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ABSTRACT

The problem of obtaining absorption probability distributions of random paths, as outcomes of sampling from finite populations either with replacement or without replacement, on any given barrier sets is considered. The absorption probability of random path at a point in the barrier sets is simply the product of an easily computable "latent", which is a function defined on the barrier sets, and the underlying probability at this point. The latent (function) is evaluated recursively on the barrier sets. This result holds when underlying probability for the random path is multinomially or multihypergeometrically distributed. Above investigations are done for the cases when absorption of paths is defined as first hitting (or second hitting and etc.) to some points in the barrier sets, as well as ordered hittings on ordered barrier sets. Applications of these methods to obtain absorption probability distributions for sequential tests in dichotomous populations, such as multiple-stage test or test of three hypotheses, are given.

1 Introduction

In many practical problems, a random variable (or vector) is under investigation. Suppose the distribution of this interested random variable (or vector) belongs to certain class $\{F_\theta,\ \theta \in \Theta\}$, where $\theta = (\theta_1, \cdots, \theta_d)$, $d \geq 1$, but the true θ is unknown to us. We are interested in testing hypothesis $H_0 : \theta \in \Theta_0$ *v.s.* $H_1 : \theta \in \Theta_1$. To make a statistical decision, a number of observations from this random number (or vector) are sampled to provide information about the underlying true θ. More observations are sampled, more information is obtained. But in real life, more observations means higher costs and longer time needed. A challenge to statisticians is to find ways to get more information from less observations. Sequentially gathering observations and making decision whenever information provided by the gathered observations is enough, this practice provides efficient means to achieve the goal mentioned above. Now suppose the observations are $\tilde{X}_1, \tilde{X}_2, \cdots$ and $\tilde{S}_n = \tilde{S}_n(\tilde{X}_1, \cdots, \tilde{X}_n)$ is a sufficient statistics for θ, where $\tilde{X}_l$ and $\tilde{S}_n$ are vectors. We can always imagine that the sampling is ever going thus the infinite sequence $\tilde{S}_1, \tilde{S}_2, \cdots$, can be observed. In sequential procedure, we have an opportunity to look at $\tilde{S}_1, \tilde{S}_2, \cdots$ in sequence one by one, stop (looking) and make decision whenever the early stages of this sequence shows strong evidence either in favor of Θ_0 or in favor of Θ_1. The observations in later stages of this sequence are ignored at all. In fixed sampling size procedure (assuming sampling size is n_0), we making our decision only depends on outcomes of $\tilde{S}_{n_0}$, so the observations in stages, up to $n_0 - 1$ and after n_0, of the sequence are ignored at all. Looking at these two kinds of decision procedures from this unique point of view, pinpoints their similarities and differences.

Next we give some definitions to summarize above idea. Let $\underset{\sim}{S} = (\tilde{S}_1, \tilde{S}_2, \cdots)$. Through out this paper, we will call $\underset{\sim}{S}$ the random path. Let $\mathcal{X}$ be the set of all sample paths of $\underset{\sim}{S}$. Let $P_\theta(\cdot)$ be a probability measure on $\mathcal{X}$, derived from F_θ. Then $\{P_\theta(\cdot), \theta \in \Theta\}$, derived from $\{F_\theta, \theta \in \Theta\}$, is a class of probability measures on $\mathcal{X}$. In order to test $H_0 : \theta \in \Theta_0$ *v.s.* $H_1 : \theta \in \Theta_1$, we divide $\mathcal{X}$ into a partition $\{H_\alpha, \alpha \in \mathcal{A}\}$ ($\mathcal{A}$ is some proper index set) according to $\{P_\theta(\cdot), \theta \in \Theta\}$, the probability measures on $\mathcal{X}$. We say $\{H_\alpha, \alpha \in \mathcal{A}\}$ is a partition of $\mathcal{X}$ if $\bigcup_{\alpha \in \mathcal{A}} H_\alpha = \mathcal{X}$ and $H_\alpha \cap H_{\alpha'} = \emptyset$ for any $\alpha \neq \alpha'$. The partition of $\mathcal{X}$, $\{H_\alpha, \alpha \in \mathcal{A}\}$, should be such that for each H_α, if $P_\theta(\underset{\sim}{S} \in H_\alpha)$ is "large" ("small") for $\theta \in \Theta_0$, then it is "small" ("large") for $\theta \in \Theta_1$ (here "large" and "small" are in a relative sense). When doing the test, we observe which H_α that $\underset{\sim}{S}$ falls in, assume $\underset{\sim}{S}$ fell into H_{α_0}, accordingly we make decision in favor of Θ_0 if $P_\theta(\underset{\sim}{S} \in H_{\alpha_0})$ is "large" for $\theta \in \Theta_0$, in favor of Θ_1 otherwise. Obviously there are many different partitions met above requirements. If the partition is made only according to the possible values of $\tilde{S}_{n_0}$, then we have a procedure of fixed sampling size n_0. If the partition is made according to possible values of $\tilde{S}_n$ for some different n's, then we have a sequential procedure. We further illustrate this idea in later discussion of random paths from finite populations.

Even though sequential procedures are more efficient than the fixed sampling size procedures, there are two difficulties which disencourage practitioners to prefer the former to the latter. One difficulty is that in most cases, it is difficult (too complicated or too tedious), sometimes impossible, to compute OC (operation characteristic) and expected sampling size for a given sequential procedure. The other difficulty is that in most cases, there doesn't exist a sequential procedure which is superior than other procedures (sequential or not) uniformly on Θ.

In this paper, we only discuss sampling from finite populations. We will develop methods of obtaining the probabilities of random paths, as outcomes of sampling from finite population either *with* or *without* replacement, being absorbed by a specified set of points. These methods, simple in formulation and easily computable, not only overcome the first difficulty mentioned above but also help to select proper procedures by providing easily computed power function and expected sampling sizes for any tentative sequential procedure.

Fisher[1952], Aroian[1968] and other authors had used direct methods to obtain probability distributions of first hitting for random walk from binomial and other distributions. In direct methods, computation of absorption probabilities is tedious because it involves, based on convolution, all continuous and barrier points. Continuous points, in most sequential tests, are much more than barrier points.

In methods introduced in this paper, absorption probabilities on barrier sets is just simply the product of the latent (function) of these sets and underlying probabilities. Evaluation of the latent (function) involves only points in barrier sets. Compared with direct methods, our methods are more convenient and simpler. Its advantage becomes more obvious when absorption of random path is defined as mth hitting or as ordered hittings. Absorption probability distribution for mth hitting or for ordered hittings are useful in multiple-stage tests and two-sided test with three decisions, examples are given in Section 4.

A drawback in our methods is large number of binomial coefficients are includeded in formula, which might greatly slowdown the computation in computer because computation of factorial is time consuming. This drawback can be overcome by computing binomial coefficients recursively by $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, and storing those binomial coefficients in memories for later usage.

2 Random Paths from Finite Populations

A finite population is defined as a population $\mathcal{P}^d_{p,N}$ consists of d classes of items such that Np_k items of which are of kth class, $k = 1, \cdots, d$ and $\sum_{k=1}^d p_k = 1$. Here and throughout this paper, N, a positive integer, denotes the population size; p_k, a number between 0 and 1, denotes the proportion of kth item in the population, and $p = (p_1, \cdots, p_d)$. Sampling with replacement from a finite population, we designate $\tilde{X}_l$ as the outcome of lth observation, *i.e.* $\tilde{X}_l = (X_{l1}, \cdots, X_{ld})$ where $X_{lk} = 1$ if lth sampled item belongs to kth class, $X_{lk} = 0$ otherwise, thus $\sum_{k=1}^d X_{lk} = 1$. Then $\tilde{X}_1, \cdots, \tilde{X}_n, \cdots$ are independent random vectors identically having p_k as the success rate for kth component for $k = 1, \cdots, d$.

Let $\tilde{S}_n = \sum_{l=1}^n \tilde{X}_l$, then $\tilde{S}_n = (S_{n1}, \cdots, S_{nd})$ where $S_{nk} = \sum_{l=1}^n X_{lk}$. Obviously, $\tilde{S}_n$ has a distribution of multinomial $\mathcal{M}_d(n; p_1, \cdots, p_d)$. Let

$$\mathcal{X} = \left\{ \underset{\sim}{s} = (\tilde{s}_1, \cdots, \tilde{s}_n, \cdots) : \begin{array}{c} \tilde{s}_n = (s_{n1}, \cdots, s_{nd}), \quad s_{nk} \text{ integer} \\ s_{nk} - s_{n-1k} = 0 \ \text{ or } \ 1, \quad \sum_{k=1}^d s_{nk} = n \end{array} \right\}.$$

Sampling with replacement, $\underset{\sim}{S} = (\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_n, \cdots)$ is a random path with increment $\tilde{X}_n$, which is independent of passage position $\tilde{S}_{n-1} = \tilde{s}_{n-1}$ for $n = 1, 2, \cdots$.

We can imagine sampling is done infinite times with replacement, then $\underset{\sim}{S}$ is the *random path*, and $\mathcal{X}$ is the set of all possible *sample paths*. In other words, $\mathcal{X}$ is the sample space of $\underset{\sim}{S}$. Random path $\underset{\sim}{S} = (\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_n, \cdots)$ can be "graphed" as a sequence of points on d-dimension space which has coordinates $(s_{n1}, \cdots, s_{nd})$. Connecting those neighboring points with lines, we have a random path starting from origin and extending infinitely to positive direction of all coordinates.

Let $\{H_1, H_2, \cdots, H_k\}$ be a partition of $\mathcal{X}$. Since elements in each H_i are sample paths, for the reason of intuitive thus easier conception, we call H_i a bunch of paths. So partitioning of $\mathcal{X}$ is to divide all paths in $\mathcal{X}$ into a number of bunchs in a way such that each path in $\mathcal{X}$ belongs to some bunch, and no single path belongs to two different bunchs.

For an example in dichotomous population($d = 2$), we have random path $\underset{\sim}{S} = (\tilde{S}_1, \tilde{S}_2, \cdots)$ where $\tilde{S}_n = (S_{n1}, S_{n2}) = (S_n, n - S_n)$, S_n is binomially distributed as $B(n, p)$ where $0 \leq p \leq 1$. A simple partition of $\mathcal{X}$ is $\{H_{n_0 0}, \cdots, H_{n_0 n_0}\}$ where $H_{n_0 i} = \{\underset{\sim}{s} \in \mathcal{X} : \tilde{s}_{n_0} = (i, n_0 - i)\}$ and n_0 is any fixed positive integer. It is easy to check that $\{H_{n_0 i},\ i = 0, \cdots, n_0\}$ is a partition of $\mathcal{X}$, and we have $P(\underset{\sim}{S} \in H_{n_0 i}) = P(S_{n_0} = i) = \binom{n_0}{i} p^i (1-p)^{n_0 - i}$ for $i = 0, \cdots, n_0$.

When sampling is carried out *without* replacement, population $\mathcal{P}^d_{p,N}$ is exhausted at the Nth step of sampling, and $\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_N$ are not *i.i.d* . $\underset{\sim}{S}_N = (\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_N)$ is a random path with increment $\tilde{X}_n$, which is dependent of passage position $\tilde{S}_{n-1} = \tilde{s}_{n-1}$ for $n = 1, 2, \cdots, N$. Let $\mathcal{X}_N$ be the set of all possible sample paths of $\underset{\sim}{S}_N$, then

$$\mathcal{X}_N = \left\{ \underset{\sim}{s}_N = (\tilde{s}_1, \cdots, \tilde{s}_n, \cdots, \tilde{s}_N) : \begin{array}{ll} \tilde{s}_n = (s_{n1}, \cdots, s_{nd}), & s_{nk} \text{ integer} \\ s_{nk} - s_{n-1k} = 0 \text{ or } 1, & \sum_{k=1}^{d} s_{nk} = n \end{array} \right\}.$$

$\mathcal{X}_N$ is the sample space of $\underset{\sim}{S}_N$.

3 Absorption Probability Distribution on Barrier Set

Our goal is to derive methods of obtaining absorption probability distributions of random paths, which arise by sampling *with* or *without* replacement from finite population, on any specified barrier sets. Obtaining absorption probability is of broad interests. Especially in sequential tests, absorption probability is critical for computation of OC and expected sampling sizes for the tests. Even though this method is motivated in sequential tests, here we treat it as general as possible in hope it can find applications widely. First the discussion is given when the absorption of a random path is as the usual sense, *i.e.* the random path first hitting a point in the specified set of points. Then we will generalize this method for the cases when absorption of random path is defined as mth hitting, as well as ordered hittings in specified barrier sets.

Let $\mathcal{B}$ be a set of some interested points on this space that $\underset{\sim}{S}$ might hit. We call $\mathcal{B}$ the set of barrier points. Hence $\mathcal{B} = \{\tilde{b} = (b_1, \cdots, b_d) : b_k \text{ is positive integer}, k = 1, \cdots, d\}$. As an example for $d = 3$, let $\mathcal{B} = \{(1,2,2), (2,1,2), (2,3,3), (3,2,3), (4,1,3), (3,4,4), (5,4,2)\}$.

Let B_n be the subset of $\mathcal{B}$, points in which random path $\underset{\sim}{S}$ might hit at time n (during the nth sampling), that is $B_n = \{\tilde{b}_n = (b_{n1}, \cdots, b_{nd}) : \sum_{k=1}^{d} b_{nk} = n\}$. Let I_n be the number of points in B_n, then $I_n = 0$ if B_n is empty. We call B_n the set of barrier points at time n. Hence $\mathcal{B} = \bigcup_n B_n$, and $B_1, B_2, \cdots$ are disjoint.

In last example, $B_5 = \{(1,2,2), (2,1,2)\}$, $B_8 = \{(2,3,3), (3,2,3), (4,1,3)\}$, $B_{11} = \{(3,4,4), (5,4,2)\}$, and $B_n = \emptyset$ for $n \neq 5, 8, 11$. $\mathcal{B}$ is the set of all barrier points which random path $\underset{\sim}{S}$ might hit during the whole process of sampling.

3.1 Absorption Defined as First Hitting

Let $\mathcal{X}^{\mathcal{B}}$ be the subset of $\mathcal{X}$, which includes all sample paths which pass through some points in $\mathcal{B}$. $\mathcal{X}^{\mathcal{B}}$ can be partitioned into bunchs of paths $\{H(\tilde{b}) : \tilde{b} \in \mathcal{B}\} = \{H(\tilde{b}_n) : \tilde{b}_n \in B_n, \; n =$

$1,2,\cdots\}$ where

$$H(\tilde{b}_n) = \Big\{\underset{\sim}{s} \in \mathcal{X} : \tilde{s}_n = \tilde{b}_n;\ \ \tilde{s}_l \notin B_l,\ \ l = 1,2,\cdots,n-1\Big\} \qquad \tilde{b}_n \in B_n;\quad n = 1,2,\cdots \tag{1}$$

It is not difficulty to check that $\Big\{H(\tilde{b}_n) : \tilde{b}_n \in B_n;\ n = 1,2,\cdots.\Big\}$ is a partition of $\mathcal{X}^{\mathcal{B}}$ and

$$P(\underset{\sim}{S} \in H(\tilde{b}_n)) = P_p(\tilde{S}_n = \tilde{b}_n,\ \tilde{S}_l \notin B_l,\ \ l = 1,\cdots,n-1). \tag{2}$$

The right side of above equation indicates $P(\underset{\sim}{S} \in H(\tilde{b}))$ is the probability that random path $\underset{\sim}{S}$ hits $\tilde{b}$ before hitting other barrier points in $\mathcal{B}$, or the absorption probability of $\underset{\sim}{S}$ at $\tilde{b}$. For this moment, absorption of $\underset{\sim}{S}$ by $\tilde{b}$ is defined as $\underset{\sim}{S}$ hits $\tilde{b}$ for the first hitting in $\mathcal{B}$. Later on, we will discuss the cases when the absorption of $\underset{\sim}{S}$ by $\tilde{b}$ is defined in other meanings.

Let $\mathcal{X}_N^{\mathcal{B}}$ be the subset of $\mathcal{X}_N$, which includes all sample paths passing through some points in $\mathcal{B}$. With same idea, $\mathcal{X}_N^{\mathcal{B}}$ can be partitioned into bunchs of paths $\{H_N(\tilde{b}),\ \tilde{b} \in \mathcal{B}\} = \{H_N(\tilde{b}_n) :\ \tilde{b}_n \in B_n,\ \ n = 1,\cdots,N\}$ where

$$H_N(\tilde{b}_n) = \Big\{\underset{\sim}{s}_N = (\tilde{s}_1,\cdots,\tilde{s}_N) :\ \tilde{s}_n = \tilde{b}_n;\ \ \tilde{s}_l \notin B_l,\ \ l = 1,2,\cdots,n-1\Big\} \qquad \tilde{b}_n \in B_n;\quad n = 1,\cdots,N. \tag{3}$$

It is easy to see that $\Big\{H_N(\tilde{b}_n) : \tilde{b}_n \in B_n; n = 1,\cdots,N.\Big\}$ is a partition of $\mathcal{X}_N^{\mathcal{B}}$ and

$$P(\underset{\sim}{S}_N \in H_N(\tilde{b}_n)) = P_{p,N}(\tilde{S}_n = \tilde{b}_n,\ \tilde{S}_l \notin B_l,\ \ l = 1,\cdots,n-1). \tag{4}$$

Definition 3.1 *Let I^d be set of all $d-$tuple of integers. Given a barrier set $\mathcal{B}$ on I^d, $\psi(\cdot)$, "the latent of $\mathcal{B}$", is a function defined on $\mathcal{B} = \cup_n B_n$ such that, for any $\tilde{b}_n \in \mathcal{B}$*

$$\psi(\tilde{b}_n) = 1 - \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l) \frac{\prod_{k=1}^d \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}, \tag{5}$$

where a convention is assumed : $\binom{m}{t} = 0$ if $m < t$ or $t < 0$. ∎

Obviously, the latent $\psi(\cdot)$ on $\mathcal{B}$ can be evaluated for all $\tilde{b}_n \in \mathcal{B}$ recursively in n by (5).

Theorem 3.1 *Given a barrier set $\mathcal{B}$, sampling from a finite population $\mathcal{P}_{p,N}^d$* with replacement, *the absorption probability at $\tilde{b}_n \in \mathcal{B}$, for any $p = (p_1,\cdots,p_d)$, $\sum_{k=1}^d p_k = 1$, $0 \le$*

$p_1, \cdots, p_d \le 1$, *is*

$$
\begin{aligned}
P_p(\underset{\sim}{S} \in H(\tilde{b}_n)) &= \psi(\tilde{b}_n) P_p(\tilde{S}_n = \tilde{b}_n) \\
&= \psi(\tilde{b}_n) \binom{n}{\tilde{b}_n} \prod_{k=1}^{d} p_k^{b_{nk}},
\end{aligned} \tag{6}
$$

where $\binom{n}{\tilde{b}_n} = \binom{n}{b_{n1}, \cdots, b_{nd}} = \frac{n!}{b_{n1}! \cdots b_{nd}!}$ *and* $\psi(\cdot)$ *is given by (5).*

For the same $\mathcal{B}$*, sampling* without replacement, *the absorption probability of* $\underset{\sim}{S}_N$ *at* $\tilde{b}_n \in \mathcal{B}$*, for any* N *and* $p = \frac{1}{N}, \cdots, \frac{N}{N}$ *, is*

$$
\begin{aligned}
P_{p,N}(\underset{\sim}{S}_N \in H_N(\tilde{b}_n)) &= \psi(\tilde{b}_n) P_{p,N}(\tilde{S}_n = \tilde{b}_n) \\
&= \psi(\tilde{b}_n) \frac{\prod_{k=1}^{d} \binom{p_k N}{b_{nk}}}{\binom{N}{n}}.
\end{aligned} \tag{7}
$$

■

Illustrative Example 1:

Assuming $\mathcal{P}_{p,N}^3$ is a population with 3 classes($d = 3$), size N and proportions of classes $p = (p_1, p_2, p_3)$. As we defined before, $\mathcal{B} = \{(1,2,2), (2,1,2), (2,3,3), (3,2,3), (4,1,3), (3,4,4), (5,4,2)\}$. $B_5 = \{(1,2,2), (2,1,2)\}$, $B_8 = \{(2,3,3), (3,2,3), (4,1,3)\}$, $B_{11} = \{(3,4,4), (5,4,2)\}$, and $B_n = \emptyset$ for $n \neq 5, 8, 11$. We have $\mathcal{B} = \bigcup_n B_n$.

The latent $\psi(\cdot)$ on $\mathcal{B}$ can be evaluated by follow formula (5).

$\psi(1,2,2) = 1$, $\psi(2,1,2) = 1$ because $B_l = \emptyset$ for $l = 1, \cdots, 4$. For $\tilde{b}_8 = (2,3,3) \in B_8$, as $B_l = \emptyset$, $l = 1,2,3,4,6,7$, we have

$$
\begin{aligned}
\psi(2,3,3) &= 1 - \sum_{l=1}^{7} \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l) \frac{\binom{2}{b_{l1}}\binom{3}{b_{l2}}\binom{3}{b_{l3}}}{\binom{8}{l}} \\
&= 1 - \sum_{\tilde{b}_5 \in B_5} \psi(\tilde{b}_5) \frac{\binom{2}{b_{l1}}\binom{3}{b_{l2}}\binom{3}{b_{l3}}}{\binom{8}{5}} \\
&= 1 - \left\{ 1 \cdot \frac{\binom{2}{1}\binom{3}{2}\binom{3}{2}}{\binom{8}{5}} + 1 \cdot \frac{\binom{2}{2}\binom{3}{1}\binom{3}{2}}{\binom{8}{5}} \right\} \\
&= \frac{29}{56}.
\end{aligned}
$$

Similarly

$$\psi(3,2,3)=1-\left\{1\cdot\frac{\binom{3}{1}\binom{2}{2}\binom{3}{2}}{\binom{8}{5}}+1\cdot\frac{\binom{3}{2}\binom{2}{1}\binom{3}{2}}{\binom{8}{5}}\right\}=\frac{29}{56},$$

$$\psi(4,1,3)=1-\left\{1\cdot\frac{\binom{4}{1}\binom{1}{2}\binom{3}{2}}{\binom{8}{5}}+1\cdot\frac{\binom{4}{2}\binom{1}{1}\binom{3}{2}}{\binom{8}{5}}\right\}=\frac{19}{28},$$

$$\begin{aligned}\psi(3,4,4) &= 1-\left\{1\cdot\frac{\binom{3}{1}\binom{4}{2}\binom{4}{2}}{\binom{11}{5}}+1\cdot\frac{\binom{3}{2}\binom{4}{1}\binom{4}{2}}{\binom{11}{5}}+\frac{11}{14}\cdot\frac{\binom{3}{2}\binom{4}{3}\binom{4}{3}}{\binom{11}{8}}+\right.\\ &\left.+\frac{29}{56}\cdot\frac{\binom{3}{3}\binom{4}{2}\binom{4}{3}}{\binom{11}{8}}+\frac{19}{28}\cdot\frac{\binom{3}{4}\binom{4}{1}\binom{4}{3}}{\binom{11}{8}}\right\}=\frac{148}{385}\end{aligned}$$

$$\begin{aligned}\psi(5,4,2) &= 1-\left\{1\cdot\frac{\binom{5}{1}\binom{4}{2}\binom{2}{2}}{\binom{11}{5}}+1\cdot\frac{\binom{5}{2}\binom{4}{1}\binom{2}{2}}{\binom{11}{5}}+\frac{11}{14}\cdot\frac{\binom{5}{2}\binom{4}{3}\binom{2}{3}}{\binom{11}{8}}+\right.\\ &\left.+\frac{29}{56}\cdot\frac{\binom{5}{3}\binom{4}{2}\binom{2}{3}}{\binom{11}{8}}+\frac{19}{28}\cdot\frac{\binom{5}{4}\binom{4}{1}\binom{2}{3}}{\binom{11}{8}}\right\}=\frac{28}{33}\end{aligned}$$

Sampling *with* replacement, the underlying distribution is multinomial with parameter $p=(p_1,p_2,p_3)$, *i.e.* $\mathcal{M}_3(n;p_1,p_2,p_3)$, $n=1,2,\cdots$. Then by (6) the probability random path $\underset{\sim}{S}$ absorbed by point $(5,4,2)$ is

$$P_p(\underset{\sim}{S}\in H((5,4,2))=\psi((5,4,2))P_p(\tilde{S}_{11}=(3,4,4))=\frac{148}{385}\binom{11}{5,4,2}p_1^5p_2^4p_3^2=4440p_1^5p_2^4p_3^2.$$

Sampling without replacement, the underlying distribution is multihypergeometric with parameter $p=(p_1,p_2,p_3)$ and N, *i.e.* $\mathcal{H}_3(n;p_1,p_2,p_3;N)$, $n=1,\cdots,N$. Then by (6) the probability random path $\underset{\sim}{S}_N$ absorbed by barrier point $(3,4,4)$ is

$$P_{p,N}(\underset{\sim}{S}_N\in H_N(3,4,4)=\psi(3,4,4)P_{p,N}(\tilde{S}_{11}=(3,4,4))=\frac{148}{385}\frac{\binom{p_1N}{5}\binom{p_2N}{4}\binom{p_3N}{2}}{\binom{N}{11}}$$

The absorption probability distributions on barrier set $\mathcal{B}$ are listed on table below for these two cases, in which the underlying distributions are multinomial or multigeometric.

n	Barrier Points		Absorption Probability Distributions on $\mathcal{B}$	
n	$\tilde{b}_n \in \mathcal{B}$	$\psi(\tilde{b}_n)$	$\mathcal{M}_3(n; p_1, p_2, p_3)$ for $\underset{\sim}{S}$	$\mathcal{H}_3(n; p_1, p_2, p_3; N)$ for $\underset{\sim}{S}_N$
5	$(1,2,2)$	1	$30p_1^1p_2^2p_3^2$	$\frac{\binom{p_1N}{1}\binom{p_2N}{2}\binom{p_3N}{2}}{\binom{N}{5}}$
5	$(2,1,2)$	1	$30p_1^2p_2^1p_3^2$	$\frac{\binom{p_1N}{2}\binom{p_2N}{1}\binom{p_3N}{2}}{\binom{N}{5}}$
8	$(2,3,3)$	$\frac{29}{56}$	$290p_1^2p_2^3p_3^3$	$\frac{29}{56}\frac{\binom{p_1N}{2}\binom{p_2N}{3}\binom{p_3N}{3}}{\binom{N}{8}}$
8	$(3,2,3)$	$\frac{29}{56}$	$290p_1^3p_2^2p_3^3$	$\frac{29}{56}\frac{\binom{p_1N}{2}\binom{p_2N}{3}\binom{p_3N}{3}}{\binom{N}{8}}$
8	$(4,1,3)$	$\frac{19}{28}$	$380p_1^4p_2^1p_3^3$	$\frac{19}{28}\frac{\binom{p_1N}{4}\binom{p_2N}{1}\binom{p_3N}{3}}{\binom{N}{11}}$
11	$(3,4,4)$	$\frac{148}{385}$	$4440p_1^3p_2^4p_3^4$	$\frac{148}{385}\frac{\binom{p_1N}{3}\binom{p_2N}{4}\binom{p_3N}{4}}{\binom{N}{11}}$
11	$(5,4,2)$	$\frac{28}{33}$	$5880p_1^5p_2^4p_3^2$	$\frac{28}{33}\frac{\binom{p_1N}{5}\binom{p_2N}{4}\binom{p_3N}{2}}{\binom{N}{11}}$

3.2 Absorption Defined as mth Hitting

We have obtained the absorption probability distribution of random path $\underset{\sim}{S}$(or $\underset{\sim}{S}_N$), as outcome of sampling *with*(or *without* replacement) from a finite population, on barrier set $\mathcal{B}$ where the event that $\underset{\sim}{S}$ is absorbed by a point $\tilde{b}$ in $\mathcal{B}$ was defined as that $\underset{\sim}{S}$(or $\underset{\sim}{S}_N$) hits $\tilde{b}$ before hitting any other points in $\mathcal{B}$, or in brief words, as that $\tilde{b}$ is the point of first hitting in $\mathcal{B}$. Now we will discuss the general case, in which absorption of $\underset{\sim}{S}$ by a point $\tilde{b}$ in $\mathcal{B}$ is defined as that $\underset{\sim}{S}$ hits $\tilde{b}$ after hitting other $m-1$ points in $\mathcal{B}$, or in other words, $\tilde{b}$ is the point of mth hitting in $\mathcal{B}$.

Let $H^m(\tilde{b}_n)$ be a bunch of sample paths (in $\mathcal{X}$) which hit $\tilde{b}_n$ as the m*th* hitting in $\mathcal{B}$, then

$$H^m(\tilde{b}_n) = \bigcup_{\substack{l_1<\cdots<l_{m-1}<n \\ B(l_t)\neq\emptyset,\ t=1,\cdots,m-1}} \left\{ \underset{\sim}{s} \in \mathcal{X} : \begin{array}{c} \tilde{s}_n = \tilde{b}_n;\ \tilde{s}_{l_t} \in B_{l_t},\ t=1,\cdots,m-1; \\ s_l \notin B_l,\ \ l<n,\ \ l \neq l_1,\cdots,l_{m-1}. \end{array} \right\}. \tag{8}$$

Let $\mathcal{X}^{\mathcal{B}}(m)$ be the subset of $\mathcal{X}$, which consists of all sample paths $\tilde{s}$ which pass through at least m points in $\mathcal{B}$. $\left\{H^m(\tilde{b}_n):\quad \tilde{b}_n \in B_n;\ n=1,2,\cdots.\right\}$ is a partition of $\mathcal{X}^{\mathcal{B}}(m)$.

Similarly let $H_N^m(\tilde{b}_n)$ be a bunch of sample paths (in $\mathcal{X}_N$) which pass $\tilde{b}_n$ as the m*th* points in $\mathcal{B}$, then

$$H_N^m(\tilde{b}_n) = \bigcup_{\substack{l_1<\cdots<l_{m-1}<n \\ B(l_t)\neq\emptyset,\ t=1,\cdots,m-1}} \left\{ \tilde{s}_N \in \mathcal{X}_N : \begin{array}{c} \tilde{s}_n = \tilde{b}_n;\ s_{l_t} \in B_l,\ t=1,\cdots,m-1; \\ s_l \notin B_l,\ \ l<n,\ \ l \neq l_1,\cdots,l_{m-1}. \end{array} \right\}. \tag{9}$$

Let $\mathcal{X}_N^{\mathcal{B}}(m)$ be the subset of $\mathcal{X}_N$, which consists of all sample paths which pass through at least m points in $\mathcal{B}$. $\left\{H_N^m(\tilde{b}_n): \quad \tilde{b}_n \in B_n;\ n = 1, 2, \cdots.\right\}$ is a partition of $\mathcal{X}_N^{\mathcal{B}}(m)$.

Before giving formula for absorption probabilities $P(\underset{\sim}{S} \in H^m(\tilde{b}_n))$ and $P_{p,N}(\underset{\sim}{S}_N \in H_N^m(\tilde{b}_n))$ for random pathes $\underset{\sim}{S}$ and $\underset{\sim}{S}_N$, we need definitions of relative latent $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ and latent-m $\psi^m(\tilde{b}_n)$, both are functions defined on $\mathcal{B}$.

Definition 3.2 *Let I^d be set of all $d-$tuple of integers. Fix $\tilde{b}_{n'}$ in I^d* ($\tilde{b}_{n'}$ not necessarily in $\mathcal{B}$), *$\varphi(\tilde{b}_{n'}, \cdot)$, "relative latent of $\mathcal{B}$ with origin $\tilde{b}_{n'}$", is given by, $\tilde{b}_n \in \mathcal{B}$, $n > n'$*

$$\varphi(\tilde{b}_{n'}, \tilde{b}_n) = 1 - \sum_{l=n'+1}^{n-1} \sum_{\tilde{b}_l \in B_l} \varphi(\tilde{b}_{n'}, \tilde{b}_l) \frac{\prod_{k=1}^d \binom{b_{nk} - b_{n'k}}{b_{lk} - b_{n'k}}}{\binom{n-n'}{l-n'}}. \tag{10}$$

We simply say $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ the "relative latent of $\mathcal{B}$" if $\tilde{b}_{n'} \in \mathcal{B}$ and $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ is evaluated for all $\tilde{b}_{n'}$, $\tilde{b}_n \in \mathcal{B}$, $n' < n$. ∎

Given $\tilde{b}_n \in I^d$, $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ can be evaluated recursively in n for all $\tilde{b}_n \in \mathcal{B}$, $n > n'$. In this way, for all pairs of $(\tilde{b}_{n'}, \tilde{b}_n)$ such that $n' < n$ and $\tilde{b}_{n'}$, $\tilde{b}_n \in \mathcal{B}$, $\varphi(\tilde{b}_n, \tilde{b}_{n'})$ is computable. The definition of $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ is analogous to that of $\psi(\tilde{b}_n)$. It's easy to check $\varphi(\tilde{o}, \tilde{b}_n) = \psi(\tilde{b}_n)$, where $\tilde{o}$ is the origin in R^d. $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ is the conditional probability that $\underset{\sim}{S}$ doesn't hit any points in $\mathcal{B}$ after time n' and before time n given $\underset{\sim}{S}$ hits $\tilde{b}_{n'}$ at time n' and $\tilde{b}_n$ at time n. Or in other words, $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ is the percentage of sample paths passing through $\tilde{b}_{n'}$ and $\tilde{b}_n$ that don't pass any points in $\mathcal{B}$ between n' and n.

If we define $\psi^{(1)}(\tilde{b}_n) = \psi(\tilde{b}_n)$ whose values are available by (5), with relative latent $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ whose values are available by (10), the latent-m (function) $\psi^m(\cdot)$ on $\mathcal{B}$ is defined as follow.

Definition 3.3 *For $m = 2, 3, \cdots$, for any point $\tilde{b}_n$ in $\mathcal{B}$, $\psi^m(\tilde{b}_n)$, the "latent-m of $\mathcal{B}$", is a function defined on $\mathcal{B}$ such that*

$$\psi^m(\tilde{b}_n) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} \psi^{m-1}(\tilde{b}_l) \varphi(\tilde{b}_l, \tilde{b}_n) \frac{\prod_{k=1}^d \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}. \tag{11}$$

where $\psi^1(\cdot) = \psi(\cdot)$, $\psi(\cdot)$ is the latent of $\mathcal{B}$ given by (5); $\varphi(\cdot, \cdot)$ is the relative latent of $\mathcal{B}$ given by (10). ∎

Above definition indicates that $\left\{\psi^m(\tilde{b}_n)\right\}_{\tilde{b}_n \in \mathcal{B}}$ can be evaluated inductively for $m = 2, 3, \cdots$.

Theorem 3.2 *Let absorption of random path $\underset{\sim}{S}$(or $\underset{\sim}{S}_N$) by points in a given set $\mathcal{B}$ be defined as* mth *hitting in $\mathcal{B}$. Sampling with replacement, the absorption probability at $\tilde{b}_n \in \mathcal{B}$, for $p = (p_1, \cdots, p_d)$, is*

$$P_p(\underset{\sim}{S} \in H^m(\tilde{b}_n)) = \psi^m(\tilde{b}_n) P_p(\tilde{S}_n = \tilde{b}_n) = \psi^m(\tilde{b}_n) \binom{n}{\tilde{b}_n} \prod_{k=1}^{d} p_k^{b_{nk}}, \tag{12}$$

where $\psi^m(\cdot)$ is the latent-m (function) on $\mathcal{B}$ given by (11).

Sampling without replacement, the absorption probability of $\tilde{S}_N$ by $\tilde{b}_n \in \mathcal{B}$, for any N and $p = \frac{1}{N}, \cdots, \frac{N}{N}$, is

$$P_{p,N}(\underset{\sim}{S}_N \in H_N^m(\tilde{b}_n)) = \psi^m(\tilde{b}_n) P_{p,N}(\tilde{S}_n = b_n) = \psi^m(\tilde{b}_n) \frac{\prod_{k=1}^{d} \binom{p_k N}{b_{nk}}}{\binom{N}{n}}. \tag{13}$$

∎

Corollary 3.1 *The probability that $\underset{\sim}{S}$ hits at least m points in $\mathcal{B}$ is*

$$P_p(\underset{\sim}{S} \in \mathcal{X}^{\mathcal{B}}(m)) = \sum_{\tilde{b}_n \in \mathcal{B}} \psi^m(\tilde{b}_n) \binom{n}{\tilde{b}_n} \prod_{k=1}^{d} p_k^{b_{nk}}, \tag{14}$$

The probability that $\underset{\sim}{S}_N$ hits at least m points in $\mathcal{B}$ is

$$P_{p,N}(\underset{\sim}{S}_N \in \mathcal{X}_N^{\mathcal{B}}(m)) = \sum_{\tilde{b}_n \in \mathcal{B}} \psi^m(\tilde{b}_n) \frac{\prod_{k=1}^{d} \binom{p_k N}{b_{nk}}}{\binom{N}{n}}. \tag{15}$$

Illustrative Example 1 Continued:

Let $\mathcal{B}$ be the barrier set given in Section 3, *i.e.* $\mathcal{B} = \{(1,2,2),(2,1,2),(2,3,3),(3,2,3),(4,1,3),(3,4,4),(5,4,2)\}$. For all $\tilde{b}_{n'}, \tilde{b}_n \in \mathcal{B}$, $n' < n$, $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$, relative latent of $\mathcal{B}$, is evaluated by (10) as follows. $\varphi((1,2,2),(2,3,3)) = \varphi((1,2,2),(3,2,3)) = \varphi((1,2,2),(4,1,3)) = \varphi((2,1,2),(2,3,3)) = \varphi((2,1,2),(3,2,3)) = \varphi((2,1,2),(4,1,3)) = 1$ because $B_6 = B_7 = \emptyset$. And similarly $\varphi((2,3,3),(3,4,4)) = \varphi((2,3,3),(5,4,2)) = \varphi((3,2,3),(3,4,4)) = \varphi((3,2,3),(5,4,2)) = \varphi((4,1,3),(3,4,4)) = \varphi((4,1,3),(5,4,2)) = 1$ because $B_9 = B_{10} = \emptyset$. For

$\tilde{b}_5 = (1,2,2)$, $\tilde{b}_{11} = (3,4,4)$ in $\mathcal{B}$

$$\begin{aligned}
\varphi((1,2,2),(3,4,4)) &= 1 - \sum_{l=6}^{10} \sum_{\tilde{b}_l \in B_l} \varphi((1,2,2),\tilde{b}_l) \frac{\binom{3-1}{b_{l1}-1}\binom{4-2}{b_{l2}-2}\binom{4-2}{b_{l3}-2}}{\binom{11-5}{l-5}} \\
&= 1 - \sum_{\tilde{b}_8 \in B_8} \varphi((1,2,2),\tilde{b}_8) \frac{\binom{3-1}{b_{81}-1}\binom{4-2}{b_{82}-2}\binom{4-2}{b_{83}-2}}{\binom{11-5}{8-5}} \\
&= 1 - \left\{ 1 \cdot \frac{\binom{2}{1}\binom{2}{1}\binom{2}{1}}{\binom{6}{3}} + 1 \cdot \frac{\binom{2}{2}\binom{2}{0}\binom{2}{1}}{\binom{6}{3}} + 1 \cdot \frac{\binom{2}{3}\binom{2}{-1}\binom{2}{1}}{\binom{6}{3}} \right\} \\
&= \frac{1}{2}.
\end{aligned}$$

Likewise for all $\tilde{b}_{n'}, \tilde{b}_n \in \mathcal{B}$, $n' < n$, $\varphi(\tilde{b}_n, \tilde{b}_{n'})$ is evaluated and listed below.

$\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ (only for $n' < n$)		$\tilde{b}_n = (b_{n1}, b_{n2}, b_{n3})$, $\sum_{k=1}^{3} b_{nk} = n$				
		(2,3,3)	(3,2,3)	(4,1,3)	(3,4,4)	(5,4,2)
$\tilde{b}_{n'} = (b_{n'1}, b_{n'2}, b_{n'3})$	(1,2,2)	1	1	1	$\frac{1}{2}$	1
	(2,1,2)	1	1	1	$\frac{2}{5}$	1
	(2,3,3)				1	1
$\sum_{k=1}^{3} b_{n'k} = n'$	(3,2,3)				1	1
	(4,1,3)				1	1

When m=2, by (11) we have

$$\psi^2(\tilde{b}_n) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l)\varphi(\tilde{b}_l, \tilde{b}_n) \frac{\prod_{k=1}^{3} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}. \tag{16}$$

The latent-2 $\psi^2(\cdot)$ is evaluated as follows.

$\psi^2(1,2,2) = \psi^2(2,1,2) = 0$ because $B_l = \emptyset$ for $l = 1, \cdots, 4$. For $\tilde{b}_8 = (2,3,3)$, we have

$$\begin{aligned}
\psi^2(2,3,3) &= \sum_{l=1}^{7} \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l)\varphi(\tilde{b}_l,(2,3,3)) \frac{\binom{2}{b_{l1}}\binom{3}{b_{l2}}\binom{3}{b_{l3}}}{\binom{8}{l}} \\
&= \sum_{\tilde{b}_5 \in B_5} \psi(\tilde{b}_5)\varphi(\tilde{b}_5,(2,3,3)) \frac{\binom{2}{b_{51}}\binom{3}{b_{52}}\binom{3}{b_{53}}}{\binom{8}{5}} \\
&= 1 \cdot 1 \cdot \frac{\binom{2}{1}\binom{3}{2}\binom{3}{2}}{\binom{8}{5}} + 1 \cdot 1 \cdot \frac{\binom{2}{2}\binom{3}{1}\binom{3}{2}}{\binom{8}{5}}
\end{aligned}$$

$$= \frac{27}{56}.$$

Similarly

$$\psi^2(3,2,3) = 1\cdot 1\cdot \frac{\binom{3}{1}\binom{2}{2}\binom{3}{2}}{\binom{8}{5}} + 1\cdot 1\cdot \frac{\binom{3}{2}\binom{2}{1}\binom{3}{2}}{\binom{8}{5}} = \frac{27}{56},$$

$$\psi^2(4,1,3) = 1\cdot 1\cdot \frac{\binom{4}{1}\binom{1}{2}\binom{3}{2}}{\binom{8}{5}} + 1\cdot 1\cdot \frac{\binom{4}{2}\binom{1}{1}\binom{3}{2}}{\binom{8}{5}} = \frac{9}{28},$$

$$\begin{aligned}\psi^2(3,4,4) &= 1\cdot\frac{1}{2}\cdot\frac{\binom{3}{1}\binom{4}{2}\binom{4}{2}}{\binom{11}{5}} + 1\cdot\frac{2}{5}\cdot\frac{\binom{3}{2}\binom{4}{1}\binom{4}{2}}{\binom{11}{5}} + \frac{29}{56}\cdot 1\cdot\frac{\binom{3}{2}\binom{4}{3}\binom{4}{3}}{\binom{11}{8}} + \\ &\quad + \frac{29}{56}\cdot 1\cdot\frac{\binom{3}{3}\binom{4}{2}\binom{4}{3}}{\binom{11}{8}} + \frac{19}{28}\cdot 1\cdot\frac{\binom{3}{4}\binom{4}{1}\binom{4}{3}}{\binom{11}{8}} = \frac{156}{385},\end{aligned}$$

$$\begin{aligned}\psi^2(5,4,2) &= 1\cdot 1\cdot\frac{\binom{5}{1}\binom{4}{2}\binom{2}{2}}{\binom{11}{5}} + 1\cdot 1\cdot\frac{\binom{5}{2}\binom{4}{1}\binom{2}{2}}{\binom{11}{5}} + \frac{29}{56}\cdot 1\cdot\frac{\binom{5}{2}\binom{4}{3}\binom{2}{3}}{\binom{11}{8}} + \\ &\quad + \frac{29}{56}\cdot 1\cdot\frac{\binom{5}{3}\binom{4}{2}\binom{2}{3}}{\binom{11}{8}} + + \frac{19}{28}\cdot 1\cdot\frac{\binom{5}{4}\binom{4}{1}\binom{2}{3}}{\binom{11}{8}} = \frac{2}{11}.\end{aligned}$$

The probabilities that point $(3,4,4)$ being hit by random paths $\underset{\sim}{S}$ or $\underset{\sim}{S}_N$ as the second hitting in $\mathcal{B}$ are

$$\begin{aligned}P_p(\underset{\sim}{S} \in H^2(3,4,4)) &= \psi^2(3,4,4)P_p(\tilde{S}_{11} = (3,4,4)) \\ &= \frac{156}{385}\binom{11}{3,4,4}p_1^3p_2^4p_3^4 \\ &= 4680p_1^3p_2^4p_3^4.\end{aligned}$$

and

$$\begin{aligned}P_{p,N}(\underset{\sim}{S}_N \in H_N^2(3,4,4)) &= \psi^2(3,4,4)P_{p,N}(\tilde{S}_{11} = (3,4,4)) \\ &= \frac{156}{385}\frac{\binom{p_1N}{3}\binom{p_2N}{4}\binom{p_3N}{4}}{\binom{N}{11}}\end{aligned}$$

When absorption is defined as the second hitting on $\mathcal{B}$, the absorption probability distributions on barrier set $\mathcal{B}$ are listed on table below.

n	Barrier Points		Absorption Probability Distributions on B	
	$\tilde{b}_n \in \mathcal{B}$	$\psi^2(\tilde{b}_n)$	$\mathcal{M}_3(n; p_1, p_2, p_3)$ for $\underline{\mathrm{S}}$	$\mathcal{H}_3(n; p_1, p_2, p_3; N)$ for $\underline{\mathrm{S}}_N$
5	$(1,2,2)$	0	0	0
5	$(2,1,2)$	0	0	0
8	$(2,3,3)$	$\frac{27}{56}$	$270p_1^2p_2^3p_3^3$	$\frac{27}{56}\frac{\binom{p_1N}{2}\binom{p_2N}{3}\binom{p_3N}{3}}{\binom{N}{8}}$
8	$(3,2,3)$	$\frac{27}{56}$	$270p_1^3p_2^2p_3^3$	$\frac{27}{56}\frac{\binom{p_1N}{2}\binom{p_2N}{3}\binom{p_3N}{3}}{\binom{N}{8}}$
8	$(4,1,3)$	$\frac{9}{28}$	$90p_1^4p_2^1p_3^3$	$\frac{9}{28}\frac{\binom{p_1N}{4}\binom{p_2N}{1}\binom{p_3N}{3}}{\binom{N}{11}}$
11	$(3,4,4)$	$\frac{156}{385}$	$4680p_1^3p_2^4p_3^4$	$\frac{156}{385}\frac{\binom{p_1N}{3}\binom{p_2N}{4}\binom{p_3N}{4}}{\binom{N}{11}}$
11	$(5,4,2)$	$\frac{2}{11}$	$1200p_1^5p_2^4p_3^2$	$\frac{2}{11}\frac{\binom{p_1N}{5}\binom{p_2N}{4}\binom{p_3N}{2}}{\binom{N}{11}}$

When m=3, by (11) we have

$$\psi^3(\tilde{b}_n) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} \psi^2(\tilde{b}_l)\varphi(\tilde{b}_l, \tilde{b}_n) \frac{\prod_{k=1}^{3} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}. \tag{17}$$

With $\varphi(\tilde{b}_l, \tilde{b}_n)$ and $\psi^2(\tilde{b}_n)$ known, the latent-3 $\psi^3(\tilde{b}_n)$ can be evaluated as following. $\psi^3(1,2,2)$ $= \psi^3(2,1,2) = 0$ because $B_l = \emptyset$ for $l = 1, \cdots, 4$. $\psi^3(2,3,3) = \psi^3(3,2,3) = \psi^3(4,1,3) = 0$ because $B_l = \emptyset$ for $l = 1,2,3,4,6,7$ and $\psi^2(\tilde{b}_5) = 0$, all $\tilde{b}_5 \in B_5$. For $\tilde{b}_{11} = (3,4,4)$, we have

$$\begin{aligned}
\psi^3(3,4,4) &= \sum_{\tilde{b}_5 \in B_5} \psi^2(\tilde{b}_5)\varphi(\tilde{b}_5,(3,4,4)) \frac{\binom{3}{b_{51}}\binom{4}{b_{52}}\binom{4}{b_{53}}}{\binom{11}{5}} + \sum_{\tilde{b}_8 \in B_8} \psi^2(\tilde{b}_8)\varphi(\tilde{b}_8,(3,4,4)) \frac{\binom{3}{b_{81}}\binom{4}{b_{82}}\binom{4}{b_{83}}}{\binom{11}{8}} \\
&= 0 \cdot \frac{1}{2} \cdot \frac{\binom{3}{1}\binom{4}{2}\binom{4}{2}}{\binom{11}{5}} + 0 \cdot \frac{2}{5} \cdot \frac{\binom{3}{2}\binom{4}{1}\binom{4}{2}}{\binom{11}{5}} + \frac{27}{56} \cdot 1 \cdot \frac{\binom{3}{2}\binom{4}{3}\binom{4}{3}}{\binom{11}{8}} + \\
&\quad + \frac{27}{56} \cdot 1 \cdot \frac{\binom{3}{3}\binom{4}{2}\binom{4}{3}}{\binom{11}{8}} + \frac{9}{28} \cdot 1 \cdot \frac{\binom{3}{4}\binom{4}{1}\binom{4}{3}}{\binom{11}{8}} = \frac{81}{385},
\end{aligned}$$

Similarly

$$\begin{aligned}
\psi^3(5,4,2) &= 0 \cdot 1 \cdot \frac{\binom{5}{1}\binom{4}{2}\binom{2}{2}}{\binom{11}{5}} + 0 \cdot 1 \cdot \frac{\binom{5}{2}\binom{4}{1}\binom{2}{2}}{\binom{11}{5}} + \frac{27}{56} \cdot 1 \cdot \frac{\binom{5}{2}\binom{4}{3}\binom{2}{3}}{\binom{11}{8}} + \\
&\quad + \frac{27}{56} \cdot 1 \cdot \frac{\binom{5}{3}\binom{4}{2}\binom{2}{3}}{\binom{11}{8}} + \frac{9}{28} \cdot 1 \cdot \frac{\binom{5}{4}\binom{4}{1}\binom{2}{3}}{\binom{11}{8}} = 0.
\end{aligned}$$

The probabilities that point $(3,4,4)$ being hit by random paths $\underset{\sim}{S}$ or $\underset{\sim}{S}_N$ as the third hitting in $\mathcal{B}$ are

$$\begin{aligned} P_p(\underset{\sim}{S} \in H^3(3,4,4)) &= \psi^3(3,4,4)P_p(\tilde{S}_{11} = (3,4,4)) \\ &= \frac{81}{385}\binom{11}{3,4,4}p_1^3p_2^4p_3^4 \\ &= 2430p_1^3p_2^4p_3^4. \end{aligned}$$

or

$$\begin{aligned} P_{p,N}(\underset{\sim}{S}_N \in H_N^3(3,4,4)) &= \psi^2(3,4,4)P_{p,N}(\tilde{S}_{11} = (3,4,4)) \\ &= \frac{81}{385}\frac{\binom{p_1N}{3}\binom{p_2N}{4}\binom{p_3N}{4}}{\binom{N}{11}} \end{aligned}$$

When absorption is defined as the third hitting on $\mathcal{B}$, the absorption probability distributions on barrier set $\mathcal{B}$ are listed on table below.

	Barrier Points		Absorption Probability Distributions on $\mathcal{B}$	
n	$\tilde{b}_n \in \mathcal{B}$	$\psi^3(\tilde{b}_n)$	$\mathcal{M}_3(n;p_1,p_2,p_3)$ for $\underset{\sim}{S}$	$\mathcal{H}_3(n;p_1,p_2,p_3;N)$ for $\underset{\sim}{S}_N$
5	$(1,2,2)$	0	0	0
5	$(2,1,2)$	0	0	0
8	$(2,3,3)$	0	0	0
8	$(3,2,3)$	0	0	0
8	$(4,1,3)$	0	0	0
11	$(3,4,4)$	$\frac{81}{385}$	$2430p_1^3p_2^4p_3^4$	$\frac{81}{385}\frac{\binom{p_1N}{3}\binom{p_2N}{4}\binom{p_3N}{4}}{\binom{N}{11}}$
11	$(5,4,2)$	0	0	0

3.3 Absorption Defined as Ordered Hittings

Suppose we have three disjoint barrier sets $\mathcal{B}^2$, $\mathcal{B}^2$ and $\mathcal{B}^3$. We are interested in obtaining the probabilities that random path $\underset{\sim}{S}$ (or $\underset{\sim}{S}_N$) hits $\mathcal{B}^1$ first, then hits $\mathcal{B}^2$ before hitting $\mathcal{B}^3$, which we call as "ordered hittings of $\{\mathcal{B}^2;\mathcal{B}^1,\mathcal{B}^3\}$".

In this sense, absorption of random path $\underset{\sim}{S}$ (or $\underset{\sim}{S}_N$) by a point $\tilde{b} \in \mathcal{B}^2$ is that, $\underset{\sim}{S}$ (or $\underset{\sim}{S}_N$) hits $\mathcal{B}^1$ first, then hits this point as the first hitting in $\mathcal{B}^2$, before hitting $\mathcal{B}^3$.

Given disjoint barrier sets $\mathcal{B}^1$, $\mathcal{B}^2$ and $\mathcal{B}^3$. Let $B_n^i = \{\tilde{b}_n \in \mathcal{B}^i : \sum_{k=1}^d b_{nk} = n\}$, then $\mathcal{B}^i = \cup_n B_n^i$, $i = 1,2,3$. For $\tilde{b}_n \in \mathcal{B}^2$, let $H_{2;1,3}(\tilde{b}_n)$ be a bunch of sample paths (in $\mathcal{X}$) which hits $\tilde{b}_n$ as the first hitting in $\mathcal{B}^2$ after hitting some points in $\mathcal{B}^1$ and before hitting any points in $\mathcal{B}^3$. Then for $\tilde{b}_n \in \mathcal{B}^2$

$$H_{2;1,3}(\tilde{b}_n) = \left\{ \underline{s} \in \mathcal{X} : \begin{array}{c} \tilde{s}_n = \tilde{b}_n;\ \tilde{s}_r \in B_r^1, \text{ for some } r < n; \\ s_l \notin B_l^2 \cup B_l^3,\ l = 1, \cdots, n-1. \end{array} \right\}. \tag{18}$$

Similarly, let $H_{N2;1,3}(\tilde{b}_n)$ be a bunch of sample paths (in $\mathcal{X}_N$) which hits $\mathcal{B}^1$ first, then hits $\tilde{b}_n$ as the first hitting in $\mathcal{B}^2$, before hitting $\mathcal{B}^3$. Then for $\tilde{b}_n \in \mathcal{B}^2$

$$H_{N2;1,3}(\tilde{b}_n) = \left\{ \underline{s}_N \in \mathcal{X}_N : \begin{array}{c} \tilde{s}_n = \tilde{b}_n;\ \tilde{s}_r \in B_r^1, \text{ for some } r < n; \\ s_l \notin B_l^2 \cup B_l^3,\ l = 1, \cdots, n-1. \end{array} \right\}. \tag{19}$$

To simplify notation, let $\mathcal{B}^{123} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, $B_n^{123} = B_n^1 \cup B_n^2 \cup B_n^3$, then $\mathcal{B}^{123} = \cup_n B_n^{123}$. Let $\psi_{123}(\cdot)$ be the latent of $\mathcal{B}^{123}$. By Definition 3.1, $\psi_{123}(\cdot)$ is evaluated by

$$\psi_{123}(\tilde{b}_n) = 1 - \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l^{123}} \psi_{123}(\tilde{b}_l) \frac{\prod_{k=1}^{d} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}} \tag{20}$$

Let $\mathcal{B}^{23} = \mathcal{B}^2 \cup \mathcal{B}^3$, $B_n^{23} = B_n^2 \cup B_n^3$, then $\mathcal{B}^{23} = \cup_n B_n^{23}$. Let $\varphi_{1,23}(\cdot,\cdot)$ be the relative latent of $\mathcal{B}^{23}$ with origins in $\mathcal{B}^1$. For $\tilde{b}_{n'} \in \mathcal{B}^1$, $n > n'$, $\tilde{b}_n \in \mathcal{B}^{23}$, $\varphi_{1,23}(\cdot,\cdot)$ can be evaluated by

$$\varphi_{1,23}(\tilde{b}_{n'}, \tilde{b}_n) = 1 - \sum_{l=n'+1}^{n-1} \sum_{\tilde{b}_l \in B_l^{23}} \varphi_{1,23}(\tilde{b}_{n'}, \tilde{b}_l) \frac{\prod_{k=1}^{d} \binom{b_{nk} - b_{n'k}}{b_{lk} - b_{n'k}}}{\binom{n-n'}{l-n'}} \tag{21}$$

Definition 3.4 *Given disjoint barrier sets $\mathcal{B}^1, \mathcal{B}^2$ and $\mathcal{B}^3$, the "ordered latent of $\{\mathcal{B}^2;\mathcal{B}^1,\mathcal{B}^3\}$", $\psi_{2;1,3}(\cdot)$, is a function defined on $\mathcal{B}^2$, for $\tilde{b}_n \in \mathcal{B}^2$*

$$\psi_{2;1,3}(\tilde{b}_n) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l^1} \psi_{123}(\tilde{b}_l) \varphi_{1,23}(\tilde{b}_l, \tilde{b}_n) \frac{\prod_{k=1}^{d} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}} \tag{22}$$

where $\psi_{123}(\cdot)$ is given by (20), $\varphi_{1,23}(\cdot,\cdot)$ is given by (21). ■

Theorem 3.3 *Given disjoint barrier sets $\mathcal{B}^1$, $\mathcal{B}^2$ and $\mathcal{B}^3$, let absorption of random path $\underline{S}$(or $\underline{S}_N$) be defined as ordered hittings of $\{\mathcal{B}^2;\mathcal{B}^1,\mathcal{B}^3\}$,* i.e. *$\underline{S}$(or $\underline{S}_N$) is absorbed by $\tilde{b} \in \mathcal{B}^2$ iff $\underline{S}$(or $\underline{S}_N$) hits $\tilde{b}$ as first hitting in $\mathcal{B}^2$ after hitting $\mathcal{B}^1$ and before hitting $\mathcal{B}^3$. The absorption* *probability of $\underline{S}$ at $\tilde{b}_n \in \mathcal{B}^2$ is*

$$P_p(\underline{S} \in H_{2;1,3}(\tilde{b}_n)) = \psi_{2;1,3}(\tilde{b}_n) P_p(\tilde{S}_n = \tilde{b}_n) = \psi_{2;1,3}(\tilde{b}_n) \binom{n}{\tilde{b}_n} \prod_{k=1}^{d} p_k^{b_{nk}}. \tag{23}$$

The absorption probability of $\underline{S}_N$ at $\tilde{b}_n \in \mathcal{B}^2$ is

$$P_p(\underline{S}_N \in H_{N2;1,3}(\tilde{b}_n)) = \psi_{2;1,3}(\tilde{b}_n) P_p(\tilde{S}_n = \tilde{b}_n) = \psi_{2;1,3}(\tilde{b}_n) \frac{\prod_{k=1}^{d} \binom{p_k N}{b_{nk}}}{\binom{N}{n}}. \tag{24}$$

■

Illustrative Example 2:

Let $\mathcal{B}^1 = \{(1,2,2),(3,2,3)\}$, $\mathcal{B}^2 = \{(4,1,3),(3,4,4),(5,4,2)\}$. $\mathcal{B}^3 = \{(2,1,2),(2,3,3)\}$. Since $\mathcal{B}^{123} = \cup_{i=1}^{3} \mathcal{B}^i$ is the same as $\mathcal{B}$ in Example 1 in Section 3.1, thus $\psi_{123}(\cdot)$ is the same as $\psi(\cdot)$ in Example 1, listed as follow.

n	5	5	8	8	8	11	11
$\tilde{b}_n \in \mathcal{B}$	(1,2,2)	(2,1,2)	(2,3,3)	(3,2,3)	(4,1,3)	(3,4,4)	(5,4,2)
$\psi_{123}(\tilde{b}_n)$	1	1	$\frac{29}{56}$	$\frac{29}{56}$	$\frac{19}{28}$	$\frac{148}{385}$	$\frac{28}{33}$

Follow (21), $\psi_{1,23}(\cdot,\cdot)$ is evaluated and listed below.

$\varphi_{1,23}(\tilde{b}_{n'}, \tilde{b}_n)$ (only for $n' < n$)		$\tilde{b}_n \in \mathcal{B}^{23}$, $\sum_{k=1}^{3} b_{nk} = n$				
		(2,1,2)	(2,3,3)	(4,1,3)	(3,4,4)	(5,4,2)
$\tilde{b}_{n'} \in \mathcal{B}^1$	(1,2,2)		1	1	$\frac{1}{2}$	1
$\sum_{k=1}^{3} b_{n'k} = n'$	(3,2,3)				1	1

Follow (22), $\psi_{2;1,3}(\cdot,\cdot)$ is evaluated and listed below. The absorption probability distributions of $\underline{S}$ and $\underline{S}_N$ on $\mathcal{B}^2$, obtained by (23) and (24), are listed below.

n	Barrier Points $\tilde{b}_n \in \mathcal{B}^2$	$\psi_{2;1,3}(\tilde{b}_n)$	Absorption Probabilities on $\mathcal{B}^2$	
			$\mathcal{M}_3(n; p_1, p_2, p_3)$ for $\underline{S}$	$\mathcal{H}_3(n; p_1, p_2, p_3; N)$ for $\underline{S}_N$
8	(4,1,3)	0	0	0
11	(3,4,4)	$\frac{74}{385}$	$2220 p_1^3 p_4^2 p_3^4$	$\frac{74}{385} \frac{\binom{p_1 N}{3}\binom{p_2 N}{4}\binom{p_3 N}{4}}{\binom{N}{11}}$
11	(5,4,2)	$\frac{5}{77}$	$450 p_1^5 p_2^4 p_3^2$	$\frac{5}{77} \frac{\binom{p_1 N}{5}\binom{p_2 N}{4}\binom{p_3 N}{2}}{\binom{N}{11}}$

4 Sequential Tests for Dichotomous Populations

As a finite population, dichotomous population is the simplest but most common one in real life applications. In Section 3, we have developed methods to obtain absorption probability distributions of $\underset{\sim}{S}$ and $\underset{\sim}{S}_N$, as random paths from finite populations, on any given barrier sets, where the absorption may be defined as first hitting, or mth hitting, or ordered hittings. In this section we apply these results in dichotomous populations to derive absorption probability distributions for varieties of sequential tests.

4.1 Sequential Tests with Simple Enclosed Boundary

We say a barrier set $\mathcal{B}$ is "finite in time" if $\mathcal{B} = \cup_{n=1}^{m} B_n$ for some $m \geq 0$. We say $\mathcal{B}$ is "complete" for random path $\underset{\sim}{S}$ if $P_p(\underset{\sim}{S} \in \mathcal{X}^{\mathcal{B}}) = 1$ for any p. We say $\mathcal{B}$ is "enclosed" for $\underset{\sim}{S}$ if $\mathcal{B}$ is "finite in time" and "complete" for $\underset{\sim}{S}$. In closed sequential tests, sampling stops whenever random path goes acrossing an enclosed boundary. In that case, the set of barrier points $\mathcal{B}$ is an enclosed boundary.

In sequential tests for dichotomous populations, if we denote S_n the number of "non-defects", then $n - S_n$ is the number of "defects", in the first n observations. Let $\tilde{S}_n = (S_n, n - S_n)$, then $\underset{\sim}{S} = (\tilde{S}_1, \tilde{S}_2, \cdots,)$ and $\underset{\sim}{S}_N = (\tilde{S}_1, \cdots, \tilde{S}_N)$ are random paths for sampling *with* and *without* replacement. A simple barrier boundary should be

$$\mathcal{B} = \Big\{(i,j) : (i,j) \in \{(a_n, n-a_n)\}_{n=1}^{m-1} \cup \{(b_n, n-b_n)\}_{n=1}^{m-1} \cup \{(k, m-k)\}_{k=a_m}^{b_m}\Big\}$$

where a_n, b_n are integers such that $a_n \leq b_n$ for $n = 1, \cdots, m$. We denote $\{(a_n, n-a_n)\}_{n=1}^{m-1}$ by $\mathcal{B}^a$, $\{(n, b_n)\}_{n=1}^{m-1}$ by $\mathcal{B}^b$, $\{(m,k)\}_{k=a_n}^{b_n}$ by $\mathcal{B}^m$. Then $\mathcal{B} = \mathcal{B}^a \cup \mathcal{B}^b \cup \mathcal{B}^m$.

To test hypothesis $H_0 : p < p^*$ *v.s.* $H_1 : p \geq p^*$, the decision rule should be defined as rejecting H_0 if random path first hits point either in $\mathcal{B}^a$ or in $\mathcal{B}^m_{k_c} = \{(k, m-k) \in \mathcal{B}^m : k \geq k_c\}$ (the cut off point is $(k_c, m-k_c)$ on $\mathcal{B}^m$), accepting H_0 otherwise. So the stopping time is $T \wedge m$ where $T = inf\{n : S_n \geq b_n \text{ or } \leq a_n\}$.

Let $\psi(\cdot)$ be the latent of $\mathcal{B}$. If we denote $\psi_{an} = \psi(a_n, n-a_n)$, $\psi_{bn} = \psi(b_n, n-b_n)$ and $\psi_{mk} = \psi(k, m-k)$, then by (5)

$$\psi_{an} = 1 - \sum_{l=1}^{n-1} \left\{ \psi_{al} \frac{\binom{a_n}{a_l}\binom{n-a_n}{l-a_l}}{\binom{n}{l}} + \psi_{bl} \frac{\binom{a_n}{b_l}\binom{n-a_n}{l-b_l}}{\binom{n}{l}} \right\}, \tag{25}$$

$$\psi_{bn} = 1 - \sum_{l=1}^{n-1} \left\{ \psi_{al} \frac{\binom{b_n}{a_l}\binom{n-b_n}{l-a_l}}{\binom{n}{l}} + \psi_{bl} \frac{\binom{b_n}{b_l}\binom{n-b_n}{l-b_l}}{\binom{n}{l}} \right\}, \tag{26}$$

for $n = 1, \cdots, m-1$;

$$\psi_{mk} = 1 - \sum_{l=1}^{m-1} \left\{ \psi_{al} \frac{\binom{k}{a_l}\binom{m-k}{l-a_l}}{\binom{m}{l}} + \psi_{bl} \frac{\binom{k}{b_l}\binom{m-k}{l-b_l}}{\binom{m}{l}} \right\}, \tag{27}$$

for $k = a_m, a_m + 1, \cdots, b_m - 1, b_m$.

Sampling with replacement, the power function is, for any $p \in [0, 1]$

$$\begin{aligned} \beta(p) &= \sum_{n=1}^{m-1} \psi_{bn} P_p(S_n = b_n) + \sum_{k=k_c}^{b_m} \psi_{mk} P_p(S_m = k) \\ &= \sum_{n=1}^{m-1} \psi_{bn} \binom{n}{b_n} p^{b_n} (1-p)^{n-b_n} + \sum_{k=k_c}^{b_m} \psi_{mk} \binom{m}{k} p^k (1-p)^{m-k} \end{aligned} \tag{28}$$

The expected sampling size $S(p)$ as a function of p on $[0, 1]$ is

$$\begin{aligned} S(p) &= E_p T \wedge m \\ &= \sum_{n=1}^{m-1} n \left\{ \psi_{bn} \binom{n}{b_n} p^{b_n} (1-p)^{n-b_n} + \psi_{an} \binom{n}{a_n} p^{a_n} (1-p)^{n-a_n} \right\} \\ &\quad + m \sum_{k=k_c}^{b_m} \psi_{mk} \binom{m}{k} p^k (1-p)^{m-k}. \end{aligned} \tag{29}$$

Sampling without replacement, the power function is, for any N and $p = \frac{0}{N}, \frac{1}{N}, \cdots, \frac{N}{N}$

$$\begin{aligned} \beta(p) &= \sum_{n=1}^{m-1} \psi_{bn} P_{p,N}(S_n = b_n) + \sum_{k=k_c}^{b_m} \psi_{mk} P_{p,N}(S_m = k) \\ &= \sum_{n=1}^{m-1} \psi_{bn} \frac{\binom{pN}{b_n}\binom{(1-p)N}{n-b_n}}{\binom{N}{n}} + \sum_{k=k_c}^{b_m} \psi_{mk} \frac{\binom{pN}{k}\binom{(1-p)N}{m-k}}{\binom{N}{m}}. \end{aligned} \tag{30}$$

The expected sampling size $S(p)$, as a function of p, is

$$\begin{aligned} S(p) &= E_{p,N} T \wedge m \\ &= \sum_{n=1}^{m-1} n \left\{ \psi_{bn} \frac{\binom{pN}{b_n}\binom{(1-p)N}{n-b_n}}{\binom{N}{n}} + \psi_{an} \frac{\binom{pN}{a_n}\binom{(1-p)N}{n-a_n}}{\binom{N}{n}} \right\} \end{aligned}$$

$$+m \sum_{k=k_c}^{b_m} \psi_{mk} \frac{\binom{pN}{k}\binom{(1-p)N}{m-k}}{\binom{N}{m}}. \tag{31}$$

4.2 Multiple-Stage Sequential Tests

Assume we want to test $H_0 : p \leq p^*$ em v.s. $H_1 : p > p^*$ with multiple-stage sequential tests for dichotomous populations. Let J denote the maximum number of stage, m_j denote the number of observations taken at jth stage, $j = 1, \cdots, J$. Let $n_j = \sum_{i=1}^{j} m_i$, and let S_{n_j} be the number of nondefects among the n_j observations taken in the first j stages. The sequential procedure is as follow: if $S_{n_j} \leq a_{n_j}$, stop and accept H_0; if $S_{n_j} \leq r_{n_j}$, stop and accept H_1; if $a_{n_j} < S_{n_j} < r_{n_j}$, continue to stage $j+1$; where a_{n_j}, r_{n_j} are given constants.

Let $\tilde{S}_n = (S_n, n - S_n)$, then $\underset{\sim}{S} = (\tilde{S}_1, \tilde{S}_2, \cdots)$ or $\underset{\sim}{S}_N = (\tilde{S}_1, \cdots, \tilde{S}_N)$ are random paths, by sampling *with* or *without* replacement, from dichotomous populations.

Let barrier set $\mathcal{B} = \cup_{j=1}^{J} B_{n_j}$, where $B_{n_j} = B_{n_j}^a \cup B_{n_j}^r$, $B_{n_j}^a = \{(i, n_j - i) : 0 \leq i \leq a_j\}$, $B_{n_j}^r = \{(i, n_j - i) : r_j \leq i \leq n_j\}$. Let $\mathcal{B}^a = \cup_{j=1}^{J} B_{n_j}^a$, $\mathcal{B}^r = \cup_{j=1}^{J} B_{n_j}^r$, then the multiple-stage test procedure is: accept H_0 if $\underset{\sim}{S}$ (or $\underset{\sim}{S}_N$) hits $\mathcal{B}^a$ before hitting $\mathcal{B}^r$, accept H_1 otherwise.

Let $\psi(\cdot)$ be the latent of $\mathcal{B}$, then by Theorem 3.1, the absorption probability distribution of $\underset{\sim}{S}$ (or $\underset{\sim}{S}_N$) on $\mathcal{B}$ is given as, for any $\tilde{b}_{n_j} \in \mathcal{B}$,

$$P_p(\underset{\sim}{S} \in H(\tilde{b}_{n_j})) = \psi(\tilde{b}_{n_j}) \binom{n_j}{b_{n_j}} p^{b_j} (1-p)^{n_j - b_{n_j}} \tag{32}$$

or

$$P_{p,N}(\underset{\sim}{S}_N \in H_N(\tilde{b}_{n_j})) = \psi(\tilde{b}_{n_j}) \frac{\binom{pN}{b_{n_j}}\binom{(1-p)N - b_{n_j}}{b_{n_j}}}{\binom{N}{n_j}} \tag{33}$$

where $\tilde{b}_n = (b_{n_j}, n_j - b_{n_j})$ and $\psi(\cdot)$ is evaluated by

$$\psi(\tilde{b}_{n_j}) \;=\; 1 - \sum_{l=1}^{j-1} \sum_{\tilde{b}_{n_l} \in B_{n_l}} \psi(\tilde{b}_{n_l}) \frac{\binom{b_{n_j}}{b_{n_l}}\binom{n_j - b_{n_j}}{n_l - b_{n_l}}}{\binom{n_j}{n_l}}. \tag{34}$$

Number of barrier points in each B_{n_j} is usually large for multiple-stage tests. To save computation, there is an alternative way to evaluate $\psi(\cdot)$, the latent of $\mathcal{B}$, by applying result in Section 3.2. This alternative way may save computation of $\psi(\cdot)$, especially when $r_{n_j} - a_{n_j}$ is much smaller than $\frac{n_j}{2}$.

Theorem 4.1 *Let $\mathcal{B}^c = \cup_{j=1}^{J-1} B_{n_j}^c$ where $B_{n_j}^c = \{(i, n_j - i) : a_{n_j} < i < r_{n_j}\}$. Let $\psi_c^j(\cdot)$ be the latent-j of $\mathcal{B}^c$ as in Definition 3.3, then for $j = 2, \cdots, J-1$, $\tilde{b}_{n_j} \in \mathcal{B}^c$*

$$\psi_c^j(\tilde{b}_{n_j}) = \sum_{\tilde{b}_{n_{j-1}} \in B_{n_{j-1}}^c} \psi_c^{j-1}(\tilde{b}_{n_{j-1}}) \frac{\binom{b_{n_j}}{b_{n_{j-1}}}\binom{n_j - b_{n_j}}{n_{j-1} - b_{n_{j-1}}}}{\binom{n_j}{n_{j-1}}} \tag{35}$$

If $\psi(\cdot)$ is the latent of $\mathcal{B} = \cup_{j=1}^{J} B_{n_j}$, $B_{n_j} = \{(i, n_j - i) : 0 \le i \le a_j \;\; or \;\; r_j \le i \le n_j\}$, then for $j = 2, \cdots, J$, $\tilde{b}_{n_j} \in \mathcal{B}$

$$\psi(\tilde{b}_{n_j}) = \sum_{\tilde{b}_{n_{j-1}} \in B_{n_{j-1}}^c} \psi_c^{j-1}(\tilde{b}_{n_{j-1}}) \frac{\binom{b_{n_j}}{b_{n_{j-1}}}\binom{n_j - b_{n_j}}{n_{j-1} - b_{n_{j-1}}}}{\binom{n_j}{n_{j-1}}} \tag{36}$$

∎

Illustrative Example 3:

Consider 3-stage test with $m_1 = 5, m_2 = 3, m_3 = 3$ as numbers of observations taken in stages 1, 2, 3; thus we have $J = 3$, $n_1 = 5$, $n_2 = 8$, $n_3 = 11$. Suppose $a_5 = 1$, $r_5 = 4$; $a_8 = 2$, $r_8 = 4$; $a_{11} = 5$, $r_{11} = 6$. Then we have barrier set $\mathcal{B} = B_5 \cup B_8 \cup B_{11}$, where $B_5 = B_5^a \cup B_5^r$, $B_8 = B_8^a \cup B_8^r$, $B_{11} = B_{11}^a \cup B_{11}^r$; $B_5^a = \{(0,5),(1,4)\}$, $B_5^r = \{(4,1),(5,0)\}$; $B_8^a = \{(0,8),(1,7),(2,6)\}$, $B_5^r = \{(6,2),(7,1),(8,0)\}$; $B_{11}^a = \{(0,11),(1,10),(2,9),(3,8),(4,7),(5,6)\}$, $B_{11}^r = \{(6,5),(7,4),(8,3),(9,2),(10,1),(11,0)\}$. Barrier points for the 3-stage test are graphed in Figure 4.2.

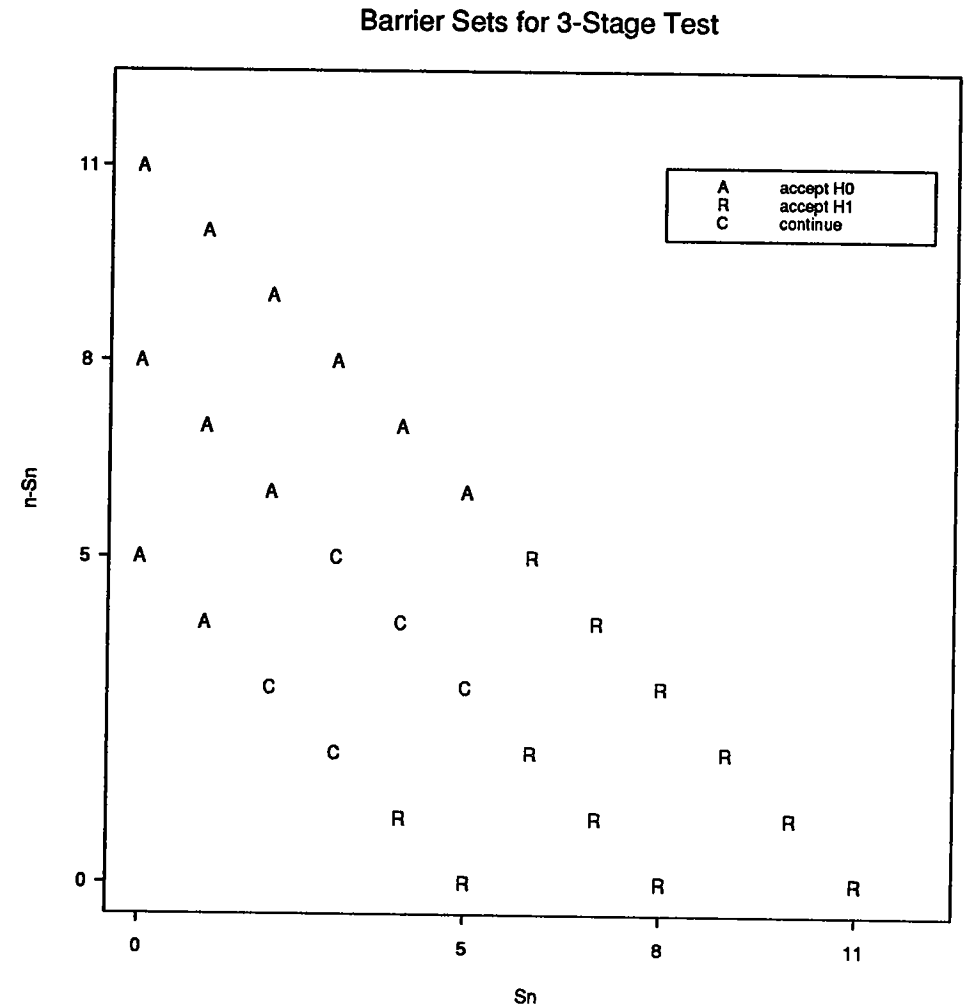

Figure 1: **3-stage Sequential Test**

To obtain absorption probability distribution on $\mathcal{B}$, we only need to evaluate $\psi(\cdot)$, the latent of $\mathcal{B}$. Though $\psi(\cdot)$ can be evaluated by (5), to save computation, we apply (35) and (36) to evaluate $\psi(\cdot)$. Let $\mathcal{B}^c = B_5^c \cup B_8^c$ where $B_5^c = \{(2,3),(3,2)\}$, $B_8^c = \{(3,5),(4,4),(5,3)\}$. Let $\psi_c(\cdot)$ be the latent, $\psi_c^j(\cdot)$ be the latent-j, of $\mathcal{B}^c$. Follow (5), $\psi_c(2,3) = \psi_c(3,2) = 1$. By definition, $\psi_c^1(\cdot) = \psi_c(\cdot)$. For $n_2 = 8$, $(3,5) \in B_8^c$, by (35)

$$\begin{aligned}\psi_c^2(3,5) &= \sum_{\tilde{b}_5 \in B_5^c} \psi_c^1(\tilde{b}_5) \frac{\binom{3}{b_5}\binom{5}{5-b_5}}{\binom{8}{5}} \\ &= 1 \cdot \frac{\binom{3}{2}\binom{5}{3}}{\binom{8}{5}} + 1 \cdot \frac{\binom{3}{3}\binom{5}{2}}{\binom{8}{5}}\end{aligned}$$

$$= \frac{5}{7} \tag{37}$$

In this way, $\psi_c^j(\cdot)$, latent-j of $\mathcal{B}^c$, is evaluated on $B_{n_j}^c$ for $j = 1, \cdots, J-1$, listed below.

$n_1 = 5$		$n_2 = 8$	
$\tilde{b}_5 \in B_5^c$	latent-1 of $\mathcal{B}^c$ $\psi_c^1(\tilde{b}_5)$ on B_5^c	$\tilde{b}_8 \in B_8^c$	latent-2 of $\mathcal{B}^c$ $\psi_c^2(\tilde{b}_8)$ on B_8^c
$(2,3)$	1	$(3,5)$	$\frac{5}{7}$
$(3,2)$	1	$(4,3)$	$\frac{6}{7}$
		$(5,3)$	$\frac{5}{7}$

Next we evaluate $\psi(\cdot)$, latent of $\mathcal{B}$. For any $\tilde{b}_5 \in B_5$, $\psi(\tilde{b}_5) = 1$ by (5). For $\tilde{b}_8 \in B_8$ or $\tilde{b}_{11} \in B_{11}$, $\psi(\cdot)$ is evaluated by applying the relation between $\psi(\cdot)$ and $\psi_c^j(\cdot)$ given by (36). For instance, $\tilde{b}_{11} = (4,7) \in B_{11}$

$$\begin{aligned} \psi(4,7) &= \sum_{\tilde{b}_8 \in B_8^c} \psi_c^2(\tilde{b}_8) \frac{\binom{4}{b_8}\binom{7}{8-b_8}}{\binom{11}{8}} \\ &= \frac{5}{7}\frac{\binom{4}{3}\binom{7}{5}}{\binom{11}{8}} + \frac{6}{7}\frac{\binom{4}{4}\binom{7}{4}}{\binom{11}{8}} + \frac{5}{7}\frac{\binom{4}{5}\binom{7}{3}}{\binom{11}{8}} \\ &= \frac{6}{11} \end{aligned} \tag{38}$$

In same way, $\psi(\cdot)$ is evaluated and listed below.

$n_1 = 5$				$n_3 = 11$			
$\tilde{b}_5 \in B_5^a$	$\psi(\tilde{b}_5)$	$\tilde{b}_5 \in B_5^r$	$\psi(\tilde{b}_5)$	$\tilde{b}_{11} \in B_{11}^a$	$\psi(\tilde{b}_{11})$	$\tilde{b}_{11} \in B_{11}^r$	$\psi(\tilde{b}_{11})$
$(0,5)$	1	$(4,1)$	1	$(0,11)$	0	$(6,5)$	$\frac{170}{231}$
$(1,4)$	1	$(5,0)$	1	$(1,10)$	0	$(7,4)$	$\frac{6}{11}$
$n_2 = 8$				$(2,9)$	0	$(8,3)$	$\frac{8}{33}$
$\tilde{b}_8 \in B_8^a$	$\psi(\tilde{b}_8)$	$\tilde{b}_8 \in B_8^r$	$\psi(\tilde{b}_8)$	$(3,8)$	$\frac{8}{33}$	$(9,2)$	0
$(0,8)$	0	$(6,2)$	$\frac{5}{14}$	$(4,7)$	$\frac{6}{11}$	$(10,1)$	0
$(1,7)$	0	$(7,1)$	0	$(5,6)$	$\frac{170}{231}$	$(11,0)$	0
$(2,6)$	$\frac{5}{14}$	$(8,0)$	0				

Once $\psi(\cdot)$ is known, the absorption probability distributions of $\underset{\sim}{S}$ and $\underset{\sim}{S}_N$ on $\mathcal{B}$ are easily available by (32) and (33).

4.3 Two-sided Sequential Tests with Three Decisions

We are interested in testing hypothesis $H_0 : p = p^*$ *v.s.* $H_1 : p > p^*$ *v.s.* $H_2 : p < p^*$. Though this hypothesis is just for one dichotomous population, an adaptation of this hypothesis testing is widely utilizeded to compare the proportions of "defects" in two dichotomous populations in double dichotomy sampling. Various closed sequential procedures for testing this hypothesis have been proposed by authors such as Bross[1952], Armitage[1957]. In their sequential schemes, a simple enclosed boundary is divided into three sections. Decision is made in favor of H_0 or H_1 or H_2 exclusively according to which section the random path hits first.

Here we discuss a class of closed sequential procedures, for testing this hypothesis, which is simple in implementation and more efficient than procedures with simple boundaries. We illustrate the ways to obtain absorption probability distributions for this class of sequential procedures by using results in Section 3.

As proposed by Soble and Wald[1949], testing $H_0 : p = p^*$ *v.s.* $H_1 : p > p^*$ *v.s.* $H_2 : p < p^*$ is equivalent to testing simultaneously hypotheses $H_0' : p \leq p^*$ *v.s.* $H_1' : p > p^*$, and hypotheses $H_0'' : p \geq p^*$ *v.s.* $H_1'' : p < p^*$. The relations among decisions for those hypotheses are: $H_0 : p = p^*$ is favored iff $H_0' : p \leq p^*$ and $H_0'' : p \geq p^*$ are favored; $H_1 : p > p^*$ is favored iff $H_1' : p > p^*$ and $H_0'' : p \geq p^*$ are favored; $H_2 : p < p^*$ is favored iff $H_0' : p \leq p^*$ and $H_1'' : p < p^*$ are favored.

Let $\mathcal{B}^1 = \mathcal{B}^{a_1} \cup \mathcal{B}^{r_1}$ be an enclosed barrier set for testing $H_0' : p \leq p^*$ *v.s.* $H_1' : p > p^*$. Decision is made in favor of H_0' if random path hits $\mathcal{B}^{a_1}$ before hitting $\mathcal{B}^{r_1}$, in favor of H_1' otherwise.

Let $\mathcal{B}^2 = \mathcal{B}^{a_2} \cup \mathcal{B}^{r_2}$ be an enclosed barrier set for testing $H_0'' : p \geq p^*$ *v.s.* $H_1'' : p < p^*$. Decision is made in favor of H_0'' if random path hits $\mathcal{B}^{a_2}$ before hitting $\mathcal{B}^{r_2}$, in favor of H_1'' otherwise.

In order to test $H_0 : p = p^*$ *v.s.* $H_1 : p > p^*$ *v.s.* $H_2 : p < p^*$, these two procedures are carried out simultaneously by observing a same random path from sequential sampling. Final decision is made as soon as both procedures are completed. Equivalently for the combined test, decision is made in favor of H_0 if random path hits $\mathcal{B}^{a_1}$ and $\mathcal{B}^{a_2}$ before hitting $\mathcal{B}^{r_1} \cup \mathcal{B}^{r_2}$; in favor of H_1 if random path hits $\mathcal{B}^{r_1}$ and $\mathcal{B}^{a_2}$ before hitting $\mathcal{B}^{a_1} \cup \mathcal{B}^{r_2}$; in favor of H_2 if random path hits $\mathcal{B}^{a_1}$ and $\mathcal{B}^{r_2}$ before hitting $\mathcal{B}^{r_1} \cup \mathcal{B}^{a_2}$.

For this combined sequential test, absorption of random path by $\tilde{b} \in \mathcal{B}^{a_1} \cup \mathcal{B}^{r_1} \cup \mathcal{B}^{a_2} \cup \mathcal{B}^{r_2}$ is defined as $\tilde{b}$ being hit by the random path when both two sequential procedures are completed. Let $\psi(\cdot)$ be the "interactive latent of $\{\mathcal{B}^{a_1}, \mathcal{B}^{r_1}; \mathcal{B}^{a_2}, \mathcal{B}^{r_2}\}$", which is defined as

$\psi(\cdot) = \psi_{a_1;a_2,r_1r_2}(\cdot)$ on $\mathcal{B}^{a_1}$; $\psi(\cdot) = \psi_{a_2;a_1,r_1r_2}(\cdot)$ on $\mathcal{B}^{a_2}$; $\psi(\cdot) = \psi_{r_1;a_2,a_1r_2}(\cdot)$ on $\mathcal{B}^{r_1}$; $\psi(\cdot) = \psi_{r_2;a_1,a_2r_1}(\cdot)$ on $\mathcal{B}^{r_2}$. $\psi_{a_1;a_2,r_1r_2}(\cdot)$ is the "ordered latent of $\{\mathcal{B}^{a_1};\mathcal{B}^{a_2},\mathcal{B}^{r_1}\cup\mathcal{B}^{r_2}\}$" as defined in (22), likewise for $\psi_{a_2;a_1,r_1r_2}(\cdot)$ or $\psi_{r_1;a_2,a_1r_2}(\cdot)$ or $\psi_{r_2;a_1,a_2r_1}(\cdot)$.

Once $\psi(\cdot)$ is known, by Theorem 3.1 and Theorem 3.3, absorption probability distributions of $\underset{\sim}{S}$ or $\underset{\sim}{S}_N$ are, for any $\tilde{b}_n \in \mathcal{B}^{a_1}\cup\mathcal{B}^{r_1}\cup\mathcal{B}^{a_2}\cup\mathcal{B}^{r_2}$

$$P_p(\underset{\sim}{S} \in H(\tilde{b}_n)) = \psi(\tilde{b}_n)\binom{n}{b_n}p^{b_n}(1-p)^{n-b_n} \tag{39}$$

or

$$P_p(\underset{\sim}{S}_N \in H_N(\tilde{b}_n)) = \psi(\tilde{b}_n)\frac{\binom{pN}{b_n}\binom{(1-p)N}{b_n}}{\binom{N}{n}} \tag{40}$$

where $H(\tilde{b}_n)$(or $H_N(\tilde{b}_n)$) is a bunch of sample paths in $\mathcal{X}$(or in $\mathcal{X}_N$) which are absorbed by $\tilde{b}_n$, where absorption is defined above.

Illustrative Example 4:

Assume we want to test $H_0 : p = \frac{1}{2}$ $v.s.$ $H_1 : p > \frac{1}{2}$ $v.s.$ $H_2 : p < \frac{1}{2}$ sequentially, the maximum sampling size is 28. Suppose it is required that $\mathcal{X}$, the sample space for random path $\underset{\sim}{S}$, be partitioned into $\{\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2\}$ such that H_i is accepted if $\underset{\sim}{S} \in \mathcal{X}_i$ $i = 0, 1, 2$. $P_{\frac{1}{2}}(\underset{\sim}{S} \in \mathcal{X}_0) \geq 0.90$, $\min_{p\leq\frac{1}{5}} P_p(\underset{\sim}{S} \in \mathcal{X}_1) = P_{\frac{1}{5}}(\underset{\sim}{S} \in \mathcal{X}_1) \geq 0.95$, $\min_{p\geq\frac{4}{5}} P_p(\underset{\sim}{S} \in \mathcal{X}_2) = P_{\frac{4}{5}}(\underset{\sim}{S} \in \mathcal{X}_2) \geq 0.95$.

For testing hypotheses $H_0' : p \leq \frac{1}{2}$ $v.s.$ $H_1' : p > \frac{1}{2}$ and hypotheses $H_0'' : p \geq \frac{1}{2}$ $v.s.$ $H_1'' : p < \frac{1}{2}$ simultaneously and independently, barrier sets $\mathcal{B}^{a_1}$, $\mathcal{B}^{r_1}$, $\mathcal{B}^{a_2}$ and $\mathcal{B}^{r_2}$ are given in Table 1 and graphed on Fig 2. It is clear that for each n there is only one, if any, $\tilde{b}_n$ in $\mathcal{B}^{a_1}$. We denote this $\tilde{b}_n$ by $\tilde{a}_{1n}$. Similarly, we denote the only $\tilde{b}_n$, if any, by $\tilde{r}_{1n}$ in $\mathcal{B}^{r_1}$; by $\tilde{a}_{2n}$ in $\mathcal{B}^{a_2}$; by $\tilde{r}_{2n}$ in $\mathcal{B}^{r_2}$. These barrier points are obtained by following procedure. Detail of this procedure is discussed in Xiong[92].

Let $a_{11} = -1$, $r_{11} = 2$, $a_{21} = 2$, $r_{21} = -1$. For $n = 2, \cdots, m$, let

$$a_{1n} = \begin{cases} a_{1n-1} + 1 & \text{if } G(\frac{a_{1n-1}+1}{m}, \frac{n-a_{1n-1}-1}{m}; p_1^*) \geq g_{11} \\ a_{1n-1} & \text{otherwise} \end{cases}, \tag{41}$$

$$r_{1n} = \begin{cases} r_{1n-1} & \text{if } G(\frac{r_{1n-1}}{m}, \frac{n-r_{1n-1}}{m}; p_1^*) \geq g_{12} \\ r_{1n-1} + 1 & \text{otherwise} \end{cases}, \tag{42}$$

	Testing Hypothesis $H_0^{'}: p \geq p^* \ v.s.\ H_1^{'}: p < p^*$					Testing Hypothesis $H_0^{''}: p \leq p^* \ v.s.\ H_1^{''}: p > p^*$			
	Favor $H_0^{'}$		Favor $H_1^{'}$			Favor $H_0^{''}$		Favor $H_1^{''}$	
n	$\tilde{a}_{1n} \in \mathcal{B}^{a_1}$	$\psi_1(\tilde{a}_{1n})$	$\tilde{r}_{1n} \in \mathcal{B}^{r_1}$	$\psi_1(\tilde{r}_{1n})$	n	$\tilde{a}_{2n} \in \mathcal{B}^{a_2}$	$\psi_2(\tilde{a}_{2n})$	$\tilde{r}_{2n} \in \mathcal{B}^{r_2}$	$\psi_2(\tilde{r}_{2n})$
1					1				
2					2				
3	(0,3)	1.000			3	(3,0)	1.000		
4	(0,4)	0.000			4	(4,0)	0.000		
5	(0,5)	0.000			5	(5,0)	0.000		
6	(1,5)	0.500			6	(5,1)	0.500		
7	(1,6)	0.000	(7,0)	1.000	7	(5,2)	0.000	(0,7)	1.000
8	(2,6)	0.429	(8,0)	0.000	8	(6,2)	0.429	(0,8)	0.000
9	(2,7)	0.000	(9,0)	0.000	9	(7,2)	0.000	(0,9)	0.000
10	(3,7)	0.358	(10,0)	0.000	10	(7,3)	0.358	(0,10)	0.000
11	(4,7)	0.458	(11,0)	0.000	11	(7,4)	0.458	(0,11)	0.000
12	(4,8)	0.000	(11,1)	0.583	12	(8,4)	0.000	(1,11)	0.583
13	(5,8)	0.295	(12,1)	0.000	13	(8,5)	0.295	(1,12)	0.000
14	(6,8)	0.396	(13,1)	0.000	14	(8,6)	0.396	(1,13)	0.000
15	(6,9)	0.000	(13,2)	0.533	15	(9,6)	0.000	(2,13)	0.533
16	(7,9)	0.241	(14,2)	0.000	16	(9,7)	0.241	(2,14)	0.000
17	(8,9)	0.341	(15,2)	0.000	17	(9,8)	0.341	(2,15)	0.000
18	(8,10)	0.000	(15,3)	0.462	18	(10,8)	0.000	(3,15)	0.462
19	(9,10)	0.201	(16,3)	0.000	19	(10,9)	0.201	(3,16)	0.000
20	(10,10)	0.298	(16,4)	0.497	20	(10,10)	0.298	(4,16)	0.497
21	(11,10)	0.343	(17,4)	0.000	21	(10,11)	0.343	(4,17)	0.000
22	(12,10)	0.362	(17,5)	0.480	22	(10,12)	0.362	(5,17)	0.480
23	(13,10)	0.367	(18,5)	0.000	23	(10,13)	0.367	(5,18)	0.000
24	(14,10)	0.364	(18,6)	0.453	24	(10,14)	0.364	(6,18)	0.453
25	(15,10)	0.356	(19,6)	0.000	25	(10,15)	0.356	(6,19)	0.000
26	(16,10)	0.345	(19,7)	0.423	26	(10,16)	0.345	(7,19)	0.423
27	(17,10)	0.329	(19,8)	0.552	27	(10,17)	0.329	(8,19)	0.552
28	(18,10)	0.305	(19,9)	0.580	28	(10,18)	0.305	(9,19)	0.580

Table 1: **Latents $\psi_1(\cdot)$ and $\psi_2(\cdot)$ on Barrier Sets**

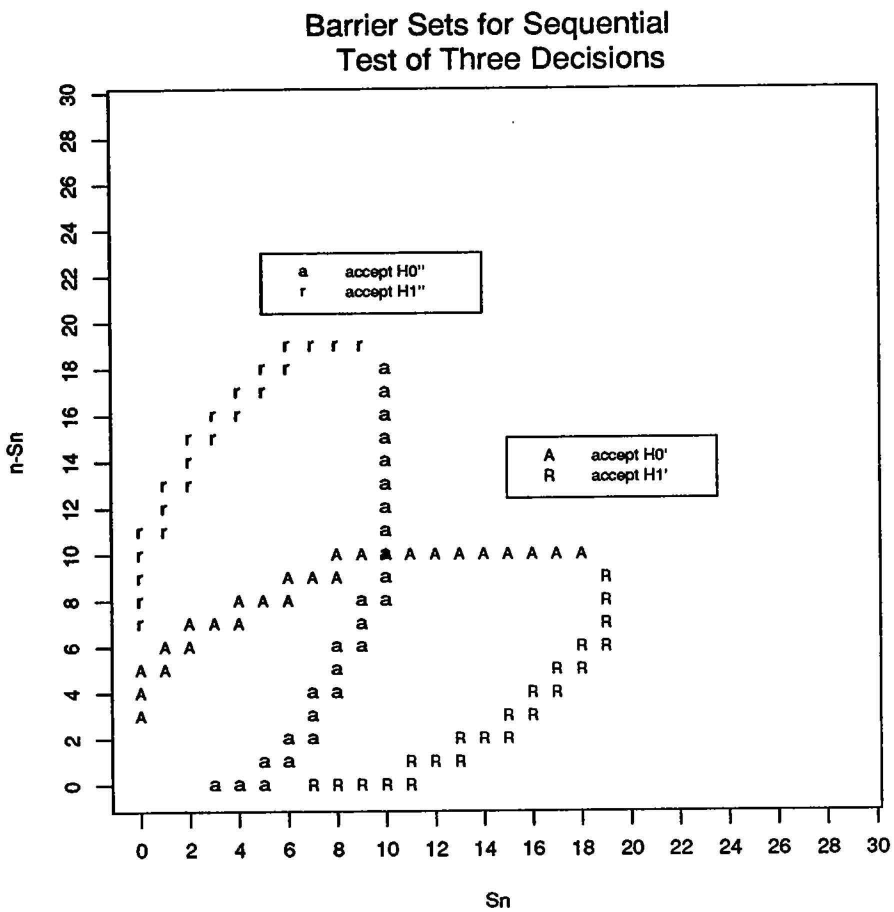

Figure 2: **Sequential Test of Three Decisions**

$$a_{2n} = \begin{cases} a_{2n-1} & \text{if } G(\frac{a_{2n-1}}{m}, \frac{n-a_{2n-1}}{m}; p_2^*) \geq g_{21} \\ a_{1n-1}+1 & \text{otherwise} \end{cases}, \tag{43}$$

$$r_{2n} = \begin{cases} r_{2n-1}+1 & \text{if } G(\frac{r_{2n-1}+1}{m}, \frac{n-r_{2n-1}-1}{m}; p_2^*) \geq g_{22} \\ r_{2n-1} & \text{otherwise} \end{cases}, \tag{44}$$

where $G(u, v; p^*)$ is called "the ratio function", in which p^* is a parameter in $[0, 1]$, for u, v such that $0 \leq u \leq p^*$ and $0 \leq v \leq 1 - p^*$

$$G(u, v; p^*) = p^* \log \frac{1}{p^*} - u \log \frac{u+v}{u} - (p^* - u) \log \frac{1-u-v}{p^*-u}$$

$$+(1-p^*)\log\frac{1}{1-p^*}-v\log\frac{u+v}{v}-(1-p^*-v)\log\frac{1-u-v}{1-p^*-v}; \qquad (45)$$

for $u \notin [0,p^*]$ or $v \notin [0,1-p^*]$

$$G(u,v;p^*) \;=\; p^*\log\frac{1}{p^*}+(1-p^*)\log\frac{1}{1-p^*}. \qquad (46)$$

In (41), (42), (43) and (44), m, p_1^*, p_2^*, g_{11}, g_{12}, g_{21}, g_{22} are called "operating parameters" for this sequential test. In this example, we let $m=28$, $p_1^*=0.65$, $p_2^*=0.35$, $g_{11}=g_{12}=g_{21}=g_{22}=0.11$.

Let $\tilde{a}_{1n}=(a_{1n},n-a_{1n})$, $\tilde{r}_{1n}=(r_{1n},n-r_{1n})$, $\tilde{a}_{2n}=(a_{2n},n-a_{2n})$, $\tilde{r}_{2n}=(r_{2n},n-r_{2n})$. Deleting all of those points which have negative components, we obtain barrier sets $\mathcal{B}^{a_1}$, $\mathcal{B}^{r_1}$, $\mathcal{B}^{r_2}$ and $\mathcal{B}^{r_2}$ which are graphed in Figure 2.

To obtain absorption probability distribution for sequential test of $H_0^{'}: p\le\frac{1}{2}$ *v.s.* $H_1^{'}: p>\frac{1}{2}$, let $\psi_1(\cdot)$ be the "latent of $\mathcal{B}^1=\mathcal{B}^{a_1}\cup\mathcal{B}^{r_1}$". $\psi_1(\cdot)$ is evaluated by, for $\tilde{b}_n=\tilde{a}_{1n}\in\mathcal{B}^{a_1}$ or $\tilde{r}_{1n}\in\mathcal{B}^{r_1}$

$$\psi_1(\tilde{b}_n)=1-\sum_{l<n}\left\{\psi_1(\tilde{a}_{1l})\frac{\binom{b_n}{a_{1l}}\binom{n-b_n}{l-a_{1l}}}{\binom{n}{l}}+\psi_1(\tilde{r}_{1l})\frac{\binom{b_n}{r_{1l}}\binom{n-b_n}{l-r_{1l}}}{\binom{n}{l}}\right\}. \qquad (47)$$

For this test, absorption is defined as first hitting in $\mathcal{B}^1$. Absorption probability distribution for this test is obtained easily by (39) or (40), in which $\psi_1(\cdot)$ is used instead of $\psi(\cdot)$.

To obtain absorption probability distribution for sequential test of $H_0^{''}: p\ge\frac{1}{2}$ *v.s.* $H_1^{''}: p<\frac{1}{2}$, let $\psi_2(\cdot)$ be the "latent of $\mathcal{B}^2=\mathcal{B}^{a_2}\cup\mathcal{B}^{r_2}$". $\psi_2(\cdot)$ is evaluated by, for $\tilde{b}_n=\tilde{a}_{2n}\in\mathcal{B}^{a_2}$ or $\tilde{r}_{2n}\in\mathcal{B}^{r_2}$

$$\psi_2(\tilde{b}_n)=1-\sum_{l<n}\left\{\psi_2(\tilde{a}_{2l})\frac{\binom{b_n}{a_{2l}}\binom{n-b_n}{l-a_{2l}}}{\binom{n}{l}}+\psi_2(\tilde{r}_{2l})\frac{\binom{b_n}{r_{2l}}\binom{n-b_n}{l-r_{2l}}}{\binom{n}{l}}\right\}. \qquad (48)$$

For this test, absorption is defined as first hitting in $\mathcal{B}^2$. Replacing $\psi(\cdot)$ by $\psi_2(\cdot)$ in (39) or (40), we obtain the absorption probability distribution for this sequential test. Latents $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are evaluated and listed on Table 1.

For the combined sequential test of $H_0: p=\frac{1}{2}$ *v.s.* $H_1: p>\frac{1}{2}$ *v.s.* $H_2: p<\frac{1}{2}$, it is clear that if random path hits $\mathcal{B}^{r_1}$ before hitting $\mathcal{B}^{a_1}$, then it did hit $\mathcal{B}^{a_2}$ and didn't hit $\mathcal{B}^{r_2}$. Therefore it is not difficult to find out that: $\psi_{a_1;a_2,r_1r_2}(\cdot)=\psi_{a_1;a_2,r_1}(\cdot)$ on $\mathcal{B}^{a_1}$; $\psi_{a_2;a_1,r_1r_2}(\cdot)=\psi_{a_2;a_1,r_2}(\cdot)$ on $\mathcal{B}^{a_2}$; $\psi_{r_1;a_2,a_1r_2}(\cdot)=\psi_1(\cdot)$ on $\mathcal{B}^{r_1}$; $\psi_{r_2;a_1,a_2r_1}(\cdot)=\psi_2(\cdot)$ on $\mathcal{B}^{r_2}$;

where $\psi_{a_1;a_2,r_1}(\cdot)$ is the "ordered latent of $\{\mathcal{B}^{a_1};\mathcal{B}^{a_2},\mathcal{B}^{r_1}\}$", $\psi_1(\cdot)$ is the latent of $\mathcal{B}^1$; $\psi_{a_2;a_1,r_2}(\cdot)$ is the "ordered latent of $\{\mathcal{B}^{a_2};\mathcal{B}^{a_1},\mathcal{B}^{r_2}\}$", $\psi_2(\cdot)$ is the latent of $\mathcal{B}^2$.

$\psi_{a_1;a_2,r_1}(\cdot)$ can be evaluated by, for $\tilde{a}_{1n} \in \mathcal{B}^{a_1}$,

$$\psi_{a_1;a_2,r_1}(\tilde{a}_{1n}) = \sum_{l<n} \left\{ \psi_{a_1a_2}(\tilde{a}_{2l})\psi_{a_2,a_1r_1}(\tilde{a}_{2l},\tilde{a}_{1n}) \frac{\binom{a_{1n}}{a_{2l}}\binom{n-a_{1n}}{l-a_{2l}}}{\binom{n}{l}} \right\} \tag{49}$$

where $\psi_{a_2,a_1r_1}(\cdot,\cdot)$ can be evaluated by, for $\tilde{a}_{2n'} \in \mathcal{B}^{a_2}$, $n > n'$, $\tilde{b}_n = \tilde{a}_{1n}$ or $\tilde{r}_{1n} \in \mathcal{B}^1$

$$\psi_{a_2,a_1r_1}(\tilde{a}_{2n'},\tilde{b}_n) = 1 - \sum_{n'<l<n} \left\{ \psi_{a_2,a_1r_1}(\tilde{a}_{2n'},\tilde{a}_{1l}) \frac{\binom{b_n-a_{2n'}}{a_{1l}-a_{2n'}}\binom{n-n'-b_n+a_{2n'}}{l-n'-a_{1l}+a_{2n'}}}{\binom{n-n'}{l-n'}} + \right.$$
$$\left. +\psi_{a_2,a_1r_1}(\tilde{a}_{2n'},\tilde{r}_{1l}) \frac{\binom{b_n-a_{2n'}}{r_{1l}-a_{2n'}}\binom{n-n'-b_n+a_{2n'}}{l-n'-r_{1l}+a_{2n'}}}{\binom{n-n'}{l-n'}} \right\}. \tag{50}$$

$\psi_{a_1a_2}(\cdot)$, latent of $\mathcal{B}^{a_1} \cup \mathcal{B}^{a_2}$, can be evaluated by, for $\tilde{b}_n = \tilde{a}_{1n} \in \mathcal{B}^{a_1}$, or $\tilde{a}_{2n} \in \mathcal{B}^{a_2}$

$$\psi_{a_1a_2}(\tilde{b}_n) = 1 - \sum_{l<n} \left\{ \psi_{a_1a_2}(\tilde{a}_{1l}) \frac{\binom{b_n}{a_{1l}}\binom{n-b_n}{n-a_{1l}}}{\binom{n}{l}} + \psi_{a_1a_2}(\tilde{r}_{1l}) \frac{\binom{b_n}{a_{1l}}\binom{n-b_n}{n-a_{1l}}}{\binom{n}{l}} \right\}. \tag{51}$$

Formula for evaluation of $\psi_{a_2;a_1,r_2}(\cdot)$ is similar to those of $\psi_{a_1;a_2,r_1}(\cdot)$.

$\psi(\cdot)$, the "interactive latent of $\{\mathcal{B}^{a_1},\mathcal{B}^{r_1};\mathcal{B}^{a_2},\mathcal{B}^{r_2}\}$", is such that for $\tilde{r}_{1n} \in \mathcal{B}^{r_1}$, $\psi(\tilde{r}_{1n}) = \psi_1(\tilde{r}_{1n})$; for $\tilde{r}_{2n} \in \mathcal{B}^{r_2}$, $\psi(\tilde{r}_{2n}) = \psi_2(\tilde{r}_{2n})$; for $\tilde{a}_{1n} \in \mathcal{B}^{a_1}$, $\psi(\tilde{a}_{1n}) = \psi_{a_1;a_2,r_1}(\tilde{r}_{1n})$; for $\tilde{a}_{2n} \in \mathcal{B}^{a_2}$, $\psi(\tilde{a}_{2n}) = \psi_{a_2;a_1,r_2}(\tilde{a}_{2n})$. $\psi(\cdot)$ is evaluated and listed in Table 2, Absorption probability distributions of $\underset{\sim}{S}$ or $\underset{\sim}{S}_N$ on $\tilde{b} \in \mathcal{B}^{a_1}\cup\mathcal{B}^{r_1}\cup\mathcal{B}^{a_2}\cup\mathcal{B}^{r_2}$ are available by simply follow (39) or (40).

Assume sampling *with* replacement, the underlying probability distribution for the random path is binomial. The acceptance probabilities for H_0, H_1, H_2 are $\beta_0(p)$, $\beta_1(p)$, $\beta_2(p)$ which are given by, for any $p \in [0,1]$

$$\beta_0(p) = \sum_{n\geq 1}^{28} \left\{ \psi(\tilde{a}_{1n}) \binom{n}{a_{1n}} p^{a_{1n}}(1-p)^{n-a_{1n}} + \psi(\tilde{a}_{2n}) \binom{n}{a_{2n}} p^{a_{2n}}(1-p)^{n-a_{2n}} \right\} \tag{52}$$

Testing Hypotheses $H_0 : p = p^* \ v.s.\ H_1 : p > p^* \ v.s.\ H_2 : p < p^*$								
	Favor H_0				Favor H_1		Favor H_2	
n	$\tilde{a}_{1n} \in \mathcal{B}^{a_1}$	$\psi(\tilde{a}_{1n})$	$\tilde{a}_{2n} \in \mathcal{B}^{a_2}$	$\psi(\tilde{a}_{2n})$	$\tilde{r}_{1n} \in \mathcal{B}^{r_1}$	$\psi(\tilde{r}_{1n})$	$\tilde{r}_{2n} \in \mathcal{B}^{r_2}$	$\psi(\tilde{r}_{2n})$
1								
2								
3	(0,3)	0.000	(3,0)	0.000				
4	(0,4)	0.000	(4,0)	0.000				
5	(0,5)	0.000	(5,0)	0.000				
6	(1,5)	0.000	(5,1)	0.000				
7	(1,6)	0.000	(6,1)	0.000	(7,0)	1.000	(0,7)	1.000
8	(2,6)	0.000	(6,2)	0.000	(8,0)	0.000	(0,8)	0.000
9	(2,7)	0.000	(7,2)	0.000	(9,0)	0.000	(0,9)	0.000
10	(3,7)	0.008	(7,3)	0.008	(10,0)	0.000	(0,10)	0.000
11	(4,7)	0.021	(7,4)	0.021	(11,0)	0.000	(0,11)	0.000
12	(4,8)	0.000	(8,4)	0.000	(11,1)	0.583	(1,11)	0.583
13	(5,8)	0.024	(8,5)	0.024	(12,1)	0.000	(1,12)	0.000
14	(6,8)	0.048	(8,6)	0.048	(13,1)	0.000	(1,13)	0.000
15	(6,9)	0.000	(9,6)	0.000	(13,2)	0.533	(2,13)	0.533
16	(7,9)	0.058	(9,7)	0.058	(14,2)	0.000	(2,14)	0.000
17	(8,9)	0.170	(9,8)	0.170	(15,2)	0.000	(2,15)	0.000
18	(8,10)	0.000	(10,8)	0.000	(15,3)	0.462	(3,15)	0.462
19	(9,10)	0.201	(10,9)	0.201	(16,3)	0.000	(3,16)	0.000
20	(10,10)	0.298	(10,10)	0.298	(16,4)	0.497	(4,16)	0.497
21	(11,10)	0.343	(10,11)	0.343	(17,4)	0.000	(4,17)	0.000
22	(12,10)	0.362	(10,12)	0.362	(17,5)	0.480	(5,17)	0.480
23	(13,10)	0.367	(10,13)	0.367	(18,5)	0.000	(5,18)	0.000
24	(14,10)	0.364	(10,14)	0.364	(18,6)	0.453	(6,18)	0.453
25	(15,10)	0.356	(10,15)	0.356	(19,6)	0.000	(6,19)	0.000
26	(16,10)	0.423	(10,16)	0.345	(19,7)	0.423	(7,19)	0.423
27	(17,10)	0.329	(10,17)	0.329	(19,8)	0.552	(8,19)	0.552
28	(18,10)	0.305	(10,18)	0.305	(19,9)	0.580	(9,19)	0.580

Table 2: **Interactive Latent $\psi(\cdot)$ on Barrier Sets**

$$\beta_1(p) = \sum_{n\geq 1}^{28} \psi(\tilde{r}_{1n}) \binom{n}{r_{1n}} p^{r_{1n}} (1-p)^{n-r_{1n}} \tag{53}$$

$$\beta_2(p) = \sum_{n\geq 1}^{28} \psi(\tilde{r}_{2n}) \binom{n}{r_{2n}} p^{r_{2n}} (1-p)^{n-r_{2n}} \tag{54}$$

Obviously it must holds $\beta_0(p) + \beta_1(p) + \beta_2(p) \equiv 1$. The expected sampling size is

$$\begin{aligned} S(p) &= \sum_{n\geq 1}^{28} n \Big\{ \psi(\tilde{a}_{1n}) \binom{n}{a_{1n}} p^{a_{1n}} (1-p)^{n-a_{1n}} + \psi(\tilde{a}_{2n}) \binom{n}{a_{2n}} p^{a_{2n}} (1-p)^{n-a_{2n}} + \\ &\quad + \psi(\tilde{r}_{1n}) \binom{n}{r_{1n}} p^{r_{1n}} (1-p)^{n-r_{1n}} + \psi(\tilde{r}_{2n}) \binom{n}{r_{2n}} p^{r_{2n}} (1-p)^{n-r_{2n}} \Big\} \end{aligned} \tag{55}$$

Acceptance probabilities $\beta_0(p)$, $\beta_1(p)$, $\beta_2(p)$ and expected sampling size $S(p)$ for the combined sequential test of three decisions are graphed in Fig 2.

Sampling *without* replacement, corresponding $\beta_0(p)$, $\beta_1(p)$, $\beta_2(p)$ and $S(p)$ can be formulated easily by replacing binomial by hypergeometric probabilities in above equations.

5 Appendix

5.1 Proof of Theorem 3.1

First let us consider the case of sampling with replacement, in which, $\tilde{S}_n$ has multinomial distribution $\mathcal{M}_d(n;p)$ for $n = 1, 2, \cdots$. Now we define, for any $\tilde{b}_n \in \mathcal{B}$,

$$\psi(\tilde{b}_n) = \begin{cases} \frac{P_p(\underset{\sim}{S} \in H(\tilde{b}_n))}{P_p(\tilde{S}_n = \tilde{b}_n)} & \text{if } P_p(\tilde{S}_n = \tilde{b}_n) > 0 \\ 1 & \text{if } P_p(\tilde{S}_n = \tilde{b}_n) = 0 \end{cases}. \tag{56}$$

By this definition, obviously equation (6) holds. We only need this definition agrees with that given in (5).

Assume $P_p(\tilde{S}_n = \tilde{b}_n) > 0$. Since $P_p(\underset{\sim}{S} \in H(\tilde{b}_n)) = P_p\left(\underset{\sim}{S} \in H(\tilde{b}_n), \tilde{S}_n = \tilde{b}_n\right)$, by (56)

$$\begin{aligned} \psi(\tilde{b}_n) &= P_p\left(\underset{\sim}{S} \in H(\tilde{b}_n) | \tilde{S}_n = \tilde{b}_n\right) \\ &= P_p\left(\tilde{S}_n = \tilde{b}_n, \tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1 | \tilde{S}_n = \tilde{b}_n\right) \\ &= P_p\left(\tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1 | \tilde{S}_n = \tilde{b}_n\right) \end{aligned}$$

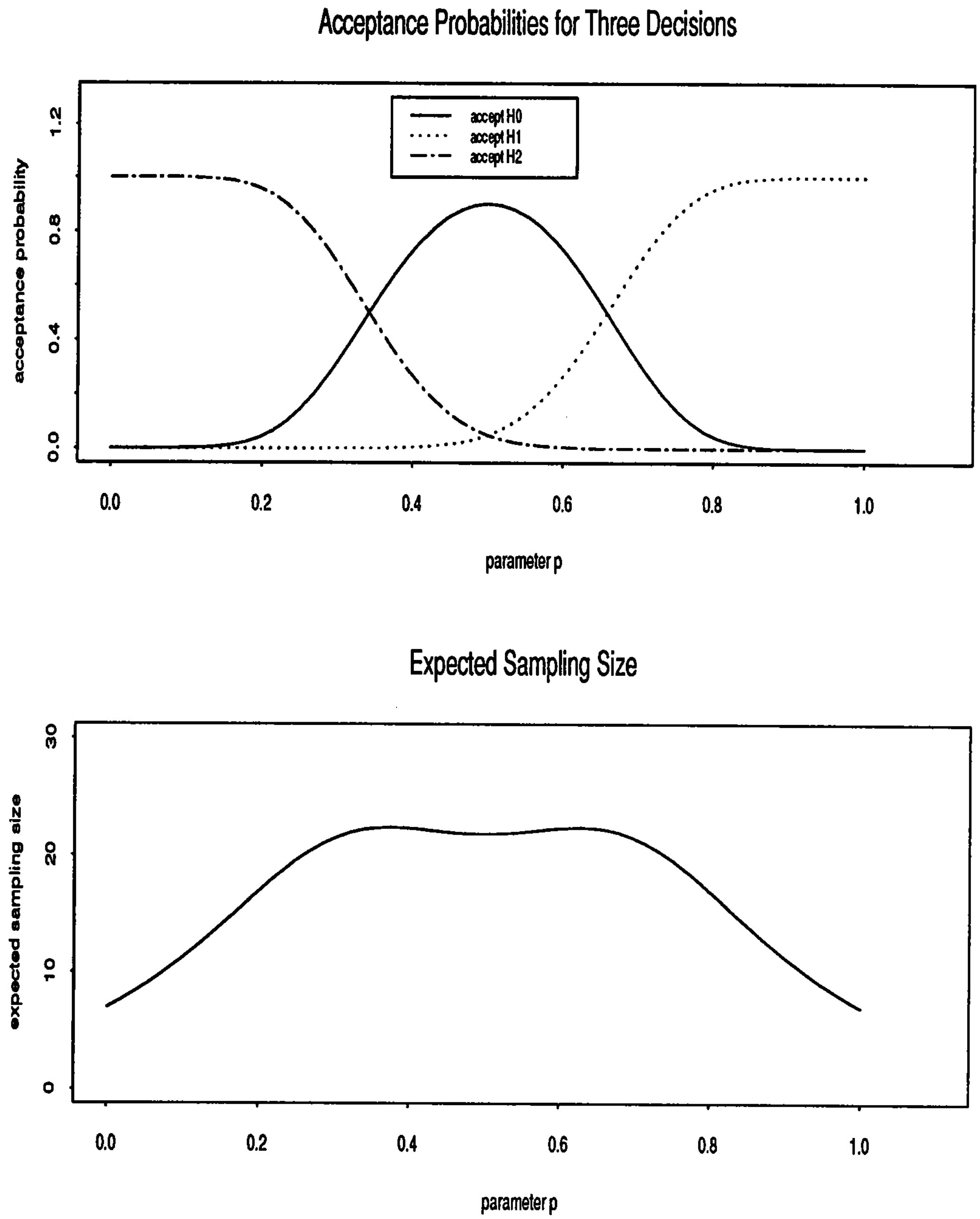

Figure 3: **Sequential Test of Three Decisions**

$$= \quad P_p\left(\underset{\sim}{X}_n \in D_n | \tilde{S}_n = \tilde{b}_n\right), \tag{57}$$

where $\underset{\sim}{X}_n = (\tilde{X}_1, \cdots, \tilde{X}_n)$, $\tilde{X}_l$ is the outcome of lth sampling we mentioned in Section 2; $D_n \in \mathcal{I}_n = \sigma\left(\{\underset{\sim}{x}_n = (\tilde{x}_1, \cdots, \tilde{x}_n) : \tilde{x}_l = (x_{l1}, \cdots, x_{ld}), x_{lk} = 0 \text{ or } 1, \sum_{k=1}^{d} x_{lk} = 1\}\right)$ such that $(\underset{\sim}{X}_n \in D_n) = \left(\tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1\right)$.

$\psi(\tilde{b}_n)$ doesn't depend on p because $\tilde{S}_n$ is a sufficient statistics of p. Hence we write $\psi(\tilde{b}_n) = P\left(\underset{\sim}{X}_n \in D_n | \tilde{S}_n = \tilde{b}_n\right)$, the intuitive meaning of which can be explained as following. The sample paths in $\mathcal{X}$ passing through $\tilde{b}_n$ can take $\binom{n}{\tilde{b}_n}$ different passages from origin $\tilde{o}$ to $\tilde{b}_n$. Each passage is taken equally likely by *random path* $\underset{\sim}{S}$. Of these passages, some encounter other barrier points in $\mathcal{B}$ before reaching $\tilde{b}_n$, while the rest didn't. Actually $\psi(\tilde{b}_n)$ is the ratio of number of passages which didn't encounter other barrier points before reaching $\tilde{b}_n$ and the number of all passages reaching $\tilde{b}_n$. Obviously, $\psi(\tilde{b}_n)$ doesn't depend on p.

In the case of sampling without replacement, $\tilde{S}_n$ has multihypergeometric distribution $\mathcal{H}_d(n; p, N)$ for $n = 1, \cdots, N$.

Claim 5.1 *For any p, N such that N is a positive integer and $p = \frac{1}{N}, \frac{2}{N}, \cdots, 1$,*

$$P_{p,N}\left(\underset{\sim}{S}_N \in H_N(\tilde{b}_n)\right) = \psi(\tilde{b}_n) P_{p,N}(\tilde{S}_n = \tilde{b}_n) \tag{58}$$

holds for any $\tilde{b}_n$ in $\mathcal{B}$.

Proof of Claim 5.1:

If $P_{p,N}(\tilde{S}_n = \tilde{b}_n) > 0$, then by definition of $H_N(\tilde{b}_n)$ in (3),

$$\begin{aligned} & P_{p,N}\left(\underset{\sim}{S}_N \in H_N(\tilde{b}_n) | \tilde{S}_n = \tilde{b}_n\right) \\ = \quad & P_{p,N}\left(\tilde{S}_n = \tilde{b}_n, \tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1 | \tilde{S}_n = \tilde{b}_n\right) \\ = \quad & P_{p,N}\left(\tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1 | \tilde{S}_n = \tilde{b}_n\right) \end{aligned}$$

$P_{p,N}\left(\tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1 | \tilde{S}_n = \tilde{b}_n\right)$ is well defined. For D_n in (57), we have

$$P_{p,N}(\tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1 | \tilde{S}_n = \tilde{b}_n) = P_{p,N}(\underset{\sim}{X}_n \in D_n | \tilde{S}_n = \tilde{b}_n).$$

If we denote $P_p(\cdot)$ as the probability measure of multinomial distribution for sampling with replacement, and denote $P_{p,N}(\cdot)$ as the probability measure of multihypergeometric distribution for sampling without replacement, then for any $\underline{\text{x}}_n \in \mathcal{I}_n$,

$$\begin{aligned} P_{p,N}(\underline{\text{X}}_n = \underline{\text{x}}_n | \tilde{S}_n = \tilde{b}_n) &= P_p(\underline{\text{X}}_n = \underline{\text{x}}_n | \tilde{S}_n = \tilde{b}_n) \\ &= \begin{cases} \frac{1}{\binom{n}{\tilde{b}_n}} & \tilde{s}_n = \tilde{b}_n; \\ 0 & \tilde{s}_n \neq \tilde{b}_n. \end{cases} \end{aligned}$$

where $\underline{\text{X}}_n = (\tilde{x}_1, \cdots, \tilde{x}_n)$ and $\tilde{s}_n = \sum_{l=1}^{n} \tilde{x}_l$. So consequently we have

$$P_{p,N}(\underline{\text{X}}_n \in D_n | \tilde{S}_n = \tilde{b}_n) = P_p(\underline{\text{X}}_n \in D_n | \tilde{S}_n = \tilde{b}_n) = \psi(\tilde{b}_n).$$

Hence $P_{p,N}(\underline{\text{S}}_N \in H_N(\tilde{b}_n) | \tilde{S}_n = \tilde{b}_n) = \psi(\tilde{b}_n)$, (58) holds.

If $P_{p,N}(\tilde{S}_n = \tilde{b}_n) = 0$, then $P_{p,N}\left(\underline{\text{S}}_N \in H_N(\tilde{b}_n)\right) = 0$ because

$$\begin{aligned} P_{p,N}\left(\underline{\text{S}}_N \in H_N(\tilde{b}_n)\right) &= P_{p,N}\left(\tilde{S}_n = \tilde{b}_n, \tilde{S}_l \notin B_l, \quad l = 1, \cdots, n-1\right) \\ &\leq P_{p,N}\left(\tilde{S}_n = \tilde{b}_n\right). \end{aligned}$$

Thus for any p, N and $\tilde{b}_n \in \mathcal{B}$, whether $P_{p,N}(\tilde{S}_n = \tilde{b}_n) > 0$ or $= 0$, it always holds

$$P_{p,N}(\underline{\text{S}}_N \in H_N(\tilde{b}_n)) = \psi(\tilde{b}_n) P_{p,N}(\tilde{S}_n = \tilde{b}_n).$$

∎

Now we want to show $\psi(\tilde{b}_n)$ defined by (56) satisfies (5). Fix n and point $\tilde{b}_n$ in $\mathcal{B}$, let $N = n$ and $p_k = \frac{b_{nk}}{n}$, then for $l = 1, \cdots, N$

$$P_{p,N}(\tilde{S}_l = \tilde{b}_l) = \frac{\prod_{k=1}^{d} \binom{p_k N}{b_{lk}}}{\binom{N}{l}} = \frac{\prod_{k=1}^{d} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}.$$

Hence

$$\begin{aligned} \sum_{\tilde{b}_l \in B_l} P_{p,N}(\underline{\text{S}}_N \in H_N(\tilde{b}_l)) &= \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l) P_{p,N}(\tilde{S}_l = \tilde{b}_l) \\ &= \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l) \frac{\prod_{k=1}^{d} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}. \end{aligned} \tag{59}$$

Let $l = N$ then (59) leads to

$$\sum_{\tilde{b}'_N \in B_N} P_{p,N}(\underline{S}_N \in H_N(\tilde{b}'_N)) = \sum_{\tilde{b}'_n \in B_n} \psi(\tilde{b}'_n) \frac{\prod_{k=1}^d \binom{b_{nk}}{b'_{nk}}}{\binom{n}{n}} = \psi(\tilde{b}_n), \tag{60}$$

which is because of $\frac{\prod_{k=1}^d \binom{b_{nk}}{b'_{nk}}}{\binom{n}{n}} = \begin{cases} 1 & \tilde{b}'_n = \tilde{b}_n; \\ 0 & \tilde{b}'_n \neq \tilde{b}_n. \end{cases}$

It is not difficult to check that if $(p_1 N, \cdots, p_d N) \in B_N$, it always holds $\sum_{l=1}^N \sum_{\tilde{b}_l \in B_l} P_{p,N}(\tilde{S}_l = \tilde{b}_l) = 1$. Thus by (59) and (60), we have

$$\psi(\tilde{b}_n) = 1 - \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} \psi(\tilde{b}_l) \frac{\prod_{k=1}^d \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}} \qquad n = 1, 2, \cdots. \tag{61}$$

$\psi(\cdot)$ can be evaluated for all $\tilde{b}_n \in \mathcal{B}$ recursively in n on $\mathcal{B}$. ■

5.2 Proof of Theorem 3.2

Similar to the definition of $\psi(\tilde{b}_n)$ given in (56) and (57), here we define

$$\begin{aligned} \psi^{(m)}(\tilde{b}_n) &= P_p\left(\underline{S} \in H^{(m)}(\tilde{b}_n) | \tilde{S}_n = \tilde{b}_n\right) \\ &= P\left(\tilde{X}_n \in D_n | \tilde{S}_n = \tilde{b}_n\right) \end{aligned} \tag{62}$$

where $H^{(m)}(\tilde{b}_n)$ were given in (8) and D_n is such that

$$(\underline{X}_n \in D_n) = \bigcup_{\substack{l_1 < \cdots < l_{m-1} < n \\ B(l_t) \neq \emptyset, \ t = 1, \cdots, m-1}} \left\{ \begin{array}{c} \tilde{S}_n = \tilde{b}_n; \ \tilde{S}_{l_t} \in B_{l_t}, \ t = 1, \cdots, m-1; \\ \tilde{S}_l \notin B_l, \ \ l < n, \ \ l \neq l_1, \cdots, l_{m-1} \end{array} \right\}.$$

By this definition, $\psi^{(m)}(\tilde{b}_n)$ satisfies equations (12). With justifications similar to those leads to (7), $\psi^{(m)}(\tilde{b}_n)$ also satisfies (13). Now we only need to show $\psi^{(m)}(\tilde{b}_n)$ defined by (62) satisfies equation (11), thus can be evaluated recursively with that equation. Before doing this, we need to give an intuitive presentation of relative barrier function $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$ which is defined in (10) and plays an important role in equation (11).

We give another definition of $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$. For $n' < n$, $\tilde{b}_{n'}$, $\tilde{b}_n \in \mathcal{B}$, let

$$\varphi(\tilde{b}_{n'}, \tilde{b}_n) \quad = \quad P_p\left(\tilde{S}_l \notin B_l,\ l = n' + 1, \cdots, n-1 | \tilde{S}_{n'} = \tilde{b}_{n'}, \tilde{S}_n = \tilde{b}_n\right). \tag{63}$$

Then we show this definition agrees with that in (10).

Conditioned on $\tilde{S}_{n'} = \tilde{b}_{n'}$, random walk $\underset{\sim}{S}^* = (\tilde{S}_{n'+1} - \tilde{S}_{n'}, \tilde{S}_{n'+2} - \tilde{S}_{n'}, \cdots)$ has same stochastic behavior as that of $\underset{\sim}{S}$. Let $l^* = l - n'$, $n^* = n - n'$, $\tilde{S}^*_{l^*} = \tilde{S}_l - \tilde{S}_{n'}$, $\tilde{b}^*_{l^*} = \tilde{b}_l - \tilde{b}_{n'}$, then equation (63) became

$$\varphi(\tilde{b}_{n'}, \tilde{b}_n) = P_p\left(\tilde{S}^*_{l^*} \notin B^*(l^*),\ l^* = 1, \cdots, n^* - 1 | \tilde{S}^*_{n^*} = \tilde{b}^*_{n^*}, \tilde{S}_{n'} = \tilde{b}_{n'}\right). \tag{64}$$

Let $\underset{\sim}{S}^* = (\tilde{S}^*_1, \tilde{S}^*_2, \cdots)$, $\mathcal{B}^* = \left\{\tilde{b}^*_{l^*} = b_l - b_{n'} :\ b_l \in \mathcal{B}, l^* = l - n', l = n' + 1, n' + 2 \cdots\right\}$, then (10) can be derived from (64) and (5) just by replacing $\psi(\tilde{b}_n)$ by $\varphi(\tilde{b}_{n'}, \tilde{b}_n)$, n by $n^* = n - n'$, $\tilde{b}_l$ by $b^*_{l^*} = \tilde{b}_l - \tilde{b}_{n'}$ *etc.* in equation (5). Justification of this derivation is analogous to the proof of (5) given in Section 5.

Now we proceed to show that $\psi^{(m)}(\tilde{b}_n)$ defined by (62) satisfies (11). Recall that $H^{(m)}(\tilde{b}_n)$ is a bunch of paths in $\mathcal{X}$ that just pass the $\tilde{b}_n$ as the mth hitting in $\mathcal{B}$.

$$\begin{aligned}
& P_p\left(\underset{\sim}{S} \in H^{(m)}(\tilde{b}_n)\right) = P_p\left(\underset{\sim}{S} \in H^{(m)}(\tilde{b}_n), \tilde{S}_n = \tilde{b}_n\right) \\
= & \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} P_p\left(\underset{\sim}{S} \in H^{(m-1)}(\tilde{b}_l); \tilde{S}_l = \tilde{b}_l; \tilde{S}_r \notin B_r, r = l+1, \cdots, n-1; \tilde{S}_n = \tilde{b}_n\right) \\
= & \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} P_p\left(\underset{\sim}{S} \in H^{(m-1)}(\tilde{b}_l) | \tilde{S}_l = \tilde{b}_l; \tilde{S}_r \notin B_r, r = l+1, \cdots, n-1; \tilde{S}_n = \tilde{b}_n\right) \cdot \\
& \qquad \cdot P_p\left(\tilde{S}_l = \tilde{b}_l;\ \tilde{S}_r \notin B_r,\ r = l + 1, \cdots, n-1;\ \tilde{S}_n = \tilde{b}_n\right).
\end{aligned} \tag{65}$$

But in (65),

$$\begin{aligned}
& P_p\left(\underset{\sim}{S} \in H^{(m-1)}(\tilde{b}_l) | \tilde{S}_l = \tilde{b}_l; \tilde{S}_r \notin B_r, r = l + 1, \cdots, n-1; \tilde{S}_n = \tilde{b}_n\right) \\
= \quad & P\left(\tilde{S} \in H^{(m-1)}(\tilde{b}_l) | \tilde{S}_l = \tilde{b}_l\right) \\
= \quad & \psi^{(m-1)}(\tilde{b}_l)
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
& P_p\left(\tilde{S}_l = \tilde{b}_l;\ \tilde{S}_r \notin B_r,\ r = l + 1, \cdots, n-1;\ \tilde{S}_n = \tilde{b}_n\right) \\
= \quad & P_p(\tilde{S}_n = \tilde{b}_n) P_p(\tilde{S}_l = \tilde{b}_l |\ \tilde{S}_n = \tilde{b}_n) \cdot
\end{aligned}$$

$$\cdot P_p\left(\tilde{S}_r \notin B_r,\ r = l+1,\cdots,n-1 |\ \tilde{S}_l = \tilde{b}_l,\ \tilde{S}_n = \tilde{b}_n\right). \qquad (67)$$

With the fact $P_p(\tilde{S}_l = \tilde{b}_l | \tilde{S}_n = \tilde{b}_n) = \frac{\prod_{k=1}^{d} \binom{b_{nk}}{b_{lk}}}{\binom{n}{l}}$ and because of (66), (67) and (63), equation (65) became

$$\begin{aligned} & P_p\left(\tilde{S} \in H^{(m)}(\tilde{b}_n)\right) \\ = & \sum_{l=1}^{n-1} \sum_{\tilde{b}_l \in B_l} \psi^{(m-1)}(\tilde{b}_n)\varphi(\tilde{b}_l, \tilde{b}_n) \frac{\binom{b_n}{b_l}\binom{n-b_n}{r-b_l}}{\binom{n}{l}} P_p(\tilde{S}_n = \tilde{b}_n). \end{aligned} \qquad (68)$$

Divided both sides by $P_p(\tilde{S}_n = \tilde{b}_n)$, equation (68) became (11). ∎

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