ESTIMATION OF CLUSTERED PARAMETERS

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Abstract

We look at probability models for the problem of estimating normal means, when the means are allowed to be equal. These models, which are called product partition models, assign probabilities to random partitions of sets of objects. Here, the objects correspond to the means. We show that when all the means are equal, the estimated number of distinct means has an asymptotic Poisson distribution. Also, when there are two sets of equal means, if they are far enough apart, then the two sets can be considered as two separate problems asymptotically. Finally, we look at simulations to see if the above results hold for moderate sample sizes.

Key words and phrases. Normal means, product partition models.

1 Introduction

We will consider the normal means problem: $X_i|\mu_i \sim N(\mu_i, 1)$ for $i = 1, \ldots, n$. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. We will look at probability models for the parameters $\mu_1, \ldots, \mu_n$ that allow for many of the μ_i's to be equal. Such models have application in clustering, mixture problems, and in the problem of multiple comparisons, where the parameters are divided into subsets such that the parameters within each subset cannot be distinguished.

Although our formal models assume normality, our method will apply more generally, generating clusters in a set of parameters. We expect similar results to hold under other models for observations given parameters.

In Section 2, there is a general description of the probability models we will use. These models, which are called product partition models, specify the probability of a random partition ρ. We consider product partition models for the normal means problem in Section 3. In Section 4, we look at the case where all the means are equal: $X_1, \ldots, X_n \sim N(0,1)$. The number of blocks B, is the number of sets in the random partition ρ. We show that $B-1$ has an asymptotic Poisson distribution, as $n \to \infty$. In Section 5 we consider two sets of equal means: $X_1, \ldots, X_{n/2} \sim N(0,1)$ and $X_{(n/2)+1}, \ldots, X_n \sim N(\theta_n, 1)$. We show that if θ_n is large enough, we can regard this problem as two separate problems. Some simulation results are given in Section 6.

For more details on simulations and on how the product partition method compares to other methods, see Crowley (1992, 1993).

2 Product partition models

Hartigan (1990) developed the idea of product partition models. For a set of objects $S_0 = \{1, 2, \ldots, n\}$, a partition $\rho = \{S_1, S_2, \ldots, S_k\}$ has the properties that $S_i \cap S_j = \emptyset$ for $i \neq j$ and $\cup_i S_i = S_0$. The probability of a partition ρ is defined by

$$P(\rho = \{S_1, S_2, \ldots, S_k\}) = K \prod_{i=1}^{k} c(S_i) \tag{1}$$

where the *cohesions* $c(S) \geq 0$ are the parameters of the product partition model and K is chosen to make the probabilities sum to one over all possible partitions.

Corresponding to each object i, we have an observation X_i. Let $X_S = \{X_i : i \in S\}$ have conditional density $p_S(X_S)$, given that $S \in \rho$. Given the random partition ρ, observations for objects in different classes are independent, so we have

$$p(\mathbf{X} | \rho = \{S_1, S_2, \ldots, S_k\}) = \prod_{i=1}^{k} p_{S_i}(X_{S_i}). \tag{2}$$

From equations (1) and (2), the posterior probability of a partition ρ is

$$P(\rho = \{S_1, S_2, \ldots, S_k\} | \mathbf{X}) = (K/\nu(\mathbf{X})) \prod_{i=1}^{k} c(S_i) p_{S_i}(X_{S_i}),$$

where $\nu(\mathbf{X})$ is the marginal density of $\mathbf{X}$. This is also a product partition model with (posterior) cohesions $c(S) p_S(X_S)$.

3 Distributions for clustered parameters

We now look at product partition models for the normal means problem. For other applications, see Hartigan (1990) and Barry and Hartigan (1992, 1993). We have $X_i|\mu_i \sim N(\mu_i, 1)$ for $i = 1, \ldots, n$. Let the prior cohesions be $c(S) = (n_S - 1)!/m^{(n_S-1)}$ where m is a parameter and n_S is the number of objects in set S. Large values of m lead to small n_S. Let μ^S be the common mean for the μ_i's with $i \in S$, that is, $\mu_i = \mu^S,\ i \in S$. Let $\mu^S \sim N(\mu_0, {\sigma_0}^2/n_S)$, where μ_0 and σ_0^2 are parameters.

Given the above choices of distributions, it follows that the joint distribution of ρ and $\mathbf{X}$, treating the parameters μ_0, σ_0^2 and m as fixed constants, is

$$P(\rho = \{S_1, S_2, \ldots, S_k\}, \mathbf{X}) = d(\mathbf{X}) \frac{\Gamma(m)}{\Gamma(n+m)} m^k \left(\prod_{r=1}^{k} (n_{S_r} - 1)!\right) (1 + \sigma_0^2)^{-k/2}$$
$$\times \exp\left(\frac{1}{2} \frac{\sigma_0^2}{1 + \sigma_0^2} \sum_{r=1}^{k} n_{S_r} (\overline{X}_{S_r} - \overline{X})^2\right) \exp\left(-\frac{1}{2} \frac{n}{1 + \sigma_0^2} (\overline{X} - \mu_0)^2\right). \quad (3)$$

where $d(\mathbf{X})^{-1} = (2\pi)^{n/2} \exp((1/2) \sum_{i=1}^{n} (X_i - \overline{X})^2)$ and $\overline{X}_S = \sum_{i \in S} X_i/n_S$. See Crowley (1993) for details.

4 Asymptotic distribution of the number of blocks when all the means are equal

Consider the case where all μ_i's are equal. The number of blocks, B, is the number of sets in the random partition ρ. The estimation of means will be more accurate, the fewer the numbers of blocks. However, if we choose the prior parameters to force the number of blocks too small, the estimation will work poorly when in fact the means are different. In order to evaluate the effect of prior parameters, we need to examine first the distribution of the number of blocks when all the means are equal. We prove the following theorem:

Theorem 1 *Let $X_1, \ldots, X_n$ be sampled from a N(0,1) distribution. Let the partition ρ be distributed according to a product partition model with the distributions specified in Section 3 and prior parameters μ_0, σ_0^2 and $m = \lambda/\ln(n)$. Then, if $\sigma_0^2 < 0.5$, $B - 1$ has an asymptotic*

Poisson distribution with mean λ, in the sense that

$$\frac{P(B-1=k|\mathbf{X})}{P(B-1=0|\mathbf{X})} \rightarrow \frac{\lambda^k}{k!}$$

in probability, as $n \rightarrow \infty$ for each fixed k.

This theorem does not say that there is exactly one block. It puts some probability on having more than one block. This probability depends on λ which in turn depends on m. Clearly, when all the means are equal, we want m to be small. However, we don't want to choose m small in general, that is, when the means are not all equal. But we can callibrate m with the one block case by deciding how much probability we are prepared to put on more than one block in order to reduce losses when the means are not all equal.

Consider the partition $\rho = \{S_1, \ldots, S_k\}$. In the results that follow, assume k a fixed integer and define the sets of partitions:

$$\begin{aligned}
T^1 &= \{S_1, \ldots, S_k | \sum_{r=1}^{k} n_{S_r} = n,\, n_{S_j} > 0,\, j = 1, \ldots, k\} \\
T^2 &= \{S_1, \ldots, S_k | \text{number of elements in } S_j \text{ is } n_{S_j}, n_{S_j} \text{ fixed}, j = 1, \ldots, k\} \\
N &= \{n_{S_1}, \ldots, n_{S_k} | \sum_{r=1}^{k} n_{S_r} = n,\, n_{S_j} > 0,\, j = 1, \ldots, k\}.
\end{aligned}$$

We will need to consider pairs of partitions: $\rho = \{S_1, \ldots, S_k\}$ and $\rho^* = \{S_1^*, \ldots, S_k^*\}$. Let $\mathbf{n} = \{n_{S_1}, \ldots, n_{S_k}\}$ and $\mathbf{n}^* = \{n_{S_1^*}, \ldots, n_{S_k^*}\}$. Define $t_{S_r S_l^*}$ to be the number of X_i's with i in both S_r and S_l^*, $r = 1, \ldots, k$, $l = 1, \ldots, k$. Let $u = \sigma_0^2/(1+\sigma_0^2)$. Then,

$$\frac{P(B=k|\mathbf{X})}{P(B=1|\mathbf{X})} = \frac{\frac{1}{k!}\sum_{\rho\in T^1} P(\rho = \{S_1, S_2, \ldots, S_k\}, \mathbf{X})}{P(\rho = \{S_0\}, \mathbf{X})}.$$

Substituting (3) in the above we get

$$m^{k-1}(1-u)^{(k-1)/2}\frac{W_n}{k!} \tag{4}$$

where

$$W_n = \sum_{\rho\in T^1} \frac{\left(\prod_{r=1}^{k}(n_{S_r}-1)!\right)}{(n-1)!} \exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2\right).$$

We will show that $W_n/EW_n \rightarrow 1$ in probability, as $n \rightarrow \infty$, using the following lemmas.

Lemma 2 *Let $X_1, \ldots, X_n$ be sampled from a N(0,1) distribution. Then, if $u < 1$,*

$$E\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2\right)\right) = (1-u)^{-(k-1)/2},$$

where $n_{S_r} > 0$ for $r = 1, \ldots, k$ and $\sum_{r=1}^{k} n_{S_r} = n$.

Proof From analysis of variance theory we have $\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2 \sim \chi^2_{k-1}$. The result follows from the formula for the moment generating function of a χ^2 with $k-1$ degrees of freedom. □

Lemma 3 *Let $X_1, \ldots, X_n$ be sampled from a N(0,1) distribution. Then, as $u \to 0$,*

$$Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*}\left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right)$$
$$= \frac{u^2}{2}\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r} n_{S_l^*}} - 1\right) + O(u^3),$$

where $n_{S_r} > 0$ for $r = 1, \ldots, k$, $\sum_{r=1}^{k} n_{S_r} = n$, $n_{S_l^*} > 0$ for $l = 1, \ldots, k$ and $\sum_{l=1}^{k} n_{S_l^*} = n$.

Proof Rewriting the exponential function as a series, we obtain

$$Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*}\left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right)$$
$$= \frac{u^2}{4}\sum_{r=1}^{k}\sum_{l=1}^{k} n_{S_r} n_{S_l^*}\, Cov\left(\left(\overline{X}_{S_r} - \overline{X}\right)^2, \left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right) + O(u^3). \tag{5}$$

The series expansion and equality of moments is justified for u small enough. We now find an expression for $Cov((\overline{X}_{S_r} - \overline{X})^2, (\overline{X}_{S_l^*} - \overline{X})^2)$. As $\overline{X}_{S_r} - \overline{X}$ and $\overline{X}_{S_l^*} - \overline{X}$ are bivariate normal random variables with $E(\overline{X}_{S_r} - \overline{X}) = E(\overline{X}_{S_l^*} - \overline{X}) = 0$, we have

$$Corr\left(\left(\overline{X}_{S_r} - \overline{X}\right)^2, \left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right) = \left(Corr(\overline{X}_{S_r} - \overline{X}, \overline{X}_{S_l^*} - \overline{X})\right)^2.$$

Using this fact, we obtain

$$Cov\left(\left(\overline{X}_{S_r} - \overline{X}\right)^2, \left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)$$
$$= \left(Corr(\overline{X}_{S_r} - \overline{X}, \overline{X}_{S_l^*} - \overline{X})\right)^2 \sqrt{Var\left(\left(\overline{X}_{S_r} - \overline{X}\right)^2\right) Var\left(\left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)}.$$

Substituting for the correlation and the variances, this becomes

$$\frac{\left(\frac{t_{S_r S_l^*}}{n_{S_r} n_{S_l^*}} - \frac{1}{n}\right)^2}{\left(\frac{1}{n_{S_r}} - \frac{1}{n}\right)\left(\frac{1}{n_{S_l^*}} - \frac{1}{n}\right)} \sqrt{2\left(\frac{1}{n_{S_r}} - \frac{1}{n}\right)^2 2\left(\frac{1}{n_{S_l^*}} - \frac{1}{n}\right)^2}$$
$$= 2\left(\frac{t_{S_r S_l^*}}{n_{S_r} n_{S_l^*}} - \frac{1}{n}\right)^2.$$

Substituting the above expression in (5), we obtain the required result. □

Lemma 4 *Let $X_1, \ldots, X_n$ be sampled from a N(0,1) distribution. Let*

$$\sum_{r=1}^{k} n_{S_r} (\overline{X}_{S_r} - \overline{X})^2 = \mathbf{X}' \mathbf{A}_1 \mathbf{X} \ \textit{and} \ \sum_{l=1}^{k} n_{S_l^*} (\overline{X}_{S_l^*} - \overline{X})^2 = \mathbf{X}' \mathbf{A}_2 \mathbf{X},$$

where $n_{S_r} > 0$ for $r = 1, \ldots, k$, $\sum_{r=1}^{k} n_{S_r} = n$, $n_{S_l^*} > 0$ for $l = 1, \ldots, k$ and $\sum_{l=1}^{k} n_{S_l^*} = n$. Then, if $0 \le u < 0.5$,

$$(a) \quad Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r} (\overline{X}_{S_r} - \overline{X})^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*} (\overline{X}_{S_l^*} - \overline{X})^2\right)\right)$$
$$= \prod_{i=1}^{2(k-1)} (1 - u\,\delta_i)^{-1/2} - (1-u)^{-(k-1)}$$

where $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{C}_{12} \mathbf{D}_{12} \mathbf{C}_{12}'$, $\mathbf{D}_{12} = \mathrm{diag}(\delta_1, \ldots, \delta_{2(k-1)}, 0, \ldots, 0)$ and $\mathbf{C}_{12}$ is orthogonal. The δ_i's satisfy the following:

$$(b) \quad 0 \le \delta_i \le 2$$
$$(c) \quad \sum_{i=1}^{2(k-1)} \delta_i = 2(k-1)$$
$$(d) \quad \sum_{i=1}^{2(k-1)} \delta_i^2 = 2\sum_{r=1}^{k}\sum_{l=1}^{k} \frac{t_{S_r S_l^*}^2}{n_{S_r} n_{S_l^*}} + 2k - 4$$

Proof Using Lemma 2 and writing the sums as quadratic forms, we obtain

$$Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r} (\overline{X}_{S_r} - \overline{X})^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*} (\overline{X}_{S_l^*} - \overline{X})^2\right)\right)$$
$$= |\mathbf{I}_n - u\,\mathbf{A}_1 - u\,\mathbf{A}_2|^{-1/2} - (1-u)^{-(k-1)}. \tag{6}$$

As $\mathbf{A}_1 + \mathbf{A}_2$ is a real, symmetric matrix, $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{C}_{12}\,\mathbf{D}_{12}\,\mathbf{C}'_{12}$, where $\mathbf{C}_{12}$ is orthogonal and $\mathbf{D}_{12} = \operatorname{diag}(\delta_1, \ldots, \delta_n)$ is diagonal. Hence,

$$\begin{aligned} |\mathbf{I}_n - u\,\mathbf{A}_1 - u\,\mathbf{A}_2|^{-1/2} &= \left(|\mathbf{C}'_{12}||\mathbf{I}_n - u\,\mathbf{A}_1 - u\,\mathbf{A}_2||\mathbf{C}_{12}|\right)^{-1/2} \\ &= |\mathbf{I}_n - u\,\mathbf{D}_{12}|^{-1/2}. \end{aligned}$$

At most $2(k-1)$ of the δ_i's differ from zero. This is because $\mathbf{A}_1$ and $\mathbf{A}_2$ have rank $k-1$ so $\mathbf{A}_1 + \mathbf{A}_2$ has rank at most $2\,(k-1)$. Without loss of generality, assume that $\mathbf{D}_{12} = \operatorname{diag}(\delta_1, \ldots, \delta_{2(k-1)}, 0, \ldots, 0)$. We now have

$$|\mathbf{I}_n - u\,\mathbf{A}_1 - u\,\mathbf{A}_2|^{-1/2} = \prod_{i=1}^{2(k-1)} (1 - u\,\delta_i)^{-1/2}.$$

Substituting in equation (6) gives (a).

Let $\mathbf{Z} = \mathbf{C}'_{12}\,\mathbf{X} = (Z_1, \ldots, Z_n)$. We have

$$\mathbf{X}'\,(\mathbf{A}_1 + \mathbf{A}_2)\,\mathbf{X} = (\mathbf{C}'_{12}\,\mathbf{X})'\,\mathbf{D}_{12}\,(\mathbf{C}'_{12}\,\mathbf{X}) = \mathbf{Z}'\,\mathbf{D}_{12}\,\mathbf{Z} = \sum_{i=1}^{2(k-1)} \delta_i\,Z_i^2.$$

Also,

$$0 \leq \mathbf{X}'\,(\mathbf{A}_1 + \mathbf{A}_2)\,\mathbf{X} \leq 2\,\mathbf{Z}'\,\mathbf{Z} = 2\sum_{i=1}^{n} Z_i^2$$

as

$$0 \leq \mathbf{X}'\,\mathbf{A}_1\,\mathbf{X} = \sum_{r=1}^{k} n_{S_r}\,(\overline{X}_{S_r} - \overline{X})^2 \leq \sum_{i=1}^{n} X_i^2 = \mathbf{X}'\,\mathbf{X} = \mathbf{Z}'\,\mathbf{Z}$$

and, similarly, $0 \leq \mathbf{X}'\,\mathbf{A}_2\,\mathbf{X} \leq \mathbf{Z}'\,\mathbf{Z}$. From the facts above, it follows that

$$0 \leq \sum_{i=1}^{2(k-1)} \delta_i\,Z_i^2 \leq \sum_{i=1}^{n} 2\,Z_i^2 \text{ for all } \mathbf{Z}.$$

For $i = 1, \ldots, 2(k-1)$, setting $Z_i = 1$ and $Z_j = 0$ for $j = 1, \ldots, n, j \neq i$, we obtain (b).

We rewrite the quantities in (6) as exponential functions.

$$\begin{aligned} |\mathbf{I}_n - u\,\mathbf{A}_1 - u\,\mathbf{A}_2|^{-1/2} &= \exp\left(\frac{1}{2}\sum_{i=1}^{2(k-1)} -\ln(1 - u\,\delta_i)\right) \\ &= \exp\left(\frac{1}{2}\sum_{j=1}^{\infty} \frac{u^j}{j}\,g(j)\right) \end{aligned}$$

where $g(j) = \sum_{i=1}^{2(k-1)} \delta_i^j = \text{trace}(\mathbf{D}_{12}^j)$. Also,

$$\begin{aligned}(1-u)^{-(k-1)} &= \exp\left(-(k-1)\ln(1-u)\right) \\ &= \exp\left((k-1)\sum_{j=1}^{\infty}\frac{u^j}{j}\right)\end{aligned}$$

Substituting for the quantities in equation (6), we obtain

$$\begin{aligned}&Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r}-\overline{X}\right)^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*}\left(\overline{X}_{S_l^*}-\overline{X}\right)^2\right)\right) \\ &= \exp\left(\frac{1}{2}\sum_{j=1}^{\infty}\frac{u^j}{j}g(j)\right) - \exp\left((k-1)\sum_{j=1}^{\infty}\frac{u^j}{j}\right).\end{aligned}$$

Expanding the exponential functions as series, the covariance is equal to

$$\begin{aligned}&1+\frac{1}{2}\sum_{j=1}^{\infty}\frac{u^j}{j}g(j)+\frac{1}{2!}\left(\frac{1}{2}\sum_{j=1}^{\infty}\frac{u^j}{j}g(j)\right)^2+\ldots \\ &-1-(k-1)\sum_{j=1}^{\infty}\frac{u^j}{j}-\frac{1}{2!}\left((k-1)\sum_{j=1}^{\infty}\frac{u^j}{j}\right)^2-\ldots\end{aligned}$$

Collecting terms, we obtain

$$u\left(\frac{g(1)}{2}-(k-1)\right)+\frac{u^2}{2}\left(\frac{g(2)}{2}+\frac{g(1)^2}{4}-(k-1)-(k-1)^2\right)+O(u^3). \qquad (7)$$

From Lemma 3, we have another expression for the covariance. Comparing coefficents of u and u^2 in (7) and Lemma 3, we obtain $g(1) = 2(k-1)$ and

$$\frac{g(2)}{2}+\frac{g(1)^2}{4}-(k-1)-(k-1)^2 = \sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}n_{S_l^*}}-1.$$

Substituting for g gives (c) and (d). □

Lemma 5

$$\sum_{\rho\in T^2}\sum_{\rho^*\in T^2}\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}n_{S_l^*}}-1\right) = \frac{n!}{\prod_{r=1}^{k} n_{S_r}!}\,\frac{n!}{\prod_{l=1}^{k} n_{S_l^*}!}\,\frac{(k-1)^2}{n-1}.$$

Proof To show this, assign n objects at random to the cells of a $k \times k$ table, subject to there being $n_{S_r} > 0$ objects in the r^{th} row and $n_{S_l^*} > 0$ objects in the l^{th} column. Note that for $r = 1, \ldots, k$ and $l = 1, \ldots, k$, when n_{S_r} and $n_{S_l^*}$ are fixed, $t_{S_r S_l^*}$ has a hypergeometric distribution, that is,

$$P(t_{S_r S_l^*} = j) = \frac{\binom{n_{S_r}}{j}\binom{n-n_{S_r}}{n_{S_l^*}-j}}{\binom{n}{n_{S_l^*}}}, j = 1, \ldots, \text{ min } (n_{S_r}, n_{S_l^*}).$$

So we have $E(t_{S_r S_l^*}) = n_{S_r}\, n_{S_l^*}/n$ and $Var(t_{S_r S_l^*}) = (n_{S_r}\, n_{S_l^*}/n)\,(1-(n_{S_r}/n))\,(n-n_{S_l^*})/(n-1)$. Hence,

$$E\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}\, n_{S_l^*}} - 1\right) = \frac{(k-1)^2}{n-1}. \tag{8}$$

The probabilities of getting any particular ρ and ρ^*, when $n_{S_r}, r = 1, \ldots, k$ and $n_{S_l^*}, l = 1, \ldots, k$ are fixed, are $\prod_{r=1}^{k} n_{S_r}!/n!$ and $\prod_{l=1}^{k} n_{S_l^*}!/n!$ respectively. It follows that

$$E\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}\, n_{S_l^*}} - 1\right) = \sum_{\rho\in T^2}\sum_{\rho^*\in T^2}\frac{\prod_{r=1}^{k} n_{S_r}!}{n!}\,\frac{\prod_{l=1}^{k} n_{S_l^*}!}{n!}\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}\, n_{S_l^*}} - 1\right)$$

Because the probabilities of getting any particular ρ and ρ^* do not depend on ρ and ρ^*, we can rewrite the above as

$$\sum_{\rho\in T^2}\sum_{\rho^*\in T^2}\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}\, n_{S_l^*}} - 1\right) = \frac{n!}{\prod_{r=1}^{k} n_{S_r}!}\,\frac{n!}{\prod_{l=1}^{k} n_{S_l^*}!}\,E\left(\sum_{r=1}^{k}\sum_{l=1}^{k}\frac{t^2_{S_r S_l^*}}{n_{S_r}\, n_{S_l^*}} - 1\right).$$

Substituting for (8), we obtain the required result. □

Lemma 6 *For fixed k,*

$$a(n,k) = \sum_{\mathbf{n}\in N}\frac{n}{n_{S_1}\ldots n_{S_k}} \sim k\,[\ln(n)]^{k-1}, \;\; \textit{as } n \to \infty,$$

where $a_n \sim b_n$ represents $a_n/b_n \to 1$ as $n \to \infty$.

Proof We use induction on k to prove the result. Trivially, $a(n,1) = 1 \sim 1\,[\ln(n)]^{1-1}$. Assume that $a(n,k) \sim k\,[\ln(n)]^{k-1}$. We have

$$a(n, k+1) = \sum \frac{n}{n_{S_1}\ldots n_{S_{k+1}}},$$

where the summation is over $\{n_{S_1}, \ldots, n_{S_{k+1}} | \sum_{i=1}^{k+1} n_{S_i} = n, n_{S_j} > 0, j = 1, \ldots, k+1\}$. Splitting up the summation, this becomes

$$\sum_{n_{S_{k+1}}=1}^{n-k} \frac{n}{n_{S_{k+1}}\,(n - n_{S_{k+1}})} \sum \frac{n - n_{S_{k+1}}}{n_{S_1} \ldots n_{S_k}},$$

where the summation is over $\{n_{S_1}, \ldots, n_{S_k} | \sum_{i=1}^{k} n_{S_i} = n - n_{S_{k+1}}, n_{S_j} > 0, j = 1, \ldots, k\}$, which we can rewrite as $\sum_{j=1}^{n-k} (n/(j\,(n-j)))\,a(n-j,k)$. By assumption,

$$\begin{aligned}
\sum_{j=1}^{n-k} \frac{n}{j\,(n-j)}\,a(n-j,k) &\sim \sum_{j=1}^{n-k} k\,[\ln(n-j)]^{k-1} \left(\frac{1}{j} + \frac{1}{n-j}\right) \\
&= \sum_{j=1}^{n-k} \frac{k\,[\ln(n-j)]^{k-1}}{j} + \sum_{j=1}^{n-k} \frac{k\,[\ln(n-j)]^{k-1}}{n-j}.
\end{aligned}$$

Note that when j is near n the assumed approximant $k\,[\ln(n-j)]^{k-1}$ will not hold, but the size of such terms is bounded by a constant times $\ln(n)$. The first term

$$\begin{aligned}
\sum_{j=1}^{n-k} \frac{k\,[\ln(n-j)]^{k-1}}{j} &\sim \int_1^{n-k+1} \frac{k\,[\ln(n-y)]^{k-1}}{y}\,dy \\
&\sim k\,[\ln(n-1)]^{k-1} \int_1^{n-k+1} \frac{dy}{y} \\
&= k\,[\ln(n-1)]^{k-1} \ln(n-k+1) \\
&\sim k\,[\ln(n)]^{k}
\end{aligned}$$

and the second term

$$\begin{aligned}
\sum_{j=1}^{n-k} \frac{k\,[\ln(n-j)]^{k-1}}{n-j} &\sim \int_1^{n-k+1} \frac{k\,[\ln(n-y)]^{k-1}}{n-y}\,dy \\
&= [\ln(n-1)]^{k} - [\ln(k-1)]^{k} \\
&\sim [\ln(n)]^{k}.
\end{aligned}$$

Hence, $a(n, k+1) \sim (k+1)\,[\ln(n)]^{k}$, as required. □

Lemma 7 *Let $X_1, \ldots, X_n$ be sampled from a N(0,1) distribution. Let*

$$\sum_{r=1}^{k} n_{S_r} \left(\overline{X}_{S_r} - \overline{X}\right)^2 = \mathbf{X}'\,\mathbf{A}_1\,\mathbf{X} \textit{ and} \sum_{l=1}^{k} n_{S_l^*} \left(\overline{X}_{S_l^*} - \overline{X}\right)^2 = \mathbf{X}'\,\mathbf{A}_2\,\mathbf{X},$$

where $n_{S_r} > 0$ *for* $r = 1, \ldots, k$, $\sum_{r=1}^{k} n_{S_r} = n$, $n_{S_l^*} > 0$ *for* $l = 1, \ldots, k$ *and* $\sum_{l=1}^{k} n_{S_l^*} = n$ *and let*

$$W_n = \sum_{\rho \in T^1} \frac{\left(\prod_{r=1}^{k}(n_{S_r} - 1)!\right)}{(n-1)!} \exp\left(\frac{u}{2} \sum_{r=1}^{k} n_{S_r} \left(\overline{X}_{S_r} - \overline{X}\right)^2\right).$$

Then, if $\sigma_0^2 < 0.5$,

$$Var(W_n) \to 0 \text{ as } n \to \infty.$$

Proof

$$Var(W_n) = \sum_{\rho \in T^1} \sum_{\rho^* \in T^1} \frac{\left(\prod_{r=1}^{k}(n_{S_r} - 1)!\right)}{(n-1)!} \frac{\left(\prod_{l=1}^{k}(n_{S_l^*} - 1)!\right)}{(n-1)!}$$
$$\times Cov\left(\exp\left(\frac{u}{2} \sum_{r=1}^{k} n_{S_r} \left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2} \sum_{l=1}^{k} n_{S_l^*} \left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right). \tag{9}$$

We need a bound for the covariance term. From Lemma 4 (a),

$$Cov\left(\exp\left(\frac{u}{2} \sum_{r=1}^{k} n_{S_r} \left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2} \sum_{l=1}^{k} n_{S_l^*} \left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right)$$
$$= \prod_{i=1}^{2(k-1)} (1 - u\,\delta_i)^{-1/2} - (1-u)^{-(k-1)}.$$

Recall that $u = \sigma_0^2/(\sigma_0^2 + 1)$. If we let $v = u/(1-u) = \sigma_0^2$ and $\epsilon_i = \delta_i - 1$, this is equal to

$$(v+1)^{k-1} \left(\prod_{i=1}^{2(k-1)} (1 - v\,\epsilon_i)^{-1/2} - 1\right). \tag{10}$$

As $0 \le v \le 0.5$ and $0 \le \delta_i \le 2$, by Lemma 4 (b), we have that $|v\,\epsilon_i| \le 0.5$. Using the fact that $-\ln(1-y) \le y + y^2$ for $|y| \le 0.5$ (which follows from Apostol, pp 181, Exercises 17(b) and 18(b)), we have $-\ln(1 - v\,\epsilon_i) \le v\,\epsilon_i + v^2\,\epsilon_i^2$. Hence,

$$\prod_{i=1}^{2(k-1)} (1 - v\,\epsilon_i)^{-1/2} = \exp\left(\frac{1}{2} \sum_{i=1}^{2(k-1)} -\ln(1 - v\,\epsilon_i)\right)$$
$$\le \exp\left(\frac{1}{2} \sum_{i=1}^{2(k-1)} \left(v\,\epsilon_i + v^2\,\epsilon_i^2\right)\right).$$

Substituting this expression in equation (10), we have

$$Cov\left(\exp\left(\frac{u}{2} \sum_{r=1}^{k} n_{S_r} \left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2} \sum_{l=1}^{k} n_{S_l^*} \left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right)$$
$$\le (v+1)^{k-1} \left(\exp\left(\frac{v}{2} \sum_{i=1}^{2(k-1)} \epsilon_i + \frac{v^2}{2} \sum_{i=1}^{2(k-1)} \epsilon_i^2\right) - 1\right) \tag{11}$$

From Lemma 4 (c), we have

$$\sum_{i=1}^{2(k-1)} \epsilon_i = \sum_{i=1}^{2(k-1)} (\delta_i - 1) = \sum_{i=1}^{2(k-1)} \delta_i - 2(k-1) = 0. \tag{12}$$

Also, by Lemma 4 (c),

$$\sum_{i=1}^{2(k-1)} \epsilon_i^2 = \sum_{i=1}^{2(k-1)} (\delta_i - 1)^2 = \sum_{i=1}^{2(k-1)} \delta_i^2 - 2(k-1),$$

which, by Lemma 4 (d), is equal to

$$2\left(\sum_{r=1}^{k}\sum_{l=1}^{k} \frac{t_{S_r S_l^*}^2}{n_{S_r} n_{S_l^*}} - 1\right). \tag{13}$$

Substituting for $\sum_{i=1}^{2(k-1)} \epsilon_i$ in (11), using (12) and using the fact that $\exp(y) \leq 1 + y\exp(y)$, for all $y \geq 0$ (Apostol, pp 359, Exercise 33), we have

$$Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*}\left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right)$$
$$\leq (v+1)^{k-1}\left(\frac{v^2}{2}\sum_{i=1}^{2(k-1)} \epsilon_i^2\right)\exp\left(\frac{v^2}{2}\sum_{i=1}^{2(k-1)} \epsilon_i^2\right).$$

This expression is less than or equal to

$$(v+1)^{k-1}\left(\frac{v^2}{2}\sum_{i=1}^{2(k-1)} \epsilon_i^2\right)\exp\left(v^2(k-1)\right),$$

as $|\epsilon_i| \leq 1$. Substituting for $\sum_{i=1}^{2(k-1)} \epsilon_i^2$, using (13), gives

$$Cov\left(\exp\left(\frac{u}{2}\sum_{r=1}^{k} n_{S_r}\left(\overline{X}_{S_r} - \overline{X}\right)^2\right), \exp\left(\frac{u}{2}\sum_{l=1}^{k} n_{S_l^*}\left(\overline{X}_{S_l^*} - \overline{X}\right)^2\right)\right)$$
$$\leq v^2(v+1)^{k-1}\left(\sum_{r=1}^{k}\sum_{l=1}^{k} \frac{t_{S_r S_l^*}^2}{n_{S_r} n_{S_l^*}} - 1\right)\exp\left(v^2(k-1)\right).$$

Substituting this bound for the covariance term in the expression for $Var(W_n)$ in (9), we obtain

$$Var(W_n) \leq v^2(v+1)^{k-1}\exp\left(v^2(k-1)\right)\sum_{\rho\in T^1}\sum_{\rho^*\in T^1} \frac{\left(\prod_{r=1}^{k}(n_{S_r}-1)!\right)}{(n-1)!}$$
$$\times\frac{\left(\prod_{l=1}^{k}(n_{S_l^*}-1)!\right)}{(n-1)!}\left(\sum_{r=1}^{k}\sum_{l=1}^{k} \frac{t_{S_r S_l^*}^2}{n_{S_r} n_{S_l^*}} - 1\right).$$

Summing first over $\rho \in T^2$, then over $\mathbf{n} \in N$, which is equivalent to summing over $\rho \in T^1$, the above is equal to

$$v^2 (v+1)^{k-1} \exp\left(v^2 (k-1)\right) \sum_{\mathbf{n} \in N} \sum_{\mathbf{n}^* \in N} \frac{\left(\prod_{r=1}^{k} (n_{S_r} - 1)!\right)}{(n-1)!}$$
$$\times \frac{\left(\prod_{l=1}^{k} (n_{S_l^*} - 1)!\right)}{(n-1)!} \sum_{\rho \in T^2} \sum_{\rho^* \in T^2} \left(\sum_{r=1}^{k} \sum_{l=1}^{k} \frac{t^2_{S_r S_l^*}}{n_{S_r} n_{S_l^*}} - 1 \right).$$

By Lemma 5, this is equal to

$$v^2 (v+1)^{k-1} \exp\left(v^2 (k-1)\right) \sum_{\mathbf{n} \in N} \sum_{\mathbf{n}^* \in N} \frac{\left(\prod_{r=1}^{k} (n_{S_r} - 1)!\right)}{(n-1)!}$$
$$\times \frac{\left(\prod_{l=1}^{k} (n_{S_l^*} - 1)!\right)}{(n-1)!} \frac{n!}{\prod_{r=1}^{k} n_{S_r}!} \frac{n!}{\prod_{l=1}^{k} n_{S_l^*}!} \frac{(k-1)^2}{n-1}$$
$$= v^2 (v+1)^{k-1} \exp\left(v^2 (k-1)\right) \frac{(k-1)^2}{n-1} a(n,k)^2,$$

Rewriting, we have

$$\begin{aligned} Var(W_n) \;\; &\leq \;\; v^2 (v+1)^{k-1} \exp\left(v^2 (k-1)\right) (k-1)^2 k^2 \\ &\quad \times \left(\frac{\ln(n)}{(n-1)^{1/2(k-1)}} \right)^{2(k-1)} \left(\frac{a(n,k)}{k \ln(n)^{k-1}} \right)^2. \end{aligned}$$

Because, by Lemma 6,

$$\frac{a(n,k)}{k \ln(n)^{k-1}} \to 1 \text{ as } n \to \infty,$$

and

$$\frac{\ln(n)}{(n-1)^{1/2(k-1)}} \to 0 \text{ as } n \to \infty,$$

the result follows. □

Proof of Theorem 1 Rewrite (4) as

$$\frac{P(B=k|\mathbf{X})}{P(B=1|\mathbf{X})} = m^{k-1} (1-u)^{(k-1)/2} \frac{1}{k!} EW_n \frac{W_n}{EW_n}.$$

First, consider W_n/EW_n. From Chebyshev's inequality and Lemma 7, we have

$$\frac{W_n}{EW_n} \to 1 \text{ in probability, as } n \to \infty, \text{ if } |EW_n| \geq 1.$$

Then consider EW_n. Replace the summation over $\rho \in T^1$ in W_n by summing first over $\rho \in T^2$, then over $\mathbf{n} \in N$. Taking expected values and using Lemma 2, we obtain

$$\begin{aligned} EW_n &= \sum_{\mathbf{n}\in N} \frac{\left(\prod_{r=1}^{k}(n_{S_r}-1)!\right)}{(n-1)!} \sum_{\rho\in T^2} (1-u)^{-(k-1)/2} \\ &= (1-u)^{-(k-1)/2}\, a(n,k). \end{aligned} \tag{14}$$

Note that this quantity is greater than or equal to one. Substituting for EW_n using (14) and for $m = \lambda/ln(n)$, we obtain

$$\begin{aligned} \frac{P(B=k|\mathbf{X})}{P(B=1|\mathbf{X})} &= \frac{\lambda^{k-1}}{(k-1)!}\frac{a(n,k)}{k\ln(n)^{k-1}}\frac{W_n}{EW_n} \\ &\to \frac{\lambda^{k-1}}{(k-1)!} \text{ in probability, as } n\to\infty, \end{aligned}$$

because

$$\frac{a(n,k)}{k\ln(n)^{k-1}} \to 1 \text{ as } n\to\infty, \text{ by Lemma 6,}$$

and

$$\frac{W_n}{EW_n} \to 1 \text{ in probability, as } n\to\infty, \text{ if } |EW_n| \geq 1.$$

We have shown that

$$\frac{P(B-1=k-1|\mathbf{X})}{P(B-1=0|\mathbf{X})} \to \frac{\lambda^{k-1}}{(k-1)!} \text{ in probability, as } n\to\infty,$$

as required. □

5 Separation into two problems when there are two sets of equal means

We will consider the case where we have two sets of equal μ_i's. Assume that n is even. Let $X_1,\ldots,X_{n/2}$ be sampled from a $N(0,1)$ distribution and $X_{(n/2)+1},\ldots,X_n$ be sampled from a $N(\theta_n,1)$ distribution. We will show that if θ_n is large enough, we can regard the above as two separate problems, one involving $H_1 = \{1,\ldots,n/2\}$ and the other involving $H_2 = \{(n/2)+1,\ldots,n\}$. This will follow from the fact that partitions with sets containing objects from both H_1 and H_2 are probabilistically negligible, which we prove in the following theorem:

Theorem 8 *Let $X_1, \ldots, X_{n/2}$ be sampled from a $N(0,1)$ distribution and $X_{(n/2)+1}, \ldots, X_n$ be sampled from a $N(\theta_n, 1)$ distribution. Let the partition ρ be distributed according to a product partition model with the distributions specified in Section 3 and prior parameters μ_0, σ_0^2 and $m = \lambda / \ln(n)$, where $0 < \lambda < 1$. Let $0 < \sigma_0^2 < 0.5$ and $\theta_n = (A + 2\sqrt{2+\epsilon})\sqrt{\ln(n)}$, where $\epsilon > 0$ and $A^2 > 4(\sigma_0^2 + 1)/\sigma_0^2$. Define*

$$C_n = \{\rho \textit{ contains at least one component intersecting both } H_1 \textit{ and } H_2\}.$$

Then

$$P(C_n | \mathbf{X}) \to 0 \textit{ in probability, as } n \to \infty.$$

So if we have two sets of equal means a distance θ_n apart, we can regard this as two separate problems. We can then apply the result of Theorem 1 to each problem (note that the conditions on the prior parameters here also satisfy the conditions on the prior parameters in Theorem 1). If we choose m to be small, this will put high probability on one block for each of the separate problems, that is, a high probability of two blocks for the overall problem (which is the true number of blocks). We can extend the above to $k > 2$ sets a distance θ_n apart. Regarding this as k separate problems and applying Theorem 1 to each, this will give high probability to the true number of blocks k. This suggests that the product partition model will work well when the sets are well separated.

The proof of Theorem 8 will be omitted.

6 Simulations

We did some simulations to check if the results of Theorem 1 hold for moderate values of n. We let $\lambda = m \ln(n) = 1$. We looked at various values of σ_0^2, some which satisfied the condition on σ_0^2 in the theorem and others which did not. The program used is similar to the Markov sampling program discussed in Crowley (1993), except that here m and σ_0^2 are fixed. We generated 50 different samples of $X_1, \ldots, X_n$. Corresponding to each sample of

$X_1, \ldots, X_n$, there were 100 Markov samples. The first 10 Markov samples were ignored. We estimated the probability that $B = k$ by the proportion, $\hat{r}_k$, of partitions with k blocks.

We estimate λ by $\hat{\lambda} = \sum_{k=1}^{n} k \, \hat{r}_k - 1$. We also compute $\hat{p}_k$, the probability that $B = k$ if $B - 1$ has a Poisson distribution with expected value $\hat{\lambda}$, and p_k, the probability that $B = k$ if $B - 1$ has a Poisson distribution with expected value λ.

For cases which satisfy the condition on σ_0^2 in Theorem 1, we find that the simulation results are consistent with the results in the theorem, that is, $\hat{r}_k$ is close to p_k. Note that $\hat{r}_k$ is usually slightly closer to $\hat{p}_k$ than to p_k. For cases not satisfying the condition on σ_0^2 in Theorem 1, we find that $B - 1$ still seems to have a Poisson distribution but with expected value $\hat{\lambda}$ instead of λ. This would suggest that there is a corresponding theorem for larger values of σ_0^2 but the methods used to prove Theorem 1 could not be applied.

The values of p_k, $\hat{p}_k$ and $\hat{r}_k$ for $k = 1, \ldots, 9$ when $n = 20$ are given in Tables 1 and 2 for $\sigma_0^2 = .25$ and 5 respectively. The three quantities are approximately zero when k is greater than 9. For more simulation results, see Crowley (1993).

We also ran some simulations to see how the probability that ρ contains at least one component intersecting both H_1 and H_2 when $\lambda = 1/2$ varied with n, θ_n and σ_0^2. The event that ρ contains at least one component intersecting both H_1 and H_2 will be called a crossover. The program used is practically the same as the program used above. Again, we generated 50 different samples of $X_1, \ldots, X_n$ and there were 100 Markov samples (of which the first 10 were not used) corresponding to each $X_1, \ldots, X_n$. We looked at two values of σ_0^2 that satisfy the condition on σ_0^2 in Theorem 8, that is, $\sigma_0^2 = .25, .5$ and also at $\sigma_0^2 = 4, 8, 12, 24$. The proportion of crossovers for $n = 20, 50, 100$, the above values of σ_0^2 and $\theta_n = 3, 6, 9$ is given in Table 3. This proportion increases with n and decreases with σ_0^2. For $\theta_n = 3$, the proportion of crossovers is greater than 0.9 for all values of n, σ_0^2 and θ_n. Even for $\theta_n = 6$, this proportion is still quite high, ranging from greater than .79 when $\sigma_0^2 = .25$ to greater than .05 when $\sigma_0^2 = 24$. For $\theta_n = 9$, the proportion of crossovers is less than .02 for $\sigma_0^2 \geq .5$ and greater than .06 for $\sigma_0^2 = .25$.

The condition $\sigma_0^2 < 0.5$ is only needed in Theorem 8 so that Theorem 1 can be applied. Hence, if Theorem 1 held even when $\sigma_0^2 > 0.5$, then so would Theorem 8. This is consistent with the results of the simulations.

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Table 1
Distribution of number of blocks for 20 observations when $\lambda = 1$, $\hat{\lambda} = 1.04$ and $\sigma_0^2 = .25$.

k	$\hat{r}_k$ [1]	$\hat{p}_k$ [2]	p_k [3]
1	0.3287	0.3522	0.3679
2	0.3896	0.3675	0.3679
3	0.2053	0.1918	0.1839
4	0.0644	0.0667	0.0613
5	0.0102	0.0174	0.0153
6	0.0016	0.0036	0.0031
7	0.0002	0.0006	0.0005
8	0.0000	0.0001	0.0001
9	0.0000	0.0000	0.0000

[1] $\hat{r}_k$ = proportion of partitions where $B = k$.
[2] $\hat{p}_k$ = poisson probability that $B = k$, if $B - 1 \sim P(\hat{\lambda})$.
[3] p_k = poisson probability that $B = k$, if $B - 1 \sim P(\lambda)$.

Table 2

Distribution of number of blocks for 20 observations when $\lambda = 1$, $\hat{\lambda} = .88$ and $\sigma_0^2 = 5$.

k	$\hat{r}_k$ [1]	$\hat{p}_k$ [2]	p_k [3]
1	0.3844	0.4165	0.3679
2	0.4040	0.3648	0.3679
3	0.1682	0.1597	0.1839
4	0.0384	0.0466	0.0613
5	0.0044	0.0102	0.0153
6	0.0004	0.0018	0.0031
7	0.0000	0.0003	0.0005
8	0.0000	0.0000	0.0001
9	0.0000	0.0000	0.0000

[1] $\hat{r}_k$ = proportion of partitions where $B = k$.

[2] $\hat{p}_k$ = poisson probability that $B = k$, if $B - 1 \sim P(\hat{\lambda})$.

[3] p_k = poisson probability that $B = k$, if $B - 1 \sim P(\lambda)$.

Table 3
The proportion of partitions containing at least one component intersecting both H_1 and H_2 for $\lambda = 1/2$ and different values of σ_0^2, θ_n and n.

		n		
σ_0^2	θ_n	20	50	100
0.25	3	1.0000	1.0000	1.0000
	6	0.7991	0.9404	0.9962
	9	0.0689	0.0942	0.1538
0.5	3	0.9989	1.0000	1.0000
	6	0.3653	0.5547	0.7787
	9	0.0060	0.0080	0.0111
4	3	0.9513	0.9976	1.0000
	6	0.0718	0.1244	0.2193
	9	0.0002	0.0002	0.0002
8	3	0.9320	0.9940	1.0000
	6	0.0589	0.1087	0.1922
	9	0.0002	0.0002	0.0004
12	3	0.9258	0.9938	1.0000
	6	0.0616	0.0989	0.1847
	9	0.0002	0.0002	0.0002
24	3	0.9267	0.9920	0.9998
	6	0.0569	0.1016	0.1787
	9	0.0000	0.0002	0.0000