SOME PROPERTIES OF SPECTRAL DENSITY FUNCTIONS IN DOWN SAMPLING A STATIONARY PROCESS AND A VERSION OF RUELLE'S GENERAL PERRON-FROEBENIUS THEOREM

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Technical Report #93-8

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January 1993

Some Properties of Spectral Density Functions in Down Sampling a Stationary Process and a Version of Ruelle's General Perron-Froebenius Theorem

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December 12, 1992

Abstract

Down sampling a stationary process consists of forming another process by taking every other entry of the original process. Passing the down sampled process through a linear filter, one still gets a stationary process. Its autocovariance function can be computed from the weights of the linear filter and the autocovariance function of the original stationary process. The relationship between the spectral density functions of the original process and of the down sampled, filtered process is obtained. In studying the use of wavelets in statistics, we encounter the situation where we repeatedly down sample a stationary process and pass it through a linear filter. It is proved in this article that the corresponding spectral density functions decay exponentially. The proof is fitted in the framework of Ruelle's general Perron-Froebenius theorem for sequence spaces.

Keywords: down sampling, spectral density, stationary process, Ruelle's Perron-Froebenius theorem.

1 Some Properties of Spectral Density Functions in Down Sampling a Stationary Process

Define the down sampling operator

$$\Theta(\mathbf{a}, t) = \sum_{j=-\infty}^{\infty} a_j F^{t+j}$$

with weights $\{a_j\}$, where F is the forward shift operator $F^j X_t = X_{t+j}$. This linear filter is not time-invariant, but it will take any stationary process to another stationary process.

Proposition 1 *If $\{X_t\}$ is any sequence of random variables such that $\sup_t E|X_t| < \infty$, and if $\sum_{j=-\infty}^{\infty} |a_j| < \infty$, then the series*

$$\Theta(\mathbf{a}, t)X_t = \sum_{j=-\infty}^{\infty} a_j F^{t+j} X_t = \sum_{j=-\infty}^{\infty} a_j X_{2t+j} \tag{1}$$

converges absolutely with probability one. If in addition $\sup_t E|X_t|^2 < \infty$ then the series converges in mean square to the same limit. If $\{X_t\}$ is a stationary process with autocovariance function $\gamma_X(\cdot)$, then the limit $\{Y_t = \Theta(\mathbf{a}, t)X_t\}$ is a stationary process with autocovariance function

$$\gamma_Y(h) = \sum_{j,k=-\infty}^{\infty} a_j a_k \gamma_X(2h + j - k). \tag{2}$$

PROOF: The proof is parallel to the standard argument which one can find in most time series books, for example in [?]. The monotone convergence theorem and the finiteness of $\sup_t E|X_t|$ give

$$\begin{aligned} E\left(\sum_{j=-\infty}^{\infty} |a_j||X_{2t+j}|\right) &= \lim_{n\to\infty} E\left(\sum_{j=-n}^{n} |a_j||X_{2t+j}|\right) \\ &\le \lim_{n\to\infty} \sum_{j=-n}^{n} |a_j| \sup_t E|X_{2t+j}| \\ &< \infty. \end{aligned}$$

It follows that $\sum_{j=-\infty}^{\infty} |a_j||X_{2t+j}|$ and $\Theta(\mathbf{a}, t)X_t = \sum_{j=-\infty}^{\infty} a_j X_{2t+j}$ are both finite with probability one.

If $\sup_t E|X_t|^2 < \infty$ and $n > m > 0$, then

$$\begin{aligned} E|\sum_{m<|j|<n} a_j X_{2t+j}|^2 &= \sum_{m<|j|,|k|<n} a_j a_k E(X_{2t+j}X_{2t+k}) \\ &\leq \sup_t E|X_t|^2 (\sum_{m<j<n} |a_j|)^2 \\ &\rightarrow 0 \quad as \;\; m, n \rightarrow \infty, \end{aligned}$$

and so by the Cauchy criterion, the series (1) converges in mean square. If Y_t denotes the mean square limit, then by Fatou's lemma,

$$\begin{aligned} E|Y_t - \Theta(\mathbf{a}, t)X_t|^2 &= E \liminf_{n\rightarrow\infty} |Y_t - \sum_{j=-n}^{n} a_j X_{2t+j}|^2 \\ &\leq \liminf_{n\rightarrow\infty} E|Y_t - \sum_{j=-n}^{n} a_j X_{2t+j}|^2 \\ &= 0 \end{aligned}$$

showing that the limit Y_t and $\Theta(\mathbf{a}, t)X_t$ are equal with probability one.

If X_t is a stationary process with autocovariance function $\gamma_X(\cdot)$, then using the mean square convergence of (1) and continuity of the inner product, we have

$$EY_t = \lim_{n\rightarrow\infty} \sum_{j=-n}^{n} a_j EX_{2t+j} = \sum_{j=-\infty}^{\infty} a_j EX_{2t}$$

and

$$\begin{aligned} E(Y_{t+h}Y_t) &= \lim_{n\rightarrow\infty} E[(\sum_{j=-n}^{n} a_j X_{2t+2h+j})(\sum_{k=-n}^{n} a_k X_{2t+k})] \\ &= \sum_{j,k=-\infty}^{\infty} a_j a_k (\gamma_X(2h + j - k) + (EX_{2t})^2). \end{aligned}$$

Thus EY_t and $E(Y_{t+h}Y_t)$ are both finite and independent of t. The autocovariance function $\gamma_Y(\cdot)$ of Y_t is given by (2). □

This result actually holds for any integer replacing the factor 2. In other words, $\Theta(\mathbf{a}, t)$ could be $\sum_{j=-\infty}^{\infty} a_j F^{kt+j}$, where $k \geq 2$.

Two autocovariance functions $\gamma_Y(\cdot)$ and $\gamma_X(\cdot)$ of the stationary processes $\{Y_t\}$ and $\{X_t\}$ defined in the proposition 1 are not only connected by (2), but also by the corresponding spectral distribution functions.

By Herglotz's theorem the autocovariance function $\gamma_X(\cdot)$ of a real stationary process X_t has a spectral representation

$$\gamma_X(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda)$$

where the spectral distribution function F is a right-continuous, nondecreasing, bounded function on $[-\pi,\pi]$ with $F(-\pi) = 0$ and $F(\pi) = \gamma_X(0)$, it is symmetric in the sense that $F(\lambda) = F(\pi^-) - F(-\lambda^-)$, $-\pi < \lambda < \pi$.

Proposition 2 *If $\{X_t\}$ is any zero-mean stationary process with spectral distribution function $F_X(\cdot)$, then the spectral distribution function $F_Y(\cdot)$ of the stationary process $\{Y_t = \Theta(\mathbf{a},t)X_t\}$ satisfies the equation*

$$(3)\quad \gamma_Y(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF_Y(\lambda) = \int_{(-\pi,\pi]} e^{i2h\lambda} g(\lambda) dF_X(\lambda) \qquad h \in Z$$

which is equivalent to

$$(4)\begin{cases} dF_Y(\lambda) = g(\frac{\lambda}{2})dF_X(\frac{\lambda}{2}) + g(\frac{\lambda}{2}-\pi)dF_X(\frac{\lambda}{2}-\pi) & 0 < \lambda \le \pi \\ \\ dF_Y(\lambda) = g(\frac{\lambda}{2})dF_X(\frac{\lambda}{2}) + g(\frac{\lambda}{2}+\pi)dF_X(\frac{\lambda}{2}+\pi) & -\pi < \lambda \le 0 \end{cases}$$

where $g(\lambda) = |\sum_{j=-\infty}^{\infty} a_j e^{ij\lambda}|^2$, $\{a_j\} \in \ell^1$.

Proof: From (2), using the spectral representation of $\gamma_X(\cdot)$, we have

$$\begin{aligned} \gamma_Y(h) &= \sum_{j,k=-\infty}^{\infty} a_j a_k \int_{(-\pi,\pi]} e^{i(2h+j-k)\lambda} dF_X(\lambda) \\ &= \int_{(-\pi,\pi]} (\sum_{j=-\infty}^{\infty} a_j e^{ij\lambda})(\sum_{k=-\infty}^{\infty} a_k e^{-ik\lambda}) e^{i2h\lambda} dF_X(\lambda) \\ &= \int_{(-\pi,\pi]} e^{i2h\lambda} g(\lambda) dF_X(\lambda). \end{aligned}$$

Let $g^*(\lambda) = g(\lambda)$ if $0 < \lambda \le \pi$ and $g^*(\lambda) = g(\lambda - 2\pi)$ if $\pi < \lambda \le 2\pi$. Let $F^*(\lambda) = F(\lambda) - F(0)$ if $0 < \lambda \le \pi$ and $F^*(\lambda) = F(\lambda - 2\pi) + F(\pi) - F(0)$ if $\pi < \lambda \le 2\pi$. Now since

$$\int_{(-\pi,\pi]} e^{ih\lambda} dF_Y(\lambda) = \int_{(0,2\pi]} e^{ih\lambda} dF_Y^*(\lambda),$$

we have

$$\begin{aligned}
& \int_{(-\pi,\pi]} e^{i2h\lambda} g(\lambda) dF_X(\lambda) \\
= & \int_{(0,2\pi]} e^{i2h\lambda} g^*(\lambda) dF_X^*(\lambda) \\
= & \int_{(0,2\pi]} e^{ih\lambda} g^*(\frac{\lambda}{2}) dF_X^*(\frac{\lambda}{2}) + \int_{(0,2\pi]} e^{ih\lambda} g^*(\frac{\lambda}{2}+\pi) dF_X^*(\frac{\lambda}{2}+\pi).
\end{aligned}$$

Thus (3) implies that

$$dF_Y^*(\lambda) = g^*(\frac{\lambda}{2}) dF_X^*(\frac{\lambda}{2}) + g^*(\frac{\lambda}{2}+\pi) dF_X^*(\frac{\lambda}{2}+\pi) \qquad 0 < \lambda \leq 2\pi$$

which is (4). It is easy to check that (4) implies (3). □

Corollary 1 *If $\{X_t\}$ is a stationary process with spectral density function $f_X(\cdot)$, then the spectral density function $f_Y(\cdot)$ of the stationary process $\{Y_t = \Theta(\mathbf{a}, t) X_t\}$ is*

$$f_Y(\lambda) = \frac{1}{2} g(\frac{\lambda}{2}) f_X(\frac{\lambda}{2}) + \frac{1}{2} g(\pi - \frac{\lambda}{2}) f_X(\pi - \frac{\lambda}{2}) \qquad 0 < \lambda \leq \pi$$

where $g(\lambda) = |\sum_{j=-\infty}^{\infty} a_j e^{ij\lambda}|^2$.

Proposition 3 *Let f_n be a function on $[0, \pi]$ defined by*

$$f_n(\lambda) = \frac{1}{2} g(\frac{\lambda}{2}) f_{n-1}(\frac{\lambda}{2}) + \frac{1}{2} g(\pi - \frac{\lambda}{2}) f_{n-1}(\pi - \frac{\lambda}{2})$$

where $f_0(\lambda)$ is a nonnegative bounded function on $[0, \pi]$ and g is a nonnegative function on $[0, \pi]$. There are constants C and θ depending on g such that $f_n(\lambda) \leq C\theta^n$.

A function sequence like the one defined above has some interesting properties. Similar function sequence has been studied in dynamic system and ergodic theory. A Ruelle's Perron-Froebenius theorem for Δ-map is proved in the next section.

2 Ruelle's Perron-Froebenius Theorem for Δ-Map

Let $\Omega = [0,1]$. A point $x \in \Omega$ has a nonterminating dyadic sequence expression

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i} = .x_1 x_2 \cdots$$

with each x_i being 0 or 1. Most time we also use x to denote its dyadic sequence $x_1 x_2 \cdots$. For definiteness, a point such as $\frac{1}{2} = .1000\cdots = .0111\cdots$, which has two expansions, takes the nonterminating one; 0 takes the expansion $.000\cdots$; and 1 takes $.111\cdots$.

Define the map σ: $\Omega \mapsto \Omega$ by $\sigma(x)_i = x_1 + (-1)^{x_1} x_{i+1}$ or

$$\sigma(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ 2 - 2x & \text{otherwise} \end{cases}$$

The map σ is a two-to-one continuous map of Ω onto itself. We call σ a Δ-map because of its shape.

We will work on the banach space $(C(\Omega), \|\cdot\|)$, where $C(\Omega)$ is the set of the real-valued continues functions on Ω and $\|f\| = \max_{x\in\Omega} |f(x)|$.

Let $M(\Omega)$ be the set of Borel probability measures on Ω. For any $\mu \in M(\Omega)$ one can define $\mathcal{L}^*\nu$ by $\mathcal{L}^*\nu(f) = \nu(\mathcal{L}f)$ for any operator $\mathcal{L} : C(\Omega) \mapsto C(\Omega)$.

For $\phi \in C(\Omega)$ define

$$Var_k\phi = \sup\{|\phi(x) - \phi(y)| : \ x_i = y_i \ \ 1 \le i \le k \ \}$$

Let $\mathcal{F}$ be the family of all continuous $\phi : \Omega \mapsto R$ for which $Var_k\phi \le b\alpha^k$ (all $k \ge 1$) for some positive constants b and $\alpha \in (0,1)$.

For $\phi \in C(\Omega)$ define the operator $\mathcal{L} = \mathcal{L}_\phi$ on $C(\Omega)$ by

$$(\mathcal{L}_\phi f)(x) = \sum_{y\in\sigma^{-1}x} e^{\phi(y)} f(y) = e^{\phi(0x)} f(0x) + e^{\phi(1\bar{x})} f(1\bar{x})$$

where $\sigma^{-1}x = \{y : \ y = 0x \text{ or } y = 1\bar{x}\}$, and $\bar{x}$ is the sequence with $(\bar{x})_i = 1 - x_i$.

Theorem 1 *Let $\phi \in \mathcal{F} \cap C(\Omega)$ and $\mathcal{L} = \mathcal{L}_\phi$ as above. There are $\lambda > 0$, $h \in C(\Omega)$ with $h > 0$ and $\nu \in M(\Omega)$ for which $\mathcal{L}h = \lambda h$, $\mathcal{L}^*\nu = \lambda\nu$, $\nu(h) = 1$ and*

$$\lim_{m\to\infty} \|\lambda^{-m}\mathcal{L}^{-m} g - \nu(g)h\| = 0 \quad \textit{for all } g \in C(\Omega).$$

PROOF: $\mathcal{L}$ is a positive operator and $\mathcal{L}1 > 0$, $G(\mu) = (\mathcal{L}^*\mu(1))^{-1}\mathcal{L}^*\mu \in M(\Omega)$ for $\mu \in M(\Omega)$. There is a $\nu \in M(\Omega)$ with $G(\nu) = \nu$ by the Schauder-Tychonoff fixed point theorem. $G(\nu) = \nu$ gives $\mathcal{L}^*\nu = \lambda\nu$ with $\lambda > 0$. Define

$$\Lambda = \{f \in C(\Omega) : \ f \geq 0, \, \nu(f) = 1, \, f(x) \leq B_m f(x') \text{ for } x_i = x_i', \, 1 \leq i \leq m \ \}$$

$B_m = \exp(\sum_{k=m+1}^{\infty} 2b\alpha^k)$, b, and α as in $Var_k\phi \leq b\alpha^k$.

We complete the proof through the following five lemmas.

Lemma 1 *$\exists\ h \in \Lambda$ with $\mathcal{L}h = \lambda h$ and $h > 0$.*

PROOF: $\lambda^{-1}\mathcal{L}f \in \Lambda$ when $f \in \Lambda$, since $\lambda^{-1}\mathcal{L}f \geq 0$ and $\nu(\lambda^{-1}\mathcal{L}f) = \lambda^{-1}\mathcal{L}^*\nu(f) = \nu(f) = 1$. Furthermore assume $x_i = x_i'$, for $1 \leq i \leq m$, then $\mathcal{L}f(x) = \exp(\phi(0x))f(0x) + \exp(\phi(1\bar{x}))f(1\bar{x})$. As $0x$ and $0x'$ agree in places 1 to $m+1$

$$e^{\phi(0x)}f(0x) \leq e^{\phi(0x')+b\alpha^{m+1}}B_{m+1}f(0x') \leq B_m e^{\phi(0x')}f(0x') \tag{5}$$

this would be also true if $0x$ were replaced by $1\bar{x}$, so $\mathcal{L}f(x) \leq B_m\mathcal{L}f(x')$.

Consider any x, $z \in \Omega$, for $f \in \Lambda$

$$\mathcal{L}f(x) \geq e^{-\|\phi\|}f(y) \geq e^{-\|\phi\|}B_1^{-1}f(z)$$

where $y = 0x$ if $z_1 = 0$, and $y = 1\bar{x}$ if $z_1 = 1$.

Let $k = \lambda e^{-\|\phi\|}B_1$, then $1 = \nu(\lambda^{-1}\mathcal{L}f) \geq k^{-1}f(z)$ gives $\|f\| \leq k$ as z was arbitrary.

Since $\nu(f) = 1$, $f(z) \geq 1$ for some z and we get $\inf(\lambda^{-1}\mathcal{L}f) \geq k^{-1}$.

If $x_i = x_i'$ for $1 \leq i \leq m$, $f \in \Lambda$, one has

$$|f(x) - f(x')| \leq (B_m - B_m^{-1})k \to 0 \qquad \text{as } m \to \infty$$

Thus Λ is equicontinuous and compact by the Arzela-Ascoli theorem. Applying the Schander-Tychonoff theorem to $\lambda^{-1}\mathcal{L}$: $\Lambda \mapsto \Lambda$ gives us $h \in \Lambda$ with $\mathcal{L}h = \lambda h$. Furthermore $\inf h = \inf \lambda^{-1}\mathcal{L}f \geq k^{-1}$. □

Lemma 2 *$\exists\ \eta \in (0,1)$ such that $\forall f \in \Lambda$, $\lambda^{-1}\mathcal{L}f = \eta h + (1-\eta)f'$ with $f' \in \Lambda$.*

PROOF: Let $g = \lambda^{-1}\mathcal{L}f - \eta h$ where η is to be determined. To show $(1-\eta)^{-1}g \in \Lambda$, first we need $\eta\|h\| \leq k^{-1}$ to get $g \geq 0$. Assume $x_i = x_i'$ for $1 \leq i \leq m$, we want to pick η so that $g(x) \leq B_m g(x')$ or equivalently

$$\eta(B_m h(x') - h(x)) \leq B_m\lambda^{-1}\mathcal{L}f(x') - \lambda^{-1}\mathcal{L}f(x) \tag{6}$$

from (5), $f \in \Lambda$, we have $\lambda^{-1}\mathcal{L}f(x) \leq B_{m+1}e^{b\alpha^{m+1}}\lambda^{-1}\mathcal{L}f(x')$. Now $h(x) \geq B_m^{-1}h(x')$ since $h \in \Lambda$, to get (6) it is enough to get

$$\eta(B_m - B_m^{-1})h(x') \leq (B_m - B_{m+1}e^{b\alpha^{m+1}})\lambda^{-1}\mathcal{L}f(x')$$

or

$$\eta(B_m - B_m^{-1})\|h\| \leq (B_m - B_{m+1}e^{b\alpha^{m+1}})k^{-1} \tag{7}$$

Since $1 \leq B_m \leq \exp\{\frac{2b\alpha^2}{1-\alpha}\}$ for all m, there is an L so that the logarithms of B_m, B_m^{-1} and $B_{m+1}e^{b\alpha^{m+1}}$ are in $[-L, L]$ for all m. Let u_1 u_2 be positive constants such that $u_1(x-y) \leq e^x - e^y \leq u_2(x-y)$ for x, $y \in [-L, L]$ For (7) to hold it is enough for $\eta > 0$ to satisfy

$$\eta\|h\|u_2(\log B_m + \log B_m) \leq k^{-1}u_1(\log B_m - \log(B_{m+1}e^{b\alpha^{m+1}}))$$

or

$$\eta\|h\|u_2\frac{4b\alpha^{m+1}}{1-\alpha} \leq k^{-1}u_1(2b\alpha^{m+1} - b\alpha^{m+1})$$

or

$$\eta \leq \frac{u_1(1-\alpha)}{4u_2\|h\|k}$$

Lemma 3 *$\exists\, A > 0$ and $\beta \in (0,1)$ such that $\|\lambda^{-n}\mathcal{L}^n f - h\| \leq A\beta^n$ for $f \in \Lambda$, $n > 0$.*

PROOF: $\lambda^{-n}\mathcal{L}^n f = (1-(1-\eta)^n)h + (1-\eta)^n f_n'$, where $f_n' \in \Lambda$. Thus

$$\|\lambda^{-n}\mathcal{L}^n f - h\| = \|-(1-\eta)^n h + (1-\eta)^n f_n'\| \leq A\beta^n$$

where $A = 2k$ and $\beta = 1-\eta$. □

Lemma 4 *Let $\mathcal{C}_r = \{f \in C(\Omega) : Var_r f = 0\}$. If $F \in \Lambda$, $f \in \mathcal{C}_r$, $f \geq 0$ and $fF \not\equiv 0$, then $\nu(fF)^{-1}\lambda^{-r}\mathcal{L}^r(fF) \in \Lambda$.*

PROOF: First show that $\mathcal{L}^r(fF)(x) \leq B_m\mathcal{L}^r(fF)(x')$ if $x_i = x_i'$ for $1 \leq i \leq m$.

$$\mathcal{L}^r(fF)(x) = \sum_{j_1j_2\cdots j_r}\exp\{\sum_{\ell=0}^{r-1}\phi(\sigma^\ell(j_1j_2\cdots j_r x^*)\}f(j_1j_2\cdots j_r x^*)\}F(j_1j_2\cdots j_r x^*)$$

where $j_1j_2\cdots j_r$ is one of 2^r distinct r-digits long binary strings, and $(x^*)_i = j_r + (-1)^{j_r}x_i$.

Now $f(j_1j_2\cdots j_rx^*) = f(j_1j_2\cdots j_rx'^*)$ as $f \in \mathcal{C}_r$. $F(j_1j_2\cdots j_rx^*) \leq B_{m+r}F(j_1j_2\cdots j_rx'^*)$ as $F \in \Lambda$. $\phi(\sigma^\ell(j_1j_2\cdots j_rx^*)) \leq \phi(\sigma(j_1j_2\cdots j_rx'^*)) + Var_{m+r-\ell}\phi$ and $B_{m+r}\exp(\sum_{\ell=0}^{r-1} Var_{m+r-\ell}\phi) \leq B_{m+r}\exp(\sum_{\ell=m+1}^{m+r} b\alpha^\ell) \leq B_m$. Hence

$$\mathcal{L}^r(fF)(x) \leq B_m\mathcal{L}^r(fF)(x')$$

We still need $\nu(fF) > 0$. For the same reason as in lemma 1,

$$\lambda^r\nu(fF) = \nu(\lambda^{-1}\mathcal{L}(\mathcal{L}^r(fF))) \geq k^{-1}\mathcal{L}^r(fF)(z)$$

$fF(w) > 0$ (for some $w = w_1w_2\cdots w_r\underline{w}$) gives us $\mathcal{L}^r(fF)(z) > 0$ for $z = \underline{w}$ if $w_r = 0$ and $z = \bar{\underline{w}}$ if $w_r = 1$. So $\nu(fF) > 0$. □

Lemma 5 *For $F \in \Lambda$, $f \in \mathcal{C}_r$, $n \geq 0$*

$$\|\lambda^{-n-r}\mathcal{L}^{n+r}(fF) - \nu(fF)h\| \leq A\nu(|fF|)\beta^n$$

For $g \in C(\Omega)$ one has

$$\lim_{m\to\infty}\|\lambda^{-m}\mathcal{L}^m g - \nu(g)h\| = 0$$

PROOF: Write $f = f^+ - f^-$, we still have f^+, $f^- \in \mathcal{C}_r$. By lemma 4, $\nu(f^\pm F)^{-1}\lambda^{-r}\mathcal{L}^r(f^\pm F) \in \Lambda$, then by lemma 3,

$$\|\lambda^{-n-r}\mathcal{L}^{n+r}(f^\pm F) - \nu(f^\pm F)h\| \leq A\nu(f^\pm F)\beta^n$$

therefore

$$\begin{aligned}
&\|\lambda^{-n-r}\mathcal{L}^{n+r}(fF) - \nu(fF)h\| \\
\leq\ &\|\lambda^{-n-r}\mathcal{L}^{n+r}(f^+F) - \nu(f^+F)h\| + \|\lambda^{-n-r}\mathcal{L}^{n+r}(f^-F) - \nu(f^-F)h\| \\
\leq\ &A\nu(f^+F)\beta^n + A\nu(f^+F)\beta^n \leq A\nu(|fF|)\beta^n
\end{aligned}$$

Given $g \in C(\Omega)$ and $\epsilon > 0$, one can find r and f_1, $f_2 \in \mathcal{C}_r$ so that $0 \leq f_2 - f_1 \leq \epsilon$, $f_1 \leq g \leq f_2$, and $|\nu(f_i) - \nu(g)| \leq \epsilon$. Take $F = 1$,

$$\begin{aligned}
\|\lambda^{-m}\mathcal{L}^m f_i - \nu(g)h\| &\leq \|\lambda^{-m}\mathcal{L}^m f_i - \nu(f_i)h\| + |\nu(f_i) - \nu(g)|\|h\| \\
&\leq A\nu(|f_i|)\beta^{m-r} + \epsilon\|h\| \\
&\leq \epsilon(1 + \|h\|)
\end{aligned}$$

for large m. Since $\|\lambda^{-m}\mathcal{L}^m f_1\| \leq \|\lambda^{-m}\mathcal{L}^m g\| \leq \|\lambda^{-m}\mathcal{L}^m f_2\|$,

$$\|\lambda^{-m}\mathcal{L}^m g - \nu(g)h\| \leq \epsilon(1 + \|h\|)$$

for large m. □

REMARK: It is pointed out by Professor Steven Lalley that there is a map on Ω such that theorem one is a special case of Ruelle's Perron-Froebenius theorem in [2].

ACKNOWLDGEMENT: Special thanks to Professor Steven Lalley for introducing to me Ruelle's theorem and for warm encouragement.

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