DISTRIBUTIONS WHICH ARE GAUSSIAN CONVOLUTIONS

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Distributions Which Are Gaussian Convolutions*

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Abstract. Let $Z \sim N(0,1)$. We consider distributions on $\mathbb{R}$ which arise as convolutions with Z. The intersection of this class of convolutions with the family of normal scale mixtures is completely characterized and the implications are discussed. We also study the tail properties of the convolutions. A domain of attraction theorem is proved. Finally, we give a characterization of all random variables Y such that the convolution $Z+Y$ is unimodal and relate the number of modes of Y to that of the convolution. One particularly surprising example is given of a harshly oscillating density which becomes unimodal when convolved with Z.

1 Introduction

Let Z have a $N(0,1)$ distribution on the real line and let Y be another random variable independent of Z. The sum $X = Z + Y$ is the convolution of Z and Y. If the standard normal CDF is denoted by Φ and the CDF of Y is denoted by G, then sometimes we will also call the CDF of X, say, $F = \Phi * G$ as the convolution of Z and Y. This article attempts to clarify some basic but as yet unresolved issues about the family of all such convolutions F. We will call this family the class of Gaussian convolutions. The questions we raise pertain to some basic issues, such as which of the standard distributions on the line are Gaussian convolutions, how rich is the convolution class, what can be said about their tail properties, when are Gaussian convolutions unimodal, etc. The probabilistic aspects are more emphasized in this article; the statistical aspects are addressed in more detail in the companion paper DasGupta (1992). We will like to remind the reader that there are well known connections of the Gaussian convolution problem to the theory of analytic continuation in function theory; (see Pollard (1943, 1953)). These are elegant but do not address the concerns of direct relevance to a statistician.

2 Outline and an Illustrative Example

Generally speaking, it is believed and indeed it is true that the convolution F has heavier tails than the standard normal CDF. For instance, if Y is any symmetric

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random variable, then for all $t \geq 0$, $P(|X| \leq t) \leq P(|Z| \leq t)$. In other words, there is more mass in the tail of the distribution of $|X|$ than there is in the distribution of $|Z|$. Perhaps the most common model for a distribution heavier tailed than normals is the t-distribution. The following illustrative example asks if the t-distribution with one degree of freedom (i.e., the Cauchy distribution) is a Gaussian convolution. The result for general degrees of freedom will be subsumed in a more general result given in Sect. 3.

Example 1. Actually, the answer is no. Let us consider the standard Cauchy distribution; the same proof sails through for the most general case. Suppose the standard Cauchy random variable X is in fact a Gaussian convolution. Then, there exists Y, independent of Z, such that

$$Z + Y \stackrel{L}{=} X$$
$$\Rightarrow e^{-t^2/2} \cdot \psi(t) = e^{-|t|},$$

where $\psi(t)$ is the characteristic function of Y. Thus, $\psi(t) = e^{\frac{t^2}{2}-|t|}$, resulting in the obvious contradiction $\lim_{|t|\to\infty} \psi(t) = \infty$. The same argument also shows that no symmetric stable law (in particular the Cauchy) can be a Gaussian convolution. This can also be proved by making an appeal to Theorem 2 in Sect. 3. The fact that the Cauchy distribution (in fact, any t-distribution) is not a Gaussian convolution seems a little puzzling. These questions are probed more deeply in the companion paper DasGupta (1992). For instance, we demonstrate there that although it is not a Gaussian convolution, there exist elements of the convolution class which are tantalizingly close to it. In particular, there is a Gaussian convolution F with the associated probability measure P such that $|P(A) - Q(A)| \leq 0.0739$ for all measurable sets A, where Q denotes the probability measure associated with the standard Cauchy distribution.

In Sect. 3, we characterize all normal scale mixtures which are Gaussian convolutions (i.e., normal location mixtures). Some examples are given for illustration. A tail property is also proved in Sect. 3. The question of unimodality of a Gaussian convolution is addressed in Sect. 4. We give a characterization result using characteristic functions. The case of a lattice valued Y is considered as one of the illustrations. We also give an example in which the density of Y is severely oscillating, but the convolution is unimodal. We then also relate the number of modes of X to that of Y. Finally, it is proved that if $U \sim u[-1, 1]$, V is independent of U and is infinitely divisible, then the symmetric unimodal random variable $X = U \cdot V$ cannot be a Gaussian convolution. The result is again illustrated by examples.

3 Intersection with Scale Mixtures

Consider an absolutely continuous distribution on the real line with density

$$f(x) = \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} d\mu(\sigma^2) \tag{3.1}$$

f is called a normal scale mixture. We have a complete characterization of normal scale mixtures which are Gaussian convolutions. We will first prove two theorems and then give a few examples. The following notations will be used:

$$\mathcal{F}_c = \{F: F = \Phi * G \text{ for some } G, \text{ where } * \text{ denotes convolution}\}$$
$$\mathcal{F}_s = \{F: F \text{ is absolutely continuous with density } f \text{ of the form (3.1) for some } \mu\} .$$

Theorem 1. *Let $F = \Phi * G \in \mathcal{F}_c$. Then $F \in \mathcal{F}_s$ iff $G \in \mathcal{F}_s$ with the associated mixing measure μ (say). Furthermore, in this case, if the scale mixture representation of the density of F is given as*

$$f(x) = \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} d\nu(\sigma^2),$$

then for any measurable set $E \subseteq [0,\infty)$, $\nu(E) = \mu(E-1)$, where $E-1 = \{x-1: x \in E\}$.

Proof. Let $G \in \mathcal{F}_s$ and denote its characteristic function by $\psi(t)$. Thus,

$$\psi(t) = \int e^{-\frac{t^2\sigma^2}{2}} d\mu(\sigma^2) \ . \tag{3.2}$$

Therefore, if $\phi(t)$ denotes the characteristic function of F, then,

$$\begin{aligned}\phi(t) &= e^{-\frac{t^2}{2}} \cdot \int e^{-\frac{t^2\sigma^2}{2}} d\mu(\sigma^2) \\ &= \int e^{-\frac{t^2}{2}(1+\sigma^2)} d\mu(\sigma^2) \\ &= \int e^{-\frac{t^2}{2}\eta^2} d\nu(\eta^2),\end{aligned}$$

where ν denotes the distribution of $1+\sigma^2$ induced by μ. This already proves everything except the assertion that $G \in \mathcal{F}_s$ if $F \in \mathcal{F}_s$. However, if $F \in \mathcal{F}_s$, then using earlier notation,

$$\begin{aligned}\phi(t) &= \int e^{-\frac{t^2\sigma^2}{2}} d\nu(\sigma^2) \\ &= e^{-\frac{t^2}{2}} \cdot \psi(t) \\ \Rightarrow \psi(t) &= e^{\frac{t^2}{2}} \cdot \int e^{-\frac{t^2\sigma^2}{2}} d\nu(\sigma^2) \\ &= \int e^{-\frac{t^2}{2}(\sigma^2-1)} d\nu(\sigma^2)\end{aligned} \tag{3.3}$$

The RHS of (3.3) converges to $+\infty$ as $|t| \to \infty$ if $\nu[0,1) > 0$; this being impossible since ψ is a genuine characteristic function, one has $\nu[0,1) = 0$. If we now define a new measure μ on $[0,\infty)$ by the formula $\mu(E) = \nu(E+1)$, then (3.3) gives by the change of variable theorem that

$$\psi(t) = \int e^{-\frac{t^2}{2}\eta^2} d\mu(\eta^2), \tag{3.4}$$

which shows that $G \in \mathcal{F}_s$ and proves the theorem.

Theorem 2. *Let $F \in \mathcal{F}_s$ with the associated mixing measure ν. Then $F \in \mathcal{F}_c$ iff $\nu[0,1) = 0$. Furthermore, in this case, if F has the representation $F = \Phi * G$, then necessarily $G \in \mathcal{F}_s$ with the associated mixing measure μ satisfying $\mu(E) = \nu(E+1)$ for all measurable sets $E \subseteq [0, \infty)$.*

This theorem has been implicitly already proved in course of proving Theorem 1. We will therefore avoid the unnecessary duplication. The following corollary, although it brings home disappointing news, is interesting.

Corollary 1. *No t, Bessel, Double Exponential, Logistic distribution or no normal distribution with a variance less than one can be a Gaussian convolution.*

Proof. The proof follows from the fact that each of these distributions is in fact a normal scale mixture and the associated mixing measure μ gives positive mass to the interval $[0,1)$. Indeed, the mixing distributions are all well known; they are the inverse gamma, gamma, exponential, Polya, and a degenerate distribution respectively.

Example 2. The Hyperbolic cosine distribution with density

$$f(x) = \frac{2}{\pi(e^x + e^{-x})}, \quad -\infty < x < \infty$$

is also a normal scale mixture by virtue of complete monotonicity of $f(\sqrt{x})$. Theorem 2 can be used to show that this distribution cannot be a Gaussian convolution by solving an appropriate Laplace transform inversion problem to find the measure μ; see Widder (1951). A simpler proof comes out of the fact that its characteristic function is sech $(\frac{t\pi}{2})$. Since this is $O(e^{-\frac{\pi}{2}|t|})$, the argument of Example 1 can be repeated.

Example 3. We now know that the standard Double Exponential distribution with density $\frac{1}{2}e^{-|x|}$ is not a Gaussian convolution. Interestingly, we will see that the convolution of the $N(0,1)$ and the standard Double Exponential has exactly Double Exponential tails. Thus the Double Exponential distribution itself cannot be obtained as a Gaussian convolution, but its tails can be.

By direct calculations, this convolution has density

$$f(x) = \frac{1}{2\sqrt{e}} \left[e^{-x} \cdot \Phi(x-1) + e^x \cdot \Phi(-x-1) \right],$$

where Φ denotes the standard normal CDF. Clearly, $f(x)e^{|x|} = O(1)$.

This example motivates the following general result which says that in some sense the tail of the convolution $Z + Y$ is the same as that of Y.

Theorem 3. *Suppose Y is in the domain of attraction of a stable law of exponent α, $0 < \alpha \leq 2$. Then so is X.*

Proof. Denote the characteristic function of Y by $\psi(t)$ and define

$$\phi(t) = e^{-t^2/2} \cdot \psi(t) \ . \tag{3.5}$$

Notice $\phi(t)$ is the characteristic function of the convolution $X = Z + Y$. We will prove the theorem for the case of a symmetric stable law. The general case entertains the same argument. By hypothesis, there exist constants $a_n > 0$, b_n, and a slowly varying function L such that

$$\left(\psi\left(\frac{t}{a_n}\right) \cdot e^{-itb_n}\right)^n \to e^{-|t|^\alpha}, \tag{3.6}$$

and $a_n = n^{1/\alpha} \cdot L(n)$.

$$\begin{aligned} &\therefore \left(\phi\left(\frac{t}{a_n}\right) e^{-itb_n}\right)^n \\ &= \left(\psi\left(\frac{t}{a_n}\right) \cdot e^{-itb_n}\right)^n \cdot e^{-\frac{nt^2}{2a_n^2}} \\ &= \left(\psi\left(\frac{t}{a_n}\right) \cdot e^{-itb_n}\right)^n \cdot e^{-\frac{t^2 \cdot n}{2n^{2/\alpha} \cdot L^2(n)}} \\ &= \left(\psi\left(\frac{t}{a_n}\right) \cdot e^{-itb_n}\right)^n \cdot e^{-\frac{t^2}{2 \cdot L^2(n)} n^{\frac{\alpha-2}{\alpha}}} \end{aligned}$$

$\to e^{-|t|^\alpha}$ if $0 < \alpha < 2$ (by taking a subsequence, if necessary). The assertion of the theorem is immediate if $\alpha = 2$. This therefore proves the theorem.

4 Unimodality of Gaussian Convolutions

Shape properties of a density function are always of natural interest. By virtue of the strong unimodality of a $N(0,1)$ distribution, it is completely obvious that the convolution $X = Z + Y$ is unimodal if Y is unimodal (not necessarily symmetric). However, since convolution is a smoothing operation, it is to be expected that X can sometimes be unimodal even if Y is not. We will first give an upper bound on the number of (local) modes of X in terms of the number of (local) modes of Y when Y is absolutely continuous.

Definition 1. Let Y be absolutely continuous with a differentiable density $g(y)$. The real number y_0 is called a local mode of Y if $g'(y_0) = 0$ and there exist intervals $I_1 = (y_0 - \varepsilon_1, y_0)$ and $I_2 = (y_0, y_0 + \varepsilon_2)$ such that $g'(y) > 0$ for y in I_1 and $g'(y) < 0$ for y in I_2.

Remark. In other words, we are defining a local mode as a point where g' changes sign from positive to negative. Notice that according to our definition, a 'shoulder' will not count as a local mode.

Theorem 4. *Let Y be absolutely continuous with a differentiable density g and suppose Y has at most k local modes. Then the convolution $X = Z + Y$ has at most $k + 1$ local modes.*

Proof. If the density of X is called f, then

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (y-x) e^{-\frac{1}{2}(y-x)^2} g(y)dy$$
$$= -\frac{1}{\sqrt{2\pi}} \int \left(\frac{d}{dy} e^{-\frac{1}{2}(y-x)^2} \right) g(y)dy$$
$$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(y-x)^2} g'(y)dy, \tag{4.1}$$

where (4.1) is obtained on integration by parts, since $e^{-\frac{1}{2}(y-x)^2} g(y) \to 0$ for any x as $|y| \to \infty$ for the kind of g we have.

At this stage, use the fact that g' can have at most $(k+1)$ sign changes because of the assumption made on g. Due to the Polya nature of the normal distributions, it now follows that therefore f' can have at most $(k+1)$ sign changes and hence f can have at most $(k+1)$ local modes, as claimed.

We will now give a characterization of all random variables Y such that the convolution $Z+Y$ is actually unimodal. A discussion of the applicability of this characterization will be given following the result.

Theorem 5. *Let Y have the characteristic function $\psi(t)$. Then the convolution $Z+Y$ is unimodal iff*

i. $\psi(t)$ is continuously differentiable for all $t \neq 0$ and $\lim_{t\to 0} t\psi'(t) = 0$.
ii. The function ϕ defined as

$$\phi(t) = \begin{cases} (1-t^2)e^{-t^2/2}\psi(t) + te^{-t^2/2}\psi'(t) & , t \neq 0 \\ 1 & , t = 0 \end{cases} \tag{4.2}$$

is a characteristic function.

Proof. The convolution $Z+Y$ is unimodal iff

$$Z + Y \stackrel{L}{=} U \cdot V, \tag{4.3}$$

where $U \sim U[0,1]$ and V is independent of U. Equation (4.3) is equivalent to the fact

$$e^{-t^2/2} \cdot \psi(t) = \int_0^1 \phi(tu)du, \tag{4.4}$$

where ϕ is the characteristic function of V.

Since ϕ is continuous, it follows from (4.4) that for all $t \neq 0$,

$$\phi(t) = \frac{d}{dt}\left(\int_0^t \phi(u)du\right)$$
$$= \frac{d}{dt}(te^{-t^2/2}\psi(t))$$
$$= (1-t^2)e^{-t^2/2}\psi(t) + te^{-t^2/2}\psi'(t) \; . \tag{4.5}$$

On the other hand, if there exists a characteristic function ϕ satisfying (4.5) for the given function ψ, then integrating both sides of (4.5), one returns to (4.4) and therefore $Z + Y$ is necessarily unimodal. We have therefore proved all assertions made in the theorem.

Discussion. Notice that the theorem does not say that ψ should be differentiable at 0. Indeed, it need not be. If Y is a Cauchy random variable, the convolution $Z + Y$ is unimodal without ψ being differentiable at 0. The applicability of the theorem depends on checking that the formula (4.2) produces a characteristic function. In general, Bochner's theorem is the only way for checking this and it is also the case that in general verification of the Bochner criterion is a hard proposition. In some cases, however, due to the presence of the term $e^{-t^2/2}$ in (4.2), the function ϕ is likely to be in $L^1(\mathbb{R})$. In such a case, one can formally invert ϕ and if the resulting function turns out to be a nonnegative L^1 function, Bochner's criterion is automatically verified. In some other cases, it may be easier to directly verify unimodality by differentiating the convolution density. We will see some examples, one rather intriguing. But first, we will see another theorem covering random variables which are symmetric and unimodal about 0. We would like to remind the reader that such a random variable X has the representation $X \stackrel{L}{=} U \cdot V$, where $U \sim U[-1, 1]$ and $V > 0$ is independent of U (Khintchine (1938)).

Theorem 6. *Consider the symmetric unimodal random variable $X = U \cdot V$, where $U \sim U[-1, 1]$, and V (independent of U) is infinitely divisible. Then X cannot be a Gaussian convolution.*

Proof. It follows from (4.4) above that if indeed X was a Gaussian convolution, then for $t > 0$,

$$|\int_0^t \phi(u)du| = |te^{-t^2/2}\psi(t)| \leq te^{-t^2/2} \to 0 \text{ as } t \to \infty \ . \tag{4.6}$$

However, since V is infinitely divisible and symmetric, it follows that $\phi(u) > 0 \ \forall \ u$, rendering $\lim_{t\to\infty} \int_0^t \phi(u)du = 0$ impossible. Hence, X cannot be a Gaussian convolution.

Finally, we will now give three illustrative examples on unimodality of Gaussian convolutions.

Example 4. Suppose Y is a discrete random variable assuming values na, $n = 0, \pm 1$ for some $a > 0$ with probabilities $1 - 2p$, p and p respectively. Clearly, the density of the convolution $X = Z + Y$ equals

$$f(x) = \frac{1}{\sqrt{2\pi}} \left[(1 - 2p)e^{-\frac{1}{2}x^2} + pe^{-\frac{1}{2}(x-a)^2} + pe^{-\frac{1}{2}(x+a)^2}\right] \tag{4.7}$$

Unimodality of X is equivalent to

$$\begin{aligned} & f'(x) \leq 0 \quad \forall \ x \geq 0 \\ \Leftrightarrow \ & (1 - 2p)x + p(x - a)e^{ax - a^2/2} + p(x + a)e^{-ax - a^2/2} \geq 0 \ \forall \ x \geq 0 \end{aligned} \tag{4.8}$$

If $p = 0$, (4.8) holds for all $a > 0$, as it should. If $p = \frac{1}{2}$, (4.8) is equivalent to

$$(x - a)e^{2ax} + (x + a) \geq 0 \; \forall \; x \geq 0 \tag{4.9}$$

Obviously (4.9) holds if $x > a$. Hence it is enough to verify it for $x \leq a$. It is possible to prove that (4.9) holds for $0 \leq x \leq a$ if and only if $a \leq 1$ as follows:

i. define $h(x) = (x - a)e^{2ax} + (x + a)$; so $h(0) = 0$,
ii. check $h'(0) \geq 0$ if $0 < a \leq 1$;
iii. check h is convex on $[0, a]$ if $0 < a \leq 1$; hence, $h'(x) \geq 0$ for $0 \leq x \leq a$ if $a \leq 1$;
iv. thus $h(x) \geq h(0) = 0$ for $0 \leq x \leq a$ if $a \leq 1$;
v. verify that for $a \geq 1$, (4.8) cannot hold for all x in $[0, a]$.

An analytic characterization of all pairs (p, a) such that (4.8) holds seems practically impossible. The following table gives the maximum possible value of a for some values of p. A plot is given in Fig. 1.

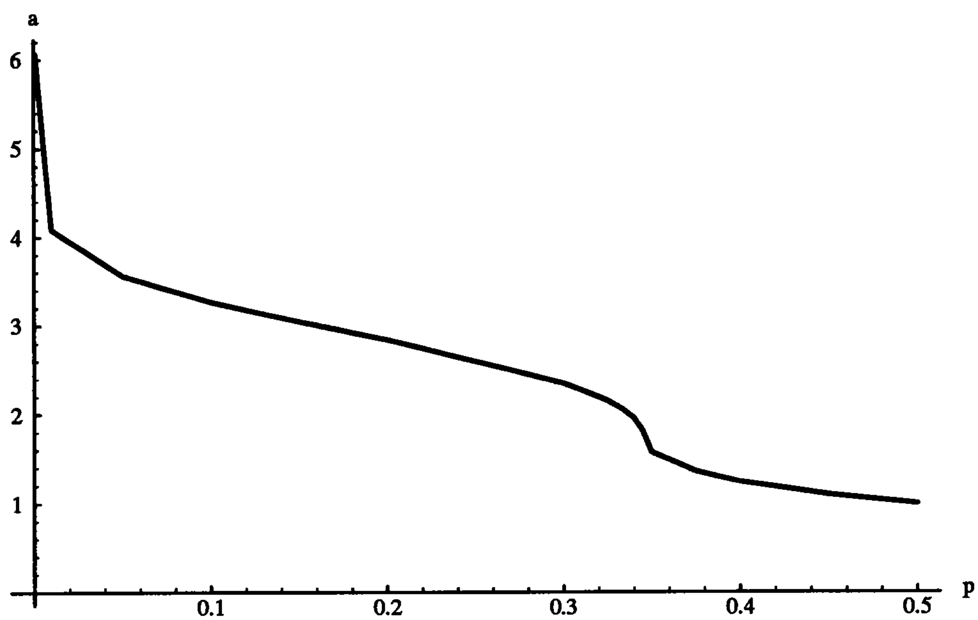

Fig. 1. Region of Unimodality

Table 1.

p	.001	.01	.05	.1	.15	.2	.25	.3	$\frac{1}{3}$	.4	.45	.5
a	4.6767	4.085	3.562	3.269	3.049	2.844	2.627	2.354	2.062	1.240	1.096	1

Example 5. The preceding example demonstrates that Y need not be unimodal for the convolution $Z+Y$ to be unimodal. Theorem 5 was not used, however. We now give an example where Theorem 5 is actually useful. Consider a random variable Y with the "inverse triangular" distribution having density

$$g(y) = \frac{1-\cos(\alpha y)}{\pi \alpha y^2}, -\infty < y < \infty \ .$$

This has a characteristic function (see Chow and Teicher (1988))

$$\begin{aligned} \psi(t) &= 1 - \tfrac{|t|}{\alpha}\ , |t| \le \alpha \\ &= 0 \qquad , |t| > \alpha \ . \end{aligned}$$

Since ψ is not differentiable at $t = \pm\alpha \neq 0$, it follows that the convolution of a $N(0,1)$ and an inverse triangular random variable can never be unimodal, however large α is.

Example 6. We close with an example that demonstrates that convolution with a $N(0,1)$ distribution can inject amazing smoothness in the density of a severely nonunimodal random variable. Towards this end, consider a random variable Y with density

$$g(y) = \frac{1}{\sqrt{2\pi}\sigma(1+e^{-\frac{w^2\sigma^2}{2}})} e^{-\frac{y^2}{2\sigma^2}}(1+\ \cos(wy)) \tag{4.10}$$

g is severely oscillating if the frequency w is large. We will now show that the convolution $Z+Y$ is unimodal *for all* w if $\sigma^2 \le 2.3$!

On direct calculation, the density of the convolution $X = Z+Y$ equals

$$f(x) = \frac{1}{\sqrt{2\pi(1+\sigma^2)}(1+e^{-\frac{w^2\sigma^2}{2}})}\left(1+\cos\left(w\cdot\frac{\sigma^2}{1+\sigma^2}x\right)e^{-\frac{w^2\sigma^2}{2(1+\sigma^2)}}\right)\cdot e^{-\frac{x^2}{2(1+\sigma^2)}} \tag{4.11}$$

If we let $0<r<1$ denote $\frac{\sigma^2}{1+\sigma^2}$, then (4.11) is unimodal if and only if

$$\begin{aligned} & f'(x) \le 0 \quad \forall\, x \ge 0 \\ \Leftrightarrow\ & \frac{1-r}{r^2}\left[e^{\frac{w^2r}{2}} + \cos\ x\right] + w^2 \sin x \ge 0 \quad \forall\, x \ge 0 \end{aligned} \tag{4.12}$$

We claim (4.12) holds for all $w \ge 0$ if $\sigma^2 \le 2.3$. For this, we require to show

$$\begin{aligned} & \inf_{w\ge0}\inf_{x\ge0}\left\{\frac{1-r}{r^2}\left[e^{\frac{w^2r}{2}}+\cos\ x\right]+w^2\frac{\sin\ x}{x}\right\} \ge 0 \\ \Leftrightarrow\ & \inf_{x\ge0}\inf_{w\ge0}\left\{\frac{1-r}{r^2}\left[e^{\frac{w^2r}{2}}+\cos\ x\right]+w^2\frac{\sin\ x}{x}\right\} \ge 0 \end{aligned} \tag{4.13}$$

For any $x \ge 0$, the quantity in flower brackets is positive for $w=0$; furthermore, the derivative with respect to w equals

$$\begin{aligned} & w\left[\frac{1-r}{r}e^{\frac{w^2r}{2}} + 2\frac{\sin\ x}{x}\right] \\ \ge\ & w\left[\frac{1-r}{r} - 0.435466\right] (\because w, r \ge 0 \text{ and } \frac{\sin\ x}{x} \ge -0.217233) \\ \ge\ & 0 \ \text{ if } \frac{1-r}{r} \ge .435466 \Leftrightarrow \sigma^2 \le 2.3 \text{ (approximately)}\ . \end{aligned}$$

Therefore, for any $x \geq 0$, the inside infimum is nonnegative and hence (4.13) holds as well. This establishes the claim.

The relation that σ and w must satisfy in general for the convolution to be unimodal can be found by numerical methods from (4.12).

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