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ABSTRACT

A new criterion for sequential testing, the conditional generalized probability likelihood ratio sequential (GCPLRS), is introduced for the closed sequential tests of composite hypotheses. Compared with fixed sampling size test rule, equally powerful sequential test rule improves efficiency by reducing expected sampling size, but its maximum sampling size is usually substantially greater than the size of fixed sampling size test. For instance, the maximum sampling size for Wald's PLRS test rule is infinity. An important feature of this new criterion is that, the maximum sampling size of sequential test rule applying this criterion is about the same as the size of equally powerful fixed sampling size test rule, while its efficiency in term of expected sampling size is close to the efficiency of Wald's PLRS test rule.

For practical purpose, the GCPLRS criterion is modified according to each family, for families of parametric distributions. In this paper, modified versions of GCPLRS criterion for several parametric families are given. GCPLRS procedures for various tests (one-sided test, two-sided test, group sequential sampling test) and estimation are scrutinized. In the case of sampling from dichotomous distribution, an approximation formula for absorption probabilities of random bridge on GCPLRS boundaries is given for easy computation of deflection factors which is critical for design of GCPLRS procedures.

Key Words and Phrases: generalized conditional probability likelihood ratio; GCPLRS; PLRS; sequential test; sequential estimation.

AMS 1991 subject classification: Primary 62L10; Secondary 62F04, 62L20.

1 Introduction

Sequential analysis, as a method of statistical inference motivated by the urgent need of improving efficiency of industrial inspections in the World War II, was introduced in early 40 this century. Early theoretic work in this field was laid down as Wald's probability likelihood ratio sequential (PLRS) test (see Wald(1947)) and various versions of its modifications.

It is well known that Wald's PLRS test has an open end which causes very large variance of expected sampling size, thus is unpractical for many testing practices in which maximum sampling size is an important factor for design of tests. Various modifications of Wald's PLRS had been made to make up this deficiency by authors such as Bross(1952), Armitage(1957), Spicer(1962), Alling(1966), Choi(1968), Breslow(1970) and *etc.*.

In all those modifications, the continuous region for Wald's PLRS which is usually bounded by two parallel lines with an open end, was truncated and reshaped by some straight lines or curves. "As Armitage(1957) remarked, once the classical open type of scheme is abandoned the range of possible tests becomes embarrassingly wide"(see Spicer(1962)). In all those modifications, maximum sampling size is still substantially greater than the size of equally powerful fixed sampling size test rule. In application, maximum sampling size for sequential test is often an important constraint for test design, *e.g.* the availability of experimental material or limit of time for experiment. A question arises that, "How to design a rule which is efficient in terms of expected sampling size as well as in terms of maximum sampling size". Obviously, any test rule having maximum sampling size less than m should be less powerful than the fixed sampling size test of size m.

To answer above question, this paper introduces a new criterion for sequential tests of composite hypotheses. This criterion is based on a new concept which is different from the concept Wald's PLRS test is based on. This criterion is promising for its efficiency in terms of maximum sampling size as well as expected sampling size. The sequential test derived with this criterion has maximum sampling size which is same as the size of equally powerful fixed sampling size test rule. The power function of this test is about same as the power function of the fixed sampling size test. For the purpose of easy implementation of test, this criterion is modified according to each family of parametric distributions for different families. Modifications make implementation much easier while keeping the merits of the original criterion.

2 GCPLRS: A New Criterion for Sequential Test

We introduce a new criterion, GCPLRS, for closed sequential tests of composite hypotheses for (one dimension in this paper) parametric distributions. The general idea of GCPLRS test procedure is that, first design a fixed sampling size test of size m, then change it into a closed sequential test (which has maximum sampling size m) to improve efficiency (in term of reducing expected sampling size) while keeping the power function (almost) unchanged.

2.1 An Introductive Example

Let $X_i = 1, 2, \cdots$ i.i.d $P_p(X_i = 1) = P_p(X_i = 0) = 1 - p$. To test hypotheses $H_0 : p \leq \frac{1}{2}$ $H_1 : p > \frac{1}{2}$, we first consider fixed sampling size test. Suppose it is required that significance level $\alpha \leq 0.05$, and testing power $\beta \geq 0.90$ to reject any $p \geq \frac{3}{4}$. The appropriate sampling size and critical value are $m = 38$ and $s_0 = 25$. Denote this test rule as δ_0 and let $S_n = \sum_{i=1}^{n} X_i$. The power function $\beta_{\delta_0}(p) = P_p(S_{38} \geq 25)$ is increasing in p, thus

$$\begin{aligned} \max_{p \leq \frac{1}{2}} \beta_{\delta_0}(p) &= \beta_{\delta_0}\left(\frac{1}{2}\right) = 0.0365 \leq 0.05, \\ \min_{p \geq \frac{3}{4}} \beta_{\delta_0}(p) &= \beta_{\delta_0}\left(\frac{3}{4}\right) = 0.929 \geq 0.90. \end{aligned}$$

Any test procedure, either fixed sampling size or sequential, may be interpreted as a random path hits (or passes over) some barrier sets. Let $\tilde{S} = ((1, S_1), (2, S_2), \cdots)$ be a *random path* from dichotomy population, where $S_n = \sum_{i=1}^{n} X_i$. We may imagine that sampling is ever going, thus *random path* $\tilde{S}$ is observable, and we make decision for hypotheses test after early part of $\tilde{S}$ is seen. A rule for testing above hypotheses should be:

$$\begin{aligned} &\text{accept } H_0 \text{ if } \tilde{S} \text{ hits } \mathcal{B}^- \text{ before hits } \mathcal{B}^+; \\ &\text{accept } H_1 \text{ if } \tilde{S} \text{ hits } \mathcal{B}^+ \text{ before hits } \mathcal{B}^-; \end{aligned}$$

where $\mathcal{B}^-$ and $\mathcal{B}^+$ are some two barrier sets. For example, the fixed sampling size test rule δ_0 above has two barrier sets

$$\begin{aligned} \mathcal{B}_0^- &= \{(38, j) : \ j = 0, \cdots, 24\}, \\ \mathcal{B}_0^+ &= \{(38, j) : \ j = 25, \cdots, 38\}. \end{aligned}$$

Next we consider a sequential test rule δ_1 for same hypotheses. Suppose barrier sets for

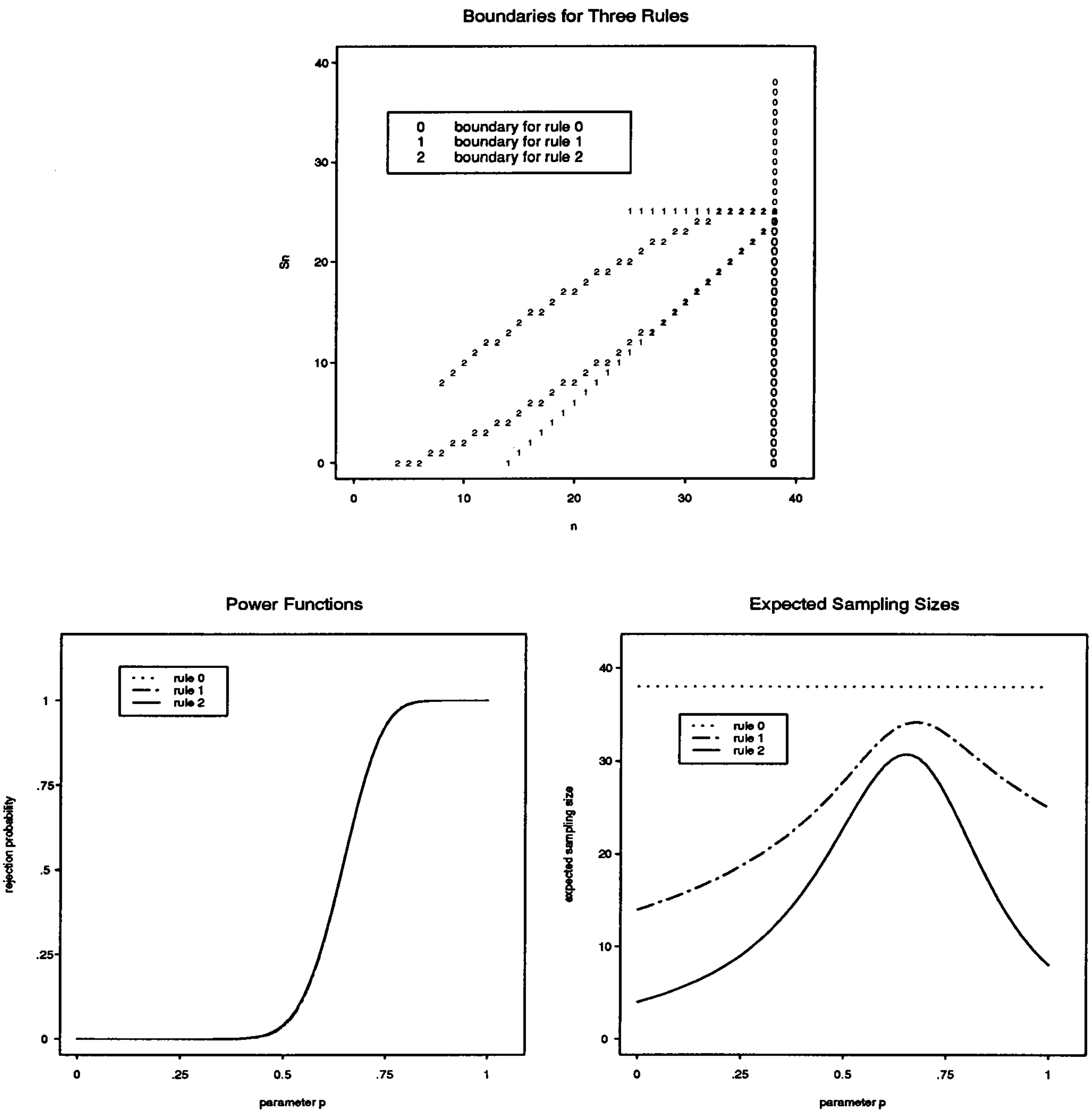

Figure 1: **Comparison of Three Test Rules**

δ_1 are

$$\mathcal{B}_1^- = \{(i,j): \ j = i - 14 \ \text{ for } \ 14 \le i \le 38\},$$
$$\mathcal{B}_1^+ = \{(i,j): \ j = 30 \ \text{ for } \ 25 \le i \le 38\}.$$

Clearly as shown in Fig 1, once $\tilde{S}$ goes across $j = 25$ at step n ($< m$), it can not go down to hit $\mathcal{B}_1^-$; once $\tilde{S}$ goes right across $j = i - 14$ for $i \le m$, it can not go up to hit $\mathcal{B}_1^+$ at step m. Therefore $\tilde{S}$ hits $\mathcal{B}_0^-$ before hits $\mathcal{B}_0^+$ iff $\tilde{S}$ hits $\mathcal{B}_1^-$ before hits $\mathcal{B}_1^+$. The power functions for rules δ_0 and δ_1 are *exactly same* because

$$\begin{aligned}
\beta_{\delta_0}(p) &= P_p(S_{38} \ge 25) \\
&= P_p(\tilde{S} \text{ hits } \mathcal{B}_0^+ \text{ before hits } \mathcal{B}_0^-) \\
&= P_p(\tilde{S} \text{ hits } \mathcal{B}_1^+ \text{ before hits } \mathcal{B}_1^-) \\
&= \beta_{\delta_1}(p). \qquad (1)
\end{aligned}$$

Let T_0 and T_1 be sampling sizes needed for making decision in δ_0 and δ_1 respectively. It is clear that $E_pT_1 < E_pT_0$ ($T_0 \equiv m$). Hence rule δ_1 is better (more efficient) than rule δ_0.

We consider another sequential test rule δ_2. Suppose $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$, as shown in Figure 1, are barrier sets for δ_2. $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$ are constructed such that the probability that $\tilde{S}$ hits $\mathcal{B}_2^-$ is very small given $S_m \ge s_0$; the probability that $\tilde{S}$ hits $\mathcal{B}_2^+$ is very small given $S_m < s_0$. Hence power functions for rule δ_2 and rule δ_0 are about same because

$$\begin{aligned}
\beta_{\delta_0}(p) &= P_p(\tilde{S} \text{ hits } \mathcal{B}_0^+ \text{ before hits } \mathcal{B}_0^-) \\
&\approx P_p(\tilde{S} \text{ hits } \mathcal{B}_2^+ \text{ before hits } \mathcal{B}_2^-) \\
&= \beta_{\delta_2}(p). \qquad (2)
\end{aligned}$$

Let T_2 be sampling size needed for making decision in δ_2, clearly $E_pT_2 < E_pT_1$. Hence δ_2 is better (more efficient) than δ_1 if the tiny difference between $\beta_{\delta_2}(p)$ and $\beta_{\delta_1}(p)$ are ignored. Comparison of rules δ_0, δ_1, δ_2 illustrated in Fig 1 clearly indicates δ_2 is superior to δ_0 and δ_1.

2.2 Criterion of GCPLRS

In example in Section 2.1, the rules δ_1, δ_2 are in fact GCPLRS test procedures (with different deflection factors) for one-sided hypotheses $H_0 : \theta \le \theta_0$ *v.s.* $H_1 : \theta > \theta_0$. In general, a GCPLRS test procedure is acquired in two steps as follow.

First Step: To design a fixed sampling size test (denoted as δ_0). For any test, experimenter has one's own requirement for sensitivity of test (in terms of significance level for null hypotheses, rejection power for alternative), under the restrictions of time and costs for experiment (in terms of sampling size, expected sampling size or maximum sampling size). Suppose after the overall consideration of these factors, experimenter has decided the sampling size m and critical value s_0 for the fixed sampling size test. This rule δ_0 is:

$$\text{accept } H_0 \text{ if } S_m < s_0;$$
$$\text{accept } H_1 \text{ if } S_m \geq s_0;$$

where $S_n = S_n(X_1, \cdots, X_n)$ is *sufficient* for θ. Without loss of generality, assuming for any s and n, $P_\theta(S_n \geq s)$ is nonincreasing in θ.

Second Step: To design a sequential test rule (denoted as δ) which satisfies two requirements:

1. $\beta_\delta(\theta) \approx \beta_{\delta_0}(\theta)$;
2. $E_\theta T_\delta$ is smaller than m as much as possible;

where $\beta_{\delta_0}(\theta)$ and $\beta_\delta(\theta)$ are power functions for rules δ_0 and δ, and T_δ is sampling size for δ. Assume $\mathcal{B}^-$ and $\mathcal{B}^+$ are barrier sets for *random path* $\tilde{S}$. $\mathcal{B}^-$ and $\mathcal{B}^+$ for δ are obtained by criterion of GCPLRS defined as below.

Definition 2.1 *Assume S_n is a sufficient statistics for θ and $E_\theta S_n = cn\theta$ for any n, where c is some known positive constant (c is called incremental scale of $\{S_n, n = 1, 2, \cdots\}$; in usual case $c = 1$). The* Generalized Conditional Probability Likelihood Ratio *(GCPLR) for $S_n = s$, given $S_m = \xi cm$, denoted by $L(n, s; m, \xi)$, is defined as*

$$L(n, s; m, \xi) = \frac{\max_{\eta > \xi} P^\eta(S_n = s)}{\max_{\eta \leq \xi} P^\eta(S_n = s)} \tag{3}$$

where $P^\eta(S_n = s) = P(S_n = s | S_m = \eta cm)$ and $n = 1, 2, \cdots, m$.

$P^\eta(S_n = s)$, the conditional probability of $S_n = s$ given $S_m = \eta cm$, doesn't depend on θ since S_n is sufficient for θ. Now we regard η as a new parameter, and $\{P^\eta(\cdot),\ \eta \in \Delta\}$ are probability measures for *a family of random bridges*

$$\left\{\tilde{S}_m = (S_1, \cdots, S_m) : S_m = \eta cm,\ \eta \in \Delta\right\} \tag{4}$$

where Δ is an appropriate set (*e.g.* $\Delta = (-\infty, \infty)$, or $\Delta = \{\text{all integers}\}$). Assume $g(\eta) = P^\eta(S_n = s)$ is increasing for $\eta \leq \frac{s}{nc}$ and decreasing for $\eta \geq \frac{s}{nc}$ (this is true for many distributions). Then we have

$$\max_{\eta>\xi} P^{\eta}(S_n = s) = \begin{cases} P^{\frac{s}{nc}}(S_n = s) & \frac{s}{nc} > \xi, \\ P^{\xi}(S_n = s) & \frac{s}{nc} \le \xi; \end{cases} \tag{5}$$

$$\max_{\eta<\xi} P^{\eta}(S_n = s) = \begin{cases} P^{\xi}(S_n = s) & \frac{s}{nc} > \xi, \\ P^{\frac{s}{nc}}(S_n = s) & \frac{s}{nc} \le \xi. \end{cases} \tag{6}$$

Hence

$$L(n, s; m, \xi) = \begin{cases} \frac{P^{\frac{s}{nc}}(S_n=s)}{P^{\xi}(S_n=s)} & \frac{s}{nc} > \xi, \\ \frac{P^{\xi}(S_n=s)}{P^{\frac{s}{nc}}(S_n=s)} & \frac{s}{nc} \le \xi. \end{cases} \tag{7}$$

In (5), (6) and (7), notation $P^{\eta}(S_n = s)$ is abused to denote density if S_n is continuously distributed.

Definition 2.2 *The GCPLRS Test rule δ for testing $H_0 : \theta \le \theta_0$ v.s. $H_1 : \theta > \theta_0$ is*

accept H_0 if $\tilde{S}$ hits $\mathcal{B}_a^+$ before hits $\mathcal{B}_b^-$;
accept H_1 if $\tilde{S}$ hits $\mathcal{B}_b^-$ before hits $\mathcal{B}_a^+$;

where

$$\mathcal{B}_a^+ = \{(n, s_n) : s_n = \inf\{s : \log L(n, s; m, \xi) \ge a\}\}, \tag{8}$$

$$\mathcal{B}_b^- = \{(n, s_n) : s_n = \sup\{s : \log L(n, s; m, \xi) \le -b\}\}. \tag{9}$$

$L(n, s; m, \xi)$ is GCPLR defined in (3) and $a, b > 0$.

Each GCPLRS test rule δ is decided by operating parameters m, ξ, a and b. m is the maximum sampling size for δ, is same as the fixed size for δ_0. ξ is the slope of center line for GCPLRS boundary, $\xi = \frac{s_0}{mc}$ in which s_0 is critical value for δ_0. It is much harder to choose a and b. If a and b are both large (enough), then power function of δ is same as the one of fixed sampling size test rule δ_0, but the expected sampling size is not reduced much from m. Smaller are a and b, smaller is the expected sampling size for δ. But if a and b are too small, the power function of δ could be changed too much from that of δ_0. For GCPLRS test, a and b should be as small as possible, but keep the power function of GCPLRS test about same as the power function of corresponding fixed sampling size test.

3 Modifications of GCPLRS according to Distributions

Experimenter implementing GCPLRS test rule might find out that formula (8), (9) generating barrier sets $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ are not practical because they are not easily computable thus it is hard to choose proper a and b. To avoid this difficulty, we modify GCPLRS, according to each family, for families of parametric distributions. We will still call each modified version the GCPLRS.

Definition 3.1 *Assume that $L(n,s;m,\xi)$ is GCPLR for parametric family $\{f_\theta, \theta \in \Theta\}$ as defined in (3). Let $l(n,s;m,\xi) = \frac{1}{m}\log L(n,s;m,\xi)$ (or $= \frac{1}{mc}\log L(n,s;m,\xi)$, it depends on distribution). We call $G(u,v;\xi)$ the "ratio function of $\{f_\theta, \theta \in \Theta\}$" if*

$$l(n,s;m,\xi) \approx \begin{cases} G\left(\frac{n}{m}, \frac{s}{cm}; \xi\right) & \frac{s}{nc} > \xi, \\ -G\left(\frac{n}{m}, \frac{s}{cm}; \xi\right) & \frac{s}{nc} \leq \xi, \end{cases} \tag{10}$$

where ξ is a parameter, u and v are two variables for $G(u,v;\xi)$, c is a known incremental scale for $\{f_\theta, \theta \in \Theta\}$.

We will see later, ratio function $G(u,v;\xi)$ is defined on $0 \leq u \leq 1$, $-\infty \leq v \leq \infty$. If $h(v) \equiv G(u,v;\xi)$ is monotone increasing for $v \leq u\xi$, monotone decreasing for $v \geq u\xi$ (this is usually true), then $\min_v G(u,v;\xi) = G(u,\xi u;\xi)$.

Definition 3.2 *Assume $G(u,v;\xi)$ is the ratio function of $\{f_\theta, \theta \in \Theta\}$. Then $\varphi_a^+(u)$ is the "upper boundary function of $\{f_\theta, \theta \in \Theta\}$" for GCPLRS if it is an unique solution satisfying*

$$\begin{cases} G(u, \varphi_a^+(u); \xi) = a, \\ \varphi_a^+(u) \geq \xi u; \end{cases} \tag{11}$$

$\varphi_a^-(u)$ is the "lower boundary function of $\{f_\theta, \theta \in \Theta\}$" for GCPLRS if it is an unique solution satisfying

$$\begin{cases} G(u, \varphi_b^-(u); \xi) = b, \\ \varphi_b^-(u) \leq \xi u; \end{cases} \tag{12}$$

where $0 < a, b < \tau$, τ is some positive constant or infinity.

In above definition, we have abused notation in (11) and (12) such that we denote $G(u,v_0;\xi) = c$ if $G(u,v;\xi) \leq c$ (or $\geq c$) for $v_0 - \epsilon < v < v_0$ and $G(u,v;\xi) \geq c$ (or $\leq c$) for $v_0 < v < v_0 + \epsilon$.

Definition 3.3 *The upper boundary for GCPLRS, denoted by* $\mathcal{B}_a^+$, *is defined by*

$$\mathcal{B}_a^+ = \{(n, a_n)\}_{n=1}^m \quad \text{where} \quad a_n = mc\varphi_a^+\left(\frac{n}{m}\right). \tag{13}$$

The lower boundary for GCPLRS, denoted by $\mathcal{B}_b^-$, *is defined by*

$$\mathcal{B}_b^- = \{(n, b_n)\}_{n=1}^m \quad \text{where} \quad b_n = mc\varphi_b^-\left(\frac{n}{m}\right). \tag{14}$$

$\mathcal{B}_a^+ \cup \mathcal{B}_b^-$ is closed for random path $\tilde{S}$ in the sense that $\tilde{S}$ passes over $\mathcal{B}_a^+ \cup \mathcal{B}_b^-$ with probability one. For testing $H_0 : \theta \leq \theta_0$ *v.s.* $H_1 : \theta > \theta_0$, the GCPLRS test rule δ is:

$$\text{accept } H_0 \text{ if } \tilde{S} \text{ passes over before } \mathcal{B}_a^+ \text{ passes over } \mathcal{B}_b^-; \tag{15}$$

$$\text{accept } H_1 \text{ if } \tilde{S} \text{ passes over before } \mathcal{B}_b^- \text{ passes over } \mathcal{B}_a^+. \tag{16}$$

We will derive modification of GCPLRS, according to each family, for several families of parametric distributions. The aim for each individual modification is to obtain ratio function $G(u, v; \xi)$, and to show existence of boundary functions φ_a^+ and φ_b^-. For the families of binomial distribution and hypergeometric distribution, there is detailed discussion in Xiong(1990) about steps to derive $G(u, v; \xi)$, properties of $G(u, v; \xi)$, existence and properties of φ_a^+ and φ_b^-. For other families which will be discussed later in this paper, steps to derive and to investigate $G(u, v; \xi)$ are about same thus omitted, only the results will be given.

3.1 For Binomial and Hypergeometric Distributions

To derive modification of GCPLRS for binomial distribution and hypergeometric distribution, first we consider binomial distribution. Let $\{f_p,\ 0 < p < 1\}$ be a family of mass functions of binomial distribution $B(k, p)$, where k is fixed and known. We will derive *ratio function* $G(u, v; \xi)$ for this family.

Let $X_i \sim B(k, p)$ i.i.d. for $i = 1, 2, \cdots$. Suppose k is known and $S_n = \sum_{i=1}^n X_i$. Then $S_n \sim B(nk, p)$; for $n \leq m$, $0 \leq s \leq \eta mk$

$$\begin{aligned} P^\eta(S_n = s) &= P(S_n = s | S_m = mk\eta) \\ &= \frac{\binom{nk}{s}\binom{(m-n)k}{mk\eta - s}}{\binom{mk}{mk\eta}}. \end{aligned} \tag{17}$$

where parameter $\eta \in \{\frac{1}{mk}, \cdots, \frac{mk-1}{mk}, 1\}$. Thus the GCPLR is, for $0 < \xi < 1$

$$
\begin{aligned}
L(n, s; m, \xi) &= \frac{\max_{\eta > \xi} \frac{\binom{(m-n)k}{mk\eta - s}}{\binom{mk}{mk\eta}}}{\max_{\eta \le \xi} \frac{\binom{(m-n)k}{mk\eta - s}}{\binom{mk}{mk\eta}}} \\
&= \frac{\max_{\eta > \xi} \frac{(mk\eta)!(mk-mk\eta)!}{(mk\eta-s)!(mk(1-\eta)-nk+s)!}}{\max_{\eta \le \xi} \frac{(mk\eta)!(mk-mk\eta)!}{(mk\eta-s)!(mk(1-\eta)-nk+s)!}}. \qquad (18)
\end{aligned}
$$

Let $l(n, s; m, k, \xi) = \frac{1}{mk} \log L(n, s; m, k, \xi)$, then

$$
l(n, s; m, k, \xi) \approx \begin{cases} G(\frac{n}{m}, \frac{s}{mk}; \xi) & \frac{s}{nk} > \xi, \\ -G(\frac{n}{m}, \frac{s}{mk}; \xi) & \frac{s}{nk} \le \xi, \end{cases} \qquad (19)
$$

where $0 < \xi < 1$ and for $0 < u < 1$, $\max\{0, u - 1 + \xi\} < v < \min\{1 - \xi, u\}$, let

$$
\begin{aligned}
G(u, v; \xi) &= v \log \frac{v}{u\xi} + (\xi - v) \log \frac{\xi - v}{(1-u)\xi} + (u - v) \log \frac{u - v}{(1-\xi)u} + \\
&\quad + (1 - \xi - u + v) \log \frac{1 - \xi - u + v}{(1-\xi)(1-u)}, \qquad (20)
\end{aligned}
$$

for $0 < u < 1$, $v < \max\{0, u - 1 + \xi\}$ or $v > \min\{1 - \xi, u\}$, let

$$
G(u, v; \xi) = \xi \log \frac{1}{\xi} + (1 - \xi) \log \frac{1}{1 - \xi}. \qquad (21)
$$

Since $\frac{\partial G}{\partial v} = \log \frac{(1-u)v}{u(\xi - v)}$ and $\frac{\partial^2 G}{\partial v^2} = \frac{\xi}{(\xi - v)v} > 0$, so $G(u, v; \xi)$ is convex in v, $\min_v G(u, v; \xi) = G(u, \xi u; \xi) = 0$. Thus for $0 < a, b < \xi \log \frac{1}{\xi} + (1-\xi) \log \frac{1}{(1-\xi)}$, the upper boundary function $\varphi_a^+(\cdot)$ exists such that $G(u, \varphi_a^+(u); \xi) = a$, $\varphi_a^+(u) \ge \xi u$; the lower boundary function $\varphi_b^-(\cdot)$ exists such that $G(u, \varphi_b^-(u); \xi) = b$, $\varphi_b^+(u) \le \xi u$. GCPLRS boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ are given by (13) and (14) in which $c = k$.

Now we consider Hypergeometric distribution. Assume sampling *without replacement* from a dichotomous population of size N, proportion of "defects" p, each time take k samples. Let X_i be the number of defects in the elements sampled at ith time, and $S_n = \sum_{i=1}^n X_i$. Then S_n has hypergeometric distribution $\mathcal{H}(nk; pN, N)$; and conditioned on S_{n-1}, X_n has hypergeometric distribution $\mathcal{H}(nk; pN - S_{n-1}, N - n + 1)$, for $n = 1, \cdots, [\frac{N}{k}]$. Suppose m is a positive integer smaller or equal than $\frac{N}{k}$. Then for $n \le m$, $0 \le s < \eta mk$

$$
P^\eta(S_n = s) = P(S_n = s | S_m = mk\eta)
$$

$$= \frac{\binom{nk}{s}\binom{(m-n)k}{mk\eta - s}}{\binom{mk}{mk\eta}}. \tag{22}$$

where parameter $\eta \in \{\frac{1}{mk}, \cdots, \frac{mk-1}{mk}, 1\}$. We may see that conditional probability in (22) is same as the one in (17) for binomial distribution. Thus the GCPLR, the ratio function $G(u, v; \xi)$, the GCPLRS boundary for hypergeometric distribution are exactly same as those for binomial distribution because all of those are derived from the same $P^\eta(S_n = s)$.

3.2 For Normal Distribution

Suppose $\{f_\mu, \ -\infty < \mu < \infty\}$ is a family of normal density functions $N(\mu, \sigma^2)$. Assume σ^2 is fixed and known. We will derive modification of GCPLRS according to this distribution.

Let $X_i \sim N(\mu, \sigma^2)$ i.i.d. for $i = 1, 2, \cdots$. Suppose σ^2 is known and $S_n = \sum_{i=1}^n X_i$. Then $S_n \sim N(n\mu, n\sigma^2)$; for $n \leq m$

$$\begin{aligned} P^\eta(S_n = s) &= P(S_n = s | S_m = \eta m \sigma) \\ &= \frac{1}{\sqrt{2\pi n(1 - \frac{n}{m})}\sigma} e^{-\frac{(s - \eta n \sigma)^2}{2n(1-\frac{n}{m})\sigma^2}}. \end{aligned} \tag{23}$$

Thus the GCPLR is

$$L(n, s; m, \xi) = \begin{cases} e^{\frac{(s-\xi n\sigma)^2}{2n(1-\frac{n}{m})\sigma^2}} & \frac{s}{n\sigma} > \xi \\ e^{-\frac{(s-\xi n\sigma)^2}{2n(1-\frac{n}{m})\sigma^2}} & \frac{s}{n\sigma} \leq \xi \end{cases}. \tag{24}$$

Let $l(n, s; m, \xi) = \frac{1}{m} \log L(n, s; m, \xi)$,

$$\begin{aligned} l(n, s; m, \xi) &= \begin{cases} \frac{(\frac{s}{m\sigma} - \frac{n}{m}\xi)^2}{2\frac{n}{m}(1-\frac{n}{m})} & \frac{s}{n\sigma} > \xi, \\ -\frac{(\frac{s}{m\sigma} - \frac{n}{m}\xi)^2}{2\frac{n}{m}(1-\frac{n}{m})} & \frac{s}{n\sigma} \leq \xi; \end{cases} \\ &= \begin{cases} G(\frac{n}{m}, \frac{s}{m\sigma}; \xi) & \frac{s}{n\sigma} > \xi, \\ -G(\frac{n}{m}, \frac{s}{m\sigma}; \xi) & \frac{s}{n\sigma} \leq \xi; \end{cases} \end{aligned} \tag{25}$$

where $-\infty < \xi < \infty$; for $0 < u < 1$ and $-\infty < v < \infty$

$$G(u, v; \xi) = \frac{(v - u\xi)^2}{2u(1-u)}. \tag{26}$$

Since $\frac{\partial G}{\partial v} = \frac{v-\xi u}{2u(1-u)}$ and $\frac{\partial^2 G}{\partial v^2} = \frac{1}{2u(1-u)} > 0$, so $G(u,v;\xi)$ is convex in v, $\min_v G(u,v;\xi) = G(u,\xi u;\xi) = 0$. Thus for $0 < a, b < \infty$, by (11), (12) and (26), boundary functions are

$$\varphi_a^+(u) = \xi u + \sqrt{2au(1-u)}, \tag{27}$$

$$\varphi_b^-(u) = \xi u - \sqrt{2bu(1-u)}, \tag{28}$$

where $0 < u < 1$. GCPLRS boundaries are given by (13), (14) in which $c = \sigma$. Thus we have $\mathcal{B}_a^+ = \left\{\left(n, \xi n\sigma + \sigma\sqrt{2bn(m-n)}\right)\right\}_{n=1}^{m}$ and $\mathcal{B}_b^- = \left\{\left(n, \xi n\sigma - \sigma\sqrt{2bn(m-n)}\right)\right\}_{n=1}^{m}$.

3.3 For Poisson Distributions

Let $\{f_\lambda, 0<\lambda<\infty\}$ be a family of density functions of poisson distribution $\mathcal{P}(\lambda)$. We will derive $G(u,v;\xi)$ for this parametric family.

Let $X_i \sim \mathcal{P}(\lambda)$ i.i.d. for $i = 1, 2, \cdots$ and $S_n = \sum_{i=1}^n X_i$. Then $S_n \sim \mathcal{P}(n\lambda)$. For $n \leq m$, $0 \leq s \leq \eta m$

$$\begin{aligned} P^\eta(S_n = s) &= P(S_n = s | S_m = \eta m) \\ &= \binom{\eta m}{s}\left(\frac{n}{m}\right)^s\left(1-\frac{n}{m}\right)^{\eta m - s}. \end{aligned} \tag{29}$$

Thus the GCPLR is

$$L(n,s;m,\xi) = \frac{\max_{\eta>\xi}\binom{\eta m}{s}(1-\frac{n}{m})^{\eta m-s}}{\max_{\eta\leq\xi}\binom{\eta m}{s}(1-\frac{n}{m})^{\eta m-s}}. \tag{30}$$

Let $l(n,s;m,\xi) = \frac{1}{m}\log L(n,s;m,\xi)$, then

$$\begin{aligned} l(n,s;m,\xi) &\approx \begin{cases} \frac{s}{m}\log\frac{\frac{s}{m}}{\frac{n}{m}\xi} + (\xi - \frac{s}{m})\log\frac{\xi - \frac{s}{m}}{(1-\frac{n}{m})\xi} & \frac{s}{n} > \xi, \\ -\left\{\frac{s}{m}\log\frac{\frac{s}{m}}{\frac{n}{m}\xi} + (\xi - \frac{s}{m})\log\frac{\xi - \frac{s}{m}}{(1-\frac{n}{m})\xi}\right\} & \frac{s}{n} \leq \xi; \end{cases} \\ &= \begin{cases} G(\frac{n}{m}, \frac{s}{m}; \xi) & \frac{s}{n} > \xi, \\ -G(\frac{n}{m}, \frac{s}{m}; \xi) & \frac{s}{n} \leq \xi; \end{cases} \end{aligned} \tag{31}$$

where $0 < \xi < \infty$, for $0 < u < 1$, $0 < v < \xi$

$$G(u,v;\xi) = v\log\frac{v}{u\xi} + (\xi - v)\log\frac{\xi - v}{(1-u)\xi}; \tag{32}$$

for $0 < u < 1$, $v \notin [0, \xi]$ let

$$G(u, v; \xi) \;=\; \infty. \tag{33}$$

Since $\frac{\partial G}{\partial v} = \log \frac{(1-u)v}{u(\xi - v)}$ and $\frac{\partial^2 G}{\partial v^2} = \frac{\xi}{(\xi - v)v} > 0$, so $G(u, v; \xi)$ is convex in v, $\min_v G(u, v; \xi) = G(u, \xi u; \xi) = 0$. Thus for $0 < a, b < \infty$, the upper and lower boundary functions $\varphi_a^+(\cdot)$ and $\varphi_b^-(\cdot)$ are implicit functions given by (11) and (12). GCPLRS boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ are given by (13) and (14) in which $c = 1$.

3.4 For Exponential Distribution and Gamma Distribution

We derive modification of GCPLRS for Exponential distribution first, then generalize it to Gamma distribution. Suppose $\{f_\lambda, 0 < \lambda < \infty\}$ is a family of density functions of exponential distribution $\mathcal{E}(\lambda)$.

Let $X_i \sim \mathcal{E}(\lambda)$ i.i.d. for $i = 1, 2, \cdots$ and $S_n = \sum_{i=1}^n X_i$. Then $S_n \sim \Gamma(n, \lambda)$. For $n \leq m$, $0 < s < \eta(m-1)$

$$\begin{aligned} P^\eta(S_n = s) &= P(S_n = s | S_m = \eta(m-1)) \\ &= \frac{\Gamma(m)}{\Gamma(n)\Gamma(m-n)} \frac{s^{n-1}((m-1)\eta - s)^{m-n-1}}{((m-1)\eta)^{m-1}}. \end{aligned} \tag{34}$$

Thus the GCPLR is

$$L(n, s; m, \xi) = \frac{\max_{\eta > \xi} \frac{(\eta - \frac{s}{m-1})^{m-n-1}}{\eta^{m-1}}}{\max_{\eta \leq \xi} \frac{(\eta - \frac{s}{m-1})^{m-n-1}}{\eta^{m-1}}}. \tag{35}$$

Let $l(n, s; m, \xi) = \frac{1}{m-1} \log L(n, s; m, \xi)$, then

$$\begin{aligned} l(n, s; m, \xi) &= \begin{cases} \log \xi + \frac{n}{m-1} \log \frac{\frac{n}{m-1}}{\frac{s}{m-1}} + (1 - \frac{n}{m-1}) \log \frac{1 - \frac{n}{m-1}}{\xi - \frac{s}{m-1}} & \text{if } \frac{s}{n} > \xi, \\ -\left\{ \log \xi + \frac{n}{m-1} \log \frac{\frac{n}{m-1}}{\frac{s}{m-1}} + (1 - \frac{n}{m-1}) \log \frac{1 - \frac{n}{m-1}}{\xi - \frac{s}{m-1}} \right\} & \text{if } \frac{s}{n} \leq \xi; \end{cases} \\ &= \begin{cases} G(\frac{n}{m-1}, \frac{s}{m-1}; \xi) & \text{if } \frac{s}{n} > \xi, \\ -G(\frac{n}{m-1}, \frac{s}{m-1}; \xi) & \text{if } \frac{s}{n} \leq \xi; \end{cases} \end{aligned} \tag{36}$$

where $0 < \xi < \infty$, for $0 < u < 1$, $0 < v < \xi$

$$G(u, v; \xi) = u \log \frac{u\xi}{v} + (1 - u) \log \frac{(1 - u)\xi}{\xi - v}; \tag{37}$$

for $0 < u < 1$, $v \notin [0,\xi]$ let

$$G(u,v;\xi) \;=\; \infty. \tag{38}$$

Since $\frac{\partial G}{\partial v} = \frac{-u\xi+v}{v(\xi-v)}$, $\frac{\partial^2 G}{\partial v^2} = \frac{(v-u\xi)^2+\xi^2(1-u)u}{v^2(\xi-v)^2} > 0$, so $G(u,v;\xi)$ is convex in v, $\min_v G(u,v;\xi) = G(u,\xi u;\xi) = 0$. Thus for $0 < a, b < \infty$, the upper and lower boundary functions $\varphi_a^+(\cdot)$ and $\varphi_b^-(\cdot)$ are implicit functions given by (11) and (12). GCPLRS boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ are given by (13) and (14), but in which $a_n = m\varphi_a^+\left(\frac{n}{m-1}\right)$, $b_n = m\varphi_b^-\left(\frac{n}{m-1}\right)$ for $n = 1,\cdots,m-1$ and $a_m = b_m = \xi(m-1)$.

Now we consider the family of Gamma distributions, $\{\Gamma(p,\lambda) : 0<p<\infty, 0<\lambda<\infty\}$. Suppose p is known, λ is unknown. Let $X_i \sim \Gamma(p,\lambda)$ i.i.d. for $i = 1,2,\cdots$ and $S_n = \sum_{i=1}^n X_i$. Then $S_n \sim \Gamma(np,\lambda)$. Suppose $mp-1 > 0$, for $n \le m$, $0 < s < \eta(mp-1)$

$$\begin{aligned} P^\eta(S_n = s) &= P(S_n = s \mid S_m = \eta(mp-1)) \\ &= \frac{\Gamma(mp)}{\Gamma(np)\Gamma((m-n)p)} \frac{s^{np-1}((mp-1)\eta - s)^{(m-n)p-1}}{((mp-1)\eta)^{mp-1}}. \end{aligned} \tag{39}$$

Thus the GCPLR is

$$L(n,s;m,\xi) = \frac{\max_{\eta>\xi} \frac{(\eta-\frac{s}{mp-1})^{(m-n)p-1}}{\eta^{mp-1}}}{\max_{\eta\le\xi} \frac{(\eta-\frac{s}{mp-1})^{(m-n)p-1}}{\eta^{mp-1}}}. \tag{40}$$

Let $l(n,s;m,\xi) = \frac{1}{mp-1}\log L(n,s;m,\xi)$, then

$$l(n,s;m,\xi) \;=\; \begin{cases} G(\frac{np}{mp-1}, \frac{s}{mp-1};\xi) & \text{if } \frac{s}{n} > \xi, \\ -G(\frac{np}{mp-1}, \frac{s}{mp-1};\xi) & \text{if } \frac{s}{n} \le \xi; \end{cases} \tag{41}$$

where $G(u,v;\xi)$ is given in (37) and (38). GCPLRS boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ are given by (13) and (14), but in which $a_n = (mp-1)\varphi_a^+\left(\frac{np}{mp-1}\right)$, $b_n = (mp-1)\varphi_b^-\left(\frac{np}{mp-1}\right)$.

3.5 Boundaries for Shrunk Random Paths and Random Bridges

One of reasons for modifying GCPLRS is to make the test rule more conceptually visible. We will illustrate the relations of ratio function $G(u,v;\xi)$ (or boundary function φ_a^+ and φ_b^-), boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ and random path $\tilde{S}$ in a intuitive way.

In Figure 2, we illustrate GCPLRS boundary functions φ_a^+ and φ_b^- which are implicit functions of $G(u,v;\xi) = a$ and $G(u,v;\xi) = b$, for families of distributions we have discussed so far. In that graph, for all boundary functions of four families of distributions, we set deflection factors $a = b = 0.15$, and set slope of centerline $\xi = 0.5$.

Definition 3.4 *A shrinkage transformation of graph with factors* $(\frac{1}{m}, \frac{1}{mc})$ *is such that the horizontal (time) axis shrinks with factor* $\frac{1}{m}$ *and the vertical (state) axis shrinks with factor* $\frac{1}{mc}$.

Definition 3.5 *Define shrunk random path*

$$\tilde{S}^* = \left(\left(\frac{1}{m}, \frac{S_1}{mc}\right), \left(\frac{2}{m}, \frac{S_2}{mc}\right), \cdots\right). \tag{42}$$

Given $\frac{S_m}{mc} = \eta$, *define shrunk random bridge*

$$\tilde{S}^*_\eta = \left(\left(\frac{1}{m}, \frac{S_1}{mc}\right), \cdots, \left(\frac{m-1}{m}, \frac{S_{m-1}}{mc}\right), (1, \eta)\right). \tag{43}$$

Clearly under shrinkage transformation, we have one-to-one relations as follows:

$$\begin{aligned} \tilde{S} \leftrightarrow \tilde{S}^*; \qquad & \tilde{S}_m \leftrightarrow \tilde{S}^*_\eta; \\ \mathcal{B}^+_a \leftrightarrow \varphi^+_a; \qquad & \mathcal{B}^-_b \leftrightarrow \varphi^-_b. \end{aligned}$$

Under shrinkage transformation, φ^+_a, φ^-_b are boundaries for shrunk random path $\tilde{S}^*$ which is defined on time $t = \frac{1}{m}, \frac{2}{m}, \cdots$; Shrunk random bridge $\tilde{S}^*_\eta$ is defined on time $t = \frac{1}{m}, \frac{2}{m}, \cdots, 1$. Clearly boundary functions φ^+_a and φ^-_b are defined on $[0, 1]$. Under this transformation, the GCPLRS test rule δ defined in Definition 2.2 should be

$$\begin{aligned} &\text{accept } H_0 \text{ if } \tilde{S}^* \text{ passes over } \varphi^+_a \text{ before passes over } \varphi^-_b \\ &\text{accept } H_1 \text{ if } \tilde{S}^* \text{ passes over } \varphi^-_b \text{ before passes over } \varphi^+_a. \end{aligned}$$

4 GCPLRS Test and Estimation Procedures

In this section, we will scrutinize GCPLRS procedures of sequential tests (one-sided and two-sided) and sequential estimation. These procedures will be illustrated with examples in dichotomous populations. For other distributions, procedures are essentially same.

Designing a GCPLRS test procedure for one-sided hypotheses is to construct an enclosed boundary which are decided by operating paramenters m, ξ, a and b, where m is the maximum sampling size; ξ is the slope of center line for GCPLRS boundary, or the cutoff values at time $t = 1$ assuming shrunk random path $\tilde{S}^*$ is the observation for the test; a and b are

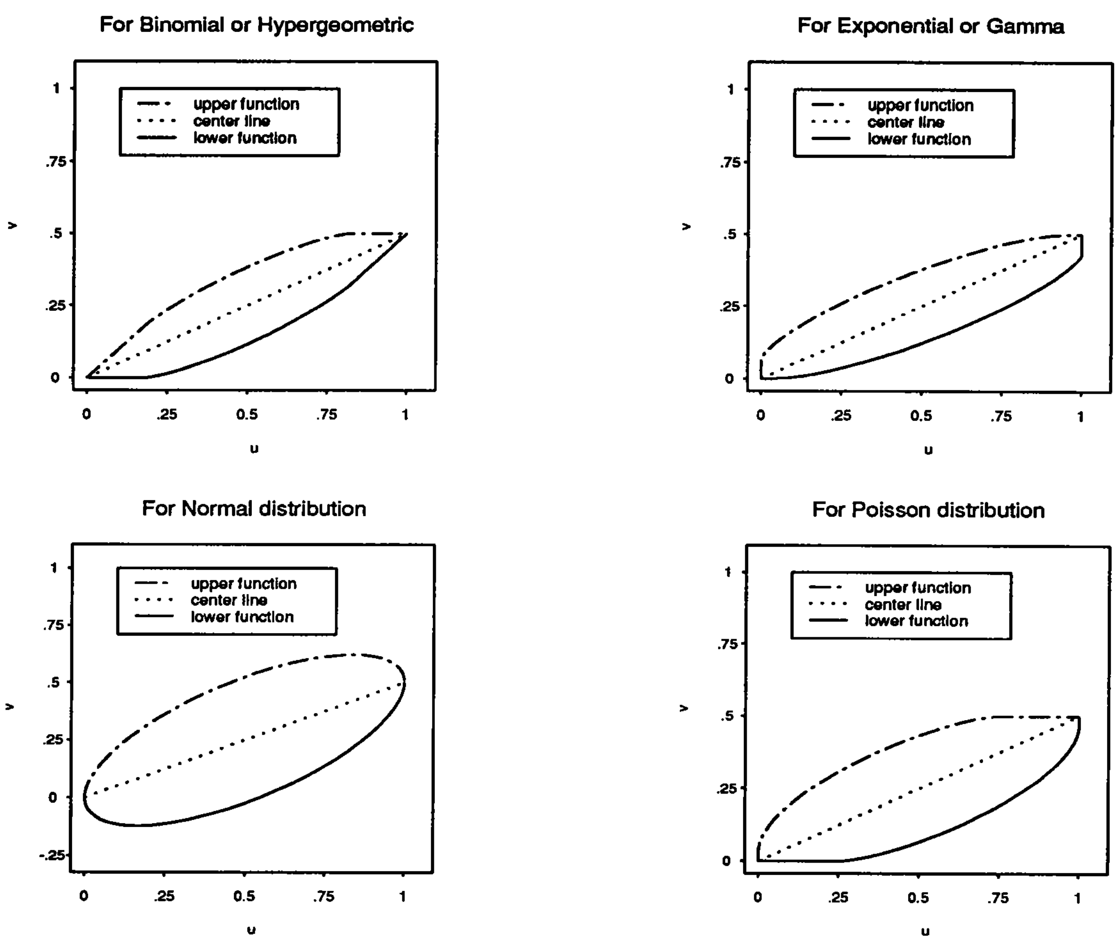

Figure 2: **GCPLRS Functions for Families of Distributions**

deflection factors determining the degrees that boundary functions $\varphi_a^+(\cdot)$, $\varphi_b^-(\cdot)$ deflect from the center line $v = \xi u$. The GCPLRS test procedure for two-sided hypotheses is in fact to construct two such enclosed boundaries. The GCPLRS estimation procedure is in fact to construct a number of such enclosed boundaries. In this section, we will show how to choose maximum sampling size m, GCPLRS boundary centerline slope ξ, deflection factors a and b for various GCPLRS testing and estimation procedures.

We assume through out this section, S_n is the sufficient statistics of first n observations for θ such that $E_\theta S_n = cn\theta$ (c known), and for any s and n, $P_\theta(S_n \geq s)$ is nondecreasing in θ.

4.1 GCPLRS Test for One-Sided Hypotheses

Assume we are interested in testing one-sided hypotheses $H_0 : \theta \leq \theta_0$ *v.s.* $H_1 : \theta > \theta_0$.

One-Sided GCPLRS Test Procedure:

Step 1. Design a fixed sampling size test which satisfies the sensitivity requirement of test. Determine sampling size m and critical value s_0. If S_m is discretely distributed, choose s_0 be nonsupport of S_m.

Step 2. Design a sequential sampling test rule by using ratio function $G(u, v; \xi)$ to generate enclosed boundary. Determine operating parameters: m is the sampling size in *Step 1*; $\xi = \frac{s_0}{cm}$; choose a and b as small as possible, but keep the power function of GCPLRS test about same as the one of fixed sampling test rule.

Example 1. Suppose in a batch of products, the proportion of defects is p, and there are N products in this batch. To test hypotheses $H_0 : p \leq 0.1$ *v.s.* $H_1 : p > 0.1$, assume sensitivity of test is required

$$\max_{p\leq 0.1} P_p(\text{ reject } H_0) = P_{0.1}(\text{ reject } H_0) \leq 0.05, \tag{44}$$

$$\min_{p\geq 0.2} P_p(\text{ reject } H_0) = P_{0.2}(\text{ reject } H_0) \geq 0.95. \tag{45}$$

Suppose $N = 100$, conduct a GCPLRS test, assume sampling one by one without replacement.

Solution:

Step 1. Let S_m be the number of defects in m sampled products. Then S_m has hypergeometric distribution $\mathcal{H}(m; pN, N)$. For this test, reject H_0 iff $S_m \geq s_0$. The power function for this test is $\beta(p) = P_p(S_m \geq s_0)$. We may determine m and s_0 by solving equations

$$\begin{cases} \beta(p_1) = \alpha_1, \\ \beta(p_2) = \alpha_2, \end{cases} \tag{46}$$

where $p_1 = 0.1$, $\alpha_1 = 0.05$ and $p_2 = 0.2$, $\alpha_2 = 0.95$. Since

$$\beta(p) = P_p\left(\frac{S_m - mp}{\sqrt{m\frac{N-m}{N-1}p(1-p)}} \geq \frac{s_0 - \frac{1}{2} - mp}{\sqrt{m\frac{N-m}{N-1}p(1-p)}}\right), \tag{47}$$

by Central Limit Theorem, approximation of m and s_0 may be obtained by solving equations

$$\begin{cases} s_0 = \frac{1}{2} + 0.1m + z_{0.95}\sqrt{m\frac{N-m}{N-1}0.1(1-0.1)}, \\ s_0 = \frac{1}{2} + 0.2m + z_{0.05}\sqrt{m\frac{N-m}{N-1}0.2(1-0.2)}; \end{cases} \tag{48}$$

where z_α is the αth quantile for standard normal distribution. Then one has $m = 57.25$ and $s_0 = 8.68$.

Step 2. Let $m = 58$, then $\xi = 0.15$. Deflection factors $a = 0.07$, $b = 0.117$ are determined by (53), (54). Because random path $\tilde{S}$ takes values only on integers, boundaries could be

$$\begin{aligned} \mathcal{B}_a^+ &= \{(n, a_n)\}_{n=1}^m, \quad \text{where} \quad a_n = \left[m\varphi_a^+\left(\frac{n}{m}\right)\right] + 1; \\ \mathcal{B}_b^- &= \{(n, b_n)\}_{n=1}^m, \quad \text{where} \quad b_n = \left[m\varphi_b^-\left(\frac{n}{m}\right)\right]; \end{aligned}$$

where $[x]$ denotes the largest integer smaller than x, $\varphi_a^+(\cdot)$ and $\varphi_b^-(\cdot)$ are boundary functions given by (11) and (12) in which $G(u, v; \xi)$ is given by (20), (21). Since this ratio function satisfies (86) and (87), $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ can be obtained directly by Computation Method (*Case One*) in Section 6.3. In Fig 3, GCPLRS boundary, power functions and expected sampling size for this test rule are illustrated. It is clear that power functions for fixed size rule and GCPLRS rule are approximately same, but expected sampling size for GCPLRS rule is much smaller than the fixed size m when parameter p is either substantially smaller or bigger than $p = \xi$.

4.2 Determine Deflection Factors for GCPLRS Boundaries

As illustrated in Section 3.5, boundary functions $v = \varphi_a^+(u)$ and $v = \varphi_b^-(u)$ form a closed boundary for shrunk random path $\tilde{S}^*$. Deflection factors a, b determine the degrees of

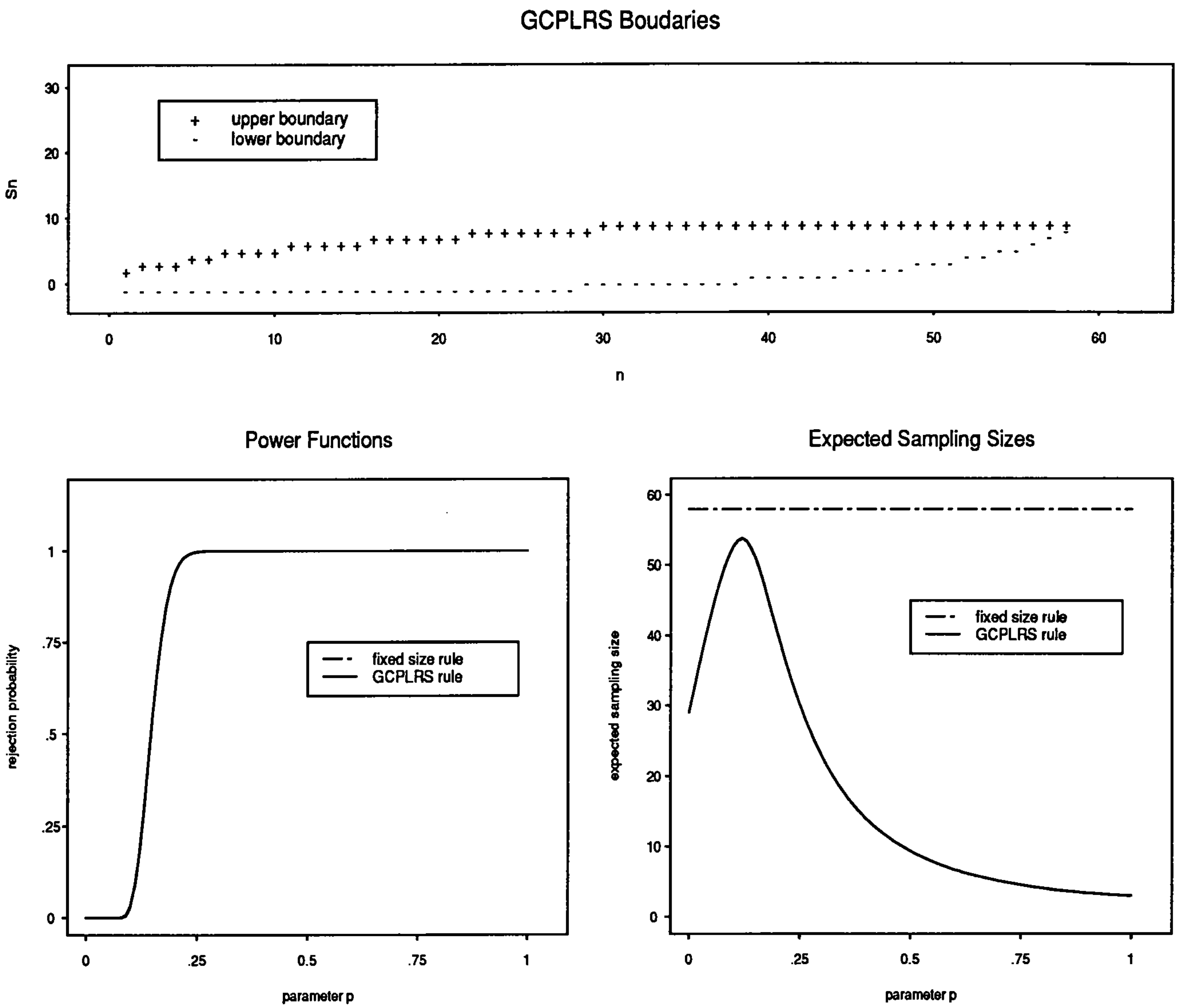

Figure 3: **GCPLRS Test for One-sided Hypotheses**

deflection that upper boundary $v = \varphi_a^+(u)$ and lower boundary $v = \varphi_b^-(u)$ are away from center line $v = \xi u$. Smaller are a and b, closer are $v = \varphi_a^+(u)$ and $v = \varphi_b^-(u)$ to $v = \xi u$, earlier $\tilde{S}^*$ passes either over $v = \varphi_b^-(u)$ or over $v = \xi u$, smaller is the expected sampling size for test.

It is important to choose suitable deflection factors for construction of GCPLRS boundaries. Since the main purpose of sequential test (compared with fixed sampling test) is to improve efficiency of test (in term of expected sampling size), so one tends to choose smaller a, b. But improvement of efficience is not supposed to be traded off by decreasing sensitivity of test. Accordingly proper deflection factors a, b for GCPLRS boundaries should be small as possible while keeping the power function of GCPLRS test about same as the power function of fixed sampling size test. There are two methods to choose deflection factors.

Method One: To compute directly the absorption probabilities on the upper and lower boundaries for random path $\tilde{S}^*$. Then choose a, b as small as possible, of which, GCPLRS test has power function about same as the power function of fixed sampling size test.

Method Two: To get approximation of absorption probability on the upper boundary for random bridge $\tilde{S}^*_{\xi_-}$, and to get approximation of absorption probability on the lower boundary for random bridge $\tilde{S}^*_{\xi_+}$, where ξ_+ and ξ_- are such that $\xi_- < \xi < \xi_+$, $\xi_- \approx \xi \approx \xi_+$. Bigger are a and b, smaller are absorption probabilities for random bridges. Then choose a, b, of which, absorption probabilities are small but not too small.

Method One is obvious; and Method Two is justified by following theorem.

Theorem 4.1 *Let $P\left(\varphi_b^-|_{\tilde{S}^*_{\xi_+}} \prec \varphi_a^+|_{\tilde{S}^*_{\xi_+}}\right)$ denotes the probability random bridge $\tilde{S}^*_{\xi_+}$ passes over boundary $v = \varphi_b^-(u)$ before passes over boundary $v = \varphi_a^+(u)$; Let $P\left(\varphi_a^+|_{\tilde{S}^*_{\xi_-}} \prec \varphi_b^-|_{\tilde{S}^*_{\xi_-}}\right)$ denotes the probability random bridge $\tilde{S}^*_{\xi_-}$ passes over boundary $v = \varphi_a^+(u)$ before passes over boundary $v = \varphi_b^-(u)$. Assume for $0 < \rho < 1$*

$$P\left(\varphi_b^-|_{\tilde{S}^*_{\xi_+}} \prec \varphi_a^+|_{\tilde{S}^*_{\xi_+}}\right) \le \rho \quad \text{and} \quad P\left(\varphi_a^+|_{\tilde{S}^*_{\xi_-}} \prec \varphi_b^-|_{\tilde{S}^*_{\xi_-}}\right) \le \rho. \tag{49}$$

Then for any $\theta \in \Theta$

$$(1-\rho)\beta_0(\theta) \le \beta(\theta) \le (1-\rho)\beta_0(\theta) + \rho \tag{50}$$

where $\beta_0(\theta)$ $(= P_\theta(S_m \ge \xi m))$ is the power function of fixed sampling size test rule δ_0; $\beta(\theta)$ $\left(= P_\theta\left(\varphi_a^+|_{\tilde{S}^*} \prec \varphi_b^-|_{\tilde{S}^*}\right)\right)$ is the power function of GCPLRS test rule δ.

Proof of this Theorem is given in Section 6.1.

The first method gives the exact absorption probability distribution on barrier sets for random path $\tilde{S}$. Accordingly exact power function and expected sampling size for GCPLRS test are available. But this method is not always practical (or convenient) because it requires intensive computation. Computation of absorption probabilities is sometimes either very tedious or difficult.

The second method is practical provided a simple approximation formula for absorption probability on GCPLRS boundaries for random bridge is available. Exact power function and expected sampling size of GCPLRS test are not available by this method, but the power function of GCPLRS test is about same as the power function of fixed sampling size test. With this property one may design a GCPLRS test rule met one's requirements. For sampling from dichotomous population (one-by-one, with or without replacement), approximation formula for absorption probability on GCPLRS boundaries for random bridges are

$$P\left(\varphi_a^+|_{\tilde{S}_\xi^*} \prec \varphi_b^-|_{\tilde{S}_\xi^*}\right) \approx \sqrt{\frac{\xi(1-\xi)m}{2\pi}}\, e^{-am}\, H(a,1-\xi), \tag{51}$$

$$P\left(\varphi_b^-|_{\tilde{S}_\xi^*} \prec \varphi_a^+|_{\tilde{S}_\xi^*}\right) \approx \sqrt{\frac{\xi(1-\xi)m}{2\pi}}\, e^{-bm}\, H(b,\xi) \tag{52}$$

where $H(\kappa,\eta)$ is called the asymptotical coefficient *w.r.t.* $G(u,v;\eta)=\kappa$. log values of $H(\kappa,\eta)$ are given in Table 1. This formula is further discussed in Section 6.2. Given ρ, we may compute deflection factors a, b by

$$a = \frac{1}{m}\left\{\log\frac{1}{\rho}+\frac{1}{2}\log\frac{\xi(1-\xi)m}{2\pi}+\log H(a,1-\xi)\right\}, \tag{53}$$

$$b = \frac{1}{m}\left\{\log\frac{1}{\rho}+\frac{1}{2}\log\frac{\xi(1-\xi)m}{2\pi}+\log H(b,\xi)\right\}. \tag{54}$$

In Example 1, we have $m=58$ and $\xi=0.15$. Let $\rho=0.02$. Then $a=0.06885+\frac{\log H(a,0.85)}{58}$, $b=0.06885+\frac{\log H(b,0.15)}{58}$. With trial and error by using Table 1, it is not difficult to get solutions $a=0.07$, $b=0.117$ (approximately). The values of a, b as solution of (53), (54) is sensitive to the choice of ρ. Though serving as a guideline to choose suitable ρ, (50) is in fact very conservative. For Binomial and Hypergeometric families, $|\beta_0(\theta)-\beta(\theta)|$ is usually less than $\frac{\rho}{5}$. This could be a working criterion for choosing ρ.

4.3 GCPLRS Test for Two-Sided Hypotheses

Suppose we are interested in testing two-sided hypotheses

$$H_0 : \theta = \theta_0 \ v.s. \ H_1 : \theta > \theta_0 \ v.s. \ H_2 : \theta < \theta_0.$$

Two-sided GCPLRS Test Procedure:

Step 1. Set up two one-sided hypotheses $H_0' : \theta \leq \theta_0$ *v.s.* $H_1' : \theta > \theta_0$ and $H_0'' : \theta \geq \theta_0$ *v.s.* $H_1'' : \theta < \theta_0$. The relations of the original two-sided hypotheses and the two one-sided hypotheses are:

$$\begin{aligned}&H_0 \text{ is favored iff both } H_0' \text{ and } H_0'' \text{ are favored;}\\&H_1 \text{ is favored iff both } H_0' \text{ and } H_1'' \text{ are favored;}\\&H_2 \text{ is favored iff both } H_1' \text{ and } H_0'' \text{ are favored.}\end{aligned} \tag{55}$$

Step 2. For each one-sided hypotheses, design a fixed sampling size test which satisfies the sensitivity requirement of test. Determine sampling sizes m', m'' and critical values s_0', s_0''.
Step 3. For each one-sided hypotheses, design a GCPLRS test rule. Determine operating parameters: m', m''; $\xi' = \frac{s_0'}{cm'}$, $\xi'' = \frac{s_0''}{cm''}$; a', b' and a'', b''.

Example 2. In the context of Example 1, suppose we want to test two-sided hypotheses of three decisions:

$$H_0 : p = 0.1 \ v.s. \ H_1 : p > 0.1 \ v.s. \ H_2 : p < 0.1. \tag{56}$$

Assume sensitivity of test is required that

$$\begin{aligned}P_{0.1}(\text{ accept } H_0) &\geq 0.95;\\ \min_{p\geq 0.15} P_p(\text{ accept } H_1) &= P_{0.15}(\text{ accept } H_1) \geq 0.95;\\ \min_{p\leq 0.05} P_p(\text{ accept } H_2) &= P_{0.05}(\text{ accept } H_2) \geq 0.95.\end{aligned} \tag{57}$$

Assuming $N = 500$, conduct a GCPLRS test by sampling one by one without replacement.
Solution:
Step 1. Testing the hypotheses of three decisions is equivalent of testing hypotheses $H_0' : p \leq 0.1$ *v.s.* $H_1' : p > 0.1$ and testing $H_0'' : p \geq 0.1$ *v.s.* $H_1'' : p < 0.1$ simultaneously start with same observations. The combined test is done if the equivalent tests are both done.

Step 2. Design a fixed sampling size test rule for H_0' *v.s.* H_1' which has sampling size m' and critical value s_0'. The test rule is such that reject H_0' iff $S_{m'} \geq s_0'$. The sensitivity requirement for this test is

$$\max_{p\leq 0.1} P_p(\text{ reject } H_0') = P_{0.1}(\text{ reject } H_0') \leq 0.025, \tag{58}$$

$$\min_{p\geq 0.15} P_p(\text{ reject } H_0') = P_{0.15}(\text{ reject } H_0') \geq 0.95. \tag{59}$$

Similarly, design a fixed sampling size test rule for H_0'' *v.s.* H_1'' which has sampling size m'' and critical value s_0''. The test rule is such that reject H_0' iff $S_{m''} \leq s_0''$. The sensitivity requirement for this test is

$$\max_{p\geq 0.1} P_p(\text{ reject } H_0'') = P_{0.1}(\text{ reject } H_0'') \leq 0.025, \tag{60}$$

$$\min_{p\leq 0.05} P_p(\text{ reject } H_0'') = P_{0.05}(\text{ reject } H_0'') \geq 0.95. \tag{61}$$

To determine sampling size m', m'' and critical value s_0', s_0'', follow steps in Example 1, we have approximations $m' = 259.3$, $s_0' = 32.8$ and $m'' = 204.8$, $s_0'' = 14.68$ by solving equations similar to (53), (54).

Step 3. For each of the fixed sampling size tests obtained in *Step 1*, design a corresponding GCPLRS rule by follow steps in Example 1. Denote those two sequential procedures δ' and δ''. The combined GCPLRS rule, denoted as δ, for testing H_0 *v.s.* H_1 *v.s.* H_2 is defined as follow. Start to do δ' and δ'' simultaneously by taking same observations sequentially. When one test is finished, continue the other one until both tests are done. Then interpret the conclusion for the combined test rule δ according to relations of hypotheses given (55).

We let $m' = 260$, then $\xi' = 0.126$; let $m'' = 205$, then $\xi'' = 0.072$. By (53) and (54) and Table.1, we have $a' = 0.0175$, $b' = 0.027$, $a'' = 0.0172$ and $b'' = 0.036$. Let $\mathcal{B}_{a'}^+$ and $\mathcal{B}_{b'}^-$ be the boundaries for δ', and let $\mathcal{B}_{a''}^+$ and $\mathcal{B}_{b''}^-$ be the boundaries for δ''. $\mathcal{B}_{a'}^+$, $\mathcal{B}_{b'}^-$, $\mathcal{B}_{a''}^+$, $\mathcal{B}_{b''}^-$ are obtained by Computation Method (*Case One*) in Section 6.3.

To test same hypotheses in (56) under the same constraint in (57), a reasonable two-stage test (which may be regarded as fixed size test) should be:

Stage 1. take sample of size $m'' = 205$. Accept H_2 if $S_{205} < 15 = s_0''$ and end the test. Other wise go to stage 2.

Stage 2. take additional $m' - m'' = 55$ observations. Accept H_0 if $S_{260} < 33 = s_0'$; accept H_1 if $S_{260} \geq 33 = s_0'$.

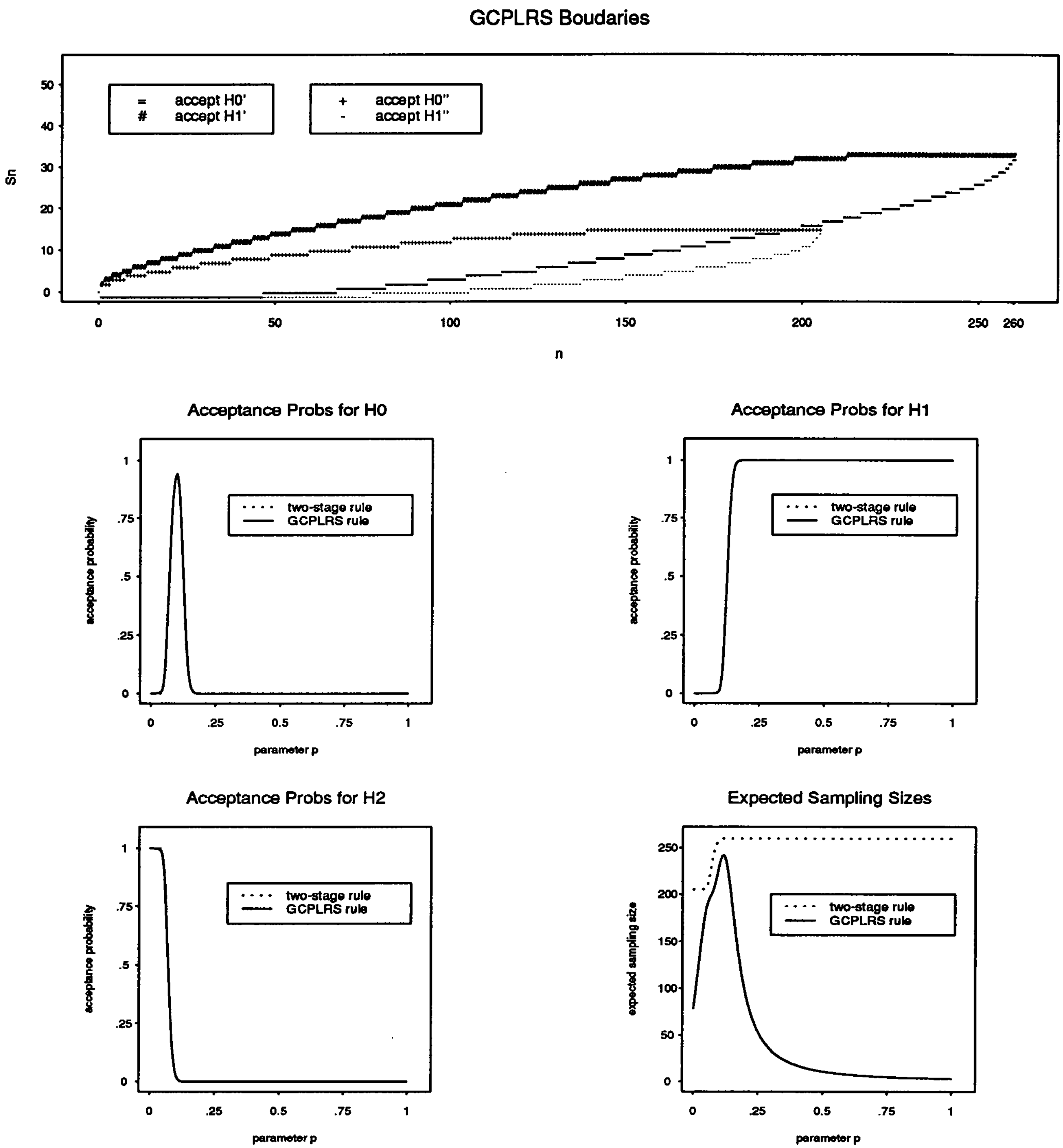

Figure 4: **GCPLRS Test for Two-sided Hypotheses**

Boundaries for the two one-sided GCPLRS test rules are graphed in Fig 4. By using a method in Xiong(1992), acceptance probabilities of H_0, H_1 and H_2, as well as the expected size for the GCPLRS test rule δ are computed and graphed in Fig 4. It is clear as illustrated, acceptance probabilities of H_0, H_1 and H_2 for GCPLRS rule δ are about the same as those for two-stage test rule. But the expected size of δ is much smaller than the one of two-stage rule when parameter p is substantially away from $p_0 = 0.1$.

4.4 Group GCPLRS Test

In real life applications, it often happens that group (multiple-stage) sequential test is preferred than one-by-one sequential sampling test for cases in which either group sequential sampling scheme is mandatory or it costs much less than one-by-one sequential sampling scheme.

For the case in which same numbers of observations are taken at each stage (in each group), the criterion of GCPLRS can be applied for group sequential tests by regarding the sufficient statistics for observations in each group as a single observation. We consider more general case in which numbers of observations taken at each stage can be different. Suppose k_i is the number of observations taken at ith group for $i = 1, 2, \cdots$.

Group GCPLRS Test Procedure:

Step 1. Design a fixed sampling test rule of size m such that this rule meets the requirements for sensitivity of test and $m = \sum_{i=1}^{m_0} k_i$ where m_0 is the number of groups possibly taken. Determine m and critical value s_0 for the fixed size test rule.

Step 2. With m and $\xi = \frac{s_0}{mc}$, design a (one-by-one sampling) GCPLRS test rule as in Section 4.1 (or as in Section 4.3 for two-sided hypotheses), which has GCPLRS boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$.

Step 3. Let $n_l = \sum_{i=1}^{l} k_i$. And let $W_l = S_{n_l}$, $a_l^* = a_{n_l}$, $b_l^* = b_{n_l}$ where S_{n_l}, a_{n_l}, b_{n_l} are from Step 2. Let $\mathcal{B}_{a^*}^+ = \{(l, a_l^*)\}_{l=1}^{m_0}$, $\mathcal{B}_{b^*}^- = \{(l, b_l^*)\}_{l=1}^{m_0}$. We say $\tilde{W} = \{(1, W_1), (2, W_2), \cdots, \}$ is the random path for group sequential test. Accept H_1 if $\tilde{W}$ passes over $\mathcal{B}_{a^*}^+$ before passes over $\mathcal{B}_{b^*}^-$; accept H_0 otherwise.

Example 3. Consider the testing problem in Example 1. To conduct a GCPLRS test by sampling in groups, taking k_i observations for each group, without replacement. Assume $N = 100$ as in Example 1, and $k_1 = 20$, $k_l = 10$ for $l = 2, \cdots$.

Solution: Let Y_l be the number of defects in the products sampled at lth group, and let $W_l = \sum_{i=1}^{l} Y_i$. Let $n_l = \sum_{i=1}^{l} k_i$. Then $S_{n_l} = W_l$ has hypergeometric distribution

$\mathcal{H}(n_l; pN, N)$, for $l = 1, 2, \cdots$.

Step 1. From Example 1, the size of fixed size rule m should be more than 57.25. So we take $m = 60$, and $s_0 = 8.5$ such that (44) and (45) hold.

Step 2. Follow steps in Section 4.1, we have $\xi = \frac{s_0}{m} = 0.142$. $a = 0.068$ and $b = 0.112$. GCPLRS boundaries $\mathcal{B}_a^+$ and $\mathcal{B}_b^-$ can be determined.

Step 3. Then we have $a_1^* = a_{20} = 7$, $b_1^* = b_{20} = -1$; $a_2^* = a_{30} = 8$, $b_2^* = b_{30} = 0$; $a_3^* = a_{40} = 9$, $b_3^* = b_{40} = 0$; $a_4^* = a_{50} = 9$, $b_4^* = b_{50} = 2$; $a_5^* = a_{60} = 9$, $b_5^* = b_{60} = 8$.

4.5 GCPLRS Estimation Procedure

We propose procedures giving coverage (confidence) intervals and estimation for parameter with criterion of GCPLRS. As we know, fixed sample size test and estimation are related such that the hypotheses $H_0 : \theta = \theta_0$ is accepted iff the confidence interval for θ captures θ_0. Follow the same idea, we introduce GCPLRS estimation procedure to derive coverage interval for true parameter.

Definition 4.1 *Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are two barrier sets for random path $\tilde{S}$. Given a random path $\tilde{S}$, if $\tilde{S}$ passes over $\mathcal{B}_1$ before or at the same time it passes over $\mathcal{B}_2$, then we denote*

$$\mathcal{B}_1|_{\tilde{S}} \prec \mathcal{B}_2|_{\tilde{S}}. \tag{62}$$

GCPLRS Estimation Procedure:

Step 1: Divide Θ into interval $(\theta_{k-1}, \theta_k]$ $\quad k = 1, \cdots, d$ where $\theta_0 < \theta_1 < \cdots < \theta_d$. Given α $(0 < \alpha < 0.5)$, define m_k, ξ_k such that

$$P_{\theta_{k-1}}(S_{m_k} \geq \xi_k m_k) < \alpha, \tag{63}$$

$$P_{\theta_k}(S_{m_k} \leq \xi_k m_k) < \alpha. \tag{64}$$

Approximation of m_k and ξ_k can be obtained by follow steps in Section 4.1.

Step 2: Construct GCPLRS barrier sets $\mathcal{B}_k^+$ and $\mathcal{B}_k^-$ by choosing proper a_k and b_k for $k = 1, \cdots, d$ such that

(i) For each k, $\forall \theta \in \Theta$ $P_\theta(\mathcal{B}_k^+|_{\tilde{S}} \prec \mathcal{B}_k^-|_{\tilde{S}}) \approx P_\theta(S_{m_k} \geq \xi_k m_k)$ (hence $P_{\theta_{k-1}}(\mathcal{B}_k^+|_{\tilde{S}} \prec \mathcal{B}_k^-|_{\tilde{S}}) \leq \alpha$, $P_{\theta_k}(\mathcal{B}_k^-|_{\tilde{S}} \prec \mathcal{B}_k^+|_{\tilde{S}}) \leq \alpha$).

(ii) For $k = 1, \cdots, d$

$$\tilde{S} \text{ can't pass over } \mathcal{B}_k^+ \text{ before it passes over } \mathcal{B}_{k-1}^+;$$

$\tilde{S}$ can't pass over $\mathcal{B}_{k-1}^-$ before it passes over $\mathcal{B}_k^-$.

Step 3: GCPLRS estimation rule is:

(1) For $k = 1, \cdots, d$

$$\begin{aligned} \text{Conclude} \quad \theta \geq \theta_{k-1} \quad \text{if} \quad \mathcal{B}_k^+|_{\tilde{S}} \prec \mathcal{B}_k^-|_{\tilde{S}}; \\ \text{Conclude} \quad \theta \leq \theta_k \quad \text{if} \quad \mathcal{B}_k^-|_{\tilde{S}} \prec \mathcal{B}_k^+|_{\tilde{S}}. \end{aligned}$$

(2) For the random path $\tilde{S}$, determine (random) integer K such that

$$\begin{aligned} &\mathcal{B}_k^+|_{\tilde{S}} \prec \mathcal{B}_k^-|_{\tilde{S}} \text{ for all } k \leq K; \\ &\mathcal{B}_k^-|_{\tilde{S}} \prec \mathcal{B}_k^+|_{\tilde{S}} \text{ for all } k > K. \end{aligned} \tag{65}$$

(3) $[\theta_{K-1}, \theta_{K+1}]$ is the $1-2\alpha$ coverage interval for θ. $(-\infty, \theta_{K+1}]$ and $[\theta_{K-1}, \infty)$ are $1-\alpha$ coverage intervals for θ

(4) Estimation of θ is $\hat{\theta} = \theta_K$. (this estimator might be biased)

Theorem 4.2 *Assuming* Step 1 *and* Step 2 *in GCPLRS Estimation Procedure hold, then claim in Step 3.(3) is true, namely, for any* $\theta \in \Theta$

$$P_\theta\left(\theta_{K-1} \leq \theta\right) \geq 1 - \alpha, \tag{66}$$

$$P_\theta\left(\theta \leq \theta_{K+1}\right) \geq 1 - \alpha, \tag{67}$$

$$P_\theta\left(\theta_{K-1} \leq \theta \leq \theta_{K+1}\right) \geq 1 - 2\alpha. \tag{68}$$

Proof: For any $\theta \in \Theta$, there is k_0 such that $\theta_{k_0-1} < \theta \leq \theta_{k_0}$. Since θ_k is increasing in k, then by difinition of K in (65)

$$\begin{aligned} P_\theta\left(\theta < \theta_{K-1}\right) &\leq P_\theta\left(\theta_{k_0-1} < \theta_{K-1}\right) \\ &= P_\theta\left(k_0 + 1 \leq K\right) \\ &\leq P_\theta\left(\mathcal{B}_{k_0+1}^+|_{\tilde{S}} \prec \mathcal{B}_{k_0+1}^-|_{\tilde{S}}\right) \\ &\leq P_{\theta_{k_0}}\left(\mathcal{B}_{k_0+1}^+|_{\tilde{S}} \prec \mathcal{B}_{k_0+1}^-|_{\tilde{S}}\right) \\ &\leq \alpha; \end{aligned} \tag{69}$$

and

$$P_\theta\left(\theta_{K+1} < \theta\right) \leq P_\theta\left(\theta_{K+1} < \theta_{k_0}\right)$$

$$
\begin{aligned}
&= P_\theta\,(K+1<k_0)\\
&\leq P_\theta\left(\mathcal{B}^-_{k_0-1}|_{\tilde{S}} \prec \mathcal{B}^+_{k_0-1}|_{\tilde{S}}\right)\\
&\leq P_{\theta_{k_0-1}}\left(\mathcal{B}^-_{k_0-1}|_{\tilde{S}} \prec \mathcal{B}^+_{k_0-1}|_{\tilde{S}}\right)\\
&\leq \alpha. \qquad (70)
\end{aligned}
$$

Therefore one has

$$
\begin{aligned}
P_\theta\,(\theta_{K-1}\leq\theta\leq\theta_{K+1}) &= 1-P_\theta\,(\theta_{K-1}<\theta)-P_\theta\,(\theta_{K+1}<\theta)\\
&\geq 1-2\alpha. \qquad (71)
\end{aligned}
$$

■

Example 4. Consider the problem in Example 1, we are interested in deriving 1–$2\alpha = 0.90$ confidence interval for θ $(= p)$ such that the length of interval is 0.2. To conduct a GCPLRS estimation by sampling one-by-one without replacement.

Solution: Divide $\Theta = (0,1]$ into interval $(\theta_{k-1}, \theta_k]$ $\;k = 1,\cdots,d$ where $d = 10$, $\theta_k = \frac{k-1}{10}$ $\;k = 1,\cdots,10$. Given $\alpha = 0.1$, define m_k, ξ_k such that

$$
\begin{aligned}
&P_{\theta_k}(S_{m_k}\geq\xi_k m_k)=0.1,\\
&P_{\theta_{k-1}}(S_{m_k}\leq\xi_k m_k)=0.1. \qquad (72)
\end{aligned}
$$

Approximation of m_k and ξ_k can be obtained by follow *Step 1* in Section 4.1. Follow *Step 2* in same section, GCPLRS barrier sets $\mathcal{B}^+_k$ and $\mathcal{B}^+_k$ can be constructed by choosing proper a_k and b_k for $k = 1,\cdots,d$. For the random path $\tilde{S}$, determine (random) integer K. Conclude that $[\theta_{K-1}, \theta_{K+1}]$ is the $1-2\alpha = 0.8$ coverage interval for θ. Estimation of θ is $\hat{\theta} = \theta_K$.

4.6 Comparison of GCPRS rule with Wald's PLRS rule

As we argued earlier, the efficiency of test rule should be judged not only in terms of expected sampling size but also in terms of maximum sampling size. GCPLRS rule is most efficient in term of maximum sampling size (same as fixed sampling size test). But what is its efficiency in term of expected sampling size.

We know that Wald's PLRS rule minimizes the expected sampling size when $\theta = \theta_0$ or $\theta = \theta_1$ $(\theta_0 < \theta_1)$ among the rules under the constraint $\beta(\theta_0) \leq \alpha_0$ and $\beta(\theta_1) \geq \alpha_1$ where $\beta(\cdot)$ is power function, θ_0, θ_1, α_0 and α_1 are given. Since expected sampling size is continuous as a function of θ, so Wald's PRLS rule should have smallest expected sampling size when the true θ is in the neighborhoods of θ_0 and θ_1.

In following example, we will compare GCPLRS rule with Wald's PLRS rule for power function and expected sampling size.

Example 5. Suppose X_i *i.i.d.* $P_p(X_i = 0) = p$ and $P_p(X_i = 1) = 1 - p$. We are interested in testing hypotheses $H_0 : p \leq 0.5$ *v.s.* $H_1 : p > 0.5$. Subject to requirement

$$\max_{p \leq 0.5} P_p(\text{ reject } H_0) = P_{0.5}(\text{ reject } H_0) \leq 0.05, \tag{73}$$

$$\min_{p \geq 0.6} P_p(\text{ reject } H_0) = P_{0.6}(\text{ reject } H_0) \geq 0.95, \tag{74}$$

to conduct GCPLRS test, and to conduct Wald's PLRS test.

Solution:

To conduct GCPLRS test, follow steps as in Example 1, solving equations

$$\begin{cases} s_0 = \frac{1}{2} + 0.5m + z_{0.95}\sqrt{m0.5(1-0.5)}, \\ s_0 = \frac{1}{2} + 0.6m + z_{0.05}\sqrt{m0.6(1-0.6)}. \end{cases} \tag{75}$$

We have $m = 266$, $\xi = 0.55$. By (53), (54) and Table 1, deflection factors $a = 0.0228$, $b = 0.0218$. Boundaries, power function and expected size are computed and graphed in Fig 5.

To conduct Wald's PLRS test, follow steps in Section 4.1.1 in McWilliams(1989), the rejection boundary is $\mathcal{B}^+ = \{(n, a_n)\}_{n=1}^{\infty}$ where $a_n = [d_n] + 1$, $d_n = h_2 + s \cdot n$; the acceptance boundary is $\mathcal{B}^- = \{(n, b_n)\}_{n=1}^{\infty}$ where $b_n = [d_n]$, $d_n = h_1 + s \cdot n$. By the requirements in this example, we have $s = 0.55034$, $h_2 = 7.2619$, $h_1 = -7.2619$. Boundary, power function and expected size are computed and graphed in Fig 5. Wald's PLRS boundary is infinitely long, only part of it is graphed.

Though power functions of two rules are similar, they are compared in following way. The better rule should have smaller power for $p \leq 0.5$, and should have bigger power for $p > 0.5$. In this sense, as shown in Fig 5, Wald's PLRS rule is better for $0.49 \leq p \leq 0.5$ or $0.55 \leq p \leq 0.61$; GCPLRS rule is better or as same good for other $p \in [0, 1]$.

Compare the expected sampling sizes of two rules, the better rule should have smaller expected sampling size (disregard maximum sampling size for this moment). In this sense, as shown in Fig 5, GCPLRS rule is better for $0 \leq p < 0.26$ or $0.82 < p \leq 1$; For $0.26 \leq p \leq 0.82$, Wald's PLRS rule is better, but efficiency of GCPLRS rule, in terms of expected sampling size, is close to efficiency of Wald's PLRS rule. As illustrated in Fig 6, the standard deviation

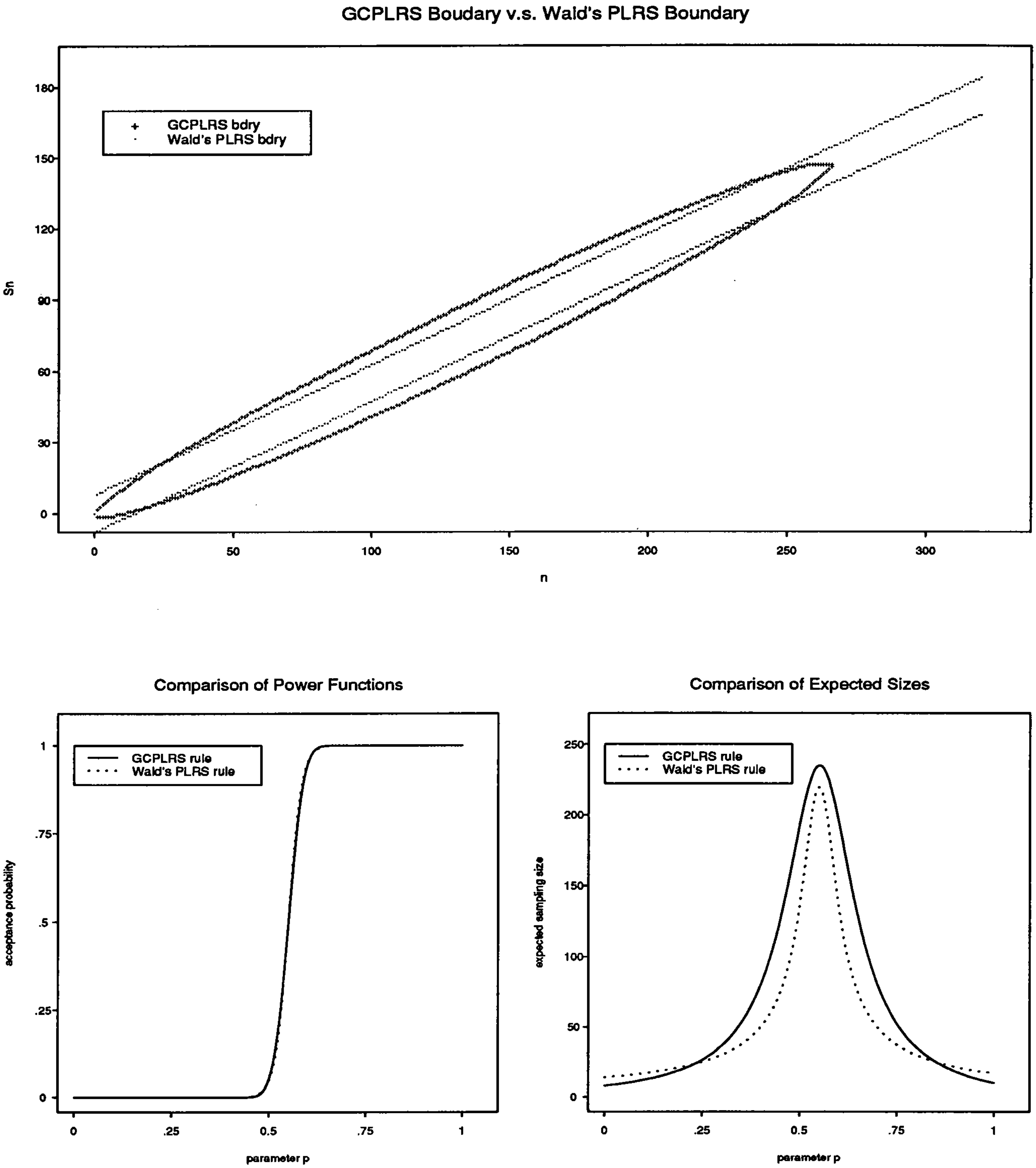

Figure 5: **Comparison of GCPLRS rule and Wald's PLRS rule**

of sampling size of Wald's PLRS rule is very large when true parameter p is close to 0.55.

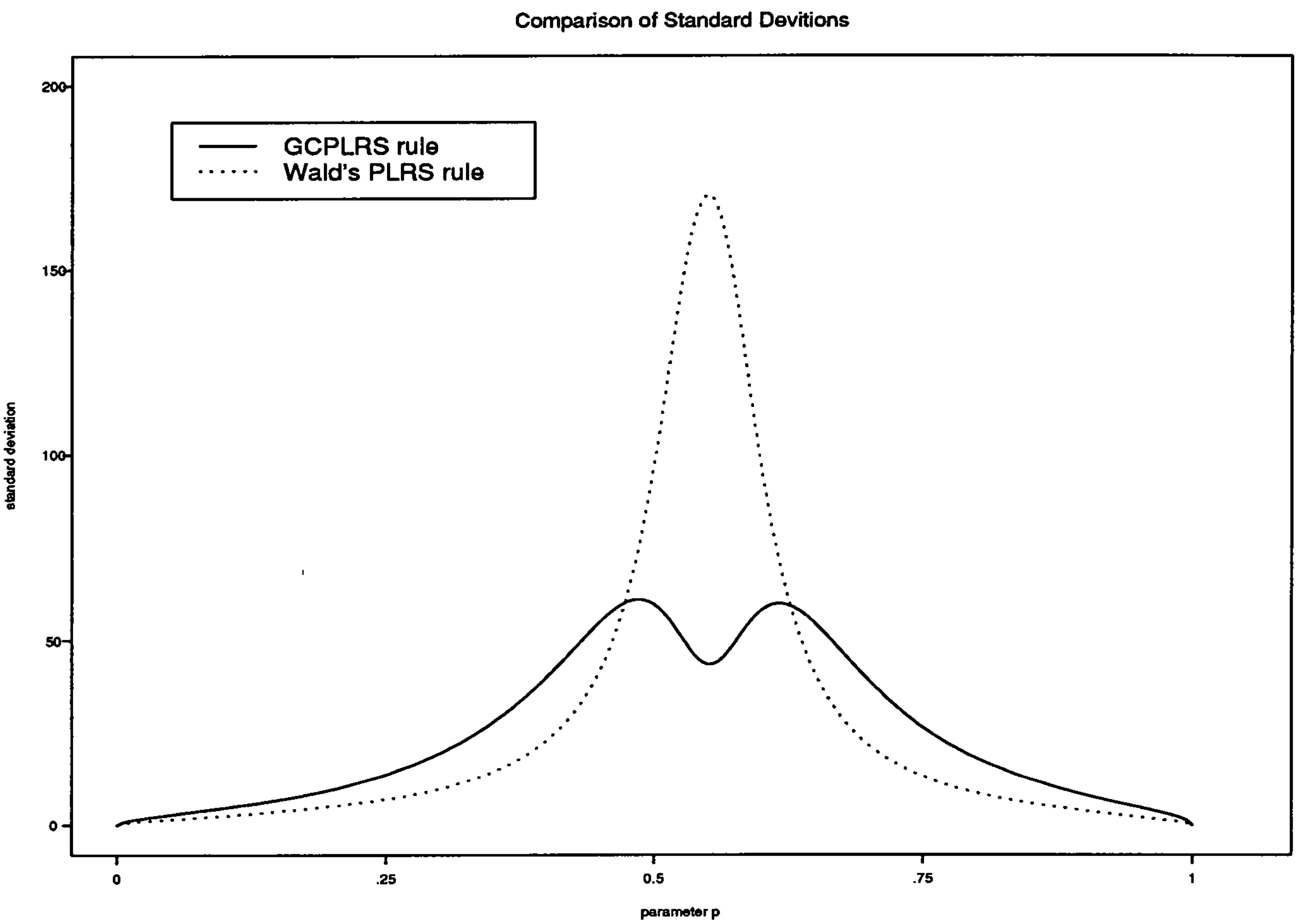

Figure 6: **Standard Deviations of Sampling sizes**

5 Appendix

5.1 Proof of Theorems

Proof of Theorem 4.1:

$$\begin{aligned}
\beta(\theta) &= P_\theta\left(\mathcal{B}_a^+|_{\tilde{S}} \prec \mathcal{B}_b^-|_{\tilde{S}}\right) \\
&= \int_{-\infty}^{\infty} P\left(\mathcal{B}_a^+|_{\tilde{S}} \prec \mathcal{B}_b^-|_{\tilde{S}} \mid S_m = \eta m\right) P_\theta(S_m = \eta m) m d\eta \\
&= \int_{-\infty}^{\infty} P\left(\varphi_a^+|_{\tilde{S}_\eta^*} \prec \varphi_b^-|_{\tilde{S}_\eta^*}\right) P_\theta(S_m = \eta m) m d\eta \qquad (76)
\end{aligned}$$

where notation $P_\theta(S_m = \eta m)$ is abused for density if S_m is continuously distributed, and notation $\int_{-\infty}^{\infty} \cdot m d\eta$ is abused for summation if S_m is discretely distributed.
But if $\eta > \xi$, then

$$P\left(\varphi_a^+|_{\tilde{S}_\eta^*} \prec \varphi_b^-|_{\tilde{S}_\eta^*}\right) \geq 1-\rho. \tag{77}$$

And if $\eta < \xi$, then

$$P\left(\varphi_a^+|_{\tilde{S}_\eta^*} \prec \varphi_b^-|_{\tilde{S}_\eta^*}\right) \leq \rho. \tag{78}$$

Let $\beta(\theta) = \mathrm{I} + \mathrm{II}$ where

$$\mathrm{I} = \int_\xi^\infty P\left(\varphi_a^+|_{\tilde{S}_\eta^*} \prec \varphi_b^-|_{\tilde{S}_\eta^*}\right) P_\theta(S_m = \eta m) m d\eta,$$
$$\mathrm{II} = \int_{-\infty}^\xi P\left(\varphi_a^+|_{\tilde{S}_\eta^*} \prec \varphi_b^-|_{\tilde{S}_\eta^*}\right) P_\theta(S_m = \eta m) m d\eta.$$

Then

$$\int_\xi^\infty (1-\rho) P_\theta(S_m = \eta m) m d\eta \leq \mathrm{I} \leq \int_\xi^\infty P_\theta(S_m = \eta m) m d\eta$$

which implies $(1-\rho)\beta_0(\theta) \leq \mathrm{I} \leq \beta_0(\theta)$. Similarly one has

$$0 \leq \mathrm{II} \leq \int_{-\infty}^\xi P_\theta(S_m = \eta m) m d\eta$$

which implies $0 \leq \mathrm{II} \leq \rho(1-\beta_0(\theta))$. Therefore one has $(1-\rho)\beta_0(\theta) \leq \beta(\theta) \leq (1-\rho)\beta_0(\theta)+\rho$.
■

5.2 Approximation of Absorption Probability for Random Bridge

To design a GCPLRS procedure, an important but difficult step is to determine deflection factors. As discussed in Section 4.2, in order to make GCPLRS procedures easily applicable, we need to provide simple approximation formula for absorption probabilities on GCPLRS boundaries for random bridges. But there doesn't exist an uniform formula for all distributions. For distributions resulted by one-by-one sampling from dichotomous population (with replacement or without replacement), the approximation formula is given in theorem below.

Theorem 5.1 *Suppose $v = \varphi_a^+(u)$, $v = \varphi_b^-(u)$ are upper and lower boundary functions derived from ratio function $G(u, v; \xi)$ which is given by (20), (21). Let $\tilde{S}_\xi^*$ be the shrunk random bridge defined in (42) where $\tilde{S}$ is random path by sampling from dichotomous population. Let $\varphi_a^+|_{\tilde{S}_\xi^*}$ ($\varphi_b^-|_{\tilde{S}_\xi^*}$) denote the event that shrunk random bridge $\tilde{S}_\xi^*$ upcross (downcross) φ_a^+ (φ_b^-). Then for large m*

$$P\left(\varphi_a^+|_{\tilde{S}_\xi^*} \prec \varphi_b^-|_{\tilde{S}_\xi^*}\right) \approx \sqrt{\frac{\xi(1-\xi)m}{2\pi}}\, e^{-am}\, H(a, 1-\xi), \tag{79}$$

$$P\left(\varphi_b^-|_{\tilde{S}_\xi^*} \prec \varphi_a^+|_{\tilde{S}_\xi^*}\right) \approx \sqrt{\frac{\xi(1-\xi)m}{2\pi}}\, e^{-bm}\, H(b, \xi) \tag{80}$$

where $H(\kappa, \eta)$ is called the asymptotical coefficient w.r.t. *$G(u, v; \eta) = \kappa$, given by*

$$H(\kappa, \eta) = \int_0^{x_0} h(x; \kappa, \eta)dx. \tag{81}$$

In above integration, the integrand is

$$h(x; \kappa, \eta) = \frac{(\eta - x)\left(\frac{t(x)}{x(1-x)(1-t(x))(\eta - xt(x))(1-\eta-(1-x)t(x))}\right)^{\frac{1}{2}}}{(1-x)\log\frac{(1-t(x))(1-x)}{1-\eta-(1-x)t(x)}} \tag{82}$$

where $t(x)$ is the implicit function of equation $G(t(x), xt(x); \eta) = \kappa$. The integration limit in (81) is $x_0 = \frac{y_0}{1-\eta+y_0}$ where y_0 is the unique solution of $G(1-\eta+y_0, y_0; \eta) = \kappa$.

This theorem is an immediate consequence of main result in Xiong(1991). The asymptotical coefficient $H(\kappa, \eta)$, as a function of κ and η is numerically evaluated and tabulated in Table 1. $H(\kappa, \eta)$ is defined for $0 < \eta < 1$, $0 < \kappa < \eta \log\frac{1}{\eta} + (1-\eta)\log\frac{1}{1-\eta}$. The proof for Theorem 6.1 and the approximation formula for other distributions discussed in Section 3 will be given in Xiong(1993).

5.3 Computation of GCPLRS Boundaries

Suppose operating parameters m, ξ, a, b are given, then the upper and lower GCPLRS boundaries $\mathcal{B}_a^+$, $\mathcal{B}_b^-$ may be obtained by (13), (14) in which $\varphi_a^+(u)$ and $\varphi_b^-(u)$ are implicit functions given by (11), (12). In most cases $\varphi_a^+(u)$ and $\varphi_b^-(u)$ have no close form, thus not easily computable. Anyhow, in some cases, there is simple way to compute $\mathcal{B}_a^+$, $\mathcal{B}_b^-$ directly without evaluation of $\varphi_a^+(u)$ and $\varphi_b^-(u)$.

Lemma 5.1 *Assume $G(u,v;\xi)$ is ratio function such that*

$$\begin{aligned} \frac{\partial G}{\partial v}(u,v;\xi) &\leq 0 \quad \textit{if} \quad v \leq \xi u, \\ &> 0 \quad \textit{if} \quad v > \xi u. \end{aligned} \tag{83}$$

Given $a, b > 0$, N positive integer, let h_n, l_n for $n = 1, \cdots, N-1$ be integers such that

$$h_n = \min\left\{k : G\left(\frac{n}{N}, \frac{k}{N}; \xi\right) > a, k > \xi n\right\}, \tag{84}$$

$$l_n = \max\left\{k : G\left(\frac{n}{N}, \frac{k}{N}; \xi\right) > b, k < \xi n\right\}. \tag{85}$$

If $\left(\frac{\partial G}{\partial u} + \frac{\partial G}{\partial v}\right)(u,v;\xi) > 0$ for $v > \xi u$, then

$$h_n \leq h_{n+1} \leq h_n + 1. \tag{86}$$

If $\frac{\partial G}{\partial u}(u,v;\xi) \leq 0$ for $v < \xi u$, then

$$l_n \leq l_{n+1} \leq l_n + 1. \tag{87}$$

Proof for this lemma is given in Xiong(1990). As a consequence of Lemma 6.1, one has

Lemma 5.2 *For $n = 1, \cdots, N-1$, if (86) holds, then*

$$h_{n+1} = \begin{cases} h_n & \textit{if} \ \ G(\frac{n+1}{N}, \frac{h_n}{N}; \xi) \geq a \\ h_n + 1 & \textit{otherwise} \end{cases}; \tag{88}$$

if (87) holds, then

$$l_{n+1} = \begin{cases} l_n + 1 & \textit{if} \ \ G(\frac{n+1}{N}; \frac{l_n+1}{N}; \xi) \geq b \\ l_n & \textit{otherwise} \end{cases}, \tag{89}$$

where l_1 and h_1 are

$$h_1 = \min\left\{k : G\left(\frac{1}{N}, \frac{k}{N}; \xi\right) > a, \quad k > \xi\right\}, \tag{90}$$

$$l_1 = \max\left\{k : G\left(\frac{1}{N}, \frac{k}{N}; \xi\right) > b, \quad k < \xi\right\}. \tag{91}$$

Computation of h_n, l_n is much easier by (88) and (89) than by (84) and (85). For sampling (one-by-one or in group) from dichotomous population, $G(u, v; \xi)$ is given by (20), (21), then (86), (87) hold. Hence (88), (89) hold and $h_1 = 2$, $l_1 = -1$.

Computation Method:

Assume (86), (87) hold, barrier sets $\mathcal{B}_a^+ = \{(n, a_n)\}_{n=1}^m$ and $\mathcal{B}_b^- = \{(n, b_n)\}_{n=1}^m$ can be obtained in following cases.

Case One: S_n is distributed on integers with increment 1 or 0 (*e.g.* one-by-one sampling from dichotomous population). $a_n = \left[m\varphi_a^+(\frac{n}{m})\right] + 1$, $b_n = \left[m\varphi_b^-(\frac{n}{m})\right]$. Then let $N = m$, compute h_i, l_i for $i = 1, \cdots, m$ by (88), (89); let $a_n = h_n$, $b_n = l_n$ for $n = 1, \cdots, m$.

Case Two: S_n is distributed on integers with in integer increment k or less (*e.g.* group sampling from dichotomous population). $a_n = \left[km\varphi_a^+(\frac{n}{m})\right] + 1$, $b_n = \left[km\varphi_b^-(\frac{n}{m})\right]$ where k is given. Then let $N = mk$, compute h_i, l_i for $i = 1, \cdots, mk$ by (88), (89); let $a_n = h_{nk}$, $b_n = l_{nk}$ for $n = 1, \cdots, m$.

Case Three: S_n is continuously distributed (*e.g.* sampling from exponential distribution). $a_n = cm\varphi_a^+(\frac{n}{m})$, $b_n = cm\varphi_b^-(\frac{n}{m})$. Obviously exact a_n, b_n are not available. Assume we want to compute a_n, b_n with accuracy of $\frac{1}{\tau}$. Then let $N = mk$ where $k = [\tau c] + 1$, compute h_i, l_i for $i = 1, \cdots, mk$ by (88), (89); let $a_n = c\frac{h_{nk}}{k}$, $b_n = c\frac{l_{nk}}{k}$ for $n = 1, \cdots, m$.

Table 1. log values of Asymptotical Coefficient $H(\kappa, \eta)$

$\eta = 0.05$		$\eta = 0.10$		$\eta = 0.15$		$\eta = 0.20$		$\eta = 0.25$	
κ	log H	κ	log H	κ	log H	κ	log H	κ	log H
.0016	3.204	.0025	2.434	.0033	1.979	.0039	1.654	.0044	1.399
.0031	3.355	.0051	2.592	.0066	2.140	.0078	1.815	.0088	1.559
.0047	3.442	.0076	2.682	.0099	2.232	.0117	1.907	.0132	1.651
.0062	3.502	.0102	2.745	.0132	2.295	.0156	1.970	.0176	1.714
.0078	3.547	.0127	2.792	.0165	2.343	.0195	2.018	.0220	1.762
.0093	3.583	.0152	2.829	.0198	2.381	.0235	2.057	.0264	1.800
.0109	3.612	.0178	2.860	.0231	2.412	.0274	2.088	.0308	1.832
.0124	3.637	.0203	2.887	.0264	2.439	.0313	2.116	.0351	1.859
.0248	3.762	.0406	3.021	.0528	2.578	.0626	2.257	.0703	2.001
.0372	3.838	.0610	3.105	.0793	2.666	.0938	2.348	.1054	2.094
.0496	3.901	.0813	3.175	.1057	2.740	.1251	2.425	.1406	2.174
.0620	3.962	.1016	3.242	.1321	2.812	.1564	2.500	.1757	2.252
.0744	4.025	.1219	3.311	.1585	2.886	.1877	2.578	.2109	2.333
.0869	4.093	.1422	3.387	.1849	2.966	.2189	2.662	.2460	2.421
.0993	4.171	.1625	3.471	.2114	3.056	.2502	2.755	.2812	2.518
.1117	4.261	.1829	3.568	.2378	3.158	.2815	2.862	.3163	2.628
.1241	4.368	.2032	3.682	.2642	3.277	.3128	2.986	.3515	2.757
.1365	4.499	.2235	3.821	.2906	3.422	.3440	3.135	.3866	2.910
.1489	4.665	.2438	3.997	.3170	3.603	.3753	3.322	.4218	3.102
.1613	4.889	.2641	4.230	.3435	3.844	.4066	3.568	.4569	3.353
.1737	5.220	.2844	4.572	.3699	4.193	.4379	3.924	.4920	3.714
.1861	5.817	.3048	5.182	.3963	4.813	.4691	4.551	.5272	4.348
.1869	5.874	.3060	5.241	.3979	4.872	.4711	4.611	.5294	4.408
.1877	5.935	.3073	5.303	.3996	4.935	.4730	4.674	.5316	4.473
.1884	6.001	.3086	5.370	.4012	5.003	.4750	4.743	.5338	4.542
.1892	6.073	.3098	5.443	.4029	5.076	.4769	4.817	.5360	4.616
.1900	6.151	.3111	5.522	.4045	5.157	.4789	4.898	.5382	4.698
.1908	6.237	.3124	5.610	.4062	5.245	.4809	4.987	.5404	4.787
.1915	6.333	.3137	5.707	.4078	5.343	.4828	5.085	.5426	4.887

Table 1. log values of Asymptotical Coefficient $H(\kappa, \eta)$

$\eta = 0.30$		$\eta = 0.35$		$\eta = 0.40$		$\eta = 0.45$		$\eta = 0.50$	
κ	log H	κ	log H	κ	log H	κ	log H	κ	log H
.0048	1.188	.0051	1.007	.0053	0.847	.0054	0.704	.0054	0.571
.0095	1.347	.0101	1.164	.0105	1.002	.0108	0.855	.0108	0.719
.0143	1.438	.0152	1.254	.0158	1.090	.0161	0.941	.0162	0.802
.0191	1.500	.0202	1.315	.0210	1.150	.0215	0.999	.0217	0.859
.0239	1.547	.0253	1.362	.0263	1.196	.0269	1.044	.0271	0.902
.0286	1.585	.0303	1.399	.0315	1.232	.0323	1.079	.0325	0.936
.0334	1.617	.0354	1.430	.0368	1.262	.0376	1.109	.0379	0.965
.0382	1.644	.0405	1.457	.0421	1.289	.0430	1.134	.0433	0.989
.0764	1.786	.0809	1.598	.0841	1.428	.0860	1.272	.0866	1.124
.1145	1.880	.1214	1.693	.1262	1.525	.1290	1.369	.1300	1.222
.1527	1.963	.1619	1.778	.1683	1.611	.1720	1.457	.1733	1.312
.1909	2.044	.2023	1.861	.2103	1.697	.2150	1.546	.2166	1.404
.2291	2.128	.2428	1.948	.2524	1.787	.2581	1.640	.2599	1.501
.2673	2.219	.2833	2.043	.2944	1.885	.3011	1.741	.3033	1.606
.3054	2.319	.3237	2.147	.3365	1.993	.3441	1.853	.3466	1.722
.3436	2.433	.3642	2.265	.3786	2.115	.3871	1.978	.3899	1.852
.3818	2.566	.4047	2.401	.4206	2.255	.4301	2.123	.4332	2.002
.4200	2.724	.4451	2.563	.4627	2.422	.4731	2.294	.4765	2.177
.4581	2.919	.4856	2.763	.5048	2.626	.5161	2.503	.5199	2.391
.4963	3.175	.5261	3.024	.5468	2.891	.5591	2.773	.5632	2.666
.5345	3.542	.5665	3.396	.5889	3.269	.6021	3.156	.6065	3.055
.5727	4.183	.6070	4.043	.6309	3.922	.6451	3.816	.6498	3.721
.5751	4.243	.6095	4.104	.6336	3.983	.6478	3.877	.6525	3.783
.5775	4.308	.6120	4.169	.6362	4.049	.6505	3.943	.6552	3.850
.5798	4.378	.6146	4.239	.6388	4.120	.6532	4.015	.6579	3.921
.5822	4.453	.6171	4.315	.6415	4.196	.6559	4.091	.6607	3.999
.5846	4.535	.6196	4.398	.6441	4.279	.6586	4.175	.6634	4.083
.5870	4.625	.6222	4.488	.6467	4.370	.6613	4.267	.6661	4.175
.5894	4.725	.6247	4.588	.6494	4.471	.6639	4.368	.6688	4.277

Table 1. log values of Asymptotical Coefficient $H(\kappa,\eta)$

$\eta = 0.55$		$\eta = 0.60$		$\eta = 0.65$		$\eta = 0.70$		$\eta = 0.75$	
κ	log H	κ	log H	κ	log H	κ	log H	κ	log H
.0054	0.448	.0053	0.330	.0051	0.215	.0048	0.100	.0044	-0.018
.0108	0.590	.0105	0.467	.0101	0.345	.0095	0.221	.0088	0.091
.0161	0.670	.0158	0.543	.0152	0.416	.0143	0.287	.0132	0.150
.0215	0.725	.0210	0.594	.0202	0.464	.0191	0.331	.0176	0.189
.0269	0.766	.0263	0.633	.0253	0.501	.0239	0.364	.0220	0.218
.0323	0.799	.0315	0.664	.0303	0.530	.0286	0.391	.0264	0.242
.0376	0.826	.0368	0.690	.0354	0.554	.0334	0.413	.0308	0.262
.0430	0.850	.0421	0.713	.0405	0.575	.0382	0.433	.0351	0.281
.0860	0.981	.0841	0.841	.0809	0.699	.0764	0.552	.0703	0.397
.1290	1.080	.1262	0.940	.1214	0.800	.1145	0.655	.1054	0.504
.1720	1.173	.1683	1.036	.1619	0.899	.1527	0.760	.1406	0.615
.2150	1.268	.2103	1.135	.2023	1.003	.1909	0.870	.1757	0.733
.2581	1.369	.2524	1.241	.2428	1.114	.2291	0.988	.2109	0.858
.3011	1.478	.2944	1.355	.2833	1.235	.2673	1.115	.2460	0.993
.3441	1.599	.3365	1.481	.3237	1.366	.3054	1.253	.2812	1.140
.3871	1.734	.3786	1.621	.3642	1.513	.3436	1.407	.3163	1.302
.4301	1.888	.4206	1.781	.4047	1.678	.3818	1.579	.3515	1.482
.4731	2.069	.4627	1.967	.4451	1.871	.4200	1.778	.3866	1.689
.5161	2.288	.5048	2.192	.4856	2.102	.4581	2.016	.4218	1.936
.5591	2.569	.5468	2.478	.5261	2.395	.4963	2.317	.4569	2.244
.6021	2.963	.5889	2.879	.5665	2.802	.5345	2.731	.4920	2.667
.6451	3.636	.6309	3.559	.6070	3.490	.5727	3.428	.5272	3.374
.6478	3.699	.6336	3.622	.6095	3.554	.5751	3.493	.5294	3.439
.6505	3.766	.6362	3.690	.6120	3.622	.5775	3.561	.5316	3.508
.6532	3.838	.6388	3.763	.6146	3.695	.5798	3.635	.5338	3.583
.6559	3.916	.6415	3.841	.6171	3.774	.5822	3.715	.5360	3.664
.6586	4.000	.6441	3.926	.6196	3.860	.5846	3.802	.5382	3.751
.6613	4.093	.6467	4.020	.6222	3.954	.5870	3.897	.5404	3.847
.6639	4.196	.6494	4.123	.6247	4.059	.5894	4.001	.5426	3.952

Table 1. log values of Asymptotical Coefficient H(κ, η)

$\eta = 0.80$		$\eta = 0.85$		$\eta = 0.90$		$\eta = 0.95$		$\eta = 0.975$	
κ	log H	κ	log H	κ	log H	κ	log H	κ	log H
.0039	-0.146	.0033	-0.295	.0025	-0.491	.0016	-0.824	.0009	-1.166
.0078	-0.052	.0066	-0.221	.0051	-0.446	.0031	-0.818	.0018	-1.173
.0117	-0.002	.0099	-0.183	.0076	-0.423	.0047	-0.808	.0027	-1.159
.0156	0.031	.0132	-0.158	.0102	-0.405	.0062	-0.795	.0037	-1.139
.0195	0.056	.0165	-0.138	.0127	-0.390	.0078	-0.780	.0046	-1.116
.0235	0.076	.0198	-0.121	.0152	-0.375	.0093	-0.763	.0055	-1.092
.0274	0.094	.0231	-0.105	.0178	-0.361	.0109	-0.745	.0064	-1.066
.0313	0.111	.0264	-0.090	.0203	-0.346	.0124	-0.725	.0073	-1.040
.0626	0.225	.0528	0.028	.0406	-0.215	.0248	-0.555	.0146	-0.821
.0938	0.339	.0793	0.153	.0610	-0.070	.0372	-0.371	.0219	-0.598
.1251	0.459	.1057	0.287	.0813	0.084	.0496	-0.181	.0292	-0.372
.1564	0.587	.1321	0.428	.1016	0.245	.0620	0.014	.0365	-0.144
.1877	0.723	.1585	0.577	.1219	0.414	.0744	0.215	.0438	0.087
.2189	0.868	.1849	0.736	.1422	0.591	.0869	0.423	.0511	0.324
.2502	1.025	2114	0.906	.1625	0.778	.0993	0.639	.0585	0.567
.2815	1.196	.2378	1.089	.1829	0.979	.1117	0.868	.0658	0.822
.3128	1.387	.2642	1.292	.2032	1.198	.1241	1.113	.0731	1.091
.3440	1.603	.2906	1.521	.2235	1.443	.1365	1.383	.0804	1.384
.3753	1.859	.3170	1.788	.2438	1.726	.1489	1.690	.0877	1.714
.4066	2.177	.3435	2.117	.2641	2.071	.1613	2.059	.0950	2.105
.4379	2.610	.3699	2.563	.2844	2.532	.1737	2.545	.1023	2.612
.4691	3.328	.3963	3.294	.3048	3.281	.1861	3.321	.1096	3.412
.4711	3.394	.3979	3.361	.3060	3.349	.1869	3.391	.1101	3.484
.4730	3.464	.3996	3.433	.3073	3.422	.1877	3.465	.1105	3.560
.4750	3.540	.4012	3.509	.3086	3.500	.1884	3.545	.1110	3.641
.4769	3.621	.4029	3.591	.3098	3.583	.1892	3.631	.1114	3.729
.4789	3.710	.4045	3.681	.3111	3.674	.1900	3.724	.1119	3.824
.4809	3.806	.4062	3.779	.3124	3.773	.1908	3.825	.1123	3.927
.4828	3.913	.4078	3.886	.3137	3.883	.1915	3.937	.1128	4.041

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