conditions, an Edgeworth expansion with remainder $o(N^{-1/2})$ is established for a U-statistic with a kernel h of degree two using weakly dependent observations. The ease of verifying these conditions is discussed in the context of three rather natural examples.

1 Introduction

Let $\{X_j : -\infty < j < \infty\}$ be a strictly stationary sequence of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. We assume that there exists a sequence $\{\mathcal{A}_j : -\infty < j < \infty\}$ of sub σ-fields of $\mathcal{A}$ such that for all j, X_j is $\mathcal{A}_{j-m}^{j+m}$ measurable where m is a fixed nonnegative integer and $\mathcal{A}_a^b$ denotes the sub σ-field of $\mathcal{A}$ generated by $\{\mathcal{A}_j : a \le j \le b\}$. We further assume that the $\mathcal{A}_j$'s satisfy an absolutely regular condition and a Markov type condition, namely that there exists a constant $\lambda > 0$ such that for all $n \ge 1$, $p \ge 0$, $-\infty < j < \infty$ and $B \in \mathcal{A}_{j-p}^{j+p}$, we have

$$(1) \qquad E[\sup_{A \in \mathcal{A}_{j+n}^{\infty}} |P(A|\mathcal{A}_{-\infty}^{j}) - P(A)|] \le \lambda^{-1} e^{-\lambda n},$$

and

$$(2) \qquad E|P(B|\mathcal{A}_k : k \ne j) - P(B|\mathcal{A}_k : 0 < |j-k| \le n+p)| \le \lambda^{-1} e^{-\lambda n}.$$

We denote the cumulative distribution function of X_j by $F(x)$, $\forall x \in R$. Next let $h : R^2 \to R$ be a measurable function symmetric in its two arguments. We shall assume throughout this paper that there exist constants $\gamma > 2$ and $M > 0$ such that

$$(3) \qquad E|h(X_1, X_j)|^{\gamma} < M, \quad \forall j > 1,$$

$$(4) \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(x,y)|^{\gamma} dF(x) dF(y) < M$$

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and, without loss of generality, that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)dF(x)dF(y) = 0. \tag{5}$$

Then $Eh(X_j, X_k)$ exists for all $j < k$. We write

$$h_{j,k}(X_j, X_k) = h(X_j, X_k) - Eh(X_j, X_k), \quad \forall j < k,$$

and for $N \geq 2$, a U-statistic with a kernel h of degree two is defined as

$$U_N = \sum_{j=1}^{N-1}\sum_{k=j+1}^{N} h_{j,k}(X_j, X_k).$$

Also we write

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} h(x,y)dF(y), \\ \psi(x,y) &= h(x,y) - g(x) - g(y), \\ \psi_{j,k}(x,y) &= h_{j,k}(x,y) - g(x) - g(y), \quad \forall j < k. \end{aligned}$$

Thus for $N \geq 2$,

$$U_N = (N-1)\sum_{j=1}^{N} g(X_j) + \sum_{a=1}^{N-1}\sum_{b=a+1}^{N} \psi_{a,b}(X_a, X_b).$$

We further assume that

$$\sigma_g^2 = E[g^2(X_1) + 2\sum_{j=2}^{\infty} g(X_1)g(X_j)] > 0, \tag{6}$$

and

$$Eg^4(X_1) < \infty. \tag{7}$$

Let σ_N^2 denote the variance of $(N-1)\sum_{j=1}^{N} g(X_j)$. Then by the stationarity of the X_j's and Lemma 1 [see Appendix], we have

$$\begin{aligned} \sigma_N^2 &= (N-1)^2 E[Ng^2(X_1) + 2\sum_{j=2}^{N}(N-j+1)g(X_1)g(X_j)] \\ &= N^3\sigma_g^2 + O(N^2), \end{aligned} \tag{8}$$

as $N \to \infty$. Next let $\{X'_j : -\infty < j < \infty\}$ be an independent replicate of $\{X_j : -\infty < j < \infty\}$ and

$$\begin{aligned}\kappa_3 &= \sigma_g^{-3} E\{g^3(X_1) + 3\sum_{j=2}^{\infty}[g^2(X_1)g(X_j) + g(X_1)g^2(X_j)] \\ &\quad +6\sum_{j=2}^{\infty}\sum_{k=j+1}^{\infty} g(X_1)g(X_j)g(X_k) \\ &\quad +3\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} g(X_j)\psi(X_1, X'_1)g(X'_k)\}. \end{aligned} \tag{9}$$

Using Lemma 1, it can be seen that $-\infty < \kappa_3 < \infty$. We observe that if $E|h(X_j, X_k)|^3 < \infty$ whenever $j < k$, then $\kappa_3 N^{-1/2}$ is an asymptotic approximation [with error $o(N^{-1/2})$] for the third cumulant of $\sigma_N^{-1}U_N$. Define

$$F_N(x) = \Phi(x) - \phi(x)\frac{\kappa_3}{6}N^{-1/2}(x^2 - 1), \quad \forall x \in R, \tag{10}$$

where ϕ and Φ denote the standard normal density and distribution function respectively.

The main aim of this paper is to establish the validity of an Edgeworth expansion for $\sigma_N^{-1}U_N$ with remainder $o(N^{-1/2})$ under mild conditions. In particular, we prove

Theorem 1 *Suppose (1)-(7) are satisfied and that for each $d > 0$, there exists a constant $0 < \delta_d < 1$ such that*

$$E|E\{e^{it[g(X_{j-m})+\cdots+g(X_{j+m})]}|\mathcal{A}_k : k \neq j\}| < \delta_d, \quad \forall j > m, \tag{11}$$

whenever $|t| \geq d$. Then

$$\sup_x |P(\sigma_N^{-1}U_N \leq x) - F_N(x)| = o(N^{-1/2}), \quad \text{as } N \to \infty.$$

Proof. Let α be a constant to be suitably chosen later for which $3/8 < \alpha < 1/2$. We define

$$T(x) = \begin{cases} x, & \text{if } |x| \leq N^\alpha, \\ xN^\alpha \hat{T}(|x|N^{-\alpha})/|x|, & \text{otherwise,} \end{cases} \tag{12}$$

where $\hat{T} \in C^\infty(0, \infty)$ satisfies $\hat{T}(x) = x$ if $x \leq 1$, $\hat{T}$ is increasing and $\hat{T}(x) = 2$ if $x \geq 2$. We write

$$Y_j = T[g(X_j)], \quad Z_j = Y_j - EY_j, \quad \forall j \geq 1. \tag{13}$$

Let $\hat{\sigma}_N^2$ denote the variance of $(N-1)\sum_{j=1}^N Z_j$, and with γ as in (3), let

$$\beta = \max\{2/(\gamma-2), 5/4\}. \tag{14}$$

We define

$$\begin{aligned}
\hat{\psi}_{j,k}(X_j, X_k) &= \psi_{j,k}(X_j, X_k) I\{|\psi_{j,k}(X_j, X_k)| \le N^\beta\}, \quad \forall 1 \le j < k \le N, \\
\hat{\Delta}_N &= \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \hat{\psi}_{j,k}(X_j, X_k), \\
\Delta_N &= \hat{\Delta}_N - E\hat{\Delta}_N,
\end{aligned}$$

where $I\{|\psi_{j,k}(X_j, X_k)| \le N^\beta\}$ denotes the indicator function of the event $\{|\psi_{j,k}(X_j, X_k)| \le N^\beta\}$. We observe that for all $x \in R$,

$$\begin{aligned}
&|P(\sigma_N^{-1} U_N \le x) - F_N(x)| \\
\le\ & |P(\hat{\sigma}_N^{-1} U_N \le \sigma_N \hat{\sigma}_N^{-1} x) - P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Y_j + \hat{\Delta}_N] \le \sigma_N \hat{\sigma}_N^{-1} x\}| \\
& + |P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N] \le y\} - F_N(y)| + |F_N(y) - F_N(x)|,
\end{aligned} \tag{15}$$

where $y = \sigma_N \hat{\sigma}_N^{-1} x - \hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N EY_j + E\hat{\Delta}_N]$. We observe from the definitions of the Y_j's and $\hat{\Delta}_N$ that

$$\begin{aligned}
& \sup_x |P(\hat{\sigma}_N^{-1} U_N \le \sigma_N \hat{\sigma}_N^{-1} x) \\
& -P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Y_j + \hat{\Delta}_N] \le \sigma_N \hat{\sigma}_N^{-1} x\}| \\
\le\ & \sum_{j=1}^N P[|g(X_j)| > N^\alpha] + \sum_{a=1}^{N-1} \sum_{b=a+1}^{N} P[|\psi_{a,b}(X_a, X_b)| > N^\beta] \\
=\ & o(N^{-1/2}), \quad \text{as } N \to \infty.
\end{aligned} \tag{16}$$

The last equality uses (3), (7) and Markov's inequality. By choosing α sufficiently close to 1/2, we observe from Lemma 3.30 of Götze and Hipp (1983) that

$$\sigma_N \hat{\sigma}_N^{-1} = 1 + o(N^{-\omega}), \tag{17}$$

for some constant $1/2 < \omega < 1$, and hence

$$\sup_x |F_N(y) - F_N(x)| = o(N^{-1/2}), \quad \text{as } N \to \infty. \tag{18}$$

Thus it follows from (15), (16) and (18) that it remains only to prove

$$\sup_x |P\{\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N] \le y\} - F_N(y)| = o(N^{-1/2}), \quad \text{as } N \to \infty.$$

To do so, we shall study the characteristic function (c.f.) of $\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N]$. Let

$$\phi_N(t) = Ee^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N]}, \quad \forall t \in R,$$

and for κ_3, as in (9), let

$$\phi_N^*(t) = e^{-t^2/2}(1 - \frac{i\kappa_3}{6} N^{-1/2} t^3), \quad \forall t \in R,$$

be the Fourier transform $\int \exp(itx) dF_N(x)$ of F_N in (10). By the smoothing lemma of Esseen [see for example, Feller (1971), p. 538], it suffices to show that

$$\text{(19)} \qquad \int_{-N^{1/2}\log N}^{N^{1/2}\log N} |\frac{\phi_N(t) - \phi_N^*(t)}{t}| dt = o(N^{-1/2}), \quad \text{as } N \to \infty.$$

However (19) is an immediate consequence of Propositions 1 and 2 whose statements and proofs are provided in Sections 2 and 3 respectively. This proves Theorem 1. □

There has been a great deal of research done on U-statistics based on independent and identically distributed (i.i.d.) observations. In this paragraph, we shall restrict our attention to i.i.d. observations. U-statistics were first discussed by Hoeffding (1948) who also showed their asymptotic normality under very mild conditions. The rate of convergence to normality was investigated by Grams and Serfling (1973) and Berry-Esseen bounds were obtained in increasing generality by Bickel (1974), Chan and Wierman (1977), Callaert and Janssen (1978) and Helmers and van Zwet (1982).

There has also been a lot of work on obtaining sufficient conditions for the asymptotic normality of U-statistics with dependent observations. The literature includes Sen (1972), Yoshihara (1976), Denker and Keller (1983) and Harel and Puri (1989). Berry-Esseen type bounds were obtained by Yoshihara (1984) for U-statistics generated by absolutely regular processes, Rhee (1988) for U-statistics based on m-dependent observations and Zhao and Chen (1987) for finite population U-statistics.

Regarding the more involved problem of Edgeworth expansions with i.i.d. observations, Callaert, Janssen and Veraverbeke (1980) established sufficient conditions for a U-statistic to have a two term Edgeworth expansion with remainder $o(N^{-1})$. This was followed by Bickel, Götze and van Zwet (1986) who gave more easily verifiable sufficient conditions for the validity of a one term [two term] Edgeworth expansion with remainder $o(N^{-1/2})$ [$o(N^{-1})$] respectively.

Under dependent observations, Kokic and Weber (1990) obtained conditions for the validity of a one term Edgeworth expansion for U-statistics based on samples from finite populations and Loh (1991) obtained an Edgeworth expansion with remainder $o(N^{-1/2})$ for a U-statistic with an m-dependent shift under very weak conditions.

The remainder of this paper is organized as follows. Sections 2 and 3 provide the statements and proofs of Propositions 1 and 2 respectively. These results are needed in the proof of Theorem 1. In Section 4, conditions (1), (2) and (11) are examined more closely. In particular, these conditions are shown to hold in three somewhat natural examples. Finally the Appendix contains a few rather technical lemmas which are used in the proofs of Propositions 1 and 2.

2 The c.f. for small values of the argument

In this section we begin by studying $\phi_N(t)$ for small values of $|t|$.

Proposition 1 *Let* $0 < \varepsilon < 1/16$ *be as in Lemma 3 (see Appendix). Then*

$$\int_{-N^\varepsilon}^{N^\varepsilon} \left|\frac{\phi_N(t) - \phi_N^*(t)}{t}\right| dt = o(N^{-1/2}), \qquad \text{as } N \to \infty.$$

Proof. It is well known that for $r \geq 0$,

$$(20) \qquad |e^{ix} - \sum_{j=0}^{r} \frac{(ix)^j}{j!}| \leq \min\{\frac{2}{r!}|x|^{r+\theta}, \frac{|x|^{r+1}}{(r+1)!}\}, \qquad \forall\theta \in [0,1).$$

Hence it follows from Lemma 2 that

$$\begin{aligned} \phi_N(t) &= Ee^{it\hat\sigma_N^{-1}(N-1)\sum_{j=1}^N Z_j}(1 + it\hat\sigma_N^{-1}\Delta_N) + O[E(t\hat\sigma_N^{-1}\Delta_N)^2] \\ (21) \qquad &= Ee^{it\hat\sigma_N^{-1}(N-1)\sum_{j=1}^N Z_j}(1 + it\hat\sigma_N^{-1}\Delta_N) + O(t^2N^{-1}), \end{aligned}$$

as $N \to \infty$ uniformly in t. We observe from (21) and Lemma 3 that for α sufficiently close to 1/2,

$$\phi_N(t) - e^{-t^2/2}(1 - \frac{i\hat\kappa_3}{6}N^{-1/2}t^3)$$

$$
\begin{aligned}
&= E it\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j} \\
&\quad + O(t^2N^{-1}) + o[(|t|^3+t^4)e^{-\epsilon t^2}N^{-1/2}],
\end{aligned} \tag{22}
$$

as $N \to \infty$ uniformly over $|t| \le N^{\epsilon}$. It remains to approximate the term $E it\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j}$. First we observe that

$$
\begin{aligned}
&E it\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j} \\
&= \sum_{a=1}^{N-1}\sum_{b=a+1}^{N} E it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j} + O(|t|N^{-3/4}),
\end{aligned} \tag{23}
$$

as $N \to \infty$ uniformly in t. Next define for $N \ge 2$,

$$u = \lceil K \log N \rceil,$$

where K is a positive constant to be suitably chosen later. Here for all $x \in R$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Define for $1 \le a < b \le N$,

$$
\begin{aligned}
S_{a,b}^{(r)} &= \hat{\sigma}_N^{-1}(N-1)\sum_{1\le j\le N, |j-a|\wedge|j-b|>ru} Z_j, \quad \forall r \ge 1, \\
S_{a,b}^{(0)} &= \hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^{N} Z_j.
\end{aligned} \tag{24}
$$

Following a method of Tikhomirov (1980), we have for sufficiently large K,

$$
\begin{aligned}
&\sum_{a=1}^{N-3u-1}\sum_{b=a+3u+1}^{N} E it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j} \\
&= \sum_{a=1}^{N-3u-1}\sum_{b=a+3u+1}^{N} E\{it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)e^{itS_{a,b}^{(1)}} \\
&\quad + it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)[e^{it(S_{a,b}^{(0)}-S_{a,b}^{(1)})}-1]e^{itS_{a,b}^{(2)}} \\
&\quad + it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)\prod_{l=1}^{2}[e^{it(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]e^{itS_{a,b}^{(2)}}\} \\
&= -\frac{i}{2}\sigma_g^{-3}t^3e^{-t^2/2}N^{-1/2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1,X_1')g(X_k') \\
&\quad + O[(|t|+t^4)N^{-1}\log^3 N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}],
\end{aligned} \tag{25}
$$

as $N \to \infty$ uniformly over $|t| \le N^\epsilon$ where $\{X'_j : -\infty < j < \infty\}$ denotes an independent replicate of $\{X_j : -\infty < j < \infty\}$. The last equality uses Lemmas 4 and 5. In a similar though less tedious way, we have for sufficiently large K,

$$\begin{aligned} &\sum_{a=1}^{N-1} \sum_{b=a+1}^{(a+3u)\wedge N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j} \\ &= \sum_{a=1}^{N-1} \sum_{b=a+1}^{(a+3u)\wedge N} E\{it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)e^{itS_{a,b}^{(1)}} \\ &\quad + it\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)[e^{it(S_{a,b}^{(0)} - S_{a,b}^{(1)})} - 1]e^{itS_{a,b}^{(1)}}\} \\ &= O[(|t| + t^2)N^{-1}\log^2 N], \end{aligned} \tag{26}$$

as $N \to \infty$ uniformly in t. Thus it follows from (23), (25) and (26) that

$$\begin{aligned} &Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^N Z_j} \\ &= -\frac{i}{2}\sigma_g^{-3}t^3 e^{-t^2/2}N^{-1/2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1, X'_1)g(X'_k) \\ &\quad + O[(|t| + t^4)N^{-1}\log^3 N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}], \end{aligned}$$

as $N \to \infty$ uniformly over $|t| \le N^\epsilon$. Hence we conclude from (9) and (22) that

$$\begin{aligned} \phi_N(t) - \phi_N^*(t) &= \phi_N(t) - e^{-t^2/2}(1 - \frac{i\kappa_3}{6}N^{-1/2}t^3) \\ &= O[(|t| + t^4)N^{-1}\log^3 N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}] \\ &\quad + o[(|t|^3 + t^4)e^{-\epsilon t^2}N^{-1/2}], \end{aligned}$$

as $N \to \infty$ uniformly over $|t| \le N^\epsilon$ and hence

$$\int_{-N^\epsilon}^{N^\epsilon} \left|\frac{\phi_N(t) - \phi_N^*(t)}{t}\right| dt = o(N^{-1/2}),$$

as $N \to \infty$. This completes the proof of Proposition 1. □

3 The c.f. for large values of the argument

We observe from (11) and Bhattacharya and Rao (1986), p. 212 that for sufficiently large N, there exists a constant $0 < \delta < 1$ such that

$$E|E[e^{it(Z_{j-m} + \cdots + Z_{j+m})}|\mathcal{A}_k : k \neq j]| \le \delta, \quad \forall j > m, \tag{27}$$

whenever $|t| \geq 1/(2\sigma_g)$. Now it follows from Lemma 3.2 of Götze and Hipp (1983) that there exists a constant $\mu > 0$ such that

$$E|E[e^{it(Z_{j-m}+\cdots+Z_{j+m})}|\mathcal{A}_k : k \neq j]| \leq e^{-\mu t^2}, \quad \forall j > m, \tag{28}$$

whenever $|t| \leq 3/(2\sigma_g)$.

Proposition 2 *Let ε be as in Section 2. Then*

$$\int_{N^\varepsilon \leq |t| \leq N^{1/2}\log N} |\frac{\phi_N(t) - \phi_N^*(t)}{t}| dt = o(N^{-1/2}), \quad \textit{as } N \to \infty.$$

Proof. It is easy to see that

$$\int_{|t| \geq N^\varepsilon} |\phi_N^*(t)/t| dt = o(N^{-1/2}),$$

as $N \to \infty$. Hence it suffices only to show

$$\int_{N^\varepsilon \leq |t| \leq N^{1/2}\log N} |\phi_N(t)/t| dt = o(N^{-1/2}), \quad \text{as } N \to \infty.$$

Let n and s be integer-valued functions of N satisfying $m < s < n < N$ such that $s \to \infty$, $n \to \infty$ and $ne^{-\lambda s/2} \to 0$ as $N \to \infty$. Define

$$\begin{aligned} \hat{\Delta}_N(n) &= \sum_{j=1}^{n} \sum_{k=j+1}^{N} \hat{\psi}_{j,k}(X_j, X_k), \\ \Delta_N(n) &= \hat{\Delta}_N(n) - E\hat{\Delta}_N(n). \end{aligned} \tag{29}$$

Then it follows from (20) and Lemma 2 that

$$\begin{aligned} |\phi_N(t)| &= |Ee^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]}[1 + it\hat{\sigma}_N^{-1}\Delta_N(n)]| \\ &\quad + O(t^2 nN^{-2}), \end{aligned} \tag{30}$$

as $N \to \infty$ uniformly in t. We shall now approximate the first term of the r.h.s. of (30). Let $1 \leq a < b \leq N$ and define $J_{a,b} = \{1, \ldots, n\} \backslash \{a, b\}$. We divide $J_{a,b}$ into blocks $B_0, A_1, B_1, \ldots, A_l, B_l$ as follows. Define $j_1, \ldots, j_l$ by

$$\begin{aligned} j_1 &= \inf\{j \in J_{a,b} : [j_1 - s, j_1 + s] \subseteq J_{a,b}\}, \\ j_{p+1} &= \inf\{j > j_p + 5s : [j_{p+1} - s, j_{p+1} + s] \subseteq J_{a,b}\}, \quad \forall 1 \leq p \leq l-1, \end{aligned}$$

where $l+1$ is the smallest integer for which the infimum is undefined. We write

$$\begin{aligned}
A_p &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : |j-j_p| \le s\}, \quad \forall 1 \le p \le l,\\
B_0 &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : j \in J_{a,b}, 1 \le j \le j_1 - s - 1\},\\
B_p &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : j \in J_{a,b}, j_p + s + 1 \le j \le j_{p+1} - s - 1\},\\
&\quad \forall 1 \le p \le l-1,\\
B_l &= \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : j \in J_{a,b}, j \ge j_l + s + 1\},
\end{aligned}$$

and

$$\begin{aligned}
R_{a,b} &= it\hat{\sigma}_N^{-1}[\hat{\psi}_{a,b}(X_a, X_b) - E\hat{\psi}_{a,b}(X_a, X_b)]e^{it\hat{\sigma}_N^{-1}[\Delta_N - \Delta_N(n)]}\\
&\quad \times \prod\{e^{it\hat{\sigma}_N^{-1}(N-1)Z_j} : 1 \le j \le N, j \notin J_{a,b}\}.
\end{aligned}$$

Using the convention that the product over an empty set is 1, we have

$$\begin{aligned}
& it\hat{\sigma}_N^{-1}[\hat{\psi}_{a,b}(X_a, X_b) - E\hat{\psi}_{a,b}(X_a, X_b)]e^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]}\\
= \; & R_{a,b} B_0 \prod_{p=1}^{l} A_p B_p.
\end{aligned}$$

Now we observe from (2) that

$$E|E(A_p|\mathcal{A}_j : j \ne j_p) - E(A_p|\mathcal{A}_j : 0 < |j-j_p| \le 2s)| \le 4\lambda^{-1}e^{-\lambda(s-m)},$$

and hence

$$\begin{aligned}
& |E[R_{a,b}B_0 \prod_{p=1}^{l} A_p B_p - R_{a,b}B_0 \prod_{p=1}^{l} B_p E(A_p|\mathcal{A}_j : 0 < |j-j_p| \le 2s)]|\\
\le \; & \sum_{q=1}^{l} |ER_{a,b}B_0(\prod_{p=1}^{q-1} A_p B_p)[E(A_q|\mathcal{A}_j : j \ne j_q)\\
& -E(A_q|\mathcal{A}_j : 0 < |j-j_q| \le 2s)]B_q\\
& \times [\prod_{p=q+1}^{l} B_p E(A_p|\mathcal{A}_j : 0 < |j-j_p| \le 2s)]|\\
\le \; & 8\lambda^{-1}|t|\hat{\sigma}_N^{-1} n N^{\beta} e^{-\lambda(s-m)}.
\end{aligned}$$

We thus conclude that

$$
\begin{aligned}
|ER_{a,b}B_0 \prod_{p=1}^{l} A_pB_p| &\le 2|t|\hat{\sigma}_N^{-1}N^{\beta}E\prod_{p=1}^{l}|E(A_p|\mathcal{A}_j : 0 < |j - j_p| \le 2s)| \\
&\quad +8\lambda^{-1}|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda(s-m)}. \qquad (31)
\end{aligned}
$$

By repeated use of Lemma 1 with $\nu = 1$, we observe that the r.h.s. of (31) is bounded by

$$
\begin{aligned}
&2|t|\hat{\sigma}_N^{-1}N^{\beta}\prod_{p=1}^{l}E|E(A_p|\mathcal{A}_j : 0 < |j - j_p| \le 2s)| \\
&+O(|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda s/2}) \\
\le\ & 2|t|\hat{\sigma}_N^{-1}N^{\beta}\prod_{p=1}^{l}E|E(A_p|\mathcal{A}_j : j \ne j_p)| + O(|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda s/2}), \qquad (32)
\end{aligned}
$$

as $N \to \infty$ uniformly in a, b and t.

Case I. Suppose that $N^{1/2} \le |t| \le N^{1/2}\log N$. For sufficiently large N, we take $n = \lceil K_1 \log^2 N\rceil$ and $s = \lceil K_2 \log N\rceil$ where K_1 and K_2 are positive constants to be suitably chosen later. Here $\lceil x\rceil$ denotes the smallest integer greater than or equal to x, whenever $x \in R$. We observe from (27), (31) and (32) that

$$|ER_{a,b}B_0 \prod_{p=1}^{l} A_pB_p| \le 2|t|\hat{\sigma}_N^{-1}N^{\beta}\delta^l + O(|t|\hat{\sigma}_N^{-1}nN^{\beta}e^{-\lambda s/2}),$$

as $N \to \infty$ uniformly over $|t| \ge N^{1/2}$ and $1 \le a < b \le N$. Note that

$$|l - n/(5s+1)| = O(1), \qquad (33)$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$. By choosing K_1 and K_2 so that $K_1K_2^{-1}$ and K_2 are both sufficiently large, we have

$$|ER_{a,b}B_0 \prod_{p=1}^{l} A_pB_p| = O(|t|N^{-5/2}),$$

as $N \to \infty$ uniformly over $|t| \ge N^{1/2}$ and $1 \le a < b \le N$. From the definitions of $R_{a,b}$, B_0, A_p and B_p, with $1 \le p \le l$, we conclude that

$$|Eit\hat{\sigma}_N^{-1}\Delta_N(n)e^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^{N} Z_j + \Delta_N - \Delta_N(n)]}| = O(|t|nN^{-3/2}), \qquad (34)$$

as $N \to \infty$ uniformly over $|t| \geq N^{1/2}$. In a similar way, it can be shown that

$$|Ee^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]}| = O(N^{-1}), \tag{35}$$

as $N \to \infty$ uniformly over $|t| \geq N^{1/2}$. Thus we conclude from (30), (34) and (35) that

$$|\phi_N(t)| = O(N^{-1} + |t|nN^{-3/2} + t^2nN^{-2}),$$

as $N \to \infty$ uniformly over $|t| \geq N^{1/2}$ and hence

$$\int_{N^{1/2}\leq|t|\leq N^{1/2}\log N} |\phi_N(t)/t|dt = o(N^{-1/2}), \quad \text{as } N \to \infty. \tag{36}$$

Case II. Suppose that $N^\epsilon \leq |t| \leq N^{1/2}$. Now for sufficiently large N, we take $n = \lceil K_1 t^{-2} N \log^2 N \rceil$ and $s = \lceil K_2 \log N \rceil$ where K_1 and K_2 are positive constants to be suitably chosen later. We observe from (28), (31) and (32) that

$$|ER_{a,b}B_0 \prod_{p=1}^{l} A_p B_p| \leq 2|t|\hat{\sigma}_N^{-1} N^\beta e^{-\mu l[t\hat{\sigma}_N^{-1}(N-1)]^2} + O(|t|\hat{\sigma}_N^{-1} n N^\beta e^{-\lambda s/2}),$$

as $N \to \infty$ uniformly over $|t| \leq N^{1/2}$ and $1 \leq a < b \leq N$. We note that

$$|l - n/(5s+1)| = O(1),$$

as $N \to \infty$ uniformly over $1 \leq a < b \leq N$ and $N^\epsilon \leq |t| \leq N^{1/2}$. Thus we conclude that by choosing K_1 and K_2 such that $K_1K_2^{-1}$ and K_2 are both sufficiently large, we get

$$|ER_{a,b}B_0 \prod_{p=1}^{l} A_p B_p| \leq O(|t|N^{-5/2}),$$

as $N \to \infty$ uniformly over $N^\epsilon \leq |t| \leq N^{1/2}$ and $1 \leq a < b \leq N$. Thus

$$|Eit\hat{\sigma}_N^{-1}\Delta_N(n)e^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]}| = O(|t|nN^{-3/2}), \tag{37}$$

as $N \to \infty$ uniformly over $N^\epsilon \leq |t| \leq N^{1/2}$. Similarly it can be shown that

$$|Ee^{it\hat{\sigma}_N^{-1}[(N-1)\sum_{j=1}^N Z_j + \Delta_N - \Delta_N(n)]}| = O(N^{-1}), \tag{38}$$

as $N \to \infty$ uniformly over $N^\varepsilon \le |t| \le N^{1/2}$. We conclude from (30), (37) and (38) that

$$|\phi_N(t)| = O(N^{-1} + |t|nN^{-3/2} + t^2 nN^{-2}),$$

as $N \to \infty$ uniformly over $N^\varepsilon \le |t| \le N^{1/2}$ and hence

$$\int_{N^\varepsilon \le |t| \le N^{1/2}} |\phi_N(t)/t| dt = o(N^{-1/2}), \tag{39}$$

as $N \to \infty$. Proposition 2 now follows from (36) and (39). □

4 Examples

In this section, we shall examine the conditions (1), (2) and (11) in the context of three somewhat natural examples.

4.1 On an m-dependent shift

Let $\{\xi_j : -\infty < j < \infty\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. We suppose that ξ_1 has a probability density function π with respect to Lebesgue measure. Let $f : R^{m+1} \to R$ be a measurable function and we define

$$X_j = f(\xi_j, \dots, \xi_{j+m}), \quad \forall -\infty < j < \infty. \tag{40}$$

The sequence $\{X_j : -\infty < j < \infty\}$ is said to be an m-dependent shift and an immediate consequence is that $(\dots, X_{j-1}, X_j)$ and $(X_{j+m+1}, X_{j+m+2}, \dots)$ are stochastically independent for all j. With g as in Section 1, we assume that $g \circ f : R^{m+1} \to R$ is continuously differentiable such that there exist real numbers $y_1, \dots, y_{2m+1}$ and an open set $\Theta \supset \{y_1, \dots, y_{2m+1}\}$ satisfying $\pi(x) > 0$ whenever $x \in \Theta$ and

$$\sum_{j=1}^{m+1} \frac{\partial}{\partial x_{m+1}} g \circ f(x_j, \dots, x_{j+m})|_{(x_1,\dots,x_{2m+1})=(y_1,\dots,y_{2m+1})} \ne 0. \tag{41}$$

To verify that conditions (1), (2) and (11) hold in this case, we choose $\mathcal{A}_j$ to be the sub σ-field of $\mathcal{A}$ generated by ξ_j whenever $-\infty < j < \infty$. Thus X_j is $\mathcal{A}_j^{j+m}$ measurable and it is clear that conditions (1) and (2) are now trivially satisfied. Next we consider the transformation

$$(x_1, \dots, x_{2m+1}) \mapsto (x_1, \dots, x_m, x_{m+2}, \dots, x_{2m+1}, \sum_{j=1}^{m+1} g \circ f(x_j, \dots, x_{j+m})).$$

From (41) we observe that there exists an open set W of R^{2m+1} satisfying $(y_1, \ldots, y_{2m+1}) \in W$ such that the Jacobian of the above transformation is nonzero on W and that $(\xi_1, \ldots, \xi_{2m+1})$ takes values in W with positive probability. Consequently we conclude that the conditional distribution of $g(X_1) + \ldots + g(X_{m+1})$ given $(\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1})$ has a nonzero absolutely continuous component with positive probability. Hence it follows from the Riemann-Lebesgue lemma that (11) holds.

Due to the special m-dependent structure, Theorem 1 can be sharpened in this example. In fact, we have

Theorem 2 *Let $\{X_j : -\infty < j < \infty\}$ be defined as in (40). With the notation of Section 1, suppose that*

$$\begin{aligned} E[g^2(X_1) + 2\sum_{j=2}^{m+1} g(X_1)g(X_j)] &> 0, \\ E|g(X_1)|^3 &< \infty, \\ E|h(X_1, X_j)|^\eta &< \infty, \quad 1 \le j \le m+2, \end{aligned}$$

for some constant $\eta > 5/3$, and

$$\limsup_{|t|\to\infty} E|E\{e^{it[g(X_1)+\cdots+g(X_{m+1})]}|\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}\}| < 1.$$

Then

$$\sup_x |P(\sigma_N^{-1} U_N \le x) - F_N(x)| = o(N^{-1/2}), \quad \text{as } N \to \infty.$$

Proof. We refer the reader to Loh (1991) for a detailed proof. □

4.2 On a homogeneous Markov chain

Let $\{\xi_j : -\infty < j < \infty\}$ be a strictly stationary homogeneous Markov chain defined on a probability space $(\Omega, \mathcal{A}, P)$. Let Ξ and $\mathcal{F}$ denote its state space and the σ-field of measurable subsets of Ξ respectively. We assume that the transition kernel $P(x, A)$ of the Markov chain satisfies

$$\sup_{x,y\in\Xi, A\in\mathcal{F}} |P(x, A) - P(y, A)| = \delta < 1. \tag{42}$$

Let f be a real-valued measurable function defined on Ξ. We write

$$X_j = f(\xi_j), \quad \forall -\infty < j < \infty.$$

With g as in Section 1, we further assume that $g(X_1)$ satisfies Cramér's condition, that is namely

$$\limsup_{|t|\to\infty} |E\exp[itg(X_1)]| < 1. \tag{43}$$

To verify that the conditions (1), (2) and (11) are satisfied in this case, we take $\mathcal{A}_j$ to be the sub σ-field of $\mathcal{A}$ generated by ξ_j whenever $-\infty < j < \infty$. Then X_j is $\mathcal{A}_j$ measurable and (2) is immediate from the Markov property. Let π denote the marginal distribution of ξ_1 and $P^n(x, A)$ the n-step transition kernel of the chain. Then from (42) and Nagaev (1961) p. 62, we have

$$\sup_{x\in\Xi, A\in\mathcal{F}} |P^n(x, A) - \pi(A)| \le \delta^n, \quad \forall n \ge 1. \tag{44}$$

Now it follows from (44) that for all $-\infty < j < \infty$ and $n \ge 1$,

$$\begin{aligned} &\sup_{A\in\mathcal{A}_{j+n}^{\infty}} |P(A|\xi_j) - P(A)| \\ = &\sup_{A\in\mathcal{A}_{j+n}^{\infty}} \left| \int_{\Xi} P(A|\xi_{n+j} = x)[P^n(\xi_j, dx) - \pi(dx)]\right| \\ \le &\ \delta^n. \end{aligned}$$

This proves (1). Finally (11) follows from (42), (43) and Lemma 2 of Statulevičius (1969), pp. 638-641.

4.3 On a stationary Gaussian process

Let $\{\xi_j : -\infty < j < \infty\}$ be a strictly stationary Gaussian process defined on a probability space $(\Omega, \mathcal{A}, P)$. As in Götze and Hipp (1983) p. 215, we suppose that this process has an absolutely continuous spectrum with a positive analytic spectral density. Let $f : R \to R$ be a function such that with g as in Section 1, $g \circ f$ is a non-constant continuously differentiable function. Define

$$X_j = f(\xi_j), \quad \forall -\infty < j < \infty.$$

As in the previous example, we let $\mathcal{A}_j$ denote the sub σ-field of $\mathcal{A}$ generated by ξ_j whenever $-\infty < j < \infty$. Then X_j is $\mathcal{A}_j$ measurable. Now (2) and (11) follow from Götze and Hipp (1983), pp. 219-220 and (1) follows from Ibragimov (1962), p. 1801 and Ibragimov and Solev (1969), p. 374.

5 Appendix

Let $\{X_j : -\infty < j < \infty\}$ be as in Section 1 and for $j_1 < \cdots < j_l$, we write

$$P_p^{(l)}(A^{(p)} \times A^{(l-p)}) = P[(X_{j_1}, \ldots, X_{j_p}) \in A^{(p)}]P[(X_{j_{p+1}}, \ldots, X_{j_l}) \in A^{(l-p)}]$$

for all $1 \leq p < l$, and

$$P_0^{(l)}(A^{(l)}) = P[(X_{j_1}, \ldots, X_{j_l}) \in A^{(l)}],$$

whenever $A^{(k)}$ is a measurable subset of R^k with $1 \leq k \leq l$.

Lemma 1 *Let $1 \leq p < l$ and $f : R^l \to R$ be a measurable function such that there exist positive constants ν and C satisfying*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_l)|^{1+\nu} dP_k^{(l)} < C, \quad k = 0, p.$$

Then for $j_{p+1} - j_p > 2m$, we have

$$\begin{aligned} & |\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_l) dP_0^{(l)} - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_l) dP_p^{(l)}| \\ \leq \; & 4C^{1/(1+\nu)}\{\lambda^{-1} \exp[-\lambda(j_{p+1} - j_p - 2m)]\}^{\nu/(1+\nu)}. \end{aligned}$$

Proof. Since X_j is $\mathcal{A}_{j-m}^{j+m}$ measurable, the result follows directly from Lemma 1 of Yoshihara (1976) and (1). □

Lemma 2 *Let $\Delta_N(n)$ be defined as in (29) with $1 < n < N$. Then*

$$E\Delta_N^2(n) = O(nN), \quad \text{as } N \to \infty.$$

In particular, $E\Delta_N^2 = O(N^2)$, as $N \to \infty$.

Proof. Since $E\Delta_N^2(n) \leq E\hat{\Delta}_N^2(n)$, it suffices to show that $E\hat{\Delta}_N^2(n) = O(nN)$, as $N \to \infty$. By Hölder's inequality, we observe from (3) and (14) that

$$\begin{aligned} & E|\psi_{a,b}(X_a, X_b)\psi_{j,k}(X_j, X_k)I\{|\psi_{a,b}(X_a, X_b)| > N^\beta\}| \\ \leq \; & [E|\psi_{a,b}(X_a, X_b)|^\gamma]^{1/\gamma}[E|\psi_{j,k}(X_j, X_k)|^\gamma]^{1/\gamma} \\ & \times P[|\psi_{a,b}(X_a, X_b)| > N^\beta]^{(\gamma-2)/\gamma} \\ = \; & O(N^{-2}), \end{aligned} \tag{45}$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $1 \le j < k \le N$. Next we observe that

$$\begin{aligned} & E\sum_{a=1}^{n}\sum_{b=a+1}^{N}\sum_{j=1}^{n}\sum_{k=j+1}^{N} \psi_{a,b}(X_a, X_b)\psi_{j,k}(X_j, X_k) \\ = & \sum_{a=1}^{n}\sum_{b=a+1}^{N}\sum_{j=1}^{n}\sum_{k=j+1}^{N} \{E\psi(X_a, X_b)\psi(X_j, X_k) \\ & -[Eh(X_a, X_b)][Eh(X_j, X_k)]\} \\ = & O(nN), \quad \text{as } N \to \infty. \end{aligned} \tag{46}$$

The last equality can be shown to be valid by using the techniques introduced by Yoshihara (1976) in the proof of his Lemma 2. Now the lemma is immediate from (45), (46) and the definition of $\hat{\Delta}_N(n)$. □

Lemma 3 *Let α, Z_j, $S_{a,b}^{(r)}$ be as in (12), (13), (24) respectively and*

$$\begin{aligned} \hat{\kappa}_3 = & \sigma_g^{-3} E\{g^3(X_1) + 3\sum_{j=2}^{\infty}[g^2(X_1)g(X_j) + g(X_1)g^2(X_j)] \\ & +6\sum_{j=2}^{\infty}\sum_{k=j+1}^{\infty} g(X_1)g(X_j)g(X_k)\}. \end{aligned}$$

Then for α sufficiently close to $1/2$, there exists a constant $0 < \varepsilon < 1/16$ such that

$$Ee^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^{N} Z_j} = e^{-t^2/2}(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3) + o[(|t|^3 + t^4)e^{-\varepsilon t^2}N^{-1/2}],$$

and

$$Ee^{itS_{a,b}^{(r)}} = e^{-t^2/2} + O(|t|N^{-1/2}\log N), \quad \forall 1 \le r \le 2,$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon}$ and $1 \le a < b \le N$.

Proof. The proof of the first statement follows from (3.36) and Lemma 3.30 of Götze and Hipp (1983). To prove the second statement, we observe that

$$\begin{aligned} & |E[e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{j=1}^{N} Z_j} - e^{itS_{a,b}^{(r)}}]| \\ \le & E|1 - e^{it\hat{\sigma}_N^{-1}(N-1)\sum_{1\le j\le N, |j-a|\wedge|j-b|\le ru} Z_j}| \\ \le & 5ru|t|\hat{\sigma}_N^{-1}(N-1)E|Z_1| \\ = & O(|t|N^{-1/2}\log N), \end{aligned} \tag{47}$$

as $N \to \infty$ uniformly in a, b and t. The second statement follows immediately from (47) and the first statement. □

Lemma 4 *With the notation of Proposition 1, for sufficiently large K we have*

$$\begin{aligned}
&\sum_{a=1}^{N-3u-1} \sum_{b=a+3u+1}^{N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)[e^{it(S_{a,b}^{(0)}-S_{a,b}^{(1)})} - 1]e^{itS_{a,b}^{(2)}} \\
= \;& -\frac{i}{2}e^{-t^2/2}t^3\sigma_g^{-3}N^{-1/2} \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1, X_1')g(X_k') \\
& +O[(|t| + t^4)N^{-1}\log N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}],
\end{aligned}$$

as $N \to \infty$ uniformly over $|t| \le N^{\epsilon}$, where $\mathcal{P}(|t|)$ is a linear combination (not depending on N) of non-negative powers of $|t|$ and $\{X_j' : -\infty < j < \infty\}$ denotes an independent replicate of $\{X_j : -\infty < j < \infty\}$.

Proof. Let $1 \le a < b \le N$ with $b - a > 3u$. Then

$$S_{a,b}^{(0)} - S_{a,b}^{(1)} = \hat{\sigma}_N^{-1}(N-1)\Big(\sum_{j=(a-u)\vee 1}^{a+u} Z_j + \sum_{k=b-u}^{(b+u)\wedge N} Z_k\Big).$$

For $1 \le c \le N$, we define

$$R_c = it\hat{\sigma}_N^{-1}(N-1)\sum_{j=(c-u)\vee 1}^{(c+u)\wedge N} Z_j. \tag{48}$$

Then from Lemma 1, for sufficiently large K we observe that

$$\begin{aligned}
& Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)e^{R_a} \\
= \;& -it\hat{\sigma}_N^{-1}[Eh(X_a, X_b)](Ee^{R_a}) + O(|t|\hat{\sigma}_N^{-1}e^{-\lambda K\log N}) \\
= \;& O(|t|N^{-3}),
\end{aligned}$$

and similarly that

$$Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)R_a = O(t^2N^{-3}),$$

as $N \to \infty$ uniformly in a, b and t. Hence

$$\begin{aligned}
& Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)[e^{it(S_{a,b}^{(0)}-S_{a,b}^{(1)})} - 1] \\
= \;& Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)[(e^{R_a} - 1 - R_a)(e^{R_b} - 1 - R_b) \\
& +R_a(e^{R_b} - 1 - R_b) + R_b(e^{R_a} - 1 - R_a) + R_aR_b] + O[(|t| + t^2)N^{-3}], \qquad (49)
\end{aligned}$$

as $N \to \infty$ uniformly in a, b and t. Next we observe from Hölder's inequality that

$$\begin{aligned} |EY_j| &= |E[Y_j - g(X_j)]I\{|g(X_j)| > N^\alpha\}| \\ &\leq 3[Eg^4(X_j)]^{1/4}P[|g(X_j)| > N^\alpha]^{3/4} \\ &= O(N^{-3\alpha}), \end{aligned}$$

as $N \to \infty$ uniformly over $1 \leq j \leq N$. Consequently we get

$$\begin{aligned} &E[Z_jZ_k\psi_{a,b}(X_a, X_b) - g(X_j)g(X_k)\psi_{a,b}(X_a, X_b)] \\ &= E\{[Y_j - g(X_j)]Z_k\psi_{a,b}(X_a, X_b) + [Y_k - g(X_k)]g(X_j)\psi_{a,b}(X_a, X_b) \\ &\quad -(EY_j)Z_k\psi_{a,b}(X_a, X_b) - (EY_k)g(X_j)\psi_{a,b}(X_a, X_b)\} \\ &= E\{[Y_j - g(X_j)]Z_k\psi_{a,b}(X_a, X_b)I\{|g(X_j)| > N^\alpha\} \\ &\quad +[Y_k - g(X_k)]g(X_j)\psi_{a,b}(X_a, X_b)I\{|g(X_k)| > N^\alpha\}\} + O(N^{-3\alpha}), \end{aligned} \tag{50}$$

as $N \to \infty$ uniformly in a, b and t. Also using Hölder's and Markov's inequalities, we observe that

$$\begin{aligned} &E|[Y_j - g(X_j)]Z_k\psi_{a,b}(X_a, X_b)I\{|g(X_j)| > N^\alpha\}| \\ &\leq \{E[Y_j - g(X_j)]^4\}^{1/4}(EZ_k^4)^{1/4}[E|\psi_{a,b}(X_a, X_b)|^\gamma]^{1/\gamma} \\ &\quad \times P[|g(X_j)| > N^\alpha]^{(1/2)-(1/\gamma)} \\ &= O(N^{-2\alpha(1-2/\gamma)}), \end{aligned}$$

and similarly,

$$\begin{aligned} &E|[Y_k - g(X_k)]g(X_j)\psi_{a,b}(X_a, X_b)I\{|g(X_k)| > N^\alpha\}| \\ &= O(N^{-2\alpha(1-2/\gamma)}), \end{aligned}$$

as $N \to \infty$ uniformly over $1 \leq j < k \leq N$ and $1 \leq a < b \leq N$. Hence we conclude from (50) that

$$\begin{aligned} &E[Z_jZ_k\psi_{a,b}(X_a, X_b) - g(X_j)g(X_k)\psi_{a,b}(X_a, X_b)] \\ &= O(N^{-3\alpha} + N^{-2\alpha(1-2/\gamma)}), \end{aligned} \tag{51}$$

as $N \to \infty$ uniformly over $1 \leq j < k \leq N$ and $1 \leq a < b \leq N$. Thus it follows from (8), (17), (51) and Lemma 1 that for sufficiently large K,

$$Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)R_aR_b$$

$$= -it^3\sigma_g^{-3}N^{-5/2}\sum_{j=(a-u)\vee 1}^{a+u}\sum_{k=b-u}^{(b+u)\wedge N} Eg(X_j)\psi_{a,b}(X_a,X_b)g(X_k)$$
$$+o(|t|^3N^{-5/2})$$
$$= -it^3\sigma_g^{-3}N^{-5/2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1,X_1')g(X_k')$$
$$+o(|t|^3N^{-5/2}), \tag{52}$$

as $N \to \infty$ uniformly in a, b and t. Furthermore it follows from (20) and Lemma 1 that for sufficiently large K,

$$E|t\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)[(e^{R_a}-1-R_a)(e^{R_b}-1-R_b)$$
$$+R_a(e^{R_b}-1-R_b)+R_b(e^{R_a}-1-R_a)]|$$
$$\leq 6E|t\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)R_aR_b^{3/2}| + 2E|t\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)R_bR_a^{3/2}|$$
$$\leq 6|t|\hat{\sigma}_N^{-1}[E|\psi_{a,b}(X_a,X_b)|^2]^{1/2}[(E|R_a|^2)^{1/2}(E|R_b|^3)^{1/2}$$
$$+(E|R_b|^2)^{1/2}(E|R_a|^3)^{1/2}] + O(|t|^{7/2}N^{-11/4})$$
$$= O(|t|^{7/2}N^{-11/4}\log^{5/2}N), \tag{53}$$

as $N \to \infty$ uniformly in a, b and t. Finally with repeated use of Lemma 1, we get for sufficiently large K,

$$\sum_{a=1}^{N-3u-1}\sum_{b=a+3u+1}^{N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)[e^{it(S_{a,b}^{(0)}-S_{a,b}^{(1)})}-1]e^{itS_{a,b}^{(2)}}$$
$$= \sum_{a=1}^{N-3u-1}\sum_{b=a+3u+1}^{N} it\hat{\sigma}_N^{-1}\{E\psi_{a,b}(X_a,X_b)[e^{it(S_{a,b}^{(0)}-S_{a,b}^{(1)})}-1]\}$$
$$\times(Ee^{itS_{a,b}^{(2)}}) + O(|t|N^{-3/2}),$$

as $N \to \infty$ uniformly in t. Thus we conclude from (49), (52), (53) and Lemma 3 that

$$\sum_{a=1}^{N-3u-1}\sum_{b=a+3u+1}^{N} Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a,X_b)[e^{it(S_{a,b}^{(0)}-S_{a,b}^{(1)})}-1]e^{itS_{a,b}^{(2)}}$$
$$= -\frac{i}{2}e^{-t^2/2}t^3\sigma_g^{-3}N^{-1/2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} Eg(X_j)\psi(X_1,X_1')g(X_k')$$
$$+O[(|t|+t^4)N^{-1}\log N] + o[|t|\mathcal{P}(|t|)e^{-t^2/2}N^{-1/2}],$$

as $N \to \infty$ uniformly over $|t| \leq N^{\epsilon}$, where $\mathcal{P}(|t|)$ is a linear combination [not depending on N] of non-negative powers of $|t|$. □

Lemma 5 *With the notation of Proposition 1, we have for sufficiently large K,*

$$\sum_{a=1}^{N-3u-1} \sum_{b=a+3u+1}^{N} |Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b) \prod_{l=1}^{2}[e^{it(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})} - 1]e^{itS_{a,b}^{(2)}}| = O[(|t| + t^4)N^{-1}\log^3 N],$$

as $N \to \infty$ uniformly in t.

Proof. Suppose that $1 \leq a < b \leq N$ such that $b - a > 3u$. Let R_a and R_b be defined as in (48). By repeated use of Lemma 1, we observe that for sufficiently large K,

$$\begin{aligned} & |Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b) \prod_{l=1}^{2}[e^{it(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})} - 1]e^{itS_{a,b}^{(2)}}| \\ = \; & |Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)[(e^{R_a} - 1 - R_a)(e^{R_b} - 1 - R_b) \\ & +R_a(e^{R_b} - 1 - R_b) + R_b(e^{R_a} - 1 - R_a) \\ & +R_aR_b][e^{it(S_{a,b}^{(1)}-S_{a,b}^{(2)})} - 1]e^{itS_{a,b}^{(2)}}| + O[(|t| + t^2)N^{-3}] \\ \leq \; & 9E|t\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b)R_aR_b[e^{it(S_{a,b}^{(1)}-S_{a,b}^{(2)})} - 1]| + O[(|t| + t^2)N^{-3}], \end{aligned} \tag{54}$$

as $N \to \infty$ uniformly over a, b and t. The last inequality uses (20). By Hölder's inequality and Lemma 1, we observe that for sufficiently large K the r.h.s. of (54) is less than or equal to

$$\begin{aligned} & 9|t|\hat{\sigma}_N^{-1}\{E|\psi_{a,b}(X_a, X_b)[e^{it(S_{a,b}^{(1)}-S_{a,b}^{(2)})} - 1]|^2\}^{1/2} \\ & \times(E|R_aR_b|^2)^{1/2} + O[(|t| + t^2)N^{-3}] \\ \leq \; & 9|t|\hat{\sigma}_N^{-1}[E|\psi_{a,b}(X_a, X_b)|^2]^{1/2}(E|R_aR_b|^2)^{1/2} \\ & \times[E|e^{it(S_{a,b}^{(1)}-S_{a,b}^{(2)})} - 1|^2]^{1/2} + O[(|t| + t^2)N^{-3}] \\ \leq \; & 9|t|\hat{\sigma}_N^{-1}[E|\psi_{a,b}(X_a, X_b)|^2]^{1/2}(E|R_aR_b|^2)^{1/2} \\ & \times[E|t(S_{a,b}^{(1)} - S_{a,b}^{(2)})|^2]^{1/2} + O[(|t| + t^2)N^{-3}], \end{aligned} \tag{55}$$

as $N \to \infty$ uniformly in a, b and t. Since

$$[E|t(S_{a,b}^{(1)} - S_{a,b}^{(2)})|^2]^{1/2} = O(|t|N^{-1/2}\log N),$$

as $N \to \infty$ uniformly in a, b and t, it follows from (54) and (55) that

$$\begin{aligned} &\sum_{a=1}^{N-3u-1} \sum_{b=a+3u+1}^{N} |Eit\hat{\sigma}_N^{-1}\psi_{a,b}(X_a, X_b) \prod_{l=1}^{2}[e^{it(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})} - 1]e^{itS_{a,b}^{(2)}}| \\ = \;& O[(|t| + t^4)N^{-1}\log^3 N], \end{aligned}$$

as $N \to \infty$ uniformly in t. This proves Lemma 5. □

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