RECENT DEVELOPMENTS IN BAYESIAN SEQUENTIAL RELIABILITY DEMONSTRATION TESTING *

by

Dongchu Sun and James O. Berger
The University of Michigan and Purdue University

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Abstract

This paper begins with general background and motivation of the Bayesian Sequential Reliability Demonstration Test (BSRDT). Two different approaches to the BSRDT, based on posterior loss and predictive loss are presented. Three testing models are considered. Various risks and features are investigated. Detailed examination for the Weibull and related distributions are developed and summarized.

Key words and Phrases. Reliability demonstration test, Bayesian analysis, predictive distribution, exponential distribution, Weibull distribution, expected stopping time, producer's risk, consumer's risk.

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1 Motivation of BSRDT Plans

1.1 Introduction

In order to produce a new product, engineers should carry out a series of experiments, which consists of prototype, development, qualification and reliability demonstration. Reliability demonstration is the last and a necessary step in the experimentation period. Reliability Demonstration Testing (RDT) is often used for the purpose of verifying whether a specified reliability has been achieved in a newly designed product. Based on a demonstration test, a decision is made to either accept the design and start formal production, or reject the design and send the product back for reengineering.

A serious problem with RDT is that a reliability test can be very expensive in terms of money and time, especially in the case of products that require very high reliability and have a long lifetime. A common solution is to take into consideration prior information, typically from engineering knowledge or knowledge of previous similar products, and to test in a sequential fashion.

There are several issues in the reliability demonstration. What are the suitable distributions of lifetimes for a product? How should we use the engineer's knowledge and knowledge of previous similar products? How many units need to be tested to reach a decision? How long does it take to make a decision? What are expected losses? What are reasonable testing procedures which are acceptable to both producers and consumers? These questions will be discussed in this paper.

1.2 History of BSRDT

It is well known that a Bayesian Sequential Test (BST) can be a useful tool, since it can take into consideration prior information, loss structure and the cost of testing. In many cases, use of a BST can significantly reduce the amount of testing required. However, BST's are often too complex to evaluate.

Several authors have thus proposed simple Bayesian Sequential Reliability Demonstration Tests (BSRDT). These simpler tests ignore any loss structure or costs of conducting the test. They are based solely on the way in which consecutive observations of failures modify the prior information on the parameter of interest to produce a posterior distribution for the parameter of interest. Testing continues until the posterior distribution is decisive, according to appropriate criterion, at which stage it terminates and a decision is taken on the quality of the product.

The first BSRDT was introduced by Schafer & Singpurwalla (1970). They introduced the following test procedure. One unit at a time is tested, where the lifetimes of units are independently identically exponentially distributed with mean θ. The unknown θ is assumed to have an inverse Gamma prior distribution. Choose a minimum acceptable value, say θ_1, and let $P_n = P(\theta \geq \theta_1|\text{data})$. The test is terminated when $P_n \geq 1 - \alpha_2$, in which case a decision to accept the product is made, or when $P_n \leq \alpha_1$, in which case a decision to reject the product is made. Schafer & Singpurwalla (1970) were primarily concerned with the acceptance probability of this procedure, and developed approximations for it. Some related computations and approximations for other risks were done in Schafer and Sheffield (1971) and Mann, Shafer and Singpurwalla (1974). The extreme difficulty of all computations of this type is discussed in Martz and Waller (1982); one of the major motivations of our recent work is to show how such computations can be done explicitly, in closed form.

The stopping rule of Schafer & Singpurwalla (1970) is discrete, in the sense that one can only stop the test when a failure occurs. This can be inefficient when observations are very expensive and/or have long lifetimes. Barnett (1972) proposed a continuous BSRDT plan for the exponential failure rate problem. By his method, one can stop the test at any time that enough information has accumulated. Again, however, closed form answers were not obtained.

Related work can be found in Chandra and Singpurwalla (1981), Epstein and Sobel (1953), Goel and Coppola (1979), Harris and Singpurwalla (1968, 1969), Lindley and Singpurwalla (1991a,b), MacFarland (1971), Martz and Waller (1979), Montagne and Singpurwalla (1984), Ray (1965), and Soland (1969).

1.3 Preview

Recently, two different approaches to the BSRDT have been considered for our purposes (see Sun and Berger (1991) and Sun (1991)). One of them, which is still called the BSRDT, is stimulated by the work of Barnett (1972). Testing continues until the posterior loss is decisive according to a desired criterion, at which time testing terminates and a decision made concerning the quality of the product. The other, the PSRDT (Predictive Sequential Reliability Demonstration Tests), is based on the predictive loss of future products. Those two procedures and their discrete versions will be described in Section 2.

In Section 3, two common testing models, with replacement and without replacement, and their generalization, the stepwise model, are considered. In Section 4, various risk criteria and other important features for the BSRDT and the PSRDT are presented. In Section 5, details about the Weibull and related distributions are developed and summarized.

2 Stopping and Decision Rules

Suppose that the underliying distribution of the new product is characterized by an unknown parameter $\theta (> 0)$. Larger θ provide larger mean time to failure, and θ is assumed to have a prior distribution $\pi(\theta)$.

2.1 The BSRDT

There are a variety of possible goals for sequential experimentation. The following BSRDT plan is the intersection of two goals.

1. Let θ_1 be the goal to begin production, in the sense that the experiment will stop and production begin if there is $100(1-\alpha_1)\%$ "confidence" that $\theta > \theta_1$.

2. Let θ_2 be the mature product goal , in the sense that the experiment will stop and the product will be rejected (sent back for reengineering) if there is $100(1-\alpha_2)\%$ "confidence" that $\theta < \theta_2$.

Here α_1 and α_2 are two usually somewhat small numbers, and $\theta_1 < \theta_2$ are two prespecified values. The region $\theta_1 < \theta < \theta_2$ is often called the indifference region.

The BSRDT also arises in formal decision models. Suppose that a product with small $\theta < (\theta_1)$ should be rejected and with large $\theta > (\theta_2)$ should be accepted. Let $l(\theta)$ be the loss for making a wrong decision, where $l(\theta)$ is nonincreasing and nondecreasing in $(0, \theta_1]$ and $[\theta_2, \infty)$, respectively, and $l(\theta) = 0$ for $\theta \in (\theta_1, \theta_2)$. The test will stop and production begin if the posterior loss of accepting the product ($\int_0^{\theta_1} l(\theta)\pi(\theta|\text{data})d\theta$) is small enough, and the test will stop and the product be rejected if the posterior loss of rejecting the product ($\int_{\theta_2}^{\infty} l(\theta)\pi(\theta|\text{data})d\theta$) is small enough. The BSRDT arises if $l(\theta)$ is constant on both $(0, \theta_1]$ and $[\theta_2, \infty)$.

It is easy to see that the testing plan is equivalent to

$$\begin{cases} \text{Stop and accept the product,} & \text{if } q^*(\alpha_1) > \theta_1; \\ \text{Stop and reject the product,} & \text{if } q^*(1-\alpha_2) \leq \theta_2, \\ \text{Continue testing,} & \text{otherwise,} \end{cases} \tag{1}$$

where $q^*(\alpha)$ is the α^{th} posterior quantile. More details about the BSRDT can be found from Sun and Berger (1991).

2.2 The PSRDT

For the BSRDT plans, inference is made based on the information about the unknown parameter in the distribution of time to failure. This is understandable and acceptable to an engineer. But from the viewpoint of a manager, it might be more natural to consider the time to failure of a future product. In fact, the purpose of reliability demonstration testing is to make a decision about a future product. Basically, it is a predictive problem. Thus we also consider an alternative sequential procedure, the PSRDT, which makes inference based on the predictive distribution of a new time to failure Z, given the current data. The idea is as follows. Choose the desired lifetime, t_2, and the minimum lifetime to begin production, t_1. A product will be rejected (accepted) if its time to fail is less then t_1 (more then t_2). Let $l(\cdot)$ be the loss function of making a wrong decision, where $l(z)$ is nondecreasing and nonincreasing in $(0, t_1]$ and $[t_2, \infty)$, respectively, and $l(z) = 0$ for $z \in (t_1, t_2)$. The plan is the intersection of following two goals:

1. The experiment will stop and production begin at time t if the predictive loss of accepting the product ($\int_0^{t_1} l(z)f(z|\text{data at t})dz$) $< \alpha_1$.

2. The experiment will stop and product will be rejected at time t (sent back for reengineering) if the predictive loss of rejecting the product ($\int_{t_2}^{\infty} l(z)f(z|\text{data at t})dz$) $< \alpha_2$.

Here α_1 and α_2 are two usually somewhat small numbers, and $f(z|\text{data at t})$ is the predictive p.d.f. For the 0-1 loss, the predictive BSRDT plan is equivalent to

$$\begin{cases} \text{Stop and accept the product at t,} & \text{if } P(Z < t_1|\text{data at t}) < \alpha_1, \\ \text{Stop and reject the product at t,} & \text{if } P(Z > t_2|\text{data at t}) < \alpha_2. \end{cases} \tag{2}$$

The BSRDT and the PSRDT describe above are actually using continuous stopping times. In contrast, there are discrete versions, the discrete BSRDT and the discrete PSRDT, in which the two design goals are examined only at each failure.

3 Testing Models

We have considered three testing models, with replacement, without replacement and stepwise models.

- *With Replacement.* Units are independently tested on m machines. Whenever a unit fails, it is replaced by a new unit and testing is continued until enough information has been obtained. This model includes the case in which m machines are tested themselves and, upon failure, a machine is repaired or rebuilt (immediately) so that the repaired machine is as good as new.

- *Without Replacement.* Units are independently tested on m machines without repair.

- *Stepwise Models.* Suppose that, at time 0, m (≥ 1) machines are used to test the items with replacement. If no decision is made at time $X_1(> 0)$, $m_1(\geq -1)$ more machines are put on test with replacement. Generally, if no decision is made at time X_j ($> X_{j-1}$), $m_j(\geq 0)$ more machines are put on test with replacement. The $X_j's$ are called the evaluation points.

The stepwise model generalizes the first two models. If $m_j = 0$ ($j \geq 0$) then the stepwise model is the model with replacement. If $m_j = -1(1 \leq j \leq m)$ and $m_j = 0(j \geq m)$ then the stepwise model is the model without replacement.

Note that one can "accept" when accumulated nonfailure time is large enough, and this could happen at any time, but rejection takes place only on the occurrence of a failure. Let

$T_1 \leq T_2 \leq \cdots \leq T_n$ be the first n ordered failure times for all the machines. A natural choice of evaluation points is

$$X_j = T_j, \; i = 1, 2, \cdots. \tag{3}$$

This is reasonable, since we would typically consider more machines when the failure time has been too long to reject, but not long enough to accept.

4 Features of the BSRDT and the PSRDT

4.1 Risks

Let $\mathcal{A}$ and $\mathcal{R}$ denote the action (or, by an abuse of notation, the region) of accepting the product and the action of rejecting the product, respectively. Several risk criteria, defined in Chapter 10 of Martz and Waller (1982), can be used to measure the goodness of the BSRDT. The following names of these risks are borrowed from related conventions in quality control.

1. Classical Producer's Risk, $\gamma = P(\mathcal{R}|\theta_2)$, and Classical Consumer's Risk, $\delta = P(\mathcal{A}|\theta_1)$. Here γ is the probability that a product at the mature product goal will fail the BSRDT and δ is the probability that a product at the goal to begin production will pass the BSRDT. If the lifetime distribution has a monotone likelihood ratio, $P(\mathcal{A}|\theta)$ is monotonically increasing in θ. In this case, $P(\mathcal{R}|\theta) < \gamma$ for $\theta > \theta_2$, and $P(\mathcal{A}|\theta) < \delta$ for $\theta < \theta_1$.

2. Average Producer's Risk, $\tilde{\gamma} = P(\mathcal{R}|\theta \geq \theta_2)$, and Average Consumer's Risk, $\tilde{\delta} = P(\mathcal{A}|\theta \leq \theta_1)$. Here $\tilde{\gamma}$ is the probability of rejecting a good product and $\tilde{\delta}$ is the probability of accepting a bad product. Note that computation of these risks involves the prior.

3. Posterior Producer's Risk, $\gamma^* = P(\theta \geq \theta_2|\mathcal{R})$, and Posterior Consumer's Risk, $\delta^* = P(\theta \leq \theta_1|\mathcal{A})$. Here γ^* is the posterior probability that a rejected product is good, and δ^* is the posterior that an accepted product is bad.

4. Rejection probability, $P(\mathcal{R}) = \int_\Theta P(\mathcal{R}|\theta)\pi(\theta)d\theta$, and Acceptance probability, $P(\mathcal{A}) = 1 - P(\mathcal{R})$. Here $P(\mathcal{A})$ is the unconditional probability of the product passing the BSRDT.

Rejection probability and Acceptance probability can also be used for the PSRDT. The choice of criteria to evaluate the BSRDT is left to the user. For the fixed sample size problem, many papers are available concerning how to choose the criteria. For example, Balaban (1975) favors the mixed classical/Bayesian pair (γ, δ^*) to determine a Bayesian reliability demonstration test. Also see Easterling (1970), Schafer and Sheffield (1971), Schick and Drnas (1972), Goel and Joglekar (1976).

4.2 Other Design Criteria

Besides various risks, the following features are also important to design:

1. N_{TU}, the total sample size or the total number of testing units put on test;
2. T, the total time on test;
3. $N = N(T)$, the number of failed units at the time when testing stops.

Finding the expected stopping time, $E(T) = \int E(T|\theta)\pi(\theta)d\theta$, and the expected sample size, $E(N_{TU}) = \int E(N_{TU}|\theta)\pi(\theta)d\theta$, is important for design. The following relationship between the total sample size and the number of failures follows immediately from the definition of the stopping rule, and allows us to consider N instead of N_{TU}.

Theorem 4.1 Consider the stepwise model. For both the BSRDT and the PSRDT,

$$N_{TU} = m + \sum_{j=0}^{N-I(\mathcal{R})} (1 + m_j). \tag{4}$$

Here $I(\cdot)$ is the indicator function. $\square$

Remark 4.1 1. Note that, for the model with replacement, $N_{TU} = m + N - I(\mathcal{R})$, and for the model without replacement, $N_{TU} = m - N + I(\mathcal{R})$.

2. For the stepwise model and the discrete versions of the BSRDT and PSRDT,

$$N_{TU} = m_0 + \sum_{j=0}^{N-1} (1 + m_j). \tag{5}$$

5 Details for the Weibull and Related Distributions

5.1 Failure Distributions

In this section, it is assumed that the lifetime probability density function of the product has the following form

$$f(x|\theta) = \frac{H'(x)}{Q(\theta)} \exp\left\{-\frac{H(x)}{Q(\theta)}\right\}, \quad t > 0. \tag{6}$$

Here $H(\cdot)$ is a known increasing function satisfying $H(0^+) = 0$ and $\lim_{x\to\infty} H(x) = \infty$, $Q(\cdot)$ is a known and strictly increasing function, and θ is the unknown characteristic life. The density of (6) is a special form of the exponential family and encompasses many common reliability distributions.

Example 5.1 If, in (6), $Q(x) \equiv H(x) = x^\beta, (x > 0)$ for some known positive constant β, the density becomes

$$f(x|\theta) = \frac{\beta x^{\beta-1}}{\theta^\beta} \exp\Big\{-\Big(\frac{x}{\theta}\Big)^\beta\Big\}, \quad t > 0, \tag{7}$$

which is the p.d.f. of the **Weibull distribution**, $\mathcal{W}(\theta,\beta)$. It is well known that the Weibull distribution encompasses both increasing (with $\beta > 1$) and decreasing (with $\beta < 1$) hazard rates, and has been successfully used to describe both initial failures and wearout failure (Von Alven (1964) and Lieblein & Zelen (1956)). It has been argued that when a system is composed of a number of components and failure is due to the most severe defect of a large number of possible defects, the Weibull distribution is often especially appropriate. It is found in practice to be suitable for data on failure strengths and also failure times. It is also one of the stable distributions of extreme value theory. The exponential and Raleigh distributions are obtained when $\beta = 1$ and 2, respectively.

Example 5.2 Assume that, for given θ, X_1, X_2 are independent random variables and X_i has reliability function $\exp\{-t^{\beta_i}/\theta^\beta\}$, where β, β_1, and β_2 are known positive constants. Then $\min\{X_1, X_2\}$ has the p.d.f.

$$f(x|\theta) = \frac{\beta_1 x^{\beta_1-1} + \beta_2 x^{\beta_2-1}}{\theta^\beta} \exp\Big\{-\frac{x^{\beta_1} + x^{\beta_2}}{\theta^\beta}\Big\}, \quad t > 0, \tag{8}$$

which is a special case of (6).

Example 5.3 Assume that, for given θ, $\{X_i\}_{i\geq 1}$ is a sequence of independent random variables and X_i has reliability function $\exp\Big\{-x^{\beta_1}/(i!\theta^\beta)\Big\}$, where β_1 and β are known positive constants. Then $\inf_{n\geq 1} X_n$ has the p.d.f.

$$f(x|\theta) = \frac{\beta_1 e^{\beta_1 x}}{\theta^\beta} \exp\Big\{-(e^{\beta_1 x} - 1)/\theta^\beta\Big\}, \quad t > 0, \tag{9}$$

which is the truncated extreme value distribution, and again a special case of (6).

Example 5.4 Let $H(x) = \ln(x+1)\,(x>0)$ in (6). Then the lifetimes $t_{ij},\ i = 1, 2, \cdots, m,$ $j = 1, 2, \cdots,$ are *iid*. Pareto distributions with p.d.f.

$$f(x|\theta) = 1\Big/\Big[Q(\theta)(x+1)^{\frac{1}{Q(\theta)}+1}\Big],\quad t > 0. \tag{10}$$

From Billingsley (1986) ((21.9) on Page 282), $E(X) = \int_0^\infty P(X > x)dx$. It follows from the assumptions on $Q(\cdot)$ and $H(\cdot)$ that larger θ provide larger expected lifetime.

5.2 Prior, Posterior and Predictive Distributions

Prior information about the unknown parameter θ is assumed available in the form of a prior density function $\pi(\cdot)$. Schafer (1969) and Schafer and Sheffield (1971) observed that the inverse Gamma prior distributions are often reasonable for exponential failure problems. Here, the prior p.d.f. of θ for the family (6) will be assumed to belong to the conjugate family

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\frac{Q'(\theta)}{Q^{a+1}(\theta)}\ \exp\Big\{-\frac{b}{Q(\theta)}\Big\},\ \text{for } \theta > 0. \tag{11}$$

Note that then $Q(\theta)$ has an inverse gamma distribution, $\mathcal{IG}(a, b)$. Methods of choosing a and b can be found from Sun and Berger (1991).

Let $N(t)$ be the total number of failures at time t and define the adjusted total time on test by

$$\begin{aligned} V(t) &= \sum_{t_j:\ \text{observed failure time before t}} H(t_j) \\ &+ \sum_{t_j^*:\ \text{observed test time not failed by t}} H(t_j^*). \end{aligned} \tag{12}$$

Then the posterior p.d.f. of θ given data at time t is

$$\pi(\theta|\text{data at t}) = \frac{(V(t)+b)^{N(t)+a}}{\Gamma(N(t)+a)}\ \frac{Q'(\theta)}{Q^{N(t)+a+1}(\theta)}\exp\Big\{-\frac{V(t)+b}{Q(\theta)}\Big\},$$

for $\theta > 0$, i.e., $Q(\theta)$ has, *a posteriori*, an inverse gamma distribution, $\mathcal{IG}(N(t)+a, V(t)+b)$. In particular, it follows that the posterior α^{th} quantile is

$$q^*(\alpha) = Q^{-1}\Big(2(V(t)+b)/\chi^2_{2(N(t)+a)}(1-\alpha)\Big),\ \text{ for } 0 < \alpha < 1, \tag{13}$$

where $Q^{-1}(\cdot)$ is the inverse function of $Q(\cdot)$ and $\chi^2_j(1-\alpha)$ is the $(1-\alpha)^{th}$ quantile of the χ^2 distribution with j degrees of freedom.

Assume that Z, a future time to failure, is independent of current data. From Berger (1985) (page 157), the predictive survival function at time t is

$$\begin{aligned} P(Z > s|\text{data at t}) &= \int_0^\infty P(Z > s|\theta)\pi(\theta|\text{data at t})d\theta \\ &= \frac{(V(t)+b)^{N(t)+a}}{\Gamma(N(t)+a)} \int_0^\infty \frac{Q'(\theta)}{(Q(\theta))^{N(t)+a+1}} \exp\left\{-\frac{V(t)+H(s)+b}{Q(\theta)}\right\} d\theta \\ &= \left\{1 + \frac{H(s)}{V(t)+b}\right\}^{-N(t)-a}, \quad s > 0. \end{aligned}$$

This implies that the α^{th} quantile of the predictive distribution is

$$q^Z(\alpha) = H^{-1}\Big(\Big[V(t)+b\Big]\Big[(1-\alpha)^{-1/(N(t)+a)} - 1\Big]\Big), \quad for \;\; 0 < \alpha < 1, \tag{14}$$

where $H^{-1}(\cdot)$ is the inverse function of $H(\cdot)$.

5.3 Representations for the BSRDT and the PSRDT

The stopping (and decision) rules for the BSRDT and the PSRDT are equivalent to

$$\begin{cases} \text{Stop and accept the product at t,} & \text{if } V(t) + b > \frac{1}{2}Q(\theta_1)\chi^2_{2(N(t)+a)}(1-\alpha_1), \\ \text{Stop and reject the product at t,} & \text{if } V(t) + b \le \frac{1}{2}Q(\theta_2)\chi^2_{2(N(t)+a)}(\alpha_2). \end{cases} \tag{15}$$

and

$$\begin{cases} \text{Stop and accept the product at t,} & \text{if } V(t) + b > H(t_1)\Big/\Big\{(1-\alpha_1)^{-1/(N(t)+a)} - 1\Big\}, \\ \text{Stop and reject the product at t,} & \text{if } V(t) + b \le H(t_2)\Big/\Big\{\alpha_2^{-1/(N(t)+a)} - 1\Big\}. \end{cases} \tag{16}$$

respectively. As with certain classical sequential tests these procedures are "semicontinuous" (see Epstein and Sobel, 1955): one can "accept" when the continuous time of accumulated nonfailure is large enough, but can "reject" only on the (discrete) occurrence of a failure.

For $i = 0, 1, 2, \cdots$, let

$$c_i = \begin{cases} \frac{1}{2}Q(\theta_2)\chi^2_{2(a+i)}(\alpha_2), & \text{for the BSRDT,} \\ H(t_2)\Big/\Big\{\alpha_2^{-1/(a+i)} - 1\Big\}, & \text{for the PSRDT,} \end{cases} \tag{17}$$

and

$$d_i = \begin{cases} \frac{1}{2}Q(\theta_1)\chi^2_{2(a+i)}(1-\alpha_1), & \text{for the BSRDT,} \\ H(t_1)\Big/\Big\{(1-\alpha_1)^{-1/(a+i)} - 1\Big\}, & \text{for the PSRDT.} \end{cases} \tag{18}$$

Then the stopping (and decision) rules for the BSRDT and the PSRDT have the common form: Stop and accept (reject) the product at t, if $V(t) + b > d_{N(t)}$ ($\le c_{N(t)}$).

5.4 Summary about Evaluation of Important Features

Sun and Berger (1991) found closed form expressions for all the important features of the BSRDT defined in Section 4, for the model with replacement. Exact expressions for the expected sample size, various risks and the distribution of total number of failures were determined. Bounds on expected testing time were given. Based on a result about a Poisson process, the exact expected testing time for exponential failure was also found. Included in these risks and expected stopping times were frequentist versions, thereof, so that the results also provided frequentist answers for a class of interesting stopping rules.

For the discrete BSRDT, Sun (1991) found the exact formulas for the various risks, and the distributions of the number of failures and sample size. If only one unit at a time is tested, the expected testing time has a closed form. Some asymptotic properties of the discrete BSRDT plan were also discussed. For a general loss, the discrete BSRDT is asymptotically equivalent to the Bayes sequential test when the cost per observation is small enough.

For the stepwise model, Sun (1991) gave closed form expressions for all the risks, the expected sample size, and the distribution of total number of failures for the BSRDT and the PSRDT.

Appendix

Here are summaries of exact form expressions which are true for both the BSRDT and the PSRDT and all the three models. Their proofs can be found in Sun and Berger (1991) and Sun(1991). We use the notation and assumptions of Section 5.

A1. Technical Preliminaries

Since both α_1 and α_2 are small for the BSRDT and the PSRDT, it can be assumed that $\alpha_2 < 1 - \alpha_1$. Then for c_i and d_i defined by (17) and (18), respectively, it can be shown that $c_i \geq d_{i-1}$ when i is large enough, as long as $\theta_1 < \theta_2$ for the BSRDT or $t_1 < t_2$ for the PSRDT. Thus there is an i_0 such that

$$i_0 = \min\{i = 1, 2, ... : c_i \geq d_{i-1}\}. \tag{19}$$

For $n = 1, \cdots, i_0$, let

$$G_n = \{(y_1, \cdots, y_n) : y_j > 0, c_j - b < y_1 + \cdots + y_j \leq d_{j-1} - b, j = 1, \cdots, n\}, \tag{20}$$

let $||G_0|| = 1$ and let $||G_n||$ denote the volume or Lebesgue measure of G_n $(n \geq 1)$. For $j = 1, 2, \cdots, i = 0, 1, 2, \cdots$, and $y \geq 0$, define

$$a_{ij}(y) = \begin{cases} (c_{i+j} \wedge d_{j-1}) \vee y - (c_j \vee y), & \text{if } i \geq 1, \\ d_{j-1} - c_j \vee y, & \text{if } i = 0, \end{cases} \tag{21}$$

where $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$. Then

$$a_{ij}(c_j) = \begin{cases} c_{i+j} \wedge d_{j-1} - c_j, & \text{if } i \geq 1, \\ d_{j-1} - c_j, & \text{if } i = 0. \end{cases} \tag{22}$$

For $n \geq 2$, define two sets of partitions of n by

$$\Psi_n = \left\{(i_1, \cdots, i_k) : \sum_{j=1}^{k} i_j = n, \ k \geq 1, \ i_j \geq 1\right\} \tag{23}$$

and

$$\Psi_n^* = \left\{(i_1, \cdots, i_k) : (i_1, \cdots, i_k) \in \Psi_n, i_k \geq 2\right\}. \tag{24}$$

For $1 \leq r \leq n, (i_1, \cdots, i_k) \in \Psi_n$, define

$$\begin{aligned} \rho_1(b; n; r; i_1, \cdots, i_k) &\equiv \rho_1(b; n; r; i_1, \cdots, i_k; c_j, d_{j-1}, 1 \leq j \leq n) \\ &= \begin{cases} \dfrac{a_{n-r,r}^{r}(b)}{r!}, & \text{if } k = 1, \\ \dfrac{a_{n-r,r}^{i_1}(c_r)}{\prod_{j=1}^{k} i_j!} \left\{\prod_{j=2}^{k-1} a_{r,j}^*(c_{r,j}^*)\right\} a_{r,k}^*(b), & \text{if } k \geq 2, \end{cases} \end{aligned} \tag{25}$$

and for $2 \leq l \leq r \leq n, (i_1, \cdots, i_k) \in \Psi_l^*$,

$$\begin{aligned} \rho_2(b; n; r; l; i_1, \cdots, i_k) &\equiv \rho_2(b; n; r; l; i_1, \cdots, i_k; c_j, d_{j-1}, 1 \leq j \leq n) \\ &= \begin{cases} \dfrac{a_{n-r,r}^{l}(d_{r-l})}{l!} ||G_{r-l}||, & \text{if } k = 1, \\ \dfrac{a_{n-r,r}^{i_1}(c_r)}{\prod_{j=1}^{k} i_j!} \left\{\prod_{j=2}^{k-1} a_{r,j}^*(c_{r,j}^*)\right\} a_{r,k}^*(d_{r-l}) ||G_{r-l}||, & \text{if } k \geq 2, \end{cases} \end{aligned} \tag{26}$$

where $i_0 = 0$, $i_{(j)} = i_0 + i_1 + \cdots + i_j$, $a_{r,j}^*(\cdot) = a_{i_{j-1}, r-i_{(j)}}^{i_j}(\cdot)$, $a_{ij}(\cdot)$ is given by (21), $c_{n,j}^* = c_{n-i_{(j)}}$, and $\prod_{j=2}^{1} \cdot = 1$. For $2 \leq n \leq i_0$ and $1 \leq r \leq n$, let

$$\xi_{n,r} = \sum_{(i_1, \cdots, i_k) \in \Psi_r} \rho_1(b; n; r; i_1, \cdots, i_k) - \sum_{l=2}^{r} \sum_{(i_1, \cdots, i_k) \in \Psi_l^*} \rho_2(b; n; r; l; i_1, \cdots, i_k). \tag{27}$$

Lemma A1 The volume of G_n is $||G_n|| = \xi_{n,n}$ $(n \geq 1)$.

A2. Exact form Expressions

Theorem A1 The rejection probability for given θ is

$$P(\mathcal{R} \mid \theta) = 1 - P(\mathcal{A} \mid \theta) = \sum_{n=0}^{i_0-1} \frac{\|G_n\|}{Q^n(\theta)} \exp\Big\{-\frac{d_n - b}{Q(\theta)}\Big\}, \ \theta > 0,$$

and the unconditional rejection probability is

$$P(\mathcal{R}) = 1 - P(\mathcal{A}) = 1 - \sum_{n=0}^{i_0-1} \frac{b^a \Gamma(a+n)}{d_n^{a+n}\Gamma(a)} \|G_n\|.$$

Theorem A2 The cumulative distribution function of N for given θ is

$$P(N \le n|\theta) = \begin{cases} \exp\big\{-\frac{d_0 - b}{Q(\theta)}\big\}, & \text{if } n = 0, \\ 1 - J_{\theta,n} + \sum\limits_{i=n-1}^{n} \frac{\|G_i\|}{Q^i(\theta)} \exp\Big\{-\frac{d_i - b}{Q(\theta)}\Big\}, & \text{if } 1 \le n < i_0; \end{cases} \tag{28}$$

the expected number of failed units for given θ is

$$E(N \mid \theta) = \sum_{n=0}^{i_0-1} J_{\theta,n} - 2\sum_{n=0}^{i_0-2} \frac{\|G_n\|}{Q^n(\theta)} \exp\Big\{-\frac{d_n - b}{Q(\theta)}\Big\} - \frac{\|G_{i_0-1}\|}{Q^{i_0-1}(\theta)} \exp\Big\{-\frac{d_{i_0-1} - b}{Q(\theta)}\Big\}; \tag{29}$$

the marginal cumulative distribution of N is

$$P(N \le n) = \begin{cases} \left(\frac{b}{d_0}\right)^a, & \text{if } n = 0, \\ 1 - J_n + \sum\limits_{i=n-1}^{n} \|G_i\| \frac{b^a \Gamma(a+i)}{d_i^{a+i}\Gamma(a)}, & \text{if } 1 \le n < i_0; \end{cases} \tag{30}$$

and the expected number of failed units is

$$E(N) = \sum_{n=0}^{i_0-1} J_n - 2\sum_{n=0}^{i_0-2} \|G_n\| \frac{b^a \Gamma(a+n)}{d_n^{a+n}\Gamma(a)} - \|G_{i_0-1}\| \frac{b^a \Gamma(a+i_0-1)}{d_{i_0-1}^{a+i_0-1}\Gamma(a)}, \tag{31}$$

where $\sum_0^{-1} \cdot = 0$, $J_{\theta,0} = J_0 = 1$,

$$J_{\theta,n} = \exp\Big\{-\frac{c_n \vee b - b}{Q(\theta)}\Big\} - \sum_{i=0}^{n-2} \frac{\exp\{-\frac{c_n \vee d_i - b}{Q(\theta)}\}}{Q^i(\theta)} \|G_i\| + \exp\Big\{-\frac{c_n - b}{Q(\theta)}\Big\} \sum_{r=1}^{n-1} \frac{\xi_{n,r}}{Q^r(\theta)}, \tag{32}$$

and

$$J_n = \frac{b^a}{(c_n \vee b)^a} - \sum_{i=0}^{n-2} \frac{b^a \Gamma(a+i)\|G_i\|}{(c_n \vee d_i)^{a+i}\Gamma(a)} + \sum_{r=1}^{n-1} \frac{b^a \Gamma(a+r)}{c_n^{a+r}\Gamma(a)} \xi_{n,r}, \tag{33}$$

for $n \ge 1$.

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