A TWO-STAGE PROCEDURE FOR SELECTING THE POPULATION WITH THE LARGEST MEAN WHEN THE COMMON VARIANCE IS UNKNOWN

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A TWO-STAGE PROCEDURE FOR SELECTING THE POPULATION WITH THE LARGEST MEAN WHEN THE COMMON VARIANCE IS UNKNOWN*

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Abstract

In this paper, a new technique to select the population associated with the largest mean from k populations with unknown locations parameters and a common unknown scale parameter is investigated. Asymptotic approximations to the distribution of the linear combination of the sample mean and the sample standard deviation have been derived. Using the approximations, an elimination type two-stage selection procedure is proposed and investigated. Furthermore, two lower bounds on the probability of the correct selection are obtained. This procedure is applied in detail to the selection of the best logistic population.

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1 Introduction

Consider $k(k \geq 2)$ populations $\Pi_1, ..., \Pi_k$, where

$$\Pi_i \sim F\left(\frac{x_i - \mu_i}{\sigma}\right),$$

and F is continuous distribution over the real line. The location parameters $\mu_1, ..., \mu_k$ and the common scale parameter σ are unknown. Our goal is to select the population associated with the largest means (a correct selection or CS). It should be pointed out that in this model the selection problems in terms of the means or the location parameters are equivalent. We adopt the indifference zone formulation of Bechhofer (1954) in which the probability of CS is required to be at least as large as a prespecified probability level whenever the largest and second largest population mean are at least some prespecified distance apart.

There are many papers for selecting the population with the largest mean from k populations having the common known variance σ^2. When σ^2 is unknown, we need to use a two-stage selection procedure. One such procedure is based on a combination of Gupta's subset selection (1956) and the indifference zone formulation. At the first stage one chooses a random number of populations to enter the second stage eliminating all those populations whose first-stage sample means indicate they are inferior. Cohen (1959), Alam (1970), Tamhane and Bechhofer (1977, 1979), Gupta and Kim (1984) and Gupta and Liang (1991) have studied two-stage elimination type procedures for normal populations with a common variance σ^2 (known and unknown), in which they used Gupta's (1956, 1965) subset selection procedure in the first stage to screen out non-contending populations and Bechhofer's(1954) indifference zone approach to select the best from among the remaining populations in the second stage.

For normal populations with a known variance σ^2, Tamhane and Bechhofer(1977, 1979) studied a two-stage elimination type rule in which they adopted a minimax criterion to choose a rule from this class. Santner and Hefferman (1992) further studied this procedure and derived the maximum expected total sample size. Gupta and Han(1991,1992) studied a two-stage procedure $\mathcal{P}_2$ for logistic populations with a common known variance σ^2 using Edgeworth expansion for the sample mean. They derived the minimum expected total sample size and relative efficiency of $\mathcal{P}_2$ with respect to a single stage procedure $\mathcal{P}_1$. For the case of normal populations, reference could be made to Gupta and Kim (1984) where other references are available.

Since the unknown variance case is very important and useful in application, we study this problem further. In many two-stage procedures, the first stage is to select a subset of random size based on sample means. Generally, we select a subset I of $\{1, ..., k\}$ by the following rule (assuming for the moment that σ^2 is known):

$$\text{select } \Pi_i, i \in I \iff \overline{x}_i^{(1)} \geq \max_{1 \leq j \leq k} \overline{x}_j^{(1)} - \frac{h\,\sigma}{\sqrt{n_1}},$$

where $\overline{x}_i^{(1)}$ is the sample mean from the ith population, and h is a positive constant. When the variance is unknown, we may naturally think of substituting sample variance S^2 for σ^2. The difficulty is how to find the distribution of $\sqrt{n}\,\overline{X} - hS$. In this paper, we will find an approximation for the distribution of $\sqrt{n}\bar{X}_i - hS_i$ where S_i^2 is the sample variance of the ith population. This is done by using the Edgeworth expansion for the function of sample means. When $h = 0$, this approximation is just the well-known Edgeworth expansion of $\sqrt{n}\,\overline{X}_i$. The approximation is accurate enough and can be used in all procedures where both means and variances are unknown.

In this paper, we especially consider an elimination type two-stage procedure $\mathcal{P}_2$ for selecting the logistic population with the largest mean from k populations when their common variance is unknown. For other distributions one can use the same method. In this paper we propose a procedure $\mathcal{P}_2$ and derive lower bounds on the probability of CS and the infimum over the preference zone of the lower bounds. Then we determine the supremum of the expected total sample size needed for $\mathcal{P}_2$ over the whole parameter space and provide tables of constants for the corresponding single-stage procedure $\mathcal{P}_1$ for two special cases of equally spaced and slippage configurations. Finally, we would like to point out that finding approximate expressions for the distribution of $\sqrt{n}\,\overline{X} - hS$ is of both theoretical and practical interest.

2 Main Theorem

2.1 General Result

In order to state our procedure, we need those notations and results. Let $X_1, ..., X_n$ be *iid* samples from a continuous distribution F with mean zero and variance unity. Denote the sample mean and variance S^2 by

$$\overline{X} = \sum_{i=1}^{n} X_i, \; S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Bhattacharya and Ghosh (1978) made fundamental contributions to Edgeworth expansion of vector function of sample means. Recently, Babu and Bai (1990) reinvestigated the functions of this type and relaxed the moment and continuity conditions on F. Using their new results, we obtain the following results for $U_n(h) \equiv \sqrt{n}\overline{X} - h(S-1)$.

Theorem 2.1 *If the rth absolute moment of F exists, then for any $h > 0$ the distribution $P_n(x,h)$ of $U_n(h)$ can be expressed as*

$$P_n(x,h) \equiv \mathbb{P}\left((\sqrt{n}\overline{X} - h(S-1)) \leq x\right) = \Phi(x) + \phi(x)\sum_{j=1}^{r-2} Q_j(x,h)/n^{j/2} + O(n^{(r-1)/2}). \tag{1}$$

where $\Phi(x)$ and $\phi(x)$ are the distribution function and the density of a standard normal random variable respectively, and $Q_j(x,h), j = 1,\cdots,r-2$, are polynomials in both x and h, whose degree in x and in h does not exceed $3r-1$ and $r-2$, respectively. These coefficients $Q_j(x,h)$ depend only on the first r moments of F and on the partial derivatives of $K(z_1,z_2) = z_1 - h\sqrt{z_2 - z_1^2}$ of order up to r evaluated at $(0,1)$.

Proof. Let $X_1, X_2\cdots$, be a sequence of independent and identically distributed random variables with mean zero and variance unity. Let f_1 and f_2 be real-valued Borel measurable functions on $\mathbb{R}$. Consider the transformation

$$W_n = \sqrt{n}\,(H(\overline{Z}_n^{(1)}, \overline{Z}_n^{(2)}) - H(\nu_1,\nu_2))/\sigma, \tag{2}$$

where H is a real-valued Borel measurable function on $\mathbb{R}^2$ and where

$$Z_n^{(1)} = f_1(X_n),\ Z_n^{(2)} = f_2(X_n),\ \nu_1 = \mathbb{E}(X_1),\ \nu_2 = \mathbb{E}(X_2),\ \sigma > 0,$$

and

$$\boldsymbol{Z}_n = (Z_n^{(1)}, Z_n^{(2)}),\ \overline{\boldsymbol{Z}}_n = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Z}_i.$$

Now, let us choose

$$H(z_1,z_2) = K(z_1,z_2) = z_1 - \frac{h}{\sqrt{n}}\sqrt{z_2 - z_1^2}. \tag{3}$$

Note that the sample variance S^2 can be expressed as

$$S^2 = \ \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(X_i - \overline{X})^2 = \tfrac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2 = \overline{Z}_n^{(2)} - (\overline{Z}_n^{(1)})^2, \tag{4}$$

where $f_1(x) = x, f_2(x) = x^2$. Therefore S is a function of vector $\boldsymbol{Z}_n$ and $\overline{X} - hS/\sqrt{n}$ can be expressed by (3). Now our statistic H satisfies all assumptions in Babu and Bai(1990), so we can use their results for K if F has finite rth absolute moment. Consequently, Theorem 2.1 holds. $\square$

Note that if F is also symmetric, the expression (1) is simplified. The polynomial Q is zero when j is an odd integer. In the following, we only discuss the logistic distribution. For other distributions the discussion is similar.

2.2 Edgeworth Expansion for $U_n(h)$ for the Logistic Population

A population is called a logistic population with mean m and variance σ^2, denoted by $L(m,\sigma^2)$, if its density function is given by

$$f(x) = (g/\sigma)\exp\{-g(x-m)/\sigma\}[1+\exp\{-g(x-m)/\sigma\}]^{-2},$$

where $g = \pi/\sqrt{3},\ m \in \mathbb{R},\ \sigma > 0$ and $x \in \mathbb{R}$.

It is easy to see that if $X \sim L(m,\sigma^2)$, then the moment generating function of standardized random variable $(X-m)/\sigma$ is given by

$$M(t) = \sqrt{3}\,t/\sin(\sqrt{3}\,t),\ |t| < 1.$$

Now let $X_1, ..., X_n$ be n *iid* observations from $L(m,\sigma^2)$. Set $Y_i = (X_i - m)/\sigma$, then $Y_i \sim L(0,1)$. Write $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}) \equiv (0,1)$ and let

$$\ell_{i_1 i_2 \ldots i_p} = D_{i_1} D_{i_2} \ldots D_{i_p} H(z_1, z_2)\,|_{(0,1)},$$

where D_i denotes partial differentiation with respect to the ith coordinate.

The key of the Edgeworth expansion of $W_n = \sqrt{n}\,(H(\overline{\boldsymbol{Z}}_n) - H(\boldsymbol{\mu}))$ is that H can be approximated by Taylor expansion H' of H at $\boldsymbol{\mu}$ up to $s-2$ degree. Then W_n may be approximated better by

$$\begin{aligned} W_n' &= \sqrt{n}\Big\{\sum_{i=1}^{2} \ell_i(\overline{Z}_n^{(i)} - \mu^{(i)}) + \frac{1}{2!}\sum_{i,j=1}^{2} \ell_{ij}(\overline{Z}_n^{(i)} - \mu^{(i)})(\overline{Z}_n^{(j)} - \mu^{(j)}) \\ &+ \frac{1}{(s-2)!}\sum_{i_1,\ldots,i_{s-2}=1}^{2} \ell_{i_1\ldots i_{s-2}}(\overline{Z}_n^{(i_1)} - \mu^{(i_1)})\cdots(\overline{Z}_n^{(i_{s-2})} - \mu^{(i_{s-2})})\Big\}. \end{aligned}$$

We note the relation between cumulants and moments of W_n', and the facts

$$\begin{aligned} \Big(\frac{\partial}{\partial z_1}\Big)^j (z_2 - z_1^2)^{-1/2}\Big|_{(0,1)} &= \frac{(-1)^j(2j-1)!!}{2^j}, \\ \Big(\frac{\partial}{\partial z_2}\Big)^j (z_2 - z_1^2)^{-1/2}\,|_{(0,1)} &= \begin{cases} 0 & \text{if } j = 2\nu+1, \\ (2k-1)!! & \text{if } j = 2\nu. \end{cases} \end{aligned}$$

In order to obtain the Edgeworth expansions of distribution function $P_n(x,h)$ and density function $p_n(x,h)$ for the statistic $U_n(h)$, we employ the function $H_j(x)$, the Hermite polynomial of degree j, which is defined by

$$(\frac{d}{dx})^j \exp(-x^2/2) = (-1)^j H_j(x) \exp(-x^2/2),\ j = 0, 1, \cdots.$$

Then the following approximations of $P_n(x,h)$ and $p_n(x,h)$ up to order n^{-3} can be computed by using Mathematica software system (Wolfram,1991):

$$P_n(x,h) = \Phi(x) - \phi(x)[C_1(x,h)/n + C_2(x,h)/n^2 + C_3(x,h)/n^3], \tag{5}$$

where

$$\begin{aligned}
C_1(x,h) &= \frac{1}{20}H_3(x) + h\Big\{\frac{9}{10} - \frac{3}{10}H_2(x)\Big\} + h^2\Big\{\frac{2}{5}H_1(x)\Big\},\\
C_2(x,h) &= \frac{1}{105}H_5(x) + \frac{1}{800}H_7(x) \ + \ h\Big\{-\frac{127}{280} + \frac{363}{700}H_2(x) - \frac{37}{700}H_4(x) - \frac{3}{200}H_6(x)\Big\}\\
&+ h^2\Big\{-\frac{127}{280}H_1(x) + \frac{19599}{5600}H_3(x) - \frac{2521}{1600}H_5(x) + \frac{105}{256}H_7(x)\Big\}\\
&+ h^3\Big\{\frac{37}{525}H_2(x) - \frac{3}{25}H_4(x)\Big\} + h^4\frac{2}{25}H_3(x),\\
C_3(x,h) &= \frac{3}{1400}H_7(x) + \frac{1}{2100}H_9(x) + \frac{1}{48000}H_{11}(x)\\
&+ h\Big\{\frac{6583}{560} - \frac{5667}{2800}H_2(x) + \frac{16951}{5600}H_4(x)\\
&+ \frac{7056473}{294000}H_6(x) - \frac{53}{8000}H_8(x) - \frac{3}{8000}H_{10}(x)\Big\}\\
&+ h^2\Big\{\frac{11649}{2800}H_1(x) - \frac{8709}{5600}H_3(x) - \frac{14423}{11200}H_5(x) + \frac{141457}{672000}H_7(x)\\
&- \frac{2537}{32000}H_9(x) + \frac{21}{1024}H_{11}(x)\Big\}\\
&+ h^3\Big\{\frac{44703}{14000}H_2(x) - \frac{30099}{14000}H_4(x) - \frac{35101}{84000}H_6(x) + \frac{54453}{64000}H_8(x) - \frac{63}{512}H_{10}(x)\Big\}\\
&+ h^4\Big\{-\frac{31}{700}H_3(x) - \frac{79}{350}H_5(x) + \frac{71}{500}H_7(x) + \frac{21}{128}H_9(x)\Big\}\\
&+ h^5\Big\{-\frac{23}{525}H_4(x) - \frac{3}{125}H_6(x)\Big\} + h^6\Big\{\frac{4}{375}H_5(x)\Big\}
\end{aligned}$$

and

$$p_n(x,h) = \phi(x)[1 + C_4(x,h)/n + C_5(x,h)/n^2 + C_6(x,h)/n^3], \tag{6}$$

where

$$C_4(x,h) = \frac{1}{20}H_4(x) + h\Big\{\frac{9}{10}H_1(x) - \frac{3}{10}H_3(x)\Big\} + h^2\frac{2}{5}H_2(x),$$

$$
\begin{aligned}
C_5(x,h) &= \frac{1}{105}H_6(x)+\frac{1}{800}H_8(x)+h\Big\{-\frac{127}{280}H_1(x)+\frac{363}{700}H_3(x)-\frac{37}{700}H_5(x)-\frac{3}{200}H_7(x)\Big\}\\
&+ h^2\Big\{-\frac{127}{280}H_2(x)+\frac{19599}{5600}H_4(x)-\frac{2521}{1600}H_6(x)+\frac{105}{256}H_8(x)\Big\}\\
&+ h^3\Big\{\frac{37}{525}H_3(x)-\frac{3}{25}H_5(x)\Big\}+h^4\frac{2}{25}H_4(x),\\
C_6(x,h) &= \frac{3}{1400}H_8(x)+\frac{1}{2100}H_{10}(x)+\frac{1}{48000}H_{12}(x)\\
&+ h\Big\{\frac{6583}{560}H_1(x)-\frac{5667}{2800}H_3(x)+\frac{16951}{5600}H_5(x)\\
&+ \frac{7056473}{294000}H_7(x)-\frac{53}{8000}H_9(x)-\frac{3}{8000}H_{11}(x)\Big\}\\
&+ h^2\Big\{\frac{11649}{2800}H_2(x)-\frac{8709}{5600}H_4(x)-\frac{14423}{11200}H_6(x)+\frac{141457}{672000}H_8(x)\\
&- \frac{2537}{32000}H_{10}(x)+\frac{21}{1024}H_{12}(x)\Big\}\\
&+ h^3\Big\{\frac{44703}{14000}H_3(x)-\frac{30099}{14000}H_5(x)-\frac{35101}{84000}H_7(x)+\frac{54453}{64000}H_9(x)-\frac{63}{512}H_{11}(x)\Big\}\\
&+ h^4\Big\{-\frac{31}{700}H_4(x)-\frac{79}{350}H_6(x)+\frac{71}{500}H_8(x)+\frac{21}{128}H_{10}(x)\Big\}\\
&+ h^5\Big\{-\frac{23}{525}H_5(x)-\frac{3}{125}H_7(x)\Big\}+h^6\Big\{\frac{4}{375}H_6(x)\Big\}.
\end{aligned}
$$

When $h=0$, we get the distribution $F_n(x)$ and the density function $f_n(x)$ of $\sqrt{n}\,\overline{X}_n$ as follows:

$$
\begin{aligned}
F_n(x) &= \Phi(x)-\phi(x)\Big\{(\frac{1}{4!})(\frac{6}{5})H_3(x)n^{-1}+\Big[(\frac{1}{6!})(\frac{48}{7})H_5(x)+(\frac{35}{8!})(\frac{6}{5})^2H_7(x)\Big]n^{-2}\\
&+ \Big[(\frac{1}{8!})(\frac{432}{5})H_7(x)+(\frac{210}{10!})(\frac{48}{7})(\frac{6}{5})H_9(x)+(\frac{5775}{12!})(\frac{6}{5})^3H_{11}(x)\Big]n^{-3}\Big\}+O(n^{-7/2}),\\
f_n(x) &= \phi(x)\Big\{1+\frac{1}{4!}(\frac{6}{5})H_4(x)n^{-1}+\Big[(\frac{1}{6!})(\frac{48}{7})H_6(x)+(\frac{35}{8!})(\frac{6}{5})^2H_8(x)\Big]n^{-2}\\
&+ \Big[(\frac{1}{8!})(\frac{432}{5})H_8(x)+(\frac{210}{10!})(\frac{48}{7})(\frac{6}{5})H_{10}(x)+(\frac{5775}{12!})(\frac{6}{5})^3H_{12}(x)\Big]n^{-3}\Big\}+O(n^{-7/2}).
\end{aligned}
$$

3 A Two-Stage Procedure for Selecting the Population with the Largest Mean from k Populations

3.1 Two-Stage Procedure

Let Π_i, $i=1,\cdots,k$, be k populations with unknown means $\mu_i, i=1,\cdots,k$, and a common unknown variance σ^2, and let

$$\Omega=\{(\boldsymbol{\mu},\sigma)=(\mu_1,...,\mu_k;\sigma): -\infty<\mu_i<\infty,\ i=1,...,k,\ \sigma>0\} \tag{7}$$

be the parameter space. Denote the ordered values of the μ_i by $\mu_{[1]}\le\cdots\le\mu_{[k]}$, and define

$$\delta'_{ij}=\mu_{[i]}-\mu_{[j]}. \tag{8}$$

We assume that one has no prior knowledge concerning the pairing of unordered and ordered $\mu_i's$. We set $\Pi_{(j)}$ as the population associated with $\mu_{[j]}$.

Our goal is to select the 'best' population which is defined as the population with the largest mean $\mu_{[k]}$. This event is referred to as a correct selection (CS). Any procedure $\mathcal{P}$, to be valid, should guarantee a specified minimum probability of a correct selection P(CS), say P^* $(1/k < P^* < 1)$, whenever the best (assumed to be unique) and the second best populations are apart at least by a specified amount, i.e. $\delta'_{k,k-1} = \mu_{[k]} - \mu_{[k-1]} \geq \delta' > 0$. Note that in this formulation, in order to select the best population, it is necessary to assume that δ'/σ must great than a positive specified constant δ. For any specified $\delta > 0$, let Ω_δ be the subset of the parameter space Ω defined by

$$\Omega_\delta = \{(\boldsymbol{\mu}, \sigma) \in \Omega \mid (\mu_{[k]} - \mu_{[k-1]})/\ \sigma \geq \delta\}. \tag{9}$$

The subset Ω_δ is called the preference zone. Letting $\mathbb{P}(CS \mid \mathcal{P})$ denote the P(CS) of a rule $\mathcal{P}$, in order to valid, it should satisfy

$$\mathbb{P}(CS \mid \mathcal{P}) \geq P^* \tag{10}$$

for all $(\mu, \sigma) \in \Omega_\delta$. Both δ and P^* are specified by the experimenter in advance.

Now we propose an elimination type two-stage procedure, namely, $\mathcal{P}_2 = \mathcal{P}_2(n_1, n_2, h)$, where n_1, n_2 are positive integers and $h > 0$: these are explained later.

Stage 1. Take n_1 independent observations

$$x_{ji}^{(1)}, i = 1, \cdots, n_1$$

from $\Pi_j, j = 1, \cdots, k$, respectively, and calculate their sample means and sample variances

$$\begin{aligned} \overline{x}_j^{(1)} &= \frac{1}{n_1}\sum_{i=1}^{n_1} x_{ji}^{(1)}, \quad j = 1, \cdots, k, \qquad (11) \\ s_j^2 &= \frac{1}{n_1}\sum_{i=1}^{n_1}(x_{ji}^{(1)} - \overline{x}_j^{(1)})^2, \quad j = 1, \cdots, k. \qquad (12) \end{aligned}$$

Next, order the k sample means, say, $\overline{x}_{[1]}^{(1)} \leq \cdots \leq \overline{x}_{[k]}^{(1)}$. Then select a subset J of $\{1, ..., k\}$ such that

$$j \in J \Longleftrightarrow \overline{x}_j^{(1)} \geq \max_{1 \leq i \leq k}(\bar{x}_i^{(1)} - hs_i/\sqrt{n_1}\ ) \tag{13}$$

where h is a specified positive number and determined later. Denote the associated populations $\Pi_j, j \in J$ by Π_j. Let S' be the number of elements in the set J.

1. If $S' = 1$, stop sampling and select the one population satisfying (13) as the best one. Note that this is the population corresponding to the largest observed mean at Stage 1.

2. If $S' > 1$, proceed to the Stage 2.

Stage 2. Take n_2 additional independent observations $x_{ji}^{(2)}, i = 1, ..., n_2, j \in J$, and compute their cumulative sample means

$$\bar{x}_j = \frac{1}{n_1 + n_2}(\sum_{i=1}^{n_1} x_{ji}^{(1)} + \sum_{i=1}^{n_2} x_{ji}^{(2)}) = \frac{1}{n_1 + n_2}(n_1 \bar{x}_j^{(1)} + n_2 \bar{x}_j^{(2)}), \quad j \in J,$$

where

$$\bar{x}_j^{(2)} = \frac{1}{n_2} \sum_{i=1}^{n_2} x_{ji}^{(2)}.$$

Then assert that the population associated with $\max_{j \in J} \bar{x}_j$ is the best.

3.2 Some Lower Bounds on the PCS for $\mathcal{P}_2$

If we do not constrain the sample size, there are an infinite number of combinations of (n_1, n_2) for given h, k, δ and P^*, which will exactly guarantee that PCS of $\mathcal{P}_2$ satisfies (10). Besides, h can be choose appropriately to reduce total sample size and satisfies (10). In the sequel, we will consider such a criteria. Let

$$S = S' I(S' > 1), \tag{14}$$

where $I(A)$ is the indicator function of the set A. Then the total sample size required by $\mathcal{P}_2$, say TSS, is given by

$$TSS = kn_1 + Sn_2 \tag{15}$$

Let $\mathbb{E}_{\boldsymbol{\mu},\sigma}(TSS \mid \mathcal{P}_2)$ denote the expected total sample size for $\mathcal{P}_2$ under $(\boldsymbol{\mu},\sigma)$.

We adopt the following criterion to select (n_1, n_2, h). For given k, δ and P^*, choose (n_1, n_2, h) to minimize

$$\sup_{(\boldsymbol{\mu},\sigma) \in \Omega} \mathbb{E}_{\boldsymbol{\mu},\sigma}(TSS \mid \mathcal{P}_2) \tag{16}$$

subject to

$$\inf_{(\boldsymbol{\mu},\sigma) \in \Omega_\delta} \mathbb{P}_{\boldsymbol{\mu},\sigma}(CS \mid \mathcal{P}_2) \geq P^*. \tag{17}$$

Since the exact probability of $\mathbb{P}(CS \mid \mathcal{P}_2)$ is too complicated to compute, we will study some lower bounds on $\mathbb{P}(CS \mid \mathcal{P}_2)$ and derive conservative two-stage procedures. For convenience, we write $n = n_1 + n_2$.

Theorem 3.1 *For any $(\mu, \sigma) \in \Omega$, we have*

$$\inf_{(\mu,\sigma)\in\Omega_\delta} \mathbb{P}_{\mu,\sigma}(CS \mid \mathcal{P}_2) \geq \left(\int P_{n_1}^{k-1}(x+\delta\sqrt{n_1}\ + h, h) f_{n_1}(x)dx\right)\left(\int F_n^{k-1}(x+\delta\sqrt{n}\ ) f_n(x)dx\right). \quad (18)$$

For proving this theorem, we need a simple lemma.

Lemma 3.1 *If the random variable U is independent of random variables V and W, then we have for any real values a and b and any $c > 0$,*

$$\mathbb{P}(V+W \leq a, cV+U \leq b) \geq \mathbb{P}(V+W \leq a)\mathbb{P}(cV+U \leq b).$$

Proof. First we prove for any random variables V and W, the following probability inequality:

$$\mathbb{P}(V+W \leq a, V \leq b) \geq \mathbb{P}(V+W \leq a)\mathbb{P}(V \leq b).$$

Using conditional expectation one gets

$$\begin{aligned}
\mathbb{P}(V+W \leq a, V \leq b) &= \mathbb{E}_W\mathbb{P}(V+W \leq a, V \leq b \mid W) \\
&= \mathbb{E}_W\mathbb{P}(V \leq a-W, V \leq b \mid W) \\
&\geq \mathbb{E}_W\{min[\mathbb{P}(V \leq a-W \mid W), \mathbb{P}(V \leq b)]\} \\
&\geq \mathbb{E}_W\mathbb{P}(V \leq a-W \mid W)\mathbb{P}(V \leq b) \\
&= \mathbb{P}(V+W \leq a)\mathbb{P}(V \leq b).
\end{aligned}$$

Since U is independent of V and W, the above probability inequality implies that

$$\begin{aligned}
\mathbb{P}(V+W \leq a, cV+U \leq b) &= \mathbb{E}_U\mathbb{P}(V+W \leq a, V \leq (b-U)/c \mid U) \\
&\geq \mathbb{E}_U\mathbb{P}(V+W \leq a)\mathbb{P}(V \leq (b-U)/c \mid U) \\
&= \mathbb{P}(V+W \leq a)\mathbb{P}(cV+U \leq b).
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 3.1. Set

$$\begin{aligned}
Y_{ji}^{(1)} &= (X_{ji}^{(1)} - \mu_j)/\sigma,\ i = 1, \cdots, n_1, j = 1, \cdots, k, \\
Y_{ji}^{(2)} &= (X_{ji}^{(2)} - \mu_j)/\sigma,\ i = n_1+1, \cdots, n_1+n_2,\ j = 1, \cdots, k, \\
\overline{Y}_j^{(1)} &= \frac{1}{n_1}\sum_{i=1}^{n_1} Y_{ji}^{(1)}, \\
S_{Yj}^2 &= \frac{1}{n_1}\sum_{i=1}^{n_1}(Y_{ji}^2 - \overline{Y}_j^{(1)})^2, \\
\overline{Y}_j &= \frac{n_1}{n}\sum_{i=1}^{n_1} Y_{ji}^{(1)} + \frac{n_2}{n}\sum_{i=1}^{n_1} Y_{ji}^{(2)}.
\end{aligned}$$

Let $P_{n_1}(\cdot \mid \mu_j)$ and $F_n(\cdot \mid \mu_j)$ denote the distributions of $\sqrt{n_1}\,\overline{Y}_j^{(1)} - hS_{Yj}$ and $\sqrt{n}\,\overline{Y}_j$ respectively, and $H(\cdot,\cdot \mid \mu_j)$ denote the joint distribution of $\sqrt{n_1}\,\overline{Y}_j^{(1)} - hS_{Yj}$ and $\sqrt{n}\,\overline{Y}_j$. Without loss of generality we assume that $\mu_1 \leq \cdots \leq \mu_k$. Noting the definitions of $\overline{Y}_j^{(1)}$ and $\overline{Y}_j$, we have

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\mu},\sigma}(CS \mid \mathcal{P}_2) &= \mathbb{P}_{\boldsymbol{\mu},\sigma}(\overline{X}_k^{(1)} \geq \max_{1\leq j\leq k}(\overline{X}_j^{(1)} - hS_j/\sqrt{n_1}), \overline{X}_k = \max_{j\in J}\overline{X}_j) \\
&\geq \mathbb{P}_{\boldsymbol{\mu},\sigma}(\overline{X}_j^{(1)} - hS_j/\sqrt{n_1} \leq \overline{X}_k^{(1)}, \overline{X}_j \leq \overline{X}_k, j = 1,\cdots,k-1) \\
&= \mathbb{E}_{\mu_k,\sigma}\prod_1^{k-1}\mathbb{P}_{\mu_j,\sigma}(\overline{X}_j^{(1)} - hS_j/\sqrt{n_1} \leq \overline{X}_k^{(1)}, \overline{X}_j \leq \overline{X}_k \mid \overline{X}_k^{(1)}, \overline{X}_k) \\
&= \mathbb{E}_{\mu_k,\sigma}\prod_1^{k-1}\mathbb{P}_{\mu_j,\sigma}(\sqrt{n_1}\,(\overline{Y}_j^{(1)} - h(S_{Yj}-1)/\sqrt{n_1}\,) \leq \sqrt{n_1}\,\overline{Y}_k^{(1)} + \sqrt{n_1}\,\delta_{kj}/\sigma + h, \\
&\qquad \sqrt{n}\,\overline{Y}_j \leq \sqrt{n}\,\overline{Y}_k + \sqrt{n}\,\delta_{kj}/\sigma \mid \sqrt{n_1}\,\overline{Y}_k^{(1)}, \sqrt{n}\,\overline{Y}_k) \\
&\geq \mathbb{E}_{\mu_k,\sigma}\prod_1^{k-1}\mathbb{P}_{\mu_j,\sigma}(\sqrt{n_1}\,(\overline{Y}_j^{(1)} - h(S_{Yj}-1)/\sqrt{n_1}\,) \leq \sqrt{n_1}\,\overline{Y}_k^{(1)} + \sqrt{n_1}\,\delta + h, \\
&\qquad \sqrt{n}\,\overline{Y}_j \leq \sqrt{n}\,\overline{Y}_k + \sqrt{n}\,\delta \mid \sqrt{n_1}\,\overline{Y}_k^{(1)},\ \sqrt{n}\,\overline{Y}_k) \\
&= \mathbb{E}_{\mu_k,\sigma}H^{k-1}(\sqrt{n_1}\,\overline{Y}_k^{(1)} + \sqrt{n_1}\,\delta + h, \sqrt{n}\,\overline{Y}_k + \sqrt{n}\,\delta), \qquad (19)
\end{aligned}$$

where the expectation is taken with respect to the joint distribution of $\sqrt{n_1}\,\overline{Y}_k^{(1)}$ and $\sqrt{n}\,\overline{Y}_k$. By (19) we have

$$\inf_{(\boldsymbol{\mu},\sigma)\in\Omega_\delta}\mathbb{P}_{\boldsymbol{\mu},\sigma}(CS \mid \mathcal{P}_2) \geq \inf_{(\boldsymbol{\mu},\sigma)\in\Omega_\delta}\mathbb{E}_{\mu_k,\sigma}H^{k-1}(\sqrt{n_1}\,(\overline{Y}_k^{(1)} + \delta) + h, \sqrt{n}\,(\overline{Y}_k + \delta)). \qquad (20)$$

Lemma 3.1 implies

$$\begin{aligned}
&H(\sqrt{n_1}\,\overline{Y}_k^{(1)} + \sqrt{n_1}\,\delta + h, \sqrt{n}\,\overline{Y}_k + \sqrt{n}\,\delta) \\
= \;& \mathbb{P}(\sqrt{n_1}\,(\overline{Y}_1^{(1)} - h(S_{Y1}-1)/\sqrt{n_1}\,) \leq \sqrt{n_1}\,\overline{Y}_k^{(1)} + \sqrt{n_1}\,\delta + h,\ \overline{Y}_1 \leq \overline{Y}_k + \delta) \\
= \;& \mathbb{P}(\overline{Y}_1^{(1)} - h(S_{Y1}-1)/sne \leq \overline{Y}_k^{(1)} + \delta + h/\sqrt{n_1}\,, \frac{n_1}{n}\overline{Y}_1^{(1)} + \frac{n_2}{n}\overline{Y}_1^{(2)} \leq \overline{Y}_k + \delta) \\
\geq \;& \mathbb{P}(\overline{Y}_1^{(1)} - h(S_{Yj}-1)/\sqrt{n_1}\ \leq \overline{Y}_k^{(1)} + \delta + h/\sqrt{n_1}\,)\ \mathbb{P}(\frac{n_1}{n}\overline{Y}_1^{(1)} + \frac{n_2}{n}\overline{Y}_1^{(2)} \leq \overline{Y}_k + \delta) \\
= \;& P_{n_1}(\sqrt{n_1}\,(\overline{Y}_k^{(1)} + \delta) + h, h)F_n(\sqrt{n}\,(\overline{Y}_k + \delta)). \qquad (21)
\end{aligned}$$

Using (21) and a version of Chebyshev's inequality (Hardy, Littlewood and Pólya, 1934) we have

$$\begin{aligned}
&\mathbb{E}_{\mu_k,\sigma}H^{k-1}(\sqrt{n_1}\,\overline{Y}_k^{(1)} + \sqrt{n_1}\,\delta + h, \sqrt{n}\,\overline{Y}_k + \sqrt{n}\,\delta) \\
\geq \;& \mathbb{E}_{\mu_k,\sigma}(P_{n_1}(\sqrt{n_1}\,(\overline{Y}_k^{(1)} + \delta) + h, h)F_n(\sqrt{n}\,(\overline{Y}_k + \delta)))^{k-1} \\
\geq \;& \mathbb{E}_{\mu_k,\sigma}P_{n_1}^{k-1}(\sqrt{n_1}\,(\overline{Y}_k^{(1)} + \delta) + h, h)\cdot\mathbb{E}_{\mu_k,\sigma}F_n^{k-1}(\sqrt{n}\,(\overline{Y}_k + \delta)). \qquad (22)
\end{aligned}$$

Combining (21) and (22) implies

$$\inf_{(\boldsymbol{\mu},\sigma)\in\Omega_\delta} \mathbb{P}_{\boldsymbol{\mu},\sigma}(CS \mid \mathcal{P}_2) \geq \int P_{n_1}^{k-1}(x+\sqrt{n_1}\,\delta+h,h)f_{n_1}(x)dx \cdot \int F_n^{k-1}(x+\sqrt{n}\,\delta)f_n(x)dx,$$

which prove the theorem. □

Since $ab \geq a+b-1$ for any $a,b \in [0,1]$, we can get another lower bound.

Corollary 3.1

$$\inf_{(\boldsymbol{\mu},\sigma)\in\Omega_\delta} \mathbb{P}_{\boldsymbol{\mu},\sigma}(CS \mid \mathcal{P}_2) \geq \int P_{n_1}^{k-1}(x+\sqrt{n_1}\,\delta+h,h)f_{n_1}(x)dx + \int F_n^{k-1}(x+\sqrt{n}\,\delta)f_n(x)dx - 1. \quad (23)$$

¿From the proof of this theorem it can be seen that the key to sharpen the lower bound is to sharpen the lower bound of the estimator of (20). We believe that the Edgeworth expansion of the vector function of sample means and the standard deviation can do it, although it is much more tedious.

4 Expected Total Sample Size for $\mathcal{P}_2$

4.1 The Supremum of $\mathbb{E}_{\mu,\sigma}(TSS|\mathcal{P}_2)$

In procedure $\mathcal{P}_2$, one important problem is to calculate the expected value of the total sample size and how to estimate it in the preference zone. We first consider the supremum of the expectated value of the total sample size over the parameter space Ω.

Theorem 4.1 *If the distribution F has the monotone likelihood ratio property, then for fixed k and (n_1, n_2, h) one has*

$$\sup_{(\boldsymbol{\mu},\sigma)\in\Omega} \mathbb{E}_{\boldsymbol{\mu},\sigma}(TSS \mid \mathcal{P}_2) = k\Big\{n_1 + n_2\Big(\int P_{n_1}^{k-1}(x+h,h)f_{n_1}(x)dx - \int F_{n_1}^{k-1}(x-h)p_{n_1}(x,h)dx\Big)\Big\}. (24)$$

Proof. For any $(\boldsymbol{\mu},\sigma)\in\Omega$, (15) implies

$$\mathbb{E}_{\boldsymbol{\mu},\sigma}(TSS \mid \mathcal{P}_2) = kn_1 + n_2\mathbb{E}_{\boldsymbol{\mu},\sigma}(S \mid \mathcal{P}_2), \quad (25)$$

where S is defined in (14). It is easy to see that

$$\mathbb{E}_{\boldsymbol{\mu},\sigma}(S \mid \mathcal{P}_2) = \mathbb{E}_{\boldsymbol{\mu},\sigma}(S' \mid \mathcal{P}_2) - P_{\boldsymbol{\mu},\sigma}(S'=1 \mid \mathcal{P}_2).$$

Noting the definition of S', we have

$$S' = \sum_{i=0}^{k} I(\overline{X}_i^{(1)} \geq \max_{1 \leq j \leq k, j \neq i} (\overline{X}_j^{(1)} - h(S_j - 1)/\sqrt{n_1})). \tag{26}$$

Equation (26) implies

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\mu},\sigma} S' &= \sum_{i=1}^{k} \mathbb{P}(\overline{X}_i^{(1)} \geq \max_{1 \leq j \leq k, j \neq i} (\overline{X}_j^{(1)} - h(S_j - 1)/\sqrt{n_1})) \\
&= \sum_{i=1}^{k} \mathbb{E}_{\mu_i,\sigma} \prod_{j \neq i} \mathbb{P}(\overline{X}_j^{(1)} - hS_j/\sqrt{n_1} \leq \overline{X}_i^{(1)} \mid \overline{X}_i^{(1)}) \\
&= \sum_{i=1}^{k} \int \prod_{j \neq i} P_{n_1}(x + \sqrt{n_1}\, \delta_{ij} + h, h) f_{n_1}(x) dx, \qquad (27)
\end{aligned}$$

where $\delta_{ij} = \delta'_{ij}/\sigma$. Also, it is seen that

$$I(S' = 1) = \sum_{i=1}^{k} I(\max_{1 \leq j \leq k, j \neq i} \overline{X}_j^{(1)} < \overline{X}_i^{(1)} - hS_i/\sqrt{n_1})$$

and

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\mu},\sigma}(S' = 1 \mid \mathcal{P}_2) &= \sum_{i=1}^{k} \mathbb{P}_{\boldsymbol{\mu},\sigma}(\max_{1 \leq j \leq k, j \neq i} \overline{X}_j^{(1)} < \overline{X}_i^{(1)} - hS_i/\sqrt{n_1}) \\
&= \sum_{i=1}^{k} \mathbb{E}_{\mu_i,\sigma} \mathbb{P}(\overline{X}_j^{(1)} < \overline{X}_i^{(1)} - hS_i/\sqrt{n_1}\ , \ \forall j \neq i) \\
&= \sum_{i=1}^{k} \mathbb{E}_{\mu_i,\sigma} \prod_{j \neq i} F_{n_1}(X + \delta_{ij}\sqrt{n_1} - h) \\
&= \sum_{i=1}^{k} \int \prod_{j \neq i} F_{n_1}(x + \delta_{ij}\sqrt{n_1} - h) p_{n_1}(x, h) dx, \qquad (28)
\end{aligned}$$

where expectation $\mathbb{E}_{\mu_i,\sigma}$ is with respect to the distribution $P_{n_1}(x)$ of $\sqrt{n_1}\, \bar{Y}_i^{(1)} - h(S_{Yi} - 1)$.

Combining (25),(27) and (28), we have

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\mu},\sigma}(TSS \mid \mathcal{P}_2) &= kn_1 + n_2 \sum_{i=1}^{k} \Bigg(\int \prod_{j \neq i} P_{n_1}(x + \sqrt{n_1}\, \delta_{ij} + h, h) f_{n_1}(x) dx \\
&\qquad - \int \prod_{j \neq i} F_{n_1}(x + \sqrt{n_1}\, \delta_{ij} - h) p_{n_1}(x, h) dx \Bigg). \qquad (29)
\end{aligned}$$

On the other hand, from Gupta (1965), it can be shown that both the supremum of $\mathbb{E}_{\boldsymbol{\mu},\sigma}(S' \mid \mathcal{P}_2)$ and the infimum of $\mathbb{P}_{\boldsymbol{\mu},\sigma}(S' = 1 \mid \mathcal{P}_2)$ are attained when $\mu_{[1]} = \cdots = \mu_{[k]}$ in the limit for any distribution with monotone likelihood ratio property. Using this result and (29), the theorem is proved. □

4.2 Optimal Design Problem

Theorem 4.1 provided an exact formula for the supremum of the total sample size over Ω, and (29) yields an exact formula for fixed $\boldsymbol{\mu}$ and σ. It is helpful to solve the optimal design problem (16) and (17). But since the exact value of $\inf_{\boldsymbol{\mu},\sigma}(CS \mid \mathcal{P}_2)$ is very complex, we replace this quantity by the conservative lower bound in Theorem 3.1 and consider such an optimal design problem: For given k, δ and P^*, choose the triple (n_1, n_2, h) such that

$$k\Big\{n_1 + n_2\Big[\int P_{n_1}^{k-1}(x+h,h)f_{n_1}(x)dx - \int F_{n_1}^{k-1}(x-h)p_{n_1}(x,h)dx\Big]\Big\} = min \tag{30}$$

and

$$\int P_{n_1}^{k-1}(x+\delta\sqrt{n_1}\ +h,h)f_{n_1}(x)dx \cdot \int F_n^{k-1}(x+\delta\sqrt{n}\ )p_n(x,h)dx \geq P^*, \tag{31}$$

where n_1 and n_2 are non-negative integers, $n = n_1 + n_2$ and $h \geq 0$.

Tables 1 and 2 provide these design constants and the expected total sample size (ETSS) for $\delta = 0.1, 0.3, 0.5, 1.0.$ and $k = 2, 3, 5, 10, 20, P^* = 0.90; k = 2, 5, 10, 20, P^* = 0.95$.

4.3 Relative Efficiency of $\mathcal{P}_2$ w.r.t. $\mathcal{P}_1$

To select the 'best' population from k populations, the simplest procedure is to take the population associated with the largest mean based on samples of common size n_r as the best. Denote this procedure by $\mathcal{P}_1$. The relative efficiency(RE) of $\mathcal{P}_2$ with respect to $\mathcal{P}_1$ is defined by the ratio $\mathbb{E}_{\boldsymbol{\mu},\sigma}(TSS \mid \mathcal{P}_2)/\ kn_s$, where n_s is the smallest integer for which

$$\int F_{n_s}^{k-1}(x+\delta\sqrt{n_s})f_{n_s}(x)dx \geq P^*.$$

Now the RE is given by

$$\begin{aligned} RE \ &= \ \frac{1}{kn_s}\Big\{kn_1 + n_2\sum_{i=1}^{k}\Big(\int\prod_{j\neq i}P_{n_1}(x+\delta_{ij}\sqrt{n_1}\ +h,h)f_{n_1}(x)dx \\ &\qquad - \int\prod_{j\neq i}F_{n_1}(x+\delta_{ij}\sqrt{n_1}\ -h)p_{n_1}(x,h)dx\Big)\Big\}. \end{aligned} \tag{32}$$

Let $(\hat{n_1}, \hat{n_2}, \hat{h})$ be the solution of the optimization problem (30) and (31) treating n_1 and n_2 as continuous positive variables. Take $([\hat{n_1}+1], [\hat{n_2}+1], \hat{h})$ as the approximate design constants, where $[u]$ denotes the greatest integer part of u, then we get an approximate value of RE if (n_1, n_2, h) is substituted by $([\hat{n}_1], [\hat{n}_2], \hat{h})$ in the formula (32).

In the sequel we consider the RE for two special cases, namely, the equally spaced and slippage configurations. In the first case, we assume that the unknown mean of Π_i are $\mu + (i-1)\delta', i = 1, \cdots, k$. Since $\delta'_{ij} = (i-j)\delta'$, set $\delta = \delta'/\sigma$, then the RE of this case is

$$\begin{aligned} RE_e &= \frac{1}{kn_s}\Big\{k\ n_1 + n_2 \sum_{i=1}^{k} \Big[\int \prod_{j\neq i} P_{n_1}(x + \sqrt{n_1}\ (i-j)\delta + h, h) f_{n_1}(x) dx \\ &- \int \prod_{j\neq i} F_{n_1}(x + \sqrt{n_1}\ (i-j)\delta - h) p_{n_1}(x,h) dx\Big]\Big\}. \end{aligned} \quad (33)$$

In the second case, we assume that $\mu_i = \mu$ for $1 \leq i \leq k-1$, $\mu_k = \mu + \delta', \delta' \geq 0$ and $\delta = \delta'/\sigma$. Then the formula (29) yields the RE of this case

$$\begin{aligned} RE_s &= \frac{1}{kn_s}\Big(kn_1 + n_2\Big\{(k-1)\Big[\int P_{n_1}^{k-2}(x+h,h)\cdot P_{n_1}(x - \delta\sqrt{n_1}\ + h, h) f_{n_1}(x) dx \\ &\qquad - \int F_{n_1}^{k-2}(x-h)\cdot F_{n_1}(x - \delta\sqrt{n_1}\ - h) p_{n_1}(x,h) dx\Big] \\ &+ \Big[\int P_{n_1}^{k-1}(x + \delta\sqrt{n_1}\ + h, h) f_{n_1}(x) dx - \int F_{n_1}^{k-1}(x + \delta\sqrt{n_1}\ - h) p_{n_1}(x,h) dx\Big]\Big\}\Big). \end{aligned} \quad (34)$$

Table 3 gives the value of RE_s for given values of $P^* = 0.90, 0.95$, $k = 2, 5, 10, 20$ and $\delta = 0.1, 0.3, 0.5, 1.0$. ¿From these tables, we see that the $\mathcal{P}_2$ is more efficient than $\mathcal{P}_1$ based on the criteria of the expected total sample sizes.

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Tabel 1: Optimal Design Constants for $\mathcal{P}_2$ for Logistic Populations

$P^* = 0.90$

k	δ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
2	0.1	177.1	163.4	2.438	651.8
	0.3	26.71	10.72	2.732	72.96
	0.5	4.771	7.146	2.980	11.93
	1.0	2.643	8.814	3.634	5.160
3	0.1	280.3	245.8	1.463	1463.5
	0.3	16.27	42.37	2.869	161.8
	0.5	8.080	7.320	9.260	46.20
	1.0	3.872	1.243	9.900	11.61
5	0.1	369.1	400.7	1.564	3153.8
	0.3	42.46	41.82	1.743	354.1
	0.5	9.718	12.97	3.180	123.6
	1.0	3.840	1.497	0.345	22.22
10	0.1	456.8	644.6	1.374	7509.2
	0.3	51.16	70.43	1.489	841.0
	0.5	13.13	23.32	1.978	275.1
	1.0	2.091	1.851	0.776	23.40
20	0.1	518.4	893.5	1.393	16490.9
	0.3	55.41	99.69	1.564	1834.6
	0.5	5.996	0.287	1.341	119.9
	1.0	3.498	2.124	1.032	47.41

Tabel 2: Optimal Design Constants for $\mathcal{P}_2$ for Logistic Populations

$P^* = 0.95$

k	δ	$\hat{n}_1$	$\hat{n}_2$	$\hat{h}$	ETSS
2	0.1	306.3	278.7	1.773	1051.6
	0.3	38.90	27.11	1.753	119.29
	0.5	4.567	11.02	2.877	9.135
	1.0	2.600	7.286	1.594	5.200
5	0.1	541.3	523.4	1.393	4265.5
	0.3	62.63	56,67	1.408	479.00
	0.5	19.03	22.62	2.025	174.10
	1.0	4.407	7.385	0.412	32.26
10	0.1	630.1	767.0	1.352	9746.6
	0.3	71.27	84.76	1.397	1091.9
	0.5	13.06	32.08	2.072	352.7
	1.0	3.923	17.02	0.1917	41.958
20	0.1	693.6	1015.9	1.413	20981.6
	0.3	76.62	113.6	1.499	2341.7
	0.5	5.330	15.581	0.608	287.73
	1.0	3.970	3.336	0.184	83.71

Tabel 3: Relative Efficiency of $\mathcal{P}_2$ w.r.t. $\mathcal{P}_1$ for Logistic Populations

Slippage Configuration

P*	k	δ			
		0.1	0.3	0.5	1.0
0.90	2	0.927	0.973	0.861	0.817
	5	0.853	0.869	0.881	0.767
	10	0.844	0.859	0.885	0.734
	20	0.855	0.868	0.850	0.707
0.95	2	0.822	0.843	0.844	0.835
	5	0.740	0.753	0.794	0.549
	10	0.677	0.689	0.627	0.525
	20	0.625	0.628	0.596	0.475