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ABSTRACT

We consider a branching Brownian motion with killing which starts with a single particle at the origin, for which the instantaneous branching and killing rates of a particle at position x are $\beta(x)$ and $k(x)$ respectively. We show that if β is continuous and bounded and $k(x) \uparrow \infty$ as $|x| \to \infty$ then R_t, the right frontier at time t, grows sublinearly as $t \to \infty$. For the class of killing functions $k_\alpha(x) = |x|^\alpha, \ \alpha > 0$ we show that $R_t \overset{a.s.}{\sim} c_\alpha t^{2/\alpha+2}$ as $t \to \infty$, for some constant $c_\alpha > 0$.

1. Introduction

A branching Brownian motion is a stochastic process that describes the evolution of a system of particles that move through space and also reproduce and die. The process starts with a single particle at the origin. At time $t = 0$ the particle starts a standard Brownian motion in $\mathbf{R}^d$; as it moves it is subject to branching and killing, with instantaneous rates $\beta(x)$ and $k(x)$ respectively, depending on its position x. The motion continues until a random time, when the particle either dies or splits into two. If it dies, the process becomes extinct. Otherwise the particle and its offspring continue along independent Brownian paths subject to the same laws of splitting and killing.

Let τ and κ denote the times to branching and killing respectively. We assume that they satisfy the following laws:

$$P(\kappa > t | \sigma(B_s)_{0 \le s \le t}) = \exp\left\{-\int_0^t k(B_s)ds\right\}$$

$$P(\tau > t | \sigma(B_s)_{0 \le s \le t} \wedge \sigma(1\{\kappa > t\})) = \exp\left\{-\int_0^t \beta(B_s)ds\right\}$$

where 1_A denotes the indicator of the event A, and B denotes standard Brownian motion.

Under suitable conditions on the branching rate function β and the killing function k, the process does not explode, survives with positive probability, and on the event of survival, "spreads out" with probability 1.

If N_t is the number of particles in existence at time t, the state of the process at time t is completely described by their positions $Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(N_t)}$, when $N_t > 0$. A natural question that arises is: what area does the process cover by time t, and how fast is this area spreading? The answer clearly depends on the nature of β and k.

In one dimension, the question reduces to the growth of the interval (L_t, R_t), where L_t and R_t are respectively $\min_{1 \le i \le N_t} Y_t^{(i)}$ ("left-most" position) and $\max_{1 \le i \le N_t} Y_t^{(i)}$ ("right-most" position). This is the case that has been studied in detail, but for branching only, i.e. when particles reproduce but do not die. In particular, the growth of the right frontier R_t as $t \to \infty$ has been the subject of several papers. [1,2,5,12,13,14,16]

When $\beta(x) \equiv \beta_0$, a constant and $k(x) \equiv 0$, the function $u(t,x) = P(R_t < x)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta_0 u(1-u),$$
$$u(0,x) = 1(x > 0).$$

Kolmogorov, Petrovski and Piscounov [11], and independently, Fisher [6], originally studied this equation (the "KPP" equation) and showed that $u(t, m_t + x) \to w(x)$ for some function $w(x)$, with $m_t \sim at$ as $t \to \infty$ for some constant $a > 0$, m_t being the "median" of u. In other words, R_t grows linearly with time, and its distribution stabilizes to a travelling wave.

The connection of "homogeneous" branching Brownian motion with the KPP equation was established by McKean [16]. Bramson [1,2] and Lalley and Sellke [12] refined the above results by giving a precise characterization for m_t as $t \to \infty$ and an expression for $w(x)$ respectively.

When the branching rate function $\beta(x)$ is not constant, we no longer have a parabolic equation such as the above for $P(R_t < x)$. However, using probabilistic methods, Erickson [5] and Lalley and Sellke [13, 14] showed that R_t exhibits behavior similar to that for the "homogeneous" branching case mentioned above, for β belonging to certain subclasses of $bC(\mathbf{R})$.

We have studied the problem of R_t in one dimension, when the model incorporates killing as well as branching. We assume that $k(x)$, the killing rate, is continuous, and $\beta(x) \in bC(\mathbf{R})$. Depending on how k grows away from the origin, the frontier should grow at a comparable or slower rate than in the pure birth case (on the event of survival, assuming the process is supercritical).

If k is bounded everywhere, then under certain conditions on $\beta - k$, the process is supercritical, and it can be shown by methods similar to those in [13] that R_t grows linearly as $t \to \infty$ on the event of survival.

The more interesting case is when k grows without bound away from the origin (but is bounded at the origin). Suppose β is bounded. If the branching rate is large enough in a neighborhood of the origin, then the process will be supercritical. In this case, the movement of particles away from the origin is "discouraged" by the high rate of killing, and R_t grows sublinearly on the event of survival. The distribution of R_t then settles to a "degenerate" wave as $t \to \infty$. Obtaining the growth rate itself involves analyzing the trajectories of killed Brownian motion.

In this paper, we consider the latter case. First, we show that R_t grows sublinearly for any such k (subject to an additional restriction on k). Then we consider the special class of killing functions $k(x) = |x|^\alpha, \ \ \alpha > 0$ and obtain the growth rate of R_t explicitly.

Our main results are as follows.

THEOREM 1: Assume $\beta(x)$ has compact support and is bounded. Suppose that $k(x)$ is bounded at the origin, $k(x) \uparrow \infty$ as $|x| \to \infty$, $\ k'(x)/k(x) \to 0$ as $x \to \infty$, and that the maximum eigenvalue of the differential operator $\frac{1}{2}D_x^2 + \beta - k$ is $\lambda_0 > 0$. Then the branching Brownian motion with birth rate function β and killing rate function k is supercritical, and on the event that the process survives,

(a) $\overline{\lim_{t\to\infty}} R_t/x_t \le 1 \quad a.s.,$ where x_t is a deterministic function of t satisfying $\int_0^{x_t} \sqrt{2k(y)}dy = \lambda_0 t$, and

(b) $\lim_{t\to\infty} R_t/t = 0 \quad a.s.$ □

THEOREM 2: If $k(x) = |x|^\alpha$, $\alpha > 0$, then under the conditions of Theorem 1, $R_t \sim x_t$ *a.s.* as $t \to \infty$ on the event of survival of the process, where $x_t = \left(\frac{\alpha+2}{2\sqrt{2}}\lambda_0\right)^{2/\alpha+2} t^{2/\alpha+2}$ is the solution of $\int_0^{x_t} \sqrt{2k(y)}dy = \lambda_0 t$. □

2. A Heuristic Argument

Assuming the conditions of Theorem 1, the rate of growth of the right frontier is determined by the interaction of two factors: the exponential growth of the number of particles in any bounded area of space (guaranteed by a theorem of S. Watanabe [17]), and the probabilities associated with certain trajectories of individual particles.

The path of any particle is that of killed Brownian motion, i.e., if $X(t)$ denotes the position of a particle at time t, then $X(t)$ takes values in $\mathbf{R} \cup \{\Delta\}$ where Δ is the "cemetery" state (say, $\Delta = -\infty$). The law of X_t is given by

$$P^z(X_t \in A) = E^z\left(e^{-\int_0^t k(B_s)ds} 1(B_t \in A)\right)$$

for any Borel set A in $\mathbf{R}$, $z \in \mathbf{R}$ being the initial position of the particle.

Let $N_t(J)$ represent the number of particles in a subset J of $\mathbf{R}$. Let λ_0 be as in Theorem 1 and let φ_0 be the corresponding eigenfunction.

WATANABE'S THEOREM: Under the conditions of Theorem 1, on the event of survival of the process,

$$N_t(J) \sim Ze^{\lambda_0 t}\nu(J) \quad a.s.$$

as $t \to \infty$ for any bounded interval J, where $\nu(J) = \int_J \varphi_0(y)dy$, and $Z = \lim_{t\to\infty} Z_t$, with $Z_t = e^{-\lambda_0 t}\sum_{i=1}^{N_t} \varphi_0(Y_t^{(i)})$. □

Remarks:

(i) $\frac{1}{2}D_x^2 + \beta - k$ is the generator of the expectation semigroup $M_t f(x) = E^x \sum_{i=1}^{N_t} f(Y_t^{(i)})$, defined for $f \in bC(\mathbf{R})$.

(ii) If β is bounded and $k(y) \to \infty$ as $|y| \to \infty$ then $\frac{1}{2}D_x^2 + \beta - k$ has a discrete spectrum $\lambda_0 > \lambda_1 \geq \ldots$. If $\max_{y\in\mathbf{R}} \beta(y)$ is large enough, then $\lambda_0 > 0$ and φ_0, the leading eigenfunction is unique and strictly positive.

(iii) $Z > 0$ with probability 1. □

Using Watanabe's theorem, we give here a heuristic argument for obtaining the rate of growth of R_t, i.e. some deterministic function x_t of t such that $R_t \overset{P}{\sim} x_t$ as $t \to \infty$.

Note that the intensity of the point process of births in any small interval I at time t is $\int_I \beta(x)N_t(dx)$, and according to Watanabe's theorem, $N_t(I) \overset{a.s.}{\sim} Ze^{\lambda_0 t}\nu(I)$ as $t \to \infty$. From this it can be argued that at large time, conditional on $Z = C$, ($C > 0$ a constant), the point process of births in space-time behaves like a Poisson process with birth intensity measure $Ce^{\lambda_0 t}\beta(y)\nu(dy)dt$ (see Lalley and Sellke [13]). It follows that the point process of the positions of particles in existence at time t behaves like a Poisson process (and therefore, $N_t(J)$ like a Poisson random variable), as $t \to \infty$ with intensity measure proportional to $\int_0^t \int_{\mathbf{R}} e^{\lambda_0 s}\varphi_0(z)\beta(x)P^z(X_{t-s} \in dx)dzds$.

A system of Brownian particles born according to a Poisson process in space-time with such a birth intensity measure was defined to be a Poisson tidal wave (PTW) in [13]. We extend the definition to allow the particles to die with instantaneous rate $k(x)$ at position x.

We prove in Section 6 that when the branching Brownian motion (BBM) is supercritical, then, on the event of survival, and conditional on the value of Z, the right frontier at large time behaves like the right frontier of a PTW with the appropriate birth intensity function. Precisely, we have

PROPOSITION 1: Assume the conditions of Theorem 1, and let C be an arbitrary positive constant. Let $\mathcal{W}$ be a Poisson tidal wave with birth intensity measure $Ce^{\lambda_0 t}\beta(x)\nu(dx)dt$ and killing rate $k(x)$, where $(t,x) \in \mathbf{R}^2$. Let R_t^* denote the position of the right-most particle of $\mathcal{W}$ at time t. Then on some probability space may be constructed a copy of the branching Brownian motion with birth and killing rates $\beta(x)$ and $k(x)$ respectively, and the Poisson tidal wave $\mathcal{W}$ such that for all positive $\delta \leq \delta_0$, for some δ_0,

(a) $R_t \leq R_t^* + \delta$ on $\{Z < C/2\}$ eventually with probability 1;

(b) $R_t \geq R_t^* - \delta$ on $\{Z > 2C\}$ eventually with probability 1; and

(c) for all t, the histories of particles in $\mathcal{W}$ born after time t are independent of the histories of all particles in the branching Brownian motion, and $\mathcal{W}$ up to time t. $\square$

Thus the problem of determining the behavior of R_t reduces to the problem of determining that of the right frontier of a PTW.

Now, if R_t^* denotes the right frontier of a PTW with birth intensity measure $Ce^{\lambda_0 s}\mu(dz)ds$ where $\mu(dz) = \beta(z)\nu(dz)$, then

$$P(R_t^* < x) = e^{-EN_t^*[x,\infty)}$$

where $EN_t^*[x,\infty) = \int_{\mathbf{R}}\int_{-\infty}^t Ce^{\lambda_0 s}P^z(X_{t-s} \geq x)ds\mu(dz)$, $N_t^*(J)$ being the number of PTW particles in $J \subseteq \mathbf{R}$ at time t.

The problem of characterizing R_t^* as $t \to \infty$ then reduces to analyzing the above double integral. In particular, if there exists x_t such that $R_t^* \overset{P}{\sim} x_t$, then x_t should be such that for any $\varepsilon > 0$, as $t \to \infty$,

$$\begin{aligned} EN_t^*[x,\infty) &\sim e^{-at} && \text{if } x > (1+\varepsilon)x_t \\ &\sim e^{at} && \text{if } x < (1-\varepsilon)x_t \\ &= O(1) && \text{if } x = x_t, \end{aligned}$$

where $a > 0$ is a constant depending on ε.

Now if we assume that the support of μ is contained in $[-M, M]$ for some $M > 0$, and $x >> M$, then as $t \to \infty$,

$$\int_{-M}^{M} \int_{-\infty}^{t} e^{\lambda_0 s} P^z(X_{t-s} \geq x) ds \mu(dz) \asymp \int_{-\infty}^{t} e^{\lambda_0 s} P^0(X_{t-s} \geq x) ds$$

where $\asymp$ denotes logarithmic equivalence ($g(t) \asymp h(t)$ means $\log g(t) \sim \log h(t)$ as $t \to \infty$).

Thus we have to determine x_t such that $P^0(X_{t-s} \geq x_t)$ will decay exponentially, and at just the right rate such that $\int_{-\infty}^{t} e^{\lambda_0 s} P^0(X_{t-s} \geq x_t) ds = O(1)$ as $t \to \infty$. This is, in effect, a large deviations problem for killed Brownian motion.

Note that when there is no killing, then for any fixed s, $P^0(X_{t-s} \geq x_t) = P^0(B_{t-s} \geq x_t)$ decays exponentially as $t \to \infty$ precisely when x_t is a linear function of t (for any $b > 0$, $P^0(B_{t-s} > \sqrt{2b}t) \asymp e^{-bt}$ as $t \to \infty$). This accounts for the linear growth of R_t [13, 14] when $k = 0$. This behavior also occurs for certain types of non-zero killing functions, for example when k is identically a constant. However, it is not true for the unbounded killing functions we are interested in.

The rest of this paper is organized as follows. In Section 3, we solve the above mentioned large deviations problem for killed Brownian motion. In Section 4, we prove analogues of Theorems 1 and 2 for Poisson tidal waves. Following this, we deduce Theorems 1 and 2 using Proposition 1, in Section 5. Finally, in Section 6, we give a proof of Proposition 1.

3. Killed Brownian Motion

Let X denote killed Brownian motion. Recall that if k is the killing function, then for any $z \in \mathbf{R}$,

$$P^z(X_t \geq x) = P^z(\kappa \geq t,\ B_t \geq x) = E^z\left(e^{-\int_0^t k(B_s)ds} 1\{B_t \geq x\}\right) \tag{3.1}$$

where B denotes standard Brownian motion. We will study such probabilities as $t \to \infty$ when $x = f_t$, an increasing function of t, and k is continuous and grows unboundedly away from the origin. Our final aim is to find, given $b > 0$, f_t such that $P^z(X_t \geq f_t) \asymp e^{-bt}$ as $t \to \infty$.

Asymptotics for expectations of the form (3.1) were studied by Donsker and Varadhan [4]. In particular, if $k(y) \to \infty$ as $|y| \to \infty$, then there exists $\gamma_0 > 0$ such that for any $z \in \mathbf{R}$, and fixed $x \in \mathbf{R}$, $P^z(X_t \geq x) \asymp e^{-\gamma_0 t}$ as $t \to \infty$ (a result originally due to Kac [9]). Our problem differs from this in that we want x to vary with t. However, we shall make use of a special case of this result, concerning the distribution of κ, the life-time of killed Brownian motion, as $t \to \infty$. We state it below for future reference.

THEOREM (DV): If $k(y) \to \infty$ as $|y| \to \infty$, then for any $z \in \mathbf{R}$,

$$P^z(\kappa > t) = E^z\left(e^{-\int_0^t k(B_s)ds}\right) \asymp e^{-\gamma_0 t}$$

as $t \to \infty$, where γ_0 is the smallest positive eigenvalue of the differential operator $-\frac{1}{2}D_x^2 + k$, and is given by $\gamma_0 = \inf_{f \in \mathcal{F}} I(f)$, where

$$I(f) = \int_{\mathbf{R}} \left\{ k(y)f(y) + \frac{1}{8}\frac{(f'(y))^2}{f(y)} \right\} dy,$$

$$\mathcal{F} = \{pdf's\ f : f \in C^2,\ \ Supp\ f = \mathbf{R} \text{ or is compact,}$$
$$\text{and } f > 0 \text{ in the interior of the support}\}. \qquad \square$$

We shall also use a slight modification of this result which follows easily from its proof in [4].

LEMMA 1: If k and γ_0 are as in Theorem (DV), then there exists $A = A_\varepsilon > 0$ such that

$$\varliminf_{t\to\infty} \frac{\log E^z\left(e^{-\int_0^t k(B_s)ds} 1\{\|B\|_t \le A\}\right)}{t} \ge -(\gamma_0 + \varepsilon)$$

where $\|\cdot\|_t$ denotes the supremum norm on $C[0,t]$.

PROOF: In [4], it is shown that $\gamma_0 = \inf_{f \in \mathcal{F}_0} I(f)$, where $\mathcal{F}_0 = \{f \in \mathcal{F} : Supp\ f$ is compact and $\int_a^b \frac{(f')^2}{f} dy < \infty\}$. For any $A > 0$, define $\mathcal{F}_0^A = \{f \in \mathcal{F}_0 : Supp\ f \subseteq [-A, A]\}$, and $\gamma_0^A = \inf_{f \in \mathcal{F}_0^A} I(f)$. Now there exists A large enough that $I(f) < \gamma_0 + \varepsilon$ for some $f \in \mathcal{F}_0^A$ and therefore, $\gamma_0 \le \gamma_0^A \le \gamma_0 + \varepsilon$. To finish the proof, note that by (2.19) and (2.20) of [4],

$$\varliminf_{t\to\infty} \frac{\log E^z\left(e^{-\int_0^t k(B_s)ds} 1\{\|B\|_t \le A\}\right)}{t} \ge -I(f)$$

for all $f \in \mathcal{F}_0^A$. $\square$

In proving the main result of this section, we shall also use the following lemma.

LEMMA 2: Let X be killed Brownian motion with killing function $k(y)$ where $k(y) \uparrow \infty$ as $|y| \to \infty$ and $k'(y)/k(y) \to 0$ as $y \to \infty$, and let $T_x = \inf\{t : X_t \ge x\}$. Then for any $x > 0$ and $z \le x$,

$$P^z(T_x < \infty) = e^{-H(x)+H(z)}$$

where $H(x) \sim \Phi(x) = \int_0^x \sqrt{2k(y)}dy$ as $x \to \infty$.

PROOF: Let $u(x,z) = P^z(T_x < \infty) = E^z(e^{-\int_0^{\tau_x} k(B_s)ds})$, where $\tau_x = \inf\{s : B_s \geq x\}$. It is known that $\frac{\partial u}{\partial z}$ exists [10]. Now for any $\delta > 0$, by the Strong Markov Property, $u(x,z) = u(z+\delta, z)u(x, z+\delta)$. Therefore

$$\lim_{\delta \to 0} \frac{\log u(z+\delta, z)}{\delta} = \lim_{\delta \to 0} \frac{\log u(x,z) - \log u(x, z+\delta)}{\delta} = -\frac{\partial}{\partial z} \log u(x,z)$$

exists. Define $h(z) = -\lim_{\delta \to 0} \frac{\log u(z+\delta,z)}{\delta}$, then we have the differential equation

$$\frac{\partial u}{\partial z} = h(z)u, \quad u(z,z) = 1.$$

The solution of this is $u(x,z) = e^{-H(x)+H(z)}$, where $H(x) = \int_0^x h(y)dy$.

To determine the behavior of $H(x)$ as $x \to \infty$, let us first consider $u(y+\delta, y)$ for $y \geq D$ and $\delta > 0$, for fixed $D > 0$. Now, $u(y+\delta, y) = E(e^{-\int_0^{\tau_\delta} k(B_s+y)ds})$, where $\tau_\delta = \inf\{s : B_s = \delta\}$. On the event $(\tau_{-D} > \tau_\delta)$, we have

$$k(y-D) \leq k(B_s+y) \leq k(y+\delta),$$

since k is increasing in $(0, \infty)$. Therefore,

$$\begin{aligned}
u(y+\delta, y) &\leq E\left(e^{-\int_0^{\tau_\delta} k(B_s+y)ds} 1\{\tau_\delta < \tau_{-D}\}\right) + P(\tau_\delta > \tau_{-D}) \\
&\leq E\left(e^{-k(y-D)\tau_\delta}\right) + \frac{\delta}{\delta + D} \\
&= e^{-\delta\sqrt{2k(y-D)}} + \frac{\delta}{\delta + D}, \qquad \text{and} \\
u(y+\delta, y) &\geq E\left(e^{-\int_0^{\tau_\delta} k(B_s+y)ds} 1\{\tau_\delta < \tau_{-D}\}\right) \\
&\geq E\left(e^{-k(y+\delta)\tau_\delta} 1\{\tau_\delta < \tau_{-D}\}\right) \\
&\geq E\left(e^{-k(y+\delta)\tau_\delta}\right) - P(\tau_\delta > \tau_{-D}) \\
&= e^{-\delta\sqrt{2k(y+\delta)}} - \frac{\delta}{\delta + D}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\overline{\lim_{\delta \to 0}} \frac{\log u(y+\delta, y)}{\delta} &\leq -\sqrt{2k(y-D)} + \frac{1}{D}, \qquad \text{and} \\
\underline{\lim_{\delta \to 0}} \frac{\log u(y+\delta, y)}{\delta} &\geq -\sqrt{2k(y)} - \frac{1}{D},
\end{aligned}$$

thus,

$$\sqrt{2k(y-D)} - \frac{1}{D} \leq h(y) \leq \sqrt{2k(y)} + \frac{1}{D} \quad \forall y \geq D.$$

From this it follows that for any $x \geq D$,

$$\Phi(x-D) - \frac{x-D}{D} + H(D) \leq H(x) = \int_0^x h(y)dy$$
$$\leq \Phi(x) - \Phi(D) + \frac{x-D}{D} + H(D).$$

Now, $\Phi(x-D) > \Phi(x) - D\Phi'(x)$, since $\Phi''(x) = \frac{k'(x)}{\sqrt{2k(x)}} > 0$; by L'Hôpital's rule $\lim\limits_{x\to\infty} \frac{\Phi'(x)}{\Phi(x)} = \lim\limits_{x\to\infty} \frac{k'(x)}{2k(x)}$, which is 0 by hypothesis. It follows that $H(x) \sim \Phi(x)$ as $x \to \infty$.
□

We now state the main result of this section.

PROPOSITION 2: Let X denote killed Brownian motion, with killing function $k(y) = |y|^\alpha$, $\alpha > 0$. Suppose $\Phi(x) = \frac{2\sqrt{2}}{\alpha+2} x^{\frac{\alpha+2}{2}}$, and $\gamma = \gamma_\alpha$ is the leading eigenvalue of $(1/2)D_x^2 - k$. Let $M > 0$. Then for any $\varepsilon > 0$, and $z \in [-M, M]$ there exists $x_\varepsilon = x_{\varepsilon,\alpha} > 0$, $t_\varepsilon = t_{\varepsilon,\alpha} > 0$ and a constant $a_1 = a_1(\varepsilon, \alpha) > 0$ such that

$$e^{-(1+\varepsilon)(\Phi(x)+\gamma t)} \leq P^z(X_t \geq x) \leq e^{-(1-\varepsilon)(\Phi(x)+\gamma t)} \tag{3.2}$$

whenever $x \geq x_\varepsilon$, $t \geq t_\varepsilon$, and in the case of $\alpha \in (0, 2)$, $t \geq a_1 x^{\frac{2-\alpha}{2}+\varepsilon}$. □

COROLLARY: If $k(y) = |y|^\alpha$, $\alpha > 0$, then for any $z \in [-M, M]$, and $b > \gamma$, $P^z(X_t > f_t) \asymp e^{-bt}$ as $t \to \infty$, where $f_t = \left((b-\gamma)\frac{\alpha+2}{2\sqrt{2}}\right)^{2/\alpha+2} t^{2/\alpha+2}$. □

Remark:

Note that $P^z(X_t \geq x) = \int_x^\infty p_t(z, y)dy$, where P_t is the heat kernel of the operator $(1/2)D_x^2 - k$. Towards the end of this work, we learned about previous results on p_t [3, 15]. In [3], Davies and Simon give very sharp upper and lower bounds for p_t in the case when $k(x) = |x|^\alpha, \alpha > 2$, which immediately imply (3.2). In the classical case of $k(x) = x^2$ there is an exact formula for p_t [7]. When $k(x) = |x|^\alpha, 0 < \alpha < 2$, we also have bounds for p_t following from the results of Li and Yau [15] (they have the restriction that k'' is bounded above). But their results appear to give weaker bounds for $P^z(t \geq x)$ than ours.

We thank Mohan Ramachandran and Rodrigo Bañuelos for pointing out out the existence of these results.

However, we note that our method of obtaining the bounds (3.2) is applicable to a wider class of killed diffusions than are permitted by the methods of either [3] or [15]. Examples of such diffusions are:

(i) X is killed brownian motion, $k(x) = x^\alpha 1\{x > 0\}$.

(ii) X is killed brownian motion with constant drift $\mu > 0, k(x) = |x|^\alpha, \alpha > 0$. □

The rest of this section is devoted to the proof of Proposition 2.

Upper Bound

To estimate $P^z(X_t \geq x)$ above, suppose $0 < t' < t$ and $0 < x' < x$. A Brownian path attaining x at time t either stays above x' throughout $[t', t]$ or dips below x' some time in $[t', t)$. Define the event $F = \{B_s \leq x' \text{ for some } s \in [t', t)\}$, and $\tau_1 = \inf\{s \geq t' : B_s \leq x'\}$. Then,

$$\begin{aligned} P^z(X_t \geq x) &= E^z\left(e^{-\int_0^t |B_u|^\alpha du} 1(B_t \geq x) 1_F\right) + E^z\left(e^{-\int_0^t |B_u|^\alpha du} 1(B_t \geq x) 1_{F^c}\right) \\ &\leq E^z\left(e^{-\int_0^{t'} |B_u|^\alpha du} P^{x'}(\tilde{X}_{t-\tau_1} > x | \tau_1)\right) + E^z\left(e^{-\int_0^{t'} |B_u|^\alpha du - \int_{t'}^t (x')^\alpha du}\right) \\ &\leq E^z\left(e^{-\int_0^{t'} |B_u|^\alpha du}\right)\left\{P^{x'}(\tilde{T}_x < \infty) + e^{-(t-t')(x')^\alpha}\right\}, \end{aligned}$$

where $\tilde{X}$ is killed Brownian motion independent of τ_1, and $\tilde{T}_x = \inf\{t : \tilde{X}_t \geq x\}$.

FIGURE 1

Suppose t' and x' are such that $t - t' = o(t)$ and $x - x' = o(x)$, then by Theorem (DV) and Lemma 2 there exist t_ε and x_ε such that

$$E^z\left(e^{-\int_0^{t'} |B_u|^\alpha du}\right) \leq e^{-(1-\varepsilon)\gamma t},$$

$$P^{x'}(\tilde{T}_x < \infty) \leq (1/2) e^{-(1-\varepsilon)\Phi(x)}$$

for all $x \geq x_\varepsilon, t \geq t_\varepsilon$. To finish the proof of the upper bound, it remains to estimate $e^{-(t-t')(x')^\alpha}$. We consider the cases $0 < \alpha < 2, \alpha = 2$ and $\alpha > 2$ separately.

(i) When $0 < \alpha < 2$, choose $x' = x^p$ and $t' = t - t^p$, where $p = \frac{\alpha+2}{\alpha+2+2\varepsilon}$. Then there exists $a_1 > 0$ such that if x_ε is large enough, for all $x \geq x_\varepsilon$ and $t \geq a_1 x^{\frac{2-\alpha}{2}+\varepsilon}$,

$$e^{-(t-t')(x')^\alpha} = e^{-(tx^\alpha)^p} \leq (1/2) e^{-(1-\varepsilon)\frac{2\sqrt{2}}{\alpha+2} x^{\frac{\alpha+2}{2}}}.$$

(ii) When $\alpha = 2$, choose $x' = ax$ and $t' = t - b$ where $a = \sqrt{\varepsilon/2}, b = \sqrt{2}(1/\varepsilon - 1/2)$. Assume x_ε is large enough that for all $x \geq x_\varepsilon$,

$$e^{-(t-t')(x')^\alpha} = e^{-\frac{x^2}{\sqrt{2}}(1-\varepsilon/2)} \leq (1/2)e^{-(1-\varepsilon)\frac{x^2}{\sqrt{2}}}.$$

(iii) When $\alpha > 2$, choose $x' = x^p$ and $t' = t - b$ where $p = \frac{1}{\alpha} + \frac{1}{2}$ and $b = \frac{4\sqrt{2}}{\alpha+2}$. Assume x_ε to be large enough that for all $x \geq x_\varepsilon$,

$$e^{-(t-t')(x')^\alpha} = e^{-\frac{4\sqrt{2}}{\alpha+2}x^{\frac{\alpha+2}{2}}} \leq (1/2)e^{-(1-\varepsilon)\frac{2\sqrt{2}}{\alpha+2}x^{\frac{\alpha+2}{2}}}. \qquad \square$$

Lower Bound

Suppose we have shown that given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ and $x_\varepsilon > 0$ such that

$$P^0(X_t \geq x) \geq e^{-(1+\varepsilon)(\gamma t + \Phi(x))} \tag{3.3}$$

for all $x \geq x_\varepsilon$, $t \geq t_\varepsilon$, where t and x also satisfy, in the case of $\alpha \in (0,2)$, $t \geq \frac{\sqrt{2}}{2-\alpha}x^{\frac{2-\alpha}{2}}$.

For any $z \in \mathbf{R}$, define $T_0 = \inf\{s > 0 : X_s = 0\}$. Then for any $\delta_0 < t$,

$$\begin{aligned} P^z(X_t \geq x) &\geq P^z(T_0 < \delta_0, \quad X_t \geq x) \\ &= E^z(1\{T_0 < \delta_0\}P^o(\tilde{X}_{t-T_0} > x | T_0)), \end{aligned}$$

where $\tilde{X}$ is killed Brownian motion independent of T_0. Therefore, for all $x \geq x_\varepsilon$ and $t \geq t_\varepsilon + \delta_0$,

$$\begin{aligned} P^z(X_t \geq x) &\geq E^z(1(T_0 < \delta_0)e^{(1+\varepsilon)T_0})e^{-(1+\varepsilon)(\gamma t + \Phi(x))} \\ &\geq f(z)e^{-(1+\varepsilon)(\gamma t + \Phi(x))}, \end{aligned} \tag{3.4}$$

where $f(z) = P^z(T_0 < \delta_0)$, assuming $t \geq \frac{\sqrt{2}}{2-\alpha}x^{\frac{2-\alpha}{2}}$ when $0 < \alpha < 2$. Note that $f(z)$ is bounded below for $z \in [-M, M]$. So we have effectively proved the lower bound asserted in Proposition 2.

Thus it remains to prove (3.3). Recall that $P^0(X_t \geq x) = E(e^{-\int_0^t |B_s|^\alpha ds} 1\{B_t \geq x\})$. The idea of the proof is that most of the value of the expectation is contributed by Brownian paths concentrated in a neighborhood of a certain trajectory $\{\varphi_s^*\}_{0 \leq s \leq t}$, where $\varphi_0^* = 0$ and $\varphi_t^* = x$, i.e.

$$E\left(e^{-\int_0^t |B_s|^\alpha ds} 1\{B_t \geq x\}\right) \approx E\left(e^{-\int_0^t |B_s|^\alpha ds} 1\{B_t \geq x\} 1\{\|B - \varphi^*\|_t \leq A\}\right)$$

where $\|\cdot\|_t$ is the supremum norm on $C[0,t]$ and A is a suitably chosen large number.

We determine φ^* as follows. Let $\mathcal{C}_{t,x}$ be the class of absolutely continuous paths $\{\varphi_s\}_{0\le x\le t}$ with $\varphi_0 = 0$, $\varphi_t = x$ and $\dot\varphi_t = \frac{d\varphi}{dt} \in \mathcal{L}^2[0,t]$. For any $\varphi \in \mathcal{C}_{t,x}$, consider the translated process $Y_u = B_u - \varphi_u$, $0 \le u \le t$. This translation induces an absolutely continuous change of measures in $\mathcal{C}[0,t]$, by Girsanov's theorem. Denoting the expectation under the induced measure as E_φ and noting that Y_u is a standard Brownian motion under the new measure, we have, by Girsanov's formula, for any $A > 0$,

$$
\begin{aligned}
&E\left(\exp\left\{-\int_0^t |B_s|^\alpha ds\right\} 1\{B_t \ge x, \|B-\varphi\|_t \le A\}\right)\\
&= E_\varphi\left(\exp\left\{-\int_0^t |\tilde B_s + \varphi_s|^\alpha ds - \int_0^t \frac{1}{2}\dot\varphi_s^2 ds - \int_0^t \dot\varphi_s d\tilde B_s\right\} 1\{\tilde B_t \ge 0, \|\tilde B\|_t \le A\}\right)\\
&= E_\varphi\left(\exp\left\{-\int_0^t H(\tilde B_s, \varphi_s)ds - \dot\varphi_t \tilde B_t\right\} 1\{\tilde B_t \ge 0, \|\tilde B\|_t \le A\}\right) e^{-J(\varphi)}
\end{aligned}
$$

where $\tilde B$ is standard Brownian motion under the new measure,

$$
J(\varphi) = \int_0^t |\varphi_s|^\alpha + \frac{1}{2}\dot\varphi_s^2 ds, \tag{3.5}
$$

$$
H(\tilde B_s, \varphi_s) = |\tilde B_s + \varphi_s|^\alpha - |\varphi_s|^\alpha - \ddot\varphi_s \tilde B_s. \tag{3.6}
$$

Thus,

$$
\begin{aligned}
&E\left(e^{-\int_0^t |B_s|^\alpha ds} 1\{B_t \ge x)\right\} \ge \\
&\sup_{\varphi\in\mathcal{C}_{t,x}} E_\varphi\left(e^{-\int_0^t H(\tilde B_s,\varphi_s)ds - \dot\varphi_t \tilde B_t} 1\{\tilde B_t \ge 0, \|\tilde B\|_t \le A\}\right) e^{-J(\varphi)}.
\end{aligned}
$$

We guess that this supremum is attained by a path $\varphi^* = \varphi^*_{t,x} \in \mathcal{C}_{t,x}$ which minimizes $J(\varphi)$. We show below that such a path exists and gives us the desired lower bound (3.3), for suitably chosen A.

Specifically, let $\varepsilon > 0$. We will show that if φ^* is the minimizing path, then for any $A > 0$ and $x > A$ there exists numbers $x_\varepsilon > 0$ and $t_\varepsilon > 0$ such that for all $x \ge x_\varepsilon$, and $t \ge t_\varepsilon$,

$$
\begin{aligned}
&E_\varphi\left(e^{-\int_0^t H(\tilde B_s,\varphi_s^*)ds - \dot\varphi_t^* \tilde B_t} 1\{\tilde B_t \ge 0, \|\tilde B\|_t \le A\}\right) e^{-J(\varphi^*)}\\
&\quad \ge E_\varphi\left(e^{-\int_0^t |\tilde B|_s^\alpha ds} 1\{\|\tilde B\|_t \le A\}\right) e^{-(1+\varepsilon)\Phi(x) - \frac{\gamma\varepsilon}{2}t}
\end{aligned} \tag{3.7}
$$

where x and t also satisfy $t \ge \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$ when $0 < \alpha < 2$. The final step of the proof is to note that by Lemma 1 there exists $A_\varepsilon > 0$ and $t'_\varepsilon > 0$ such that

$$
E_\varphi\left(e^{-\int_0^t |\tilde B|_s^\alpha ds} 1\{\|\tilde B\|_t \le A_\varepsilon\}\right) \ge e^{-(1+\varepsilon/2)\gamma t}
$$

for all $t \geq t'_\varepsilon$.

The proof of (3.7) is given in a sequence of Lemmas. First we show the existence of φ^*, and obtain an expression for $J(\varphi^*)$. Then we estimate $\int_0^t H(\tilde{B}_s, \varphi_s)ds + \dot{\varphi}_t^* B_t$ on the event $\{\|\tilde{B}\|_t \leq A\}$.

LEMMA 3: $\Psi(x,t) = \inf_{\varphi \in \mathcal{C}_{t,x}} J(\varphi)$ exists and is attained by a path φ^* which is non-decreasing, and non-negative.

PROOF: If a minimizing path exists, it has to be non-decreasing. To see this, note that if $\varphi_b < \varphi_a$ where $(a,b) \subseteq [0,t]$, then we can construct a path $\bar{\varphi}$ as follows, with $J(\bar{\varphi}) \leq J(\varphi)$:

$$\bar{\varphi}_u = \begin{cases} \varphi_u & , u \geq b \\ \varphi_u \wedge \varphi_b & , u < b. \end{cases}$$

It follows that a minimizing path, if it exists, must also be non-negative and bounded.

Now suppose $\inf_{\varphi \in \mathcal{C}_{t,x}} J(\varphi) = s \geq 0$. Let $\tilde{\mathcal{C}}_{t,x}$ be the sub-class of $\mathcal{C}_{t,x}$ consisting of non-decreasing paths; then $\inf_{\varphi \in \hat{\mathcal{C}}_{t,x}} J(\varphi) = s$ as well. Therefore, $\mathcal{C}_n = \{\varphi \in \tilde{\mathcal{C}}_{t,x} : J(\varphi) \leq s + \varepsilon_n\}$, where $\varepsilon_n \downarrow 0$, is non-empty. $\mathcal{C}_n$ is also point-wise bounded ($\sup_{\varphi \in \mathcal{C}_n} \{\varphi_u\} = x \ \forall \ n \geq 1$, $\forall \ u \in [0,t]$), and equicontinuous. The latter follows form Lemma 2.1(b) of Freidlin and Wentzell [8], since for any $\varphi \in \mathcal{C}_n$, $J(\varphi)$ is bounded above.

Now pick a sequence $\{\varphi_n\}_{n \geq 1}$, $\varphi_n \in \mathcal{C}_n$. Then by Ascoli's theorem, there exists a subsequence $\{\varphi_{n_k}\}$ which converges uniformly to some φ^*. It follows that $J(\varphi^*) = s$, since J is lower semi-continuous. □

LEMMA 4: φ^* is the unique solution of the differential equation $\dot{\varphi}_u^2 = 2\varphi_u^\alpha + c$ with boundary conditions $\varphi_0 = 0$, and $\varphi_t = x$. Here $c = c(x,t) = \dot{\varphi}_0^2$ satisfies

(i) $c = 0$ when $\alpha < 2$ and $t > \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$;

(ii) $c = 4x^2 / \sin h^2 \sqrt{2} t$ when $\alpha = 2$;

(iii) $c = O(\frac{1}{t^{\frac{2\alpha}{\alpha-2}}})$ as $t \to \infty$, uniformly in x when $\alpha > 2$, if x is bounded below.

Remark:

We therefore obtain

(a) $\varphi_u^* = \left\{ \left(\frac{2-\alpha}{\sqrt{2}}\right)(u - t_1) \right\}^{2/2-\alpha} 1\{t_1 \leq u \leq t\}$, where $t_1 = t - \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$, when $0 < \alpha < 2$, and $t > \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$.

(b) $\varphi_u^* = \frac{x \sinh \sqrt{2} u}{\sinh \sqrt{2} t} 1\{0 \leq u \leq t\}$, when $\alpha = 2$.

A closed-form solution to the boundary value problem apparently does not exist when $\alpha > 2$.

PROOF: $J(\varphi^*) = \int_0^t F(\varphi_s^*, \dot\varphi_s^*)du$, where $F(w,v) = w^\alpha + (1/2)v^2$ is in C^2. Therefore the minimizing path must satisfy the Euler-Lagrange equation $\ddot\varphi_u = \alpha\varphi_u^{\alpha-1}$ (the integration of which gives $\dot\varphi_u^2 = 2\varphi_u^\alpha + c$) with boundary conditions $\varphi_0 = 0, \varphi_t = x$. The constant $c = c(x,t) = \dot\varphi_0^2$ is necessarily positive for $\alpha \geq 2$ (otherwise the equation is not integrable).

(i) When $0 < \alpha < 2$, setting $c = 0$ we have $\dot\varphi_u = \sqrt{2}\varphi_u^{\alpha/2} \geq 0$ since the solution must be nondecreasing. Hence, for some $t_1 \in [0,t]$,

$$\dot\varphi_u = \begin{cases} 0, & u \leq t_1 \ ; \\ \sqrt{2}\varphi_u^{\alpha/2} \neq 0, & u \in (t_1, t]. \end{cases}$$

Therefore, $\varphi_u = \left[\frac{2-\alpha}{\sqrt{2}}(u - t_1)\right]^{2/2-\alpha} 1(t_1 \leq u \leq t)$; using the condition $\varphi_t = x$ we obtain t_1. Note that this solution is valid only if $t \geq \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$.

(ii) When $\alpha = 2$, the solution of the Euler-Lagrange equation $\dot\varphi_u^2 = 2\varphi_u^2 + c$ is straight-forward, subject to the boundary conditions.

(iii) When $\alpha > 2$, we cannot solve $\dot\varphi_u^2 = 2\varphi_u^\alpha + c$ explicitly, but we may determine c uniquely through the equation

$$\int_0^x \frac{dy}{\sqrt{2y^\alpha + c}} = t \qquad (3.8)$$

This may be rewritten as

$$\int_0^{mx} \frac{dy}{\sqrt{y^\alpha + 1}} = \frac{\sqrt{2}t}{m^{\frac{\alpha-2}{2}}} \qquad (3.9)$$

where $m = (2/c)^{1/\alpha}$. Now we use the fact that

$$d_1 \leq \frac{(y+1)^\alpha}{y^\alpha + 1} \leq d_2 \qquad (3.10)$$

for certain constants $d_1, d_2 > 0$, for all $y \geq 0$. From (3.9) and (3.10) we obtain

$$\frac{\alpha - 2}{\sqrt{2}d_2}\left(1 - \frac{1}{(1+mx)^{\frac{\alpha-2}{2}}}\right)^{-1} t \leq m^{\frac{\alpha-2}{2}} \leq \frac{\alpha-2}{\sqrt{2}d_1}\left(1 - \frac{1}{(1+mx)^{\frac{\alpha-2}{2}}}\right)^{-1} t,$$

from which we can deduce that, for some constants $A_i > 0$, $1 \leq i \leq 3$,

$$\left(1 - \frac{A_1}{tx^{\frac{\alpha-2}{2}}}\right)\frac{A_2}{t^{\frac{2\alpha}{\alpha-2}}} \leq c \leq \frac{A_3}{t^{\frac{2\alpha}{\alpha-2}}}.$$

Thus if x is bounded below, $c = O\left(\frac{1}{t^{\frac{2\alpha}{\alpha-2}}}\right)$ as $t \to \infty$ uniformly in x. □

LEMMA 5: If φ^* is the minimizing path and $\Phi(x) = \frac{2\sqrt{2}}{\alpha+2} x^{\frac{\alpha+2}{2}}$, then $\Psi(x,t) = J(\varphi^*)$ satisfies, for $x > 0$,

(i) $\Psi(x,t) = \Phi(x)$ when $0 < \alpha < 2$, and $t \geq \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$;

(ii) $\Psi(x,t) = \Phi(x)(1 + O(e^{-2\sqrt{2}t}))$ when $\alpha = 2$, as $t \to \infty$;

(iii) $\Psi(x,t) = \Phi(x)(1 + O(1/t^{2\alpha/\alpha-2})) + O(1/t^{\frac{\alpha+2}{\alpha-2}})$ when $\alpha > 2$ and x is bounded below, as $t \to \infty$.

PROOF: φ^* satisfies the equation $\dot{\varphi}_u = \sqrt{2\varphi_u^\alpha + c}$, $\varphi_0 = 0$, $\varphi_t = x$, where $c = c(x,t) = \dot{\varphi}_0^2$ depends on α as in Lemma 4. Therefore, setting $\varphi = \varphi^*$, we have

$$J(\varphi) = \int_0^t (\varphi_s^\alpha + \frac{1}{2}\dot{\varphi}_s^2) ds = \int_0^x \{y^\alpha + \frac{1}{2}(2y^\alpha + c)\} \frac{dy}{\sqrt{2y^\alpha + c}}$$
$$= \frac{2}{\alpha+2} \int_0^x \{d(y\sqrt{2y^\alpha + c}) + \frac{\alpha-2}{4} \frac{dy}{\sqrt{2y^\alpha + c}}\} = \frac{2x\sqrt{2x^\alpha + c}}{\alpha+2} + \frac{\alpha-2}{2(\alpha+2)} ct,$$

making use of (3.8). Thus, by Lemma 4,

$$\Psi(x,t) = \begin{cases} \Phi(x), & \alpha < 2,\ \forall\, t \geq \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}, x > 0; \\ \Phi(x)\left(\frac{1+e^{-2\sqrt{2}t}}{1-e^{-2\sqrt{2}t}}\right), & \alpha = 2,\ \forall\, t > 0, x > 0; \\ \Phi(x)\sqrt{1 + \frac{c}{2x^\alpha}} + \frac{\alpha-2}{2(\alpha+2)} ct, & \alpha > 2,\ \forall\, t > 0, x > 0. \end{cases} \qquad \square$$

LEMMA 6: If $H(y,z) = H_\alpha(y,z) = |y+z|^\alpha - z^\alpha - \alpha z^{\alpha-1} y$, where $\alpha > 0$, then for any $y \in (-A, A)$ and $z > 0$ we have

(i) When $0 < \alpha < 1, H(y,z) \leq (A^\alpha + \alpha A/z^{1-\alpha}) 1\{0 < z < A/(1+\rho)\}$, where ρ is such that $H(-(1+\rho)z, z) = 0$;

(ii) When $\alpha = 1, H(y,z) \leq 2A1\{0 < z < A\}$;

(iii) When $\alpha \in (1,2), H(y,z) \leq (A^\alpha + \alpha A z^{\alpha-1}) 1\{0 < z < A\} + \frac{\alpha(\alpha-1)}{2} \frac{A^2}{z^{2-\alpha}} 1\{z \geq A\}$;

(iv) When $\alpha = 2, H(y,z) = y^2$ for all $z > 0$; and

(v) When $\alpha > 2, H(y,z) \leq (\beta_1 z + |y|^a) 1\{0 < z < A\} + \beta_2 A z^{\alpha-1} 1\{z \geq A\}$, for some positive constants β_1 and β_2.

PROOF: (i) When $0 < \alpha < 1, H(y,z) = |y+z|^\alpha - z^\alpha - \alpha(y/z^{1-\alpha})$. Fix $z > 0$. It is easy to check that as a function of y, H is decreasing in $(-\infty, -z)$, increasing in $(-z, 0)$ and decreasing in $(0, \infty)$. Also $H \to \infty$ as $y \to -\infty$.

FIGURE 2

Since $H(-z,z) = -(1-\alpha)z^\alpha < 0$, and $H(0,z) = 0$, there exists $\rho > 0$ such that $H(-(1+\rho)z,z) = 0$, and $H \le 0$ for all $y \ge -(1+\rho)z$ (ρ is independent of z). Therefore, if $z > A/(1+\rho), H(y,z) \ge 0$ for all $y \in (-A,A)$, while if $0 < z < A/(1+\rho)$, then $\max_{-A\le y\le A} H(y,z) = H(-A,z) = (A-z)^\alpha - z^\alpha + \alpha A/z^{1-\alpha} \le A^\alpha + \alpha A/z^{1-\alpha}$.

(ii) When $\alpha = 1, H(y,z) = -\alpha(y+z)1(y < -z)$. For fixed $z, H \to \infty$ as $y \to -\infty$ and $H = 0 \ \forall\, y \ge -z$. Clearly then, if $0 < z < A$ then $\max_{-y<A<y} H(y,z) = H(-A,z) = 2(A-z)$, while if $z > A$ then $\max_{-y<A<y} H(y,z) = 0$. Thus $H(y,z) \le H(-A,z)1\{0 < Z < A\} \le 2A1\{0 < z < A\}$.

(iii) In this case, for fixed $z > 0, H(y,z) = |y+z|^\alpha - z^\alpha - \alpha y z^{\alpha-1}$ is increasing in $(0,\infty)$, decreasing in $(-\infty,0)$, and $H(0,z) = 0$. Also $H \to \infty$ as $|y| \to \infty$.

FIGURE 3

Thus for fixed z, $\max_{-A\le y\le A} H(y,z) = \max\{H(A,z), H(-A,z)\}$. It can be easily shown that $H(-A,z) \ge H(A,z)$ for all $z > 0$. When $0 < z < A$, $H(-A,z) = (A-z)^\alpha - z^\alpha - \alpha A z^{\alpha-1} \le A^\alpha + aAz^{\alpha-1}$, while if $z \ge A$, $H(-A,z) = (z-A)^\alpha - z^\alpha + aAz^{\alpha-1} \le \frac{\alpha(\alpha-1)}{2}A^2/z^{2-\alpha}$.

(iv) When $\alpha = 2, H(y,z) = |y+z|^2 - z^2 - 2yz = y^2$.

(v) First, fix $z \in (0, A)$ and consider $G(y,z) = H(y,z) - |y|^\alpha$. G is increasing in $(-\infty, -z)$, decreasing in $(-z, 0)$ and increasing in $(0, \infty)$. Also, $G \to -\infty$ as $y \to -\infty$.

FIGURE 4

Thus when $z \in (0, A)$, $\max_{-A \le y \le A} G(y,z) = \max\{G(-z), G(A)\} = \max\{(\alpha - 2)z^\alpha, (A+z)^\alpha - z^\alpha - \alpha z^{\alpha-1}A - A^\alpha\} \le (2^\alpha - 2)A^{\alpha-1}z = \beta_1 z$, say. Thus $H(y,z) \le \beta_1 z + |y|^\alpha$ for all $y \in [-A, A]$.

Now let $z \ge A$. For fixed $z, H(y,z)$ is increasing in $(0,\infty)$, decreasing in $(-\infty, 0)$, has a zero at 0, and $H(y,z) \to \infty$ as $|y| \to \infty$.

FIGURE 5

Thus $\max_{-A \le y \le A} H(y,z) = \max\{H(-A,z), H(A,z)\} = \max\{(z+A)^\alpha - \alpha z^{\alpha-1}A, (z-A)^\alpha + \alpha z^{\alpha-1}A\} - z^\alpha = \{(1+\frac{A}{z})^\alpha - \alpha(\frac{A}{z}) - 1\}z^\alpha \le \{\beta_2(\frac{A}{z})\}z^\alpha$, for some constant $\beta_2 > 0$. $\square$

LEMMA 7: Let $\varphi = \varphi^*$ and define the random variable $Y_{x,t} = Y^\alpha_{x,t} = \int_0^t H(\varphi_s, B_s)ds + \dot{\varphi}_t B_t, \alpha > 0$, where $H(B_s, \varphi_s) = |B_s + \varphi_s|^\alpha - \varphi_s^\alpha - \ddot{\varphi}_s B_s$. Let $A > 0$. Then, given $\varepsilon > 0$ there exists $t'_\varepsilon = t'_{\varepsilon,\alpha} > 0$ and $x'_\varepsilon = x'_{\varepsilon,\alpha} > 0$ such that on the event $\{||B||_t < A\}$,

(a) $Y_{x,t} \leq \int_0^t |B_s|^\alpha ds + (\varepsilon/2)\Phi(x)$ for all $x \geq \max\{x'_\varepsilon, A\}$ and $t \geq \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$, when $0 < \alpha < 2$;

(b) $Y_{x,t} \leq \int_0^t |B_s|^2 ds + (\varepsilon/2)\Phi(x)$ for all $x \geq \max\{x'_\varepsilon, A\}$ and $t \geq t'_\varepsilon$, when $\alpha = 2$; and

(c) $Y_{x,t} \leq \int_0^t |B_s|^\alpha ds + (\varepsilon/2)(\Phi(x) + \gamma t)$ for all $x \geq \max\{x'_\varepsilon, A\}$ and $t \geq t'_\varepsilon$, when $\alpha > 2$.

PROOF: (a) Recall from Lemma 4 that if $t \geq \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$ then $\varphi_s = \varphi_s^* = \left(\frac{2-\alpha}{\sqrt{2}}\right)^{2/2-\alpha} (s - t_1)^{2/2-\alpha} 1\{s \geq t_1\}$, where $t_1 = t - \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$. Note that $\ddot{\varphi}_s = \alpha \varphi_s^{\alpha-1} 1\{t_1 \leq s \leq t\}$ and $\dot{\varphi}_s = \sqrt{2}\varphi_s^{\alpha/2} 1\{t_1 \leq s \leq t\}$. Therefore, on the event $\{||B||_t < A\}$,

$$
\begin{aligned}
Y_{x,t} &= \int_0^{t_1} |B_s|^\alpha ds + \int_{t_1}^t (|B_s + \varphi_s|^\alpha - \varphi_s^\alpha - \alpha B_s \varphi_s^{\alpha-1}) ds + \sqrt{2}\varphi_t^{\alpha/2} B_t \\
&\leq \int_0^t |B_s|^\alpha ds + \int_{t_1}^t (|B_s + \varphi_s|^\alpha - \varphi_s^\alpha - \alpha B_s \varphi_s^{\alpha-1}) ds + \sqrt{2} A x^{\alpha/2}
\end{aligned}
$$

We use the estimates from Lemma 6 for the function $H(y,z) = |y+z|^\alpha - z^\alpha - \alpha y z^{\alpha-1}$ to show that $\int_{t_1}^t H(B_s, \varphi_s) ds \leq b_1(\alpha)$ for some constant $b_1(\alpha) > 0$ on the event $\{||B||_t < A\}$, when $A < x$. Then we will have, for some $x'_\varepsilon > 0$,

$$
\begin{aligned}
Y_{x,t} &\leq \int_0^t |B_s|^\alpha ds + b_1(\alpha) + \sqrt{2} A x^{\alpha/2} \\
&\leq \int_0^t |B_s|^\alpha ds + (\varepsilon/2)\Phi(x)
\end{aligned}
$$

for all $x \geq x'_\varepsilon$ and $t \geq \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$.

In estimating $\int_{t_1}^t H(B_s, \varphi_s) ds$, we note that φ is strictly increasing in $[t_1, t]$ with $\varphi_{t_1} = 0$ and $\varphi_t = x$, so for any $d \in [0, x]$ there exists a unique $t_2 \in [t_1, t]$ such that $\varphi_{t_2} = d$; $t_2 = \frac{\sqrt{2}}{2-\alpha} d^{\frac{2-\alpha}{2}} + t_1$.

We consider the cases $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$ separately.

When $0 < \alpha < 1$, from Lemma 6, on $\{||B||_t < A\}, A < x$,

$$
\int_{t_1}^t H(B_s, \varphi_s) ds \leq \int_{t_1}^{t_2} \left(A^\alpha + \frac{\alpha A}{\varphi_s^{1-\alpha}}\right) ds,
$$

where t_2 is such that $\varphi_{t_2} = \frac{A}{1+p}$. Thus,

$$
\int_{t_1}^t H(B_s, \varphi_s) ds \leq A^\alpha (t_2 - t_1) + A\dot{\varphi}_{t_2} = A^\alpha \left(\frac{A}{1+p}\right)^{\frac{2-\alpha}{2}} \left(\frac{\sqrt{2}}{2-\alpha}\right) + \sqrt{2} A \left(\frac{A}{1+p}\right)^{\alpha/2}
$$

which is a finite constant.

When $\alpha = 1$, let t_2 be such that $\varphi_{t_2} = A$, then from Lemma 6,

$$\int_{t_1}^{t} H(B_s, \varphi_s) ds \leq \int_{t_1}^{t_2} 2A ds = 2A(t_2 - t_1) = 2A\sqrt{2A}.$$

When $1 < \alpha < 2$, let t_2 be once again such that $\varphi_{t_2} = A$, then by Lemma 6,

$$\begin{aligned}
\int_{t_1}^{t} H(B_s, \varphi_s) ds &\leq \int_{t_1}^{t_2} (A^\alpha + \alpha A \varphi_s^{\alpha-1}) ds + \int_{t_2}^{t} \frac{\alpha(\alpha-1)}{2} \frac{A^2}{\varphi_s^{1-\alpha}} ds \\
&\leq A^\alpha (t_2 - t_1) + \sqrt{2} A^{\frac{\alpha+2}{2}} + \frac{\alpha(\alpha-1)}{2} A^2 \int_{t_2}^{t} \left(\frac{\sqrt{2}}{2-\alpha} \right)^{\frac{2-\alpha}{2}} \frac{1}{(s-t_1)^2} ds \\
&= \left(\frac{\sqrt{2}}{2-\alpha} + \sqrt{2} + \frac{\alpha(\alpha-1)}{2} \left(\frac{\sqrt{2}}{2-\alpha} \right)^{\frac{\alpha}{2-\alpha}} \right) A^{\frac{\alpha+2}{2}}.
\end{aligned}$$

(b) When $\alpha = 2$, on $\{\|B\|_t < A\}$, $A < x$,

$$\begin{aligned}
Y_{x,t} &= \int_0^t H(B_s, \varphi_s) ds + \dot{\varphi}_t B_t = \int_0^t |B_s|^2 ds + \sqrt{2\varphi_t^2 + c} B_t \\
&\leq \int_0^t |B_s|^2 ds + \sqrt{2} x A \left(1 + O(e^{-2\sqrt{2}t}) \right).
\end{aligned}$$

Therefore, there exists x'_ε, $t'_\varepsilon > 0$ such that for all $x \geq \max\{A, x'_\varepsilon\}$ and $t \geq t'_\varepsilon$,

$$Y_{x,t} \leq \int_0^t |B_s|^2 ds + \frac{\varepsilon}{2} \left(\frac{x^2}{\sqrt{2}} \right) = \int_0^t |B_s|^2 ds + \frac{\varepsilon}{2} \Phi(x)$$

(c) When $\alpha > 2$, let t_2 be such that $\varphi_{t_2} = A$. From Lemma 6 we have, on $\{\|B\|_t < A\}$, where $A < x$,

$$\begin{aligned}
Y_{x,t} &= \int_0^{t_2} (\beta_1 \varphi_s + |B_s|^\alpha) ds + \int_{t_2}^{t} A \beta_2 \varphi_s^{\alpha-1} ds + \dot{\varphi}_t B_t \\
&\leq \int_0^t |B_s|^\alpha ds + \beta_1 \int_0^{t_2} \varphi_s ds + \left(\frac{\beta_2}{\alpha} + 1 \right) A \dot{\varphi}_t.
\end{aligned}$$

To estimate $\int_0^{t_2} \varphi_s ds$, recall that φ satisfies $\dot{\varphi} = \sqrt{2\varphi^\alpha + c}$ where $c = c(x,t) = O(\frac{1}{t^{2\alpha/\alpha-2}})$ for all $x > A$. Therefore,

$$\int_0^{t_2} \varphi_s ds = \int_0^A \frac{y dy}{\sqrt{2y^\alpha + c}} = \frac{m^{\frac{\alpha-4}{2}}}{\sqrt{2}} \int_0^{mA} \frac{y dy}{\sqrt{1 + y^\alpha}}$$

where $m = (2/c)^{1/\alpha}$. Note $m = O(t^{2/\alpha-2})$ as $t \to \infty$ for all $x > A$ by Lemma 4. Making use of inequality (3.8), we have

$$\int_0^{t_2} \varphi_s ds \le \frac{m^{\frac{\alpha-4}{2}}}{\sqrt{2}} \int_0^{mA} \frac{d_2 y}{(1+y)^{\alpha/2}} dy = \begin{cases} \frac{2\sqrt{2}}{(\alpha-2)(4-\alpha)} \left(\frac{1+mA}{m}\right)^{\frac{4-\alpha}{2}} , & 2 < \alpha < 4 \\ \frac{d_2}{\sqrt{2}} \log(1+mA) , & \alpha = 4 \\ \frac{2\sqrt{2} d_2 m^{\frac{\alpha-4}{2}}}{(\alpha-2)(\alpha-4)} , & \alpha > 4. \end{cases}$$

Therefore, whenever $A < x$, there exists positive constants $D_i, \quad 1 \le i \le 6$ such that

$$\int_0^{t_2} \varphi_s ds \le \begin{cases} D_1 + \frac{D_2}{t^{\frac{2}{\alpha-2}}} , & 2 < \alpha < 4 \\ D_3 + D_4 \log t + D_5/t , & \alpha = 4 \\ D_6 t^{\frac{\alpha-4}{\alpha-2}} , & \alpha > 4 \end{cases}$$

Thus, when $\alpha > 2$, there exists constants $x'_\varepsilon,\ t'_\varepsilon > 0$ such that for all $x \ge x'_\varepsilon,\ t \ge t'_\varepsilon$,

$$\begin{aligned} Y_{x,t} &\le \int_0^t |B_s|^\alpha ds + \beta_1 \int_0^{t_2} \varphi_s ds + \left(\frac{\beta_2}{\alpha} + 1\right) A\sqrt{2x^\alpha + c} \\ &\le \int_0^t |B_s|^\alpha ds + \sqrt{2}\left(\frac{\beta_2}{\alpha} + 1\right) A x^{\frac{\alpha}{2}} + \frac{\varepsilon}{2}\gamma t \\ &\le \int_0^t |B_s|^\alpha ds + \frac{\varepsilon}{2}(\Phi(x) + \gamma t). \qquad \Box \end{aligned}$$

LEMMA 8: Let $\varphi = \varphi^*$. Then, given $\varepsilon > 0$ and $\alpha > 0$ there exists $x_\varepsilon = x_{\varepsilon,\alpha} > 0$ and $t_\varepsilon = t_{\varepsilon,\alpha} > 0$ such that for any $A > 0$,

$$\begin{aligned} &E\left(e^{-\int_0^t |B_s|^\alpha ds} 1\{B_t \ge x,\ \|B - \varphi^*\|_t \le A\}\right) \\ &\quad \ge E_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t \le A\}\right) e^{-\Phi(x)(1+\varepsilon) - \frac{\gamma\varepsilon}{2} t} \end{aligned}$$

for all $x \ge \max\{A, x_\varepsilon\}$ and $t \ge t'_\varepsilon$, with the additional condition that $t \ge \frac{\sqrt{2}}{2-\alpha} x^{\frac{\alpha-2}{2}}$ when $0 < \alpha < 2$. Here $\tilde{B}$ is standard Brownian motion under P_φ, the measure on $\mathcal{C}[0,t]$ induced by the transformation $B_u \to B_u - \varphi_u$.

PROOF: Recall from (3.1), that

$$\begin{aligned} &E\left(e^{-\int_0^t |B_s|^\alpha ds} 1\{B_t \ge x, \|B - \varphi^*\|_t \le A\}\right) \\ &\quad = E_\varphi\left(e^{-\int_0^t H(\tilde{B}_s, \varphi_s) ds - \dot{\varphi}_t \tilde{B}_t} 1\{\tilde{B}_t \ge 0, \|\tilde{B}\|_t \le A\}\right) e^{-\Psi(x,t)} \end{aligned}$$

We consider the different cases $0 < \alpha < 2, \quad \alpha = 2$ and $\alpha > 2$.

(i) When $0 < \alpha < 2$, from Lemmas 5 and 7 there exists $x'_\varepsilon > 0$ such that

$$\begin{aligned}
&E_\varphi\left(e^{-\int_0^t H(\tilde{B}_s,\varphi_s)ds - \dot{\varphi}_t \tilde{B}_t} 1\{\tilde{B}_t \geq 0, \|\tilde{B}\|_t \leq A\}\right) e^{-\Psi(x,t)} \\
&= E_\varphi\left(e^{-Y_{x,t}} 1\{\tilde{B}_t \geq 0, \|\tilde{B}\|_t \leq A\}\right) e^{-\Psi(x,t)} \\
&\geq \frac{1}{2} E_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t \leq A\}\right) e^{-\Phi(x) - \frac{\varepsilon}{2}\Phi(x)} \\
&\geq E_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t \leq A\}\right) e^{-(1+\varepsilon)\Phi(x)}
\end{aligned}$$

for all $x > \max\{A, x_\varepsilon\}$ and $t > \frac{\sqrt{2}}{2-\alpha} x^{\frac{2-\alpha}{2}}$.

(ii) When $\alpha = 2$, from Lemmas 5 and 7 there exists $x'_\varepsilon,\ t'_\varepsilon > 0$ such that

$$\begin{aligned}
&E_\varphi\left(e^{-Y_{x,t}} 1\{\tilde{B}_t > 0, \|\tilde{B}\|_t < A\}\right) e^{-\Psi(x,t)} \\
&\geq E_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t < A\}\right) e^{-\frac{\varepsilon}{2}\Phi(x) - \Phi(x)(1 + O(e^{-2\sqrt{2}t}))} \\
&\geq E_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t < A\}\right) e^{-(1+\varepsilon)\Phi(x)}
\end{aligned}$$

for all $x > \max\{A, x_\varepsilon\}$ and $t > t'_\varepsilon$.

(iii) When $\alpha > 2$, from Lemmas 5 and 7 there exists $x_\varepsilon,\ t_\varepsilon > 0$ such that

$$\begin{aligned}
&\tilde{E}_\varphi\left(e^{-Y_{x,t}} 1(\tilde{B}_t > 0) 1\{\|\tilde{B}\|_t < A\}\right) e^{-\Psi(x,t)} \\
&\leq \tilde{E}_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t < A\}\right) e^{-\frac{\varepsilon}{2}(\Phi(x) + \gamma t) - \Phi(x)(1 + O(t^{-\frac{2\alpha}{2-\alpha}}) + O(t^{-\frac{\alpha+2}{\alpha-2}}))} \\
&\leq \tilde{E}_\varphi\left(e^{-\int_0^t |\tilde{B}_s|^\alpha ds} 1\{\|\tilde{B}\|_t < A\}\right) e^{-\frac{\varepsilon}{2}\gamma t - (1+\varepsilon)\Phi(x)}
\end{aligned}$$

for all $x > \max(A, x_\varepsilon),\ t > t_\varepsilon$. □

This finishes the proof of (3.7).

4. Poisson Tidal Waves

In this section we will prove analogues of theorems 1 and 2 for Poisson tidal waves. Consider a PTW with birth intensity measure $Ce^{\lambda t}\mu(dy)dt$ where $C > 0$, $\lambda > 0$, μ is finite and has compact support, say contained in $[-M, M]$. Let k be the continuous killing function, satisfying $k(y) \uparrow \infty$ as $|y| \to \infty$, and $k'(y)/k(y) \to 0$ as $y \to \infty$.

If R_t^* is the position of the rightmost particle at time t, then

$$P(R_t^* < x) = P(N_t^*[x,\infty) = 0),$$

where $N_t^*(J)$, $J \subseteq \mathbf{R}$ represents the number of PTW particles in J time t, and is a Poisson random variable, with $EN_t^*(J) = \int_{-M}^{M} \int_{-\infty}^{t} Ce^{\lambda s} P^z(X_{t-s} \in J) ds\mu(dz)$. Here X denotes killed Brownian motion.

An analogue of Watanabe's theorem holds for the PTW (proved in Section 6), and thus $R_t^* \to \infty$ a.s. as $t \to \infty$.

THEOREM 1′: $\overline{\lim_{t\to\infty}} R_t^*/x_t \le 1$ a.s. where x_t is such that $\Phi(x_t) = \int_0^{x_t} \sqrt{2k(y)}dy = \lambda t$.

PROOF: For $x > 0$, define $T_x = \inf\{s : X_s \ge x\}$. By Lemma 2,

$$\begin{aligned} EN_t^*[x,\infty) &= C \int_{-M}^{M} \int_{-\infty}^{t} e^{\lambda s} P^z(X_{t-s} \ge x) ds\mu(dz) \\ &\le C \int_{-M}^{M} \int_{-\infty}^{t} e^{\lambda s} P^z(T_x < \infty) ds\mu(dz) \\ &\le (C/\lambda)\mu(-M,M) e^{H(M)} e^{\lambda t - H(x)} \le A_1 e^{\lambda t - H(x)} \end{aligned} \tag{4.1}$$

where $H(x) \sim \Phi(x) = \int_0^x \sqrt{2k(y)}dy$ as $x \to \infty$, and $A_1 > 0$ is some constant.

Now, $\Phi(x)$ is strictly increasing for $x > 0$, and continuous, so there exists x_t such that $\Phi(x_t) = \lambda t$. Note that for any $\varepsilon > 0$,

$$\begin{aligned} \Phi((1+\varepsilon)x_t) = \int_0^{(1+\varepsilon)x_t} \sqrt{2k(y)}dy &= \lambda t + \int_{x_t}^{x_t(1+\varepsilon)} \sqrt{2k(y)}dy \\ &\ge \lambda t + \varepsilon x_t \sqrt{2k(x_t)} \\ &\ge \lambda t + \varepsilon \int_0^{x_t} \sqrt{2k(y)}dy = \lambda t(1+\varepsilon). \end{aligned}$$

Therefore, there exists $t_\varepsilon > 0$ such that $H((1+\varepsilon)x_t) \ge (1+\frac{\varepsilon}{2})\lambda t \ \forall\, t \ge t_\varepsilon$, and we have $\forall\, t \ge t_\varepsilon$

$$\begin{aligned} P(R_t^* > (1+\varepsilon)x_t) &= P(N_t^*[(1+\varepsilon)x_t,\infty) \ge 1) \\ &\le EN_t^*[(1+\varepsilon)x_t,\infty) \\ &\le A_1 e^{\lambda t - H((1+\varepsilon)x_t)} \\ &\le A_1 e^{-\frac{\varepsilon\lambda t}{2}}. \end{aligned} \tag{4.2}$$

From this it follows immediately that $\overline{\lim_{t\to\infty}} R_t^*/x_t \overset{p}{\le} 1$.

To show this holds true almost surely, consider the event

$$F_n = \{R_t^* > (1+\varepsilon)x_t \text{ for some } t \in (n, n+1)\}, \ n \in \mathbb{Z}^+.$$

Now

$$\begin{aligned} P(F_n) &\le P(R_t^* > (1+\varepsilon)x_n \text{ for some } t \in (n, n+1)) \\ &\le P(Y_n \ge 1) \le EY_n \end{aligned}$$

where Y_n, $n \ge 1$ is the number of PTW particles whose positions exceed $(1+\varepsilon)x_n$ some time in $(n, n+1)$. Y_n is a Poisson random variable with mean

$$\begin{aligned} EY_n &= \int_{-M}^{M}\int_{-\infty}^{n+1} Ce^{\lambda s}P^z(X_{t-s} > (1+\varepsilon)x_n \text{ for some } t \in (n, n+1))ds\mu(dz) \\ &\le \int_{-M}^{M}\int_{-\infty}^{n+1} Ce^{\lambda s}P^z(T_{(1+\varepsilon)x_n} < \infty)ds\mu(dz). \end{aligned}$$

Therefore, by (4.1), for all $n > t_\varepsilon$,

$$EY_n \le A_1 e^{-\varepsilon\lambda n/2}. \tag{4.3}$$

Therefore, $\sum_{n=1}^{\infty} P(F_n) < \infty$, which implies that $P(F_n \ i.o.) = 0$, by the Borel-Cantelli Lemma. Thus with probability 1 $\exists \ t_\varepsilon < \infty$ such that $\forall \ t \ge t_\varepsilon$, $R_t^* \le (1+\varepsilon)x_t$. This concludes the proof. $\square$

THEOREM 2′: If $k(y) = |y|^\alpha$, $\alpha > 0$ then $R_t^* \sim x_t$ a.s., where x_t satisfies $\Phi(x_t) = \int_0^{x_t} \sqrt{2k(y)}dy = \lambda t$ and thus equals $c_\alpha t^{2/\alpha+2}$, where $c_\alpha = (\frac{\alpha+2}{2\sqrt{2}}\lambda)^{2/\alpha+2}$.

PROOF: By Theorem 1′ it suffices to show that $\overline{\lim_{t\to\infty}} R_t^*/x_t \ge 1$ a.s., i.e. to show that given any $\varepsilon > 0$, with probability 1 there exists t_ε such that $R_t^* \ge (1-\varepsilon)x_t \ \forall \ t \ge t_\varepsilon$.

We consider the cases $0 < \alpha < 2$ and $\alpha \ge 2$ separately.

(i) $0 < \alpha < 2$. Recall from (3.4) that for any $\delta > 0 \ \exists \ x_\delta > 0$ such that for any $z \in [-M, M]$,

$$P^z(X_s \ge x) \ge f(z)e^{-(\Phi(x)+\gamma s)(1+\delta)}$$

$\forall \ x > x_\delta$ and $s > \frac{\sqrt{2}}{2-\alpha}x^{\frac{2-\alpha}{2}}$, for some $f(z)$ bounded below on $[-M, M]$.

Therefore, there exists $t'_\varepsilon > 0$ and positive constants a_2 and γ_1 such that for all $t \ge t_\varepsilon$, and $s \ge a_2 t^{\frac{2-\alpha}{2+\alpha}}$,

$$P^z(X_s \ge (1-\varepsilon)x_t) \ge f(z)e^{-(1-\varepsilon/2)\lambda t - \gamma_1 s}.$$

Thus there exists $t_\varepsilon > t'_\varepsilon$ such that $\forall\, t \geq t_\varepsilon$

$$\begin{aligned} EN_t^*[(1-\varepsilon)x_t, \infty) &= Ce^{\lambda t}\int_{-M}^{M}\int_0^\infty e^{-\lambda s}P^z(X_s \geq (1-\varepsilon)x_t)ds\mu(dz) \\ &\geq Ce^{\varepsilon\lambda t/2}(\int_{-M}^{M} f(z)\mu(dz))\int_{a_2 t^{\frac{2-\alpha}{2+\alpha}}}^\infty e^{-(\lambda+\gamma_1)s}ds \\ &\geq A_2 e^{\varepsilon\lambda t/4} \end{aligned}$$

for some constant $A_2 > 0$.

Therefore, for all $t \geq t_\varepsilon$,

$$P(R_t^* < (1-\varepsilon)x_t) \leq e^{-A_2 e^{\frac{\varepsilon\lambda t}{4}}} \tag{4.4}$$

This implies that $\sum_{n=1}^{\infty} P(R_n^* < (1-\varepsilon)x_n) < \infty$, so by the Borel-Cantelli Lemma,

$$P(R_n^* < (1-\varepsilon)x_n \ i.o.) = 0 \tag{4.5}$$

Now define W_n to be the number of PTW particles in existence at time $n+1$ whose position exceeds $(1-\varepsilon/2)x_{n+1}$, and whose position at some $t \in [n, n+1)$ is to the left of $(1-\varepsilon)x_{n+1}$. Then W_n is Poisson with mean

$$\begin{aligned} EW_n = &\int_{-M}^{M}\int_{-\infty}^{n+1} Ce^{\lambda s}P^z(X_{n+1-s} > (1-\varepsilon/2)x_{n+1}, \quad X_{t-s} < (1-\varepsilon)x_{n+1} \\ &\qquad \text{for some } t \in (n, n+1))ds\mu(dz) \\ \leq &\left(\frac{C}{\lambda}\right)e^{\lambda(n+1)} \cdot \mu(-M, M) \cdot P^{(1-\varepsilon)x_{n+1}}(\max_{0\leq s\leq 1} B_s > (1-\varepsilon/2)x_{n+1}) \cdot \\ \leq &A_3 e^{\lambda(n+1)-(\varepsilon^2/16)x_{n+1}^2} = A_3 e^{\lambda(n+1)-\frac{\varepsilon^2 c_\alpha^2}{16}(n+1)^{4/\alpha+2}} \end{aligned} \tag{4.6}$$

$\forall\, n > N_\varepsilon$, for some $N_\varepsilon > 0$, and constant $A_3 > 0$.

Therefore, if we define the event

$$E_n = (R_{n+1}^* > (1-\varepsilon/2)x_{n+1}, R_t^* < (1-\varepsilon)x_t \text{ for some } t \in (n, n+1)),$$

then

$$\sum_{n\geq 1} P(E_n) \leq \sum_{n\geq 1} P(W_n \geq 1) \leq \sum_{n\geq 1} EW_n < \infty$$

and therefore $P(E_n \ i.o.) = 0$ by the Borel-Cantelli Lemma. Together with (4.5) this gives the desired conclusion.

(ii) $\alpha \geq 2$. In this case, from (3.4) we have for any $\delta > 0$, x_δ, $s_\delta > 0$ such that $\forall\, x \geq x_\delta$ and $s \geq s_\delta$

$$P^z(X_s \geq x) \geq f(z)e^{-(\Phi(x)+\gamma s)(1+\delta)}.$$

Therefore there exists t'_ε and $s_\varepsilon > 0$ and a constant $\gamma_2 > 0$ such that for any $z \in [-M, M]$,

$$P^z(X_s \geq (1-\varepsilon)x_t) \geq f(z)e^{-(1-\varepsilon/2)\lambda t - \gamma_2 s}$$

for all $t \geq t'_\varepsilon$, $s \geq s_\varepsilon$.

Thus, there exists $t_\varepsilon > \max\{t'_\varepsilon, s_\varepsilon\}$ such that

$$\begin{aligned} EN_t^*[(1-\varepsilon)x_t, \infty) &= Ce^{\lambda t}\int_{-M}^{M}\int_0^\infty e^{-\lambda s}P^z(X_s \geq (1-\varepsilon)x_t)ds \\ &\geq Ce^{\varepsilon/2\lambda t}\Big(\int_{-M}^{M} f(z)\mu(dz)\Big)\int_{s_\varepsilon}^\infty e^{-(\lambda+\gamma_1)s}ds \\ &\geq A_4 e^{\varepsilon\lambda t/2} \end{aligned}$$

for all $t \geq t_\varepsilon$, and some constant $A_4 > 0$, so for all $t \geq t_\varepsilon$,

$$P(R_t^* < (1-\varepsilon)x_t) \leq e^{-A_4 e^{\varepsilon\lambda t/2}}. \tag{4.7}$$

Now define a sequence of times $t_n \uparrow \infty$ by marking all integers $m \geq 1$ and also points in $[m, m+1]$ with spacing $1/m$. Then, from (4.7)

$$\begin{aligned} \sum_{t_n \geq t_\varepsilon} P(R_{t_n}^* < (1-\varepsilon)x_{t_n}) &\leq \sum_{m > t_\varepsilon}\ \sum_{m < t_n \leq m+1} e^{-A_4 e^{\varepsilon\lambda t_n/2}} \\ &\leq \sum_{m > t_\varepsilon} m e^{-A_4 e^{\varepsilon\lambda m/2}} < \infty. \end{aligned}$$

Therefore, by the Borel-Cantelli Lemma,

$$P(R_{t_n}^* < (1-\varepsilon)x_{t_n}\ i.o.) = 0. \tag{4.8}$$

Next, let U_n be the number of PTW particles whose positions at time t_{n+1} are to the right of $(1-\varepsilon)x_{t_{n+1}}$ and whose positions are to the left of $(1-\varepsilon)x_{t_{n+1}}$ some time in $[t_n, t_{n+1})$. Then U_n is a Poisson random variable with mean

$$\begin{aligned} EU_n = \int_{-M}^{M}\int_{-\infty}^{t_{n+1}} & Ce^{\lambda s}P^z(X_{t_{n+1}-s} > (1-\varepsilon/2)x_{t_{n+1}}, \\ & X_{t-s} < (1-\varepsilon)x_{t_{n+1}} \quad \text{for some} \quad t \in (t_n, t_{n+1}))ds\mu(dz) \\ \leq\ & Ce^{\lambda t_{n+1}}P^{(1-\varepsilon)x_{t_{n+1}}}\big(\max_{0 \leq s \leq t_{n+1}-t_n} > (1-\frac{\varepsilon}{2})x_{t_{n+1}}\big)\cdot\mu(-M, M) \\ \leq\ & A_5 e^{\{\lambda t_{n+1} - (\varepsilon^2/16)\frac{x_{t_{n+1}}^2}{t_{n+1}-t_n}\}} \end{aligned} \tag{4.9}$$

for all $t_{n+1} \geq t_\varepsilon$, for some $t_\varepsilon > 0$, and some $A_5 > 0$. Now let $E'_n = (R_{t_{n+1}}^* > (1-\varepsilon/2)x_{t_{n+1}}$, $R_t^* < (1-\varepsilon)x_t$ for some $t \in (t_n, t_{n+1}))$, then

$$\sum_{n \geq 1} P(E'_n) \leq \sum_{n \geq 1} P(U_n \geq 1) \leq \sum_{n \geq 1} EU_n.$$

Now,

$$\sum_{t_{n+1}>t_\varepsilon} EU_n \le \sum_{m\ge t_\varepsilon} \sum_{m<t_{n+1}\le m+1} A_5 e^{\lambda t_{n+1} - \frac{\varepsilon^2 c_\alpha^2}{16} m t_{n+1}^{4/2+\alpha}}$$
$$\le \sum_{m\ge t_\varepsilon} A_5 m e^{\lambda(m+1) - \frac{\varepsilon^2 c_\alpha^2}{16} m^{1+\frac{4}{2+\alpha}}}$$
$$< \infty.$$

Therefore, $\sum_{n\ge 1} P(E_n') < \infty$, and by the Borel-Cantelli Lemma once more we have $P(E_n' \ i.o.) = 0$. Thus for any $\varepsilon > 0$ with probability 1 there exists t_ε such that $R_t^* > (1-\varepsilon)x_t \ \forall\, t \ge t_\varepsilon$. □

5. Convergence Results For R_t

We use theorems 1′ and 2′, and Proposition 1 to establish the main results of the paper.

Let $C > 0$ be arbitrary and let R_t^* now denote the right frontier at time t of a Poisson tidal wave with birth intensity measure $Ce^{\lambda_0 t}\beta(y)\nu(dy)dt$, where λ_0 and ν are as in Watanabe's Theorem.

By Proposition 1, there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ then

$$P(R_t \le R_t^* + \delta \text{ eventually } |Z < C/2) = 1 \tag{5.1}$$
$$P(R_t \ge R_t^* - \delta \text{ eventually } |Z > 2C) = 1. \tag{5.2}$$

Proof of Theorem 1

(a) By Theorem 1′, given any $\varepsilon > 0$,

$$P(R_t^* \le (1+\varepsilon/2)x_t \text{ eventually}) = 1. \tag{5.3}$$

Now for any $\delta \le \delta_0$ and $C > 0$,

$$P(R_t \le (1+\varepsilon)x_t \text{ eventually } |Z < C/2)$$
$$\ge P(R_t \le R_t^* + \delta, R_t^* < (1+\varepsilon)x_t - \delta \text{ eventually } |Z < C/2)$$
$$\ge P(R_t \le R_t^* + \delta, R_t^* < (1+\varepsilon/2)x_t \text{ eventually } |Z < C/2).$$

It immediately follows from (5.1) and (5.3) that

$$P(R_t \le (1+\varepsilon)x_t \text{ eventually } |Z < C/2) = 1$$

and therefore, since C is arbitrary,

$$P(R_t \le (1+\varepsilon)x_t \text{ eventually}) = 1.$$

(b) Recall that $\int_0^{x_t} \Phi(y)dy = \lambda t$. Since $k(y) \uparrow \infty$ as $|y| \to \infty, x_t/t \to 0$ as $t \to \infty$. It is then immediate from (a) that $R_t/t \to 0$ a.s. as $t \to \infty$.

Proof of Theorem 2

It is sufficient to show that $\varliminf_{t\to\infty} R_t/x_t \geq 1$ a.s. since k satisfies the conditions of Theorem 1. From Theorem 2′, we have for any $\varepsilon > 0$,

$$P(R_t^* \geq (1-\varepsilon/2)x_t \text{ eventually }) = 1. \tag{5.4}$$

Now for any positive $\delta \leq \delta_0$,

$$\begin{aligned} &P(R_t \geq (1-\varepsilon)x_t \text{ eventually } |Z > 2C) \\ &\geq P(R_t \geq R_t^* - \delta, R_t^* > (1-\varepsilon)x_t + \delta \text{ eventually } |Z > 2C) \\ &\geq P(R_t \geq R_t^* - \delta, R_t^* > (1-\varepsilon/2)x_t \text{ eventually } |Z > 2C). \end{aligned}$$

Therefore, by (5.2) and (5.4), for any $C > 0$ we have

$$P(R_t > (1-\varepsilon)x_t \text{ eventually } |Z > 2C) = 1.$$

The desired conclusion follows. □

6. Coupling Argument

Before proving Proposition 1, we prove an analogue of Watanabe's theorem for Poisson tidal waves. Let, $\beta(y)$, $k(y)$, λ_0, φ_0 and ν be as in Watanabe's theorem. Consider a PTW with birth intensity measure $Ce^{\lambda_0 t}\beta(y)\nu(dy)dt$ and killing rate function $k(y)$, where $(t,y) \in \mathbf{R}^2$ and $C > 0$ is a constant. Recall that the support of β is assumed to be contained in $[-M, M]$ for some M, $\nu(J) = \int_J \varphi_0(y)dy$ for any $J \subseteq \mathbf{R}$, and that $N_t^*(J)$ denotes the number of PTW particles in J at time t.

PROPOSITION 3: If J is a bounded interval, then $N_t^*(J) \sim Ce^{\lambda_0 t}\nu(J)$ a.s. as $t \to \infty$.

PROOF: $N_t^*(J)$ is a Poisson random variable with mean

$$\begin{aligned} \eta_t = \eta_t(J) &= \int_{-\infty}^{t} \int_{-M}^{M} Ce^{\lambda_0 s} P^z(X_{t-s} \in J)\beta(z)\nu(dz)ds \\ &= \int_{-\infty}^{t} \int_{-M}^{M} Ce^{\lambda_0 s} E^z(e^{-\int_0^{t-s} k(B_u)du} 1(B_{t-s} \in J))\beta(z)\varphi_0(z)dzds, \end{aligned}$$

where B is standard Brownian motion. Recall that λ_0 and φ_0 are the (leading) eigenvalue and eigenvector of the differential operator $\frac{1}{2}D_x^2 + \beta - k$. The above expression therefore reduces to $Ce^{\lambda_0 t} \int_J \varphi_0(z)dz$ i.e. $\eta_t = Ce^{\lambda_0 t}\nu(J)$ for all $t \in \mathbf{R}$.

To see this, note that if $w(z) = \int_0^\infty e^{-\lambda_0 s} E^z(e^{-\int_0^s k(B_u)du} 1\{B_s \in J\})ds$ then $(\mathcal{L}w)(z) = -1(z \in J)$, where $\mathcal{L} = \frac{1}{2}D_z^2 - \lambda_0 - k$, and

$$\eta_t = Ce^{\lambda_0 t} \int_{-M}^{M} w(z)\beta(z)\varphi_0(z)dz = -Ce^{\lambda_0 t} \int_{-M}^{M} w(z)(\mathcal{L}\varphi_0)(z)dz.$$

But $\mathcal{L}$ is self-adjoint in $\mathcal{L}^2$, so $\int_{-M}^{M} w(z)(\mathcal{L}\varphi_0)(z)dz = \int_{-M}^{M} \varphi_0(z)(\mathcal{L}w)(z)dz = -\int_{-M}^{M} 1\{z \in J\}\varphi_0(z)dz = -\nu(J)$.

Now write $N_t = N_t^*(J)$. We prove that $N_t \sim \eta_t$ a.s. as $t \to \infty$ by producing a sequence of times $0 < t_n \uparrow \infty$ such that

(i) $N_{t_n} \sim \eta_{t_n}$ a.s. as $n \to \infty$, and

(ii) for any $\varepsilon > 0$, with probability 1 the events

$$F_n = F_n^\varepsilon = \{N_{t_n} < (1+\varepsilon/2)\eta_{t_n},\ N_t > (1+\varepsilon)\eta_t \text{ for some } t \in (t_n, t_{n+1})\} \text{ and}$$
$$G_n = G_n^\varepsilon = \{N_{t_{n+1}} > (1-\varepsilon/2)\eta_{t_{n+1}},\ N_t < (1-\varepsilon)\eta_t \text{ for some } t \in (t_n, t_{n+1})\}$$

occur at most finitely often. This finishes the proof of the Proposition.

Suppose we define $\{t_n\}_{n\ge1}$ by marking points in $\mathbb{Z}^+$, and also points in $[m, m+1]$, $m \in \mathbb{Z}^+$ by sub-dividing it into m equally spaced intervals. Thus if $[t_n, t_{n+1}] \subseteq [m, m+1]$ then $t_{n+1} - t_n = 1/m$.

We show now that this sequence satisfies (i) and (ii). First, since N_t is Poisson with mean η_t, for any $\varepsilon > 0$ we have

$$\sum_{n\ge1} P(|N_{t_n} - \eta_{t_n}| > \varepsilon\eta_{t_n}) \le \sum_{n\ge1} 1/\varepsilon^2\eta_{t_n} = \sum_{n\ge1} \frac{e^{-\lambda_0 t_n}}{\varepsilon^2 C\nu(J)} \le \sum_{m\ge1} \frac{me^{-\lambda_0 m}}{\varepsilon^2 C\eta(J)} < \infty.$$

By the Borel-Cantelli Lemma, therefore, $P(|N_{t_n} - \eta_{t_n}| > \varepsilon\eta_{t_n}\ i.o.) = 0$. Thus with probability 1 there exists $n_\varepsilon > 0$ such that $|N_{t_n}/\eta_{t_n} - 1| < \varepsilon$ for all $n \ge n_\varepsilon$.

Next, by Lemma 9, proved below, there exists constants θ, δ and A_0 such that

$$P(F_n) \le \exp\left\{-\frac{\theta\varepsilon}{4}\eta_{t_n} + \{4(1-\phi(\delta/\sqrt{t_{n+1}-t_n})) + e^{\lambda_0(t_{n+1}-t_n)} - 1\}A_0(1+\varepsilon/2)\theta\eta_{t_n}\right\},$$

and

$$P(G_n) \le \exp\left\{-\frac{\theta\varepsilon}{4}\eta_{t_{n+1}} + 4A_0\left(1-\phi\left(\delta/\sqrt{t_{n+1}-t_n}\right)\right)\left(1+\frac{\varepsilon}{2}\right)\theta\eta_{t_{n+1}}\right\},$$

where $\phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy$.

Suppose $[t_n, t_{n+1}] \subseteq [m, m+1]$ and $m \geq m_\varepsilon$ where m_ε is large enough that

$$\begin{aligned}&(1+\varepsilon/2)\{4(1-\phi(\delta/\sqrt{t_{n+1}-t_n})) + e^{(t_{n+1}-t_n)\lambda_0} - 1\}A_0 \\ &= (1+\varepsilon/2)\{4(1-\phi(\delta\sqrt{m}) + e^{\lambda_0/m} - 1\} \\ &\leq \frac{\varepsilon}{8}.\end{aligned}$$

Then

$$\begin{aligned}P(F_n) &\leq e^{-\frac{\theta\varepsilon}{8}\eta_{t_n}} \leq e^{-\frac{\theta\varepsilon}{8}\eta_m}, \\ P(G_n) &\leq e^{-\frac{\theta\varepsilon}{8}\eta_{t_{n+1}}} \leq e^{-\frac{\theta\varepsilon}{8}\eta_m}.\end{aligned}$$

Thus, recalling that $\eta_m = C\nu(J)e^{\lambda_0 m}$, we have

$$\sum_{n\geq 1} P(F_n) = \sum_{m=1}^{\infty} \sum_{m\leq t_n<m+1} P(F_n) \leq \sum_{m<m_\varepsilon} \sum_{m\leq t_n<m+1} P(F_n) + \sum_{m\geq m_\varepsilon} m e^{-\frac{\theta\varepsilon}{8}\eta_m} < \infty$$

Similarly $\sum_{n\geq 1} P(G_n) < \infty$. Therefore, by another application of the Borel-Cantelli Lemma, we have $P(F_n\ i.o.) = 0$, and $P(G_n\ i.o.) = 0$. □

LEMMA 9. Fix t, s and $\varepsilon > 0$, and define events

$$\begin{aligned}F &= \{N_t < (1+\varepsilon/2)\eta_t,\ N_u > (1+\varepsilon)\eta_u \text{ for some } u \in (t, t+s)\}, \\ G &= \{N_{t+s} > (1-\varepsilon/2)\eta_{t+s},\ N_u < (1-\varepsilon)\eta_u \text{ for some } u \in (t, t+s)\}\end{aligned}$$

Then there exists constants θ, δ and A_0 such that

$$\begin{aligned}P(F) &\leq \exp\left\{-\frac{\theta\varepsilon}{4} + (4(1-\phi(\delta/\sqrt{s})) + e^{\lambda_0 s} - 1)A_0\theta(1+\varepsilon/2)\right\}\eta_t \\ P(G) &\leq \exp\left\{-\frac{\theta\varepsilon}{4} + 4(1-\phi(\delta/\sqrt{s}))A_0\theta(1+\varepsilon/2)\right\}\eta_t.\end{aligned}$$

PROOF: Let Y be the number of PTW particles which enter J some time during $(t, t+s]$ but were either in $\mathbf{R}\backslash J$ at time t or were not in existence at time t, and let W be the number of PTW particles in J at time $t+s$ which were outside J sometime during $[t, t+s)$.

We prove the Lemma in 3 steps.

(1) There exists $\theta > 0$ such that
(a) $P(F) \leq e^{-\frac{\theta\varepsilon}{2}\eta_t + \theta(1+\varepsilon/2)EY}$,
(b) $P(G) \leq e^{-\frac{\theta\varepsilon}{2}\eta_{t+s} + \theta(1+\varepsilon/2)EW}$.

Proof: We use Chebyshev's exponential inequality. For any $\theta > 0$,

$$\begin{aligned} P(F) &\leq P(N_t < (1+\varepsilon/2)\eta_t,\ N_u > (1+\varepsilon)\eta_t \text{ for some } u \in (t,t+s)) \\ &\leq P(N_t < (1+\varepsilon/2)\eta_t,\ N_t + Y > (1+\varepsilon)\eta_t) \\ &\leq P(Y > \varepsilon\eta_t/2) \\ &\leq e^{-\frac{\theta\varepsilon}{2}\eta_t} E(e^{\theta Y}) = e^{-\frac{\theta\varepsilon}{2}\eta_t + (e^\theta - 1)EY}, \end{aligned}$$

and

$$\begin{aligned} P(G) &\leq P(N_{t+s} > (1-\varepsilon/2)\eta_{t+s},\ N_u < (1-\varepsilon)\eta_{t+s} \text{ for some } u \in (t,t+s)) \\ &\leq P(W \geq \varepsilon\eta_{t+s}/2) \\ &\leq e^{-\frac{\theta\varepsilon}{2}\eta_{t+s} + (e^\theta - 1)EW}. \end{aligned}$$

Now choose θ such that $e^\theta - 1 = (1 + (\varepsilon/2))\theta$.

(2) There exists constants δ and A_0 such that

$$EY \leq \eta_t\{4A_0(1-\phi(\delta/\sqrt{s})) + A_0(e^{\lambda_0 s} - 1) + \varepsilon/6\}$$

Proof: Suppose $J = [a, b]$ for some finite constants a, b. Let $\delta > 0$ be arbitrary, and define $J_1^\delta = [a-\delta, a) \cup (b, b+\delta]$.

Now $Y = Y^{(1)} + Y^{(2)}$ where $Y^{(1)}$ is the number of PTW particles in $\mathbf{R}\backslash J$ at time t which enter J during $(t, t+s]$, and $Y^{(2)}$ is the number of PTW particles born during $(t, t+s)$ which enter J before $t+s$.

If X is the trajectory of a single particle, then for any $u > 0$, $v > 0$ and $z \in \mathbf{R}$,

$$\begin{aligned} &P^z(X_u \notin J,\ X_r \in J \text{ for some } r \in (u, u+v)) \\ &= P^z(X_u > b+\delta,\ X_r < b \text{ for some } r \in (u,u+v)) \\ &\quad + P^z(X_u < a-\delta,\ X_r > a \text{ for some } r \in (u,u+v)) \\ &\quad + P^z(X_u \in J_1^\delta,\ X_r \in (a,b) \text{ for some } r \in (u,u+v)) \\ &\leq P(\max_{0\leq r\leq v} B_r > \delta) + P(\min_{0\leq r\leq v} B_r < -\delta) + P^z(X_u \in J_1^\delta) \\ &= 4P(B_v > \delta) + P^z(X_u \in J_1^\delta). \end{aligned}$$

Therefore,

$$\begin{aligned} EY^{(1)} &= \int_{-M}^{M}\int_{-\infty}^{t} Ce^{\lambda_0 u} P^z\left(X_{t-u} \notin J,\ X_{r-u} \in J \text{ for some } r \in (s, t+s)\right)\beta(z)du\nu(dz) \\ &\leq 4CP(B_s > \delta)\int_{-M}^{M}\int_{-\infty}^{t} e^{\lambda_0 u}\beta(z)ds\nu(dz) \\ &\quad + \int_{-M}^{M}\int_{-\infty}^{t} Ce^{\lambda_0 u}P^z(X_{t-u} \in J_1^\delta)\beta(z)ds\nu(dz) \\ &= \left(\frac{4C}{\lambda_0}\int_{-M}^{M}\beta(z)\nu(dz)\right)e^{\lambda_0 t}P(B_s > \delta) + Ce^{\lambda_0 t}\nu(J_1^\delta) \end{aligned}$$

Define $A_0 = (1/\lambda_0\nu(J))\int_{-M}^{M}\beta(z)\nu(dz)$, and choose δ such that $\nu(J_1^\delta) < \varepsilon\nu(J)/6$. Then we have $EY_t^{(1)} \le (4A_0P(B_s > \delta) + \varepsilon/6)\eta_t$. Finally,

$$
\begin{aligned}
EY^{(2)} &= \int_{-M}^{M}\int_t^{t+s} Ce^{\lambda_0 u}P^z(X_{r-u}\in J \text{ for some } r\in(u,t+s))\beta(z)ds\nu(dz) \\
&\le \left(\frac{C}{\lambda_0}\right)\int_{-M}^{M}\beta(z)\nu(dz))(e^{\lambda_0 s}-1)e^{\lambda_0 t} = A_0(e^{\lambda_0 s}-1)\eta_t.
\end{aligned}
$$

(3) $EW \le \eta_t(4A_0(1-\phi(\delta/\sqrt{s})) + \varepsilon/6)$ where A_0 and δ are as in (2).

Proof: Suppose $J=[a,b]$ as before, and suppose the δ chosen in (2) is small enough that $\delta < \frac{b-a}{2}$ and $\nu(J_2^\delta) < \varepsilon\nu(J)/6$ where $J_2^\delta = [a,a+\delta)\cup(b-\delta,b]$.

Write $W = W^{(1)} + W^{(2)}$ where $W^{(1)}$ and $W^{(2)}$ are the number of PTW particles in $[a+\delta, b-\delta]$ and in J_2^δ respectively at time $t+s$, that were outside $[a,b]$ some time in $[t,t+s)$. Clearly

$$EW^{(2)} \le EN_{t+s}^*(J_2^\delta) = C\nu(J_2^\delta)e^{\lambda_0(t+s)} \le \varepsilon\eta_{t+s}/6,$$

while

$$
\begin{aligned}
EW^{(1)} = \int_{-M}^{M}\int_{-\infty}^{t+s} & Ce^{\lambda_0 u}P^z(X_{t+s-u}\in[a+\delta,b-\delta], \\
& X_{r-u}\notin[a,b] \text{ for some } r\in[t,t+s))\beta(z)ds\nu(dz).
\end{aligned}
$$

Now for any u, $v<u$, and $z\in\mathbf{R}$,

$$
\begin{aligned}
& P^z(X_u\in[a+\delta,b-\delta],\ X_r\notin[a,b] \text{ for some } r\in(u-v,u)) \\
&= P^z(X_u > a+\delta, X_r < a \text{ for some } r\in(u-v,u)) + \\
&\quad P^z(X_u < b-\delta,\ X_r > b \text{ for some } r\in(u-v,u)) \\
&\le P(\max_{0\le r\le v} B_r > \delta) + P(\min_{0\le r\le v} B_r \le -\delta) \\
&= 4P(B_v > \delta)
\end{aligned}
$$

Thus

$$
\begin{aligned}
EW^{(1)} &\le \int_{-M}^{M}\int_{-\infty}^{t+s} 4Ce^{\lambda_0 u}P(B_s>\delta)\beta(z)ds\nu(dz) \\
&= 4A_0C\nu(J)P(B_s>\delta)e^{\lambda_0(t+s)} \\
&= 4A_0P(B_s>\delta)\eta_{t+s} \qquad \square
\end{aligned}
$$

Proof of Proposition 1

Start with a copy of the branching Brownian motion (BBM) and an independent Poisson process on $\mathbf{R}^2$ with intensity measure $Ce^{\lambda_0 t}\beta(x)\nu d(x)dt$. This is to be the point process of births for a Poisson tidal wave which we shall denote as $\mathcal{W}$. Assume that the same killing rate $k(x)$ applies to both of the above processes.

An individual particle in the $\mathcal{W}$ process executes a killed Brownian motion (starting at its birth point) independent of the BBM, the birth process and the motions of all other particles in $\mathcal{W}$, until the instant it is "paired" with a BBM particle, or it dies, whichever happens first. In the former case, it "shadows" the BBM particle, i.e. follows it keeping a constant distance, until the latter dies or enters a region where it will be "unpaired" (specified below). In either case, from the instant it becomes "unpaired", the $\mathcal{W}$-particle continues to move until the next time it is paired or itself dies. Note that the changes in the path of the $\mathcal{W}$-particle happen at stopping times. Thus its movement is a Brownian motion until the time when it is killed. Thus $\mathcal{W}$ is a Poisson Tidal wave.

We have two different pairing laws depending on whether we condition on (a) $\{Z < C/2\}$ or (b) $\{Z > 2C\}$. The pairing laws are constructed so that in the first case, a $\mathcal{W}$-particle "shadowing" a BBM particle dies before the BBM particle does, and in the latter case, survives longer.

We first consider case (a). First, note that no BBM or $\mathcal{W}$- particles are born outside $[-M, M]$ (which contains the support of β). Any BBM particle born in $[-M, M]$ is allowed to wander until it hits $2M$. At this instant, it is paired with the closest unpaired $\mathcal{W}$-particle in $[2M - \delta, 2M)$ if there is one, otherwise it continues moving unpaired. A paired BBM particle remains thus until it hits $2M - \delta/2$ from the right or it dies, whichever happens first. In the case of the former, at that instant it is uncoupled from its $\mathcal{W}$-shadow. If after this time the BBM particle again returns to $2M$, it is paired with a $\mathcal{W}$-particle (if available), and so on.

FIGURE 6

To summarize:

(i) Any unpaired BBM particle hitting $2M$ is immediately paired with a free $\mathcal{W}$-particle in $[2M-\delta, 2M)$ (if there is one) and,

(ii) any paired BBM particle hitting $2M-\delta/2$ is unpaired at that instant.

Thus no BBM particles in the region $(-\infty, 2M-\delta/2)$ are paired, and we will show that if δ is small enough, eventually, all BBM particles in $[2M, \infty)$ are accompanied by $\mathcal{W}$-particles within δ of their positions (by showing that (i) happens eventually with probability 1).

Since such a BBM particle is always to the right of its shadow, its rate of dying is greater in the region $[2M-\delta/2, \infty)$. We know R_t surpasses $2M$ eventually with probability 1, thus this scheme ensures that $R_t^* > R_t - \delta$ eventually with probability 1.

It remains to show that at large time, at the instant an (unpaired) BBM particle hits $2M$, we can find an unpaired $\mathcal{W}$-particle in $[2M-\delta, 2M)$ almost surely, for δ small enough.

To do this, for any $\delta > 0$, consider, at such an instant, all the $\mathcal{W}$-particles in $[2M-\delta, 2M)$ which are already paired. The BBM particles which are the partners of these $\mathcal{W}$-particles must lie in $[2M-\delta/2, 2M+\delta)$ (recall that BBM particles in $(-\infty, 2M-\delta/2)$ are unpaired according to our scheme).

We claim that there exists δ_0 such that for all $\delta \leq \delta_0$, $N_t^*[2M-\delta, 2M) > N_t[2M-\delta/2,\ 2M+\delta)$ eventually with probability 1, where $N_t^*(J)$ denotes the number of $\mathcal{W}$-particles in J at time t.

Now, by Proposition 3 and Watanabe's Theorem, given any $\varepsilon > 0$, with probability 1 there exists $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$,

$$\begin{aligned}&N_t^*[2M-\delta, 2M) - N_t[2M-\delta/2, 2M+\delta)\\&> (1-\varepsilon)C\nu[2M-\delta, 2M)e^{\lambda_0 t} - (1+\varepsilon)Z\nu[2M-\delta/2, 2M+\delta)e^{\lambda_0 t}\end{aligned}$$

Recall that $\nu(J) = \int_J \varphi_0(x)dx$ where φ_0 is continuous. Fix $\varepsilon > 0$ and let $\delta_0 > 0$ be such that $|\varphi_0(2M) - \varphi_0(x)| < \varepsilon$ for all $x \in [2M-\delta_0, 2M+\delta_0]$. Then for all $\delta \leq \delta_0$,

$$\nu[2M-\delta, 2M) > (\varphi_0(2M) - \varepsilon)\delta$$

and

$$\nu[2M-\delta/2, 2M+\delta) < (\varphi_0(2M)+\varepsilon)(\delta+\delta/2) = \frac{3\delta}{2}(\varphi_0(2M)+\varepsilon).$$

Also, we are conditioning on $Z < C/2$. So for all $t \geq t_\varepsilon$,

$$\begin{aligned}&N_t^*[2M-\delta, 2M) - N_t[2M-\delta/2, 2M+\delta)\\&> \{(1-\varepsilon)(\varphi_0(2M)-\varepsilon) - 3/2(1+\varepsilon)(\varphi_0(2M)+\varepsilon)\}C\delta e^{\lambda_0 t}\\&\geq \frac{1}{2}\{\varphi_0(2M) - 7\varepsilon(\varphi_0(2M)+1)\}C\delta e^{\lambda_0 t}\end{aligned}$$

Since we can choose $\varepsilon < \frac{\varphi_0(2M)}{7(\varphi_0(2M)+1)}$, we conclude that with probability 1, there exists T such that at all times $t \geq T$, an unpaired BBM particle hitting $2M$ immediately finds an unpaired $\mathcal{W}$-particle in $[2M - \delta, 2M)$. Indeed, there is an infinite "surplus" of such particles as $t \to \infty$.

Note that before time T there can be at most a finite number of BBM particles hitting $2M$ which do not find a $\mathcal{W}$-particle to pair with. These particles will not affect R_t as $t \to \infty$ as they will die in finite time.

A complementary pairing scheme can be given in case (b) i.e. when we condition on $(Z > 2C)$ and have to show that $R_t \geq R_t^* - \delta$ eventually with probability 1. We reverse the roles of the PTW and the BBM in the above argument. Any unpaired $\mathcal{W}$-particle crossing $2M$ from the left is paired with an available BBM particle within δ of it, to its left, i.e. in $(2M - \delta, 2M]$, and a paired $\mathcal{W}$-particle crossing $2M - \delta/2$ from the right is uncoupled from its BBM partner. The rest of the argument proceeds as above.

Finally, note that the pairing scheme is such that for any $s > 0$, the paths followed by particles in $\mathcal{W}$ born after times s are independent of the positions of particles of $\mathcal{W}$ and the BBM at time s.

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