COMPARISON OF THE P-VALUE AND POSTERIOR PROBABILITY OF A SHARP NULL HYPOTHESIS*

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Abstract

For iid observations from a multivariate normal distribution in p dimensions with an unknown mean and a covariance matrix proportional to the Identity, we revisit the issue of the apparent irreconcilability of the classical test for a point null and the standard Bayesian formulation for testing such a point null. With appropriate families of priors on the alternative, we consider the threshold value of the apriori probability of the point null required for the smallest (over priors on the alternative) posterior probability and the classical P-value to coincide. We also consider, for an arbitrary but fixed apriori probability of the point null, the ratio of the minimum posterior probability and the classical P-value. The main results emphasize properties of the null distributions of these two quantities, such as their quartiles, the effect of the dimension by means of formal dimensional asymptotics, etc. Among many theorems proved in the article are the results that regardless of the dimension p, the threshold prior probability as defined above has a median exactly equal to 0.5 in many cases, and the ratio as described above has a median exactly equal to twice the apriori probability assigned to the null. These and other results are an attempt to clarify the issue of typicality : how often the Bayes - classical conflict will arise and in what magnitude.

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1 Introduction

1.1 The goal of this article

It is very well known that in parametric testing problems where the null hypothesis is sharp, the standard Bayesian method and the classical method of testing the null hypothesis are sometimes hard to reconcile. There are many results to this effect in the literature; the phenomenon known as Lindley's paradox is one of the first results in this direction. More recently, inspired by the writings of Edwards,Lindman and Savage(1963), Berger and Sellke (1987) made an illuminating contribution wherein they show that for testing a sharp null hypothesis about the mean of a univariate normal distribution with a known variance, even the minimum posterior probability of the null hypothesis over really large classes of priors on the two-sided alternative can be significantly larger than the P-value of the common classical test, regardless of the sample size, provided the sharp null is assigned,apriori, a probability of 0.5. This result is then given the interpretation that classical P-values tend to understate the plausibility of the sharp null. This work was followed by an array of similar results in different problems; Casella and Berger(1987) is a notable contribution that showed such a conflict appears to be germane to the sharp null.

The results in Berger and Sellke(1987) were widely discussed in the profession. Many were convinced, others skeptical. Some were skeptical because they had a very fundamental objection to the standard Bayesian formulation of the sharp null testing problem, some others because they did not actually believe that sharp null hypotheses ever occur. We will not go into these particular points in this article. Our intention is to explore, among other things, a third obviously natural issue within the structure of the problem : the relevance and the importance of the assumption made in all or most of these articles that apriori the sharp null has a probability of 0.5. It is fair to say that this assumption was also troublesome, and in any case, it seems natural and necessary that one explores in greater depth the role of this assumption in the Berger-Sellke phenomenon. It is clear that these authors were undoubtedly aware of this issue, but did not pursue it in their writings. The literature on this and the conditional-classical conflict includes Berger and Delampady(1987), Brown(1967,1978), Kiefer(1977), Olshen(1973), Robinson(1979), Dickey(1977), Good(1967), Lindley(1957), Delampady(1989 a,b,1990), Casella(1988), Hall and Selinger(1986), Neyman(1976) and Lehmann(1993).

1.2 An illustrative example

Consider the case of an i.i.d. sample of size n from the $N(\theta, 1)$ distribution ; we want to test $H_0 : \theta = 0$ $vs.$ $H_1 : \theta \neq 0$. We assign a prior probability of π_0 to the null H_0, and use an arbitrary symmetric and unimodal prior g on the alternative (symmetric and unimodal in the same sense as in Berger and Sellke).

If the data are such that the P-value of the usual classical test is .10, then the infimum (over all g of the above type) of the posterior probability of H_0 equals .390 if we use $\pi_0 = .5$, equals 0.175 if we use $\pi_0 = .25$, and equals 0.03 if we use $\pi_0 = .05$. It is evident that there is a (unique) value of the apriori probability such that the two quantities, namely the classical P-value and the minimum posterior probability, are exactly equal. For these particular data, this threshold value of π_0 is easily seen to be (approximately) 0.148. Naturally, the threshold value of π_0 will be different for different data(i.e., different values of the sample mean). The (frequentist) question of typicality will then be the following : typically, what kind of an apriori probability is needed for the Berger-Sellke phenomenon to just arise ? In other words, what is the distribution of the threshold prior probability ? One can also ask related questions such as what are its median, quartiles etc. Notice the important fact that the threshold value of π_0, as we define it, is in close philosophical resemblance to the P-value itself : the P-value is exactly the threshold-level (type 1 error) at which the null hypothesis is just rejected. In any event, the threshold prior probability will clearly shed light on the issue of the criticality of the $\pi_0 = .5$ assumption made in the existing literature.

1.3 Description of main results and notation

We will , in general, consider the multivariate case : $X_1, \ldots, X_n$ are iid random vectors with the $N(\underset{\sim}{\theta}, \sigma^2 I)$ distribution. A point null hypothesis $H_0 : \underset{\sim}{\theta} = 0$ is tested against a two-sided alternative $H_1 : \underset{\sim}{\theta} \neq 0$. π_0 will continue to denote the Bayesian's apriori probability on H_0. The family of priors on the alternative will have the generic notation Γ. Several choices of Γ will be used; the intention is to show that the same intriguing phenomena hold over many commonly used choices of Γ. In particular, the following families are used :

$\Gamma_{\mathcal{MVN}}$ = All multivariate normal $N(0, \tau^2 I)$ priors (Γ_{NOR} = All normal $N(0, \tau^2)$ priors in one dimension);

Γ_{SSM} = All scaled versions of a fixed functional type , with a Monotone likelihood ratio property(the precise definition is given later);

Γ_{USS} = All spherically symmetric unimodal priors.

The Berger-Sellke case is therefore covered as a special case. Throughout the article, the following notation will be used :

$f(x|\theta)$ = The likelihood function

$m_g(x) = \int_{\theta \neq 0} f(x|\theta) g(\theta) d\theta$

$m(x) = f(x|0)\pi_0 + (1-\pi_0)m_g(x)$, the overall marginal

$B_g(x)$ = The Bayes factor with respect to a given g

$\underline{P}(H_0|x,\Gamma) = inf_{g\in\Gamma} P(H_0|x)$

P = The P-value of the usual classical test

$\pi_0(x)$ = The threshold apriori probability in the sense defined earlier

$R(x) = \frac{\underline{P}(H_0|x,\Gamma_{NOR})}{P}$ (notice the implicit understanding in this line that this is with respect to a specified apriori probability π_0).

Finally, since everything depends on the sample data through the standardized $z = \sqrt{n}\bar{x}$, we will treat z as the observation itself.

In section 2, we consider the null distribution of the quantity $\pi_0(z)$. In section 3, we consider the null distribution of the quantity $R(z)$, for an arbitrary but fixed π_0 ; note that $R(z)$ can be regarded as the magnitude of the Bayes-classical conflict (differences rather than ratios are also sensible; such results are available in the Ph.D. dissertation, Oh(1993)). Section 4 gives an overall discussion and in addition some results which are specialized to the case of one dimension : for instance, the mathematically interesting result that $\pi_0(z)$ has a unimodal distribution is sketched. Due to the substantial evidence that the conflict is less pronounced in high dimensions, dimensional asymptotics are also considered in section 4, although we have kept it brief due to length considerations. The message is that even as the dimension tends to infinity, the conflict does not completely go away. A large number of the proofs are technically complex, and therefore for easy comprehension of the results, we have deferred practically all the proofs to an appendix in section 5. Two particular results among others are that *in any dimension*, for any of the families described above, the median of $\pi_0(z)$ equals exactly 1/2 and the median of $R(z)$, for an arbitrary given π_0, equals exactly $2\pi_0$. The implication, for example, is then that with the choice $\pi_0 = .5$ as in Berger and Sellke (1987), the Berger-Sellke phenomenon will just arise in

50% of sharp testing problems. Many other interpretations will emerge and will be provided in the appropriate sections. Using the theoretical results we prove, we also provide tables for $P_{H_0}(R(\underset{\sim}{Z}) > 1)$ and $P_{H_0}(\frac{1}{k} \leq R(\underset{\sim}{Z}) \leq k)$. These tables provide valuable information about the frequency with which a Bayes-classical conflict arises for general π_0 and p, not just $\pi_0 = 0.5, p = 1$. In fact, we are able to prove analytically that for any $\pi_0 \geq P(\chi_p^2 \geq p)$ for any $p \geq 1$, $P_{H_0}(R(\underset{\sim}{Z}) > 1)$ is exactly equal to π_0. So for instance, with $\pi_0 = 0.4$, a Bayes-classical conflict in the sense of Berger and Sellke(1987) arises exactly 40% of the times when $p = 1$. One can evidently make the argument on the basis of Table 2 that in a strictly frequentist sense, the Berger-Sellke phenomenon will not occur too often even with $\pi_0 = 0.3$. We would like to clarify that arguably one should (or can) consider the null distributions of π_0, R conditioned on an event of the type $.001 < P\!-\!value < .3$ (other values are statistically unprovocative). Our methods apply with modifications to this case, although the neat nature of the results we present are clearly lost in the process. The principal results in this article are of the following type :

a. investigation of quartiles, means and standard deviations of $\pi_0(\underset{\sim}{Z})$ and $R(\underset{\sim}{Z})$;

b. investigation of their distributions, in particular shape properties ;

c. dimensional asymptotics ;

d. brief investigation of large sample behavior under the alternative ;

e. numerical illustration of the theorems proved.

We believe the mathematics of this article may work for nonnormal problems ; distributions under the alternative or the marginal distributions are also of interest but not treated here. Figure 1 is given for visual illustration of the difference between the P-value and $\underline{P}(H_0|\mathbf{x}, \Gamma_{NOR})$ for $\pi_0 = 0.5$.

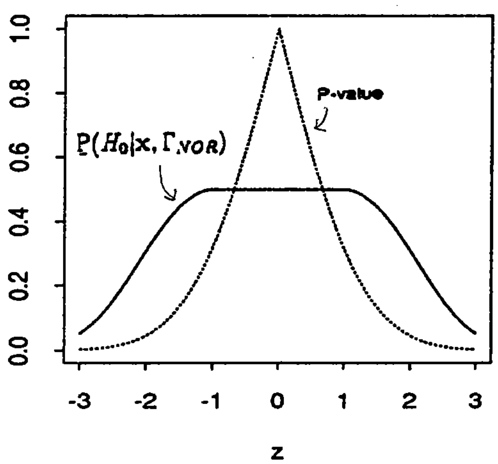

Figure 1(a)

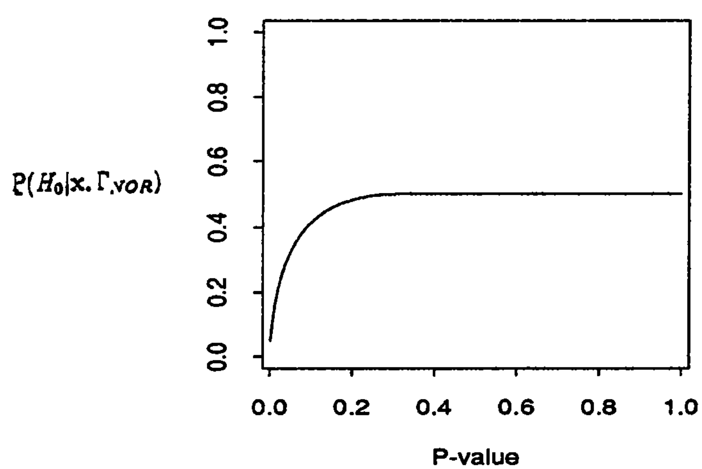

Figure 1(b)

Figure 1: (a) Plots of the P-value and $\underline{P}(H_0|\mathbf{x}, \Gamma_{NOR})$ (b) The Plot of the $\underline{P}(H_0|\mathbf{x}, \Gamma_{NOR})$ vs. the P-value

2 Threshold Prior Probability

2.1 Normal distributions

Let us consider a family of Normal priors on H_1. That is , $g(\underset{\sim}{\theta}) \in \Gamma_{\mathcal{MVN}}$, where

$$\Gamma_{\mathcal{MVN}} = \{\text{all } \mathcal{MVN}(\underset{\sim}{0}, \tau^2 \mathrm{I}_p) \text{ distributions}, 0 < \tau^2 < \infty\}.$$

Since a sufficient statistic for $\underset{\sim}{\theta}$ is $\overline{\underset{\sim}{X}}$ which is distributed as $\mathcal{MVN}(\underset{\sim}{\theta}, 1/n\mathrm{I}_p)$, $m_g(\overline{\underset{\sim}{X}})$ is a $\mathcal{MVN}(\underset{\sim}{0}, (\tau^2 + 1/n)\mathrm{I}_p)$ distribution. Then $m_g(\overline{\underset{\sim}{\mathbf{X}}})$ is maximized at $\hat{\tau}^2$, where $\hat{\tau}^2 = \max\{0, (\overline{\underset{\sim}{x}}^t \overline{\underset{\sim}{x}})/p - 1/n\}$. Let $\underset{\sim}{z} = \sqrt{n}\overline{\underset{\sim}{x}}$. We then have

$$\sup_{g \in \Gamma_{\mathcal{MVN}}} \frac{m_g(\overline{\underset{\sim}{x}})}{f(\overline{\underset{\sim}{x}}|0)} = \begin{cases} 1 & \text{if } \|z\|^2 \leq p \\ \frac{e^{\frac{1}{2}\|z\|^2}}{(e\|z\|^2/p)^{p/2}} & \text{if } \|z\|^2 > p. \end{cases}$$

Thus the infimum of the posterior probability of H_0 is

$$\underline{\mathrm{P}}(H_0|\underset{\sim}{x}, \Gamma_{\mathcal{MVN}}) = \begin{cases} \pi_0 & \text{if } \|z\|^2 \leq p \\ (1 + \frac{1-\pi_0}{\pi_0} \frac{e^{\frac{1}{2}\|z\|^2}}{(e\|z\|^2/p)^{p/2}})^{-1} & \text{if } \|z\|^2 > p. \end{cases}$$

Note that since $\|Z\|^2 = n\overline{\underset{\sim}{X}}^t\overline{\underset{\sim}{X}}$ has a *Chisquare* distribution with p degrees of freedom, the P-value is given by

$$P = P(\chi^2_p \geq \|z\|^2).$$

Solving the equation $\underline{\mathrm{P}}(H_0|\underset{\sim}{x}, \Gamma_{\mathcal{MVN}})$ =P-value for π_0 produces

$$\pi_0(\underset{\sim}{z}) = \begin{cases} P(\chi^2_p \geq \|z\|^2) & \text{if } \|z\|^2 \leq p \\ \frac{P(\chi^2_p \geq \|z\|^2)}{P(\chi^2_p \geq \|z\|^2) + (e\|z\|^2/p)^{p/2} e^{-1/2\|z\|^2} P(\chi^2_p \leq \|z\|^2)} & \text{if } \|z\|^2 > p. \end{cases}$$

Hence $\pi_0(\underset{\sim}{z})$ is a function of $\|z\|^2$ alone, as expected. Let

$$\eta(t) = (1 + \frac{(et/p)^{p/2} e^{-t/2} P(\chi^2_p \leq t)}{P(\chi^2_p \geq t)})^{-1}.$$

so that $\pi_0(\underset{\sim}{z}) = \eta(\|z\|^2)$ for $\|z\|^2 > p$. Let us now consider some interesting features of the distribution of $\pi_0(\underset{\sim}{Z})$ under the null hypothesis such as the median, first and third quantiles. To derive these quantiles, we will use the monotonicity of $\pi_0(\underset{\sim}{z})$ in $\|z\|^2$, for proving which the following lemma is needed (see Barlow, Proschan and Hunter (1965)).

Lemma 1. *Let $f_p(x)$ and $F_p(x)$ denote the density function and the cumulative distribution function of the Chisquare distribution with p degrees of freedom, respectively. Then for any $p \geq 1$, $f_p(x)/(1-F_p(x))$ is increasing in $x > 0$.*

Now, the monotonicity of $\pi_0(\underset{\sim}{z})$ in $\|z\|^2$ is shown as follows.

Lemma 2. *$\pi_0(\underset{\sim}{z})$ is decreasing in $\|z\|^2$ for any given p.*

Proof :

Clearly, $P(\chi_p^2 \geq \|z\|^2)$ is decreasing in $\|z\|^2$. To show that $\eta(\|z\|^2)$ is decreasing in $\|z\|^2$, it is sufficient to show that

$$\frac{t^{p/2}e^{-t/2}P(\chi_p^2 \leq t)}{P(\chi_p^2 \geq t)}$$

is increasing in t for $t > p$. But from Lemma 1,

$$\frac{t^{p/2-1}e^{-t/2}}{P(\chi_p^2 \geq t)}$$

is increasing in t for $t > 0$, for any $p \geq 1$. Also, it is obvious that $tP(\chi_p^2 \leq t)$ is increasing in t for $t > 0$. Combining these facts, the result follows immediately. $\square$

By using the fact that the P-value $=P(\chi_p^2 \geq \|Z\|^2)$ is distributed as *Uniform(0,1)* and by Lemma 2, we can derive following result. The proof is outlined in section 5.

Theorem 1. *For the null distribution of $\pi_0(\underset{\sim}{Z})$ in any given dimension p, the median is 1/2 and the third quartile is 3/4. Also, the first quartile is given by $\eta(\chi^2_{p,\frac{3}{4}})$, where $\chi^2_{p,\frac{3}{4}}$ is third quartile of the Chisquare distribution with p degrees of freedom.*

Thus if the prior probability on H_0 is $\pi_0 = 0.5$, then in a large sequence of experiments the infimum of posterior probability over the family of Normal distributions on H_1 will be less than the P-value 50% of the times. This result helps put the results in Berger and Sellke(1987) and the subsequent results on this issue in perspective and aids in a clear understanding of the role of their assumptions that π_0 equals 0.5.

2.2 Spherically Symmetric Unimodal Priors

Let us consider $g(\underset{\sim}{\theta}) \in \Gamma_{USS}$, where

$$\Gamma_{USS} = \{\beta(\underset{\sim}{\theta}^t\underset{\sim}{\theta}) | \beta \text{ is nonegative and } \int_{\Re^p} \beta(\underset{\sim}{\theta}^t\underset{\sim}{\theta})d\underset{\sim}{\theta} = 1\}.$$

The following result can be found in Berger and Delampady(1987).

Lemma 3. *(Berger and Delampady (1987))*

$$\inf_{g\in\Gamma_{USS}} B_g(\underset{\sim}{z}) = \exp(-\|z\|^2/2)[\sup_{r\geq 0} \frac{1}{V(r)} \int_{\|\theta\|\leq r} \exp(-\|\theta - z\|^2/2)d\theta]^{-1},$$

where $V(r) = \frac{2\pi^{p/2}}{p\Gamma(p/2)} r^p$ *is the volume of a sphere of radius* r *in p-dimension.*

The above result can be significantly strengthened as follows. It is instrumental for proving Theorem 2 following thereafter.

Lemma 4.

$$\inf_{g\in\Gamma_{USS}} B_g(\underset{\sim}{z}) = 1, \quad \textit{if } \|z\|^2 \leq p.$$

Thus by Lemma 3 and 4,

$$\underset{\sim}{P}(H_0|\underset{\sim}{x}, \Gamma_{USS}) = \begin{cases} \pi_0 & \text{if } \|z\|^2 \leq p \\ \{1 + \frac{1-\pi_0}{\pi_0} \exp(-\|z\|^2/2) \\ \quad [\sup_{r\geq 0} \frac{1}{V(r)} \int_{\|\theta\|\leq r} \exp(-\|\theta - z\|^2/2)d\theta]^{-1}\}^{-1} & \text{if } \|z\|^2 > p. \end{cases}$$

By solving the equation $\underset{\sim}{P}(H_0|\underset{\sim}{x}, \Gamma_{US}) =$ P-value in π_0, we have

$$\pi_0(\underset{\sim}{z}) = \begin{cases} P(\chi_p^2 \geq \|z\|^2) & \text{if } \|z\|^2 \leq p \\ \frac{P(\chi_p^2\geq\|z\|^2)}{P(\chi_p^2\geq\|z\|^2)+\exp(-\|z\|^2/2)P(\chi_p^2\leq\|z\|^2)/Q(\|z\|^2)} & \text{if } \|z\|^2 > p, \end{cases}$$

where

$$Q(\|z\|^2) = \sup_{r\geq 0} \frac{1}{V(r)} \int_{\|\theta\|\leq r} \exp(-\|\theta - z\|^2/2)d\theta.$$

Define

$$\delta(t) = \frac{e^{-t/2}P(\chi_p^2 \leq t)}{P(\chi_p^2 \geq t)Q(t)}, \quad t = \|z\|^2 \tag{1}$$

so that $\pi_0(\underset{\sim}{z}) = (1 + \delta(t))^{-1}$ for $\|z\|^2 > p$. By showing that $\pi_0(\underset{\sim}{z}) < 1/2$ (i.e., $\delta(t) > 1$) for $\|z\|^2 > p$ and by the Uniformity of the distribution of the P-value, we have the following result.

Theorem 2. *For the null distribution of* $\pi_0(Z)$*, the median is* $1/2$ *and the third quartile is* $3/4$*.*

2.3 Symmetric Scale-Parameter distributions with M.L.R. property in one dimension

We begin with a definition of the needed concepts.

Definition 1. The real-parameter family of densities $p_\theta(x)$ is said to have monotone likelihood ratio(m.l.r.) in $T(x)$ if there exists a real-valued function $T(x)$ such that for any $\theta < \theta'$ the distributions P_θ and $P_{\theta'}$ are distinct, and the ratio $p_{\theta'}(x)/p_\theta(x)$ is a nondecreasing function of $T(x)$.

Note that any Normal prior on H_1 with mean 0 and variance σ^2 is the form of $\frac{1}{\sigma}f(\frac{\theta}{\sigma})$, where f is the standard Normal density and $f(\frac{\theta}{\sigma})$ has m.l.r. in $|\theta|$. Clearly therefore, this is a very special case of

$$\Gamma_{SSM} = \{\frac{1}{\sigma}f(\frac{\theta}{\sigma})|\sigma > 0, f \text{ is symmetric about 0 and } f(\tfrac{\theta}{\sigma}) \text{ has m.l.r. in } |\theta|\}.$$

In the above, f is treated as a fixed function and σ varies freely in $(0, \infty)$. Such priors were considered by earlier authors in the area : in particular, see Casella and Berger(1987). Let us derive the infimum of the posterior probability $\underset{\sim}{P}(H_0|\mathbf{x}, \Gamma_{SSM})$. By symmetry of f,

$$\sup_{g \in \Gamma_{SSM}} m_g(\mathbf{x}) = \sup_{\sigma > 0} H(\sigma),$$

where

$$H(\sigma) = \sup_{\sigma>0} \int_0^\infty \frac{1}{\sigma} f(\frac{\theta}{\sigma})(e^{-(\theta-z)^2/2} + e^{-(\theta+z)^2/2})d\theta.$$

Let

$$u(\theta) = e^{-(\theta-z)^2/2} + e^{-(\theta+z)^2/2}.$$

Then by a change of variable, we get

$$\begin{aligned} H(\sigma) &= \int_0^\infty \frac{1}{\sigma} f(\frac{\theta}{\sigma}) u(\theta) d\theta \\ &= \int_0^\infty f(y) u(\sigma y) dy. \end{aligned}$$

So

$$\begin{aligned}\frac{\partial}{\partial\sigma}H(\sigma) &= \int_0^\infty f(y)u'(\sigma y)y\,dy \\ &= \int_0^\infty \frac{x}{\sigma^2}f(\frac{x}{\sigma})u'(x)dx.\end{aligned}$$

By investigating the sign changes of $u'(x)$, we have the following lemma which is proved in the appendix.

Lemma 5. *For $|z| \le 1$, $\frac{\partial}{\partial\sigma}H(\sigma) \le 0$ and for $|z| > 1$, $\frac{\partial}{\partial\sigma}H(\sigma)$ changes sign exactly once.*

Thus for $|z| \le 1$, $H(\sigma)$ is decreasing in σ for $\sigma > 0$ and for $|z| > 1$, $H(\sigma)$ has a unique extremum. To show that the extremum of $H(\sigma)$ is the maximum, i.e., $\frac{\partial}{\partial\sigma}H(\sigma)$ changes sign once in direction from the positive to the negative, for $|z| > 1$, the following well-known lemma due to Karlin(1957) is used.

Lemma 6. *(Karlin(1957)) Let $p_\theta(x)$ be a family of densities on the Real line and suppose $\{ p_\theta(x) \}$ is m.l.r. in x.*
(i) If ψ is a nondecreasing function of x, then $E_\theta(\psi(X))$ is a nondecreasing function of θ.
(ii) Assume that p_θ can be differentiated n times with respect to θ for all x. Suppose ψ changes sign n times. Then $E_\theta(\psi(X))$ changes sign at most n times. Moreover, if $E_\theta(\psi(X))$ changes sign exactly n times, then $\psi(x)$ and $E_\theta(\psi(X))$ change signs in the same order.

Note that

$$\frac{\partial}{\partial\sigma}H(\sigma) = \int_0^\infty \frac{x}{\sigma^2}f(\frac{x}{\sigma})u'(x)dx.$$

and since $f(x/\sigma)$ has the m.l.r. property, we have that for $\sigma_1 < \sigma_2$,

$$\frac{(x/\sigma_1^2)f(x/\sigma_1)}{(x/\sigma_2^2)f(x/\sigma_2)} = \frac{\sigma_2^2}{\sigma_1^2}\frac{f(x/\sigma_1)}{f(x/\sigma_2)}$$

is nonincreasing in x. Thus $C\frac{x}{\sigma^2}f(x/\sigma)$ has m.l.r. with respect to σ for $x > 0$, where the constant C is given by

$$\begin{aligned}C &= (\int_0^\infty \frac{x}{\sigma^2}f(x/\sigma)dy)^{-1} \\ &= (\int_0^\infty yf(y)dy)^{-1}.\end{aligned}$$

It is proved that $u'(x)$ changes sign once in the direction from the positive to the negative for $|z| > 1$ in the proof of Lemma 5. Thus by Lemma 5 and 6, $H(\sigma)$ has a unique maximum for $|z| > 1$ and so we have the following proposition.

Proposition 1. For $|z| \leq 1$,

$$\begin{aligned} \sup_{\sigma>0} H(\sigma) &= \lim_{\sigma\to 0} H(\sigma) \\ &= \frac{1}{2}u(0) \\ &= e^{-z^2/2} \end{aligned}$$

and for $|z| > 1$, $H(\sigma)$ is maximized at a unique point such that $\frac{\partial}{\partial\sigma}H(\sigma) = 0$.

Thus

$$\underline{P}(H_0|\mathbf{x}, \Gamma_{SSM}) = \begin{cases} \pi_0 & \text{if } |z| \leq 1 \\ (1 + \frac{1-\pi_0}{\pi_0} H(\hat{\sigma})e^{z^2/2})^{-1} & \text{if } |z| > 1, \end{cases}$$

where $\hat{\sigma}$ satisfies that $\frac{\partial}{\partial\sigma}H(\sigma)|_{\sigma=\hat{\sigma}} = 0$. Since the P-value is $2(1-\Phi(|z|))$ in one dimension, solving the equation $\underline{P}(H_0|\mathbf{X}, \Gamma_{SSM})$ =P-value in π_0 produces

$$\pi_0(Z) = \begin{cases} 2(1-\Phi(|Z|) & \text{if } |Z| \leq 1 \\ \frac{1-\Phi(Z)}{1-\Phi(Z)+e^{-z^2/2}(2\Phi(|Z|)-1)/(2H(\hat{\sigma}))} & \text{if } |Z| > 1. \end{cases}$$

For the proof of Theorem 3 following shortly, we require the following Lemma. Let us define as a notational convenience

$$U(x, z) = e^{-(x-z)^2/2} + e^{-(x+z)^2/2}.$$

Lemma 7.

$$\max_{x>0} U(x, z) = \max_{x>0}\{e^{-(x-z)^2/2} + e^{-(x+z)^2/2}\}$$

is decreasing in z for $z > 1$.

Proof :

For a given $z > 1$, $\frac{\partial}{\partial x}U(x, z) = 0$ implies that $e^{2xz} = \frac{x+z}{x-z}$ where e^{2xz} is positive for $x > 0$. So the maximizing value of x has to be greater than

z. It will be proved that $\max_{x>0} U(x, z_1) \geq \max_{x>0} U(x, z_2)$ for any given $1 < z_1 < z_2$. Note that

$$e^{2xz} = \frac{x+z}{x-z} \Leftrightarrow x - z = r(x, z),$$

where $r(x, z) = (x+z)e^{-2xz}$. Since $r(x, z)$ is decreasing in x for $x > z$ and $x - z$ is increasing in x, the maximizing value of x for a given z is unique. Let x_i maximize $U(x, z_i)$ for a given z_i, $i = 1, 2$, respectively. First, it will be shown that $x_1 \leq x_2$.

(i) Suppose that $x_1 \leq z_2$. Then since $x_2 > z_2$, it is clear that $x_1 < x_2$.

(ii) Suppose that $x_1 > z_2$. Now, we assume that $x_1 > x_2$. Then since for a given $x > 0$, $r(x, z)$ is decreasing in z for $z \leq x$, $r(z_2, z_1) > r(z_2, z_2)$. Also, $x - z_2$ is on the right side of $x - z_1$ with the same slope. Consequently, $r(x, z_2)$ decreases at most as fast as $r(x, z_1)$ decreases in x. But

$$\frac{\partial}{\partial z}\frac{\partial}{\partial x} r(x, z) = 4e^{-2xz}(x+z)(xz-1),$$

which is positive for $1 < z < x$. That is,

$$\frac{\partial}{\partial x} r(x, z_1) < \frac{\partial}{\partial x} r(x, z_2).$$

Since $r(x, z)$ is decreasing in x for $x > z$, $r(x, z_1)$ decreases faster than $r(x, z_2)$. So we have a contradiction. Hence $x_1 \leq x_2$.

Next, it will be proved that $U(x_1, z_1) \geq U(x_2, z_2)$. Note that

$$\begin{aligned} \max_{x>0} U(x, z) &= e^{-z^2/2} e^{-x^2/2} (e^{xz} + e^{-xz}) \\ &= e^{-z^2/2} e^{-x^2/2} \left(\sqrt{\frac{x+z}{x-z}} + \sqrt{\frac{x-z}{x+z}}\right) \\ &= e^{-z^2/2} e^{-x^2/2} \left(\frac{2x}{\sqrt{x^2 - z^2}}\right), \end{aligned}$$

which is decreasing in z for $z > x$ when $x > 0$ is given and is decreasing in x for $x > z$ when $z > 1$ is given. This completes proof. □

Theorem 3. *When the family of priors on H_1 is Γ_{SSM}, the median of the null distribution of $\pi_0(Z)$ is 1/2.*

Remark : Using the closed form formula for $\pi_0(Z)$, it is easy to find the density of $\pi_0(Z)$ itself. A typical picture is presented in Figure 3. The visual impression that the density of $\pi_0(Z)$ is unimodal is justified in an important special case in the appendix.

3 Distribution of the ratio of $\underline{P}(H_0|\underline{x}, \Gamma_{\mathcal{MVN}})$ and the P-value

In the previous section, we investigated the prior probability on H_0, π_0, required for the conflict between the infimum of the posterior probability of H_0 and the P-value to just arise. In a different approach to this, they will now be compared by their ratio for a fixed prior probability π_0. Let $R(\underline{z})$ denote the ratio of $\underline{P}(H_0|\underline{x}, \Gamma_{\mathcal{MVN}})$ to the P-value, i.e.,

$$R(\underline{z}) = \frac{\underline{P}(H_0|\underline{x}, \Gamma_{\mathcal{MVN}})}{P(\chi^2_p \geq \|z\|^2)}.$$

It is important to keep in mind that π_0 is now held fixed. Using the formula of section 2.1, one has

$$R(\underline{z}) = \begin{cases} \frac{\pi_0}{P(\chi^2_p \geq \|z\|^2)} & \text{if } \|z\|^2 \leq p \\ \{\{1 + \frac{1-\pi_0}{\pi_0}(e\|z\|^2/p)^{-p/2}\exp(\|z\|^2/2)\}P(\chi^2_p \geq \|z\|^2)\}^{-1} & \text{if } \|z\|^2 > p. \end{cases}$$

By using Lemma 1, it can be shown that $R(\underline{z})$ is increasing in $\|z\|^2$. As a consequence, one has the following result in Theorem 4. The plot of $R(z)$ is given in Figure 2 when $p = 1$.

Theorem 4. *For the null distribution of $R(\underline{Z})$ for any given π_0, the median is $2\pi_0$ and the first quartile is $\frac{4}{3}\pi_0$. Also, the third quartile is given by*

$$4\{1 + \frac{1-\pi_0}{\pi_0}(e\chi^2_{p,\frac{3}{4}}/p)^{-p/2}e^{\chi^2_{p,\frac{3}{4}}/2}\}^{-1},$$

where $\chi^2_{p,\frac{3}{4}}$ is the third quartile of the Chisquare distribution with p degrees of freedom.

The assertion of this result is intriguing. In particular, with $\pi_0 = 0.5$, the ratio will thus be larger than 1 in exactly 50% of a long sequence of experiments. Table 1 at the end gives the three quartiles, the means and the

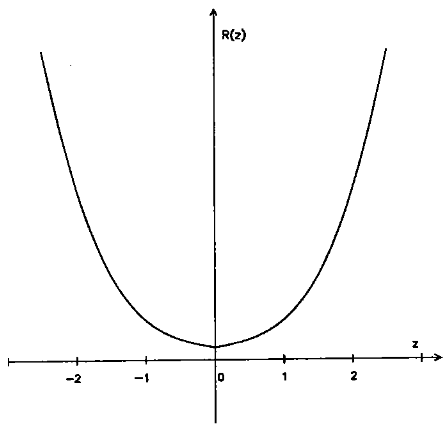

Figure 2: The Plot of ratio of the $\underset{\sim}{P}(H_0|\mathbf{x}, \Gamma_{NOR})$ to the P-value

standard deviations of $R(\underset{\sim}{Z})$ for various π_0 and p. Again, more is said later of the distribution of $R(Z)$ itself ; in particular, see Figure 4 and Proposition 6 which gives the result that the density of $R(Z)$ is monotone decreasing in an important special case. Table 2,3 and Table 4 give values for the important quantities $P_{H_0}(R(\underset{\sim}{Z}) > 1)$ and $P_{H_0}(\frac{1}{k} \leq R(\underset{\sim}{Z}) \leq k)$. In addition, the following very interesting result holds exactly.

Proposition 2. For any π_0 and any $p \geq 1$ such that $\pi_0 \geq P(\chi^2_p \geq p)$, one has $P_{H_0}(R(\underset{\sim}{Z}) > 1) = \pi_0$. In particular, for any $\pi_0 \geq 1/2$ and any $p \geq 1$, $P_{H_0}(R(\underset{\sim}{Z}) > 1) = \pi_0$.

Proof :

Immediate on noticing the fact that for $\|z\|^2 = p$,

$$\begin{aligned} R(\underset{\sim}{z}) &= \frac{\pi_0}{P(\chi^2_p \geq p)} \\ &> 1, \end{aligned}$$

for all $p \geq 1$ and $\pi_0 \geq 1/2$, since $P(\chi^2_p \geq p) < 1/2$ for all $p \geq 1$ and the fact that $R(\underset{\sim}{z})$ is monotone increasing in $\|z\|^2$. □

4 Discussion

4.1 Effect of dimension

It has been noticed that the Bayes-frequentist conflict is less in high dimensions (see Berger & Delampady (1987) and Delampady (1990)). To explore this issue a little, let us consider the limiting distribution of $\underset{\sim}{P}(H_0|\underset{\sim}{X}, \Gamma_{\mathcal{MVN}})$ under the null hypothesis as $p \to \infty$. Since the P-value is distributed *Uniformly* on [0, 1], we can let $\|Z\|^2 = F_p^{-1}(U)$ where F_p denotes the distribution function of the Chisquare distribution with p degrees of freedom and U denotes a $Uniform[0,1]$ random variable. Then (*in law*)

$$\underset{\sim}{P}(H_0|\underset{\sim}{X}, \Gamma_{\mathcal{MVN}}) = \begin{cases} \pi_0 & \text{if } U \leq F_p(p) \\ (1 + \frac{1-\pi_0}{\pi_0} \frac{\exp(F_p^{-1}(U)/2)}{(eF_p^{-1}(U)/p)^{p/2}})^{-1} & \text{if } U > F_p(p). \end{cases}$$

Let $f(u) = \frac{\exp(F_p^{-1}(u)/2)}{(eF_p^{-1}(u)/p)^{p/2}}$. Since the distribution of $(\chi_p^2 - p)/\sqrt{2p}$ converges to the standard Normal distribution,

$$F_p^{-1}(u) = \sqrt{2p}\Phi^{-1}(u) + p, \quad asymptotically \ \ as \ \ p \to \infty.$$

By using a *Taylor* expansion,

$$\lim_{p\to\infty} \log f(u) = (\Phi^{-1}(u))^2/2.$$

Also, $\lim_{p\to\infty} F_p(p) = 1/2$. Hence we have the following property.

Proposition 3.

$$\underset{\sim}{P}(H_0|\underset{\sim}{X}, \Gamma_{\mathcal{MVN}}) \to_{\mathcal{D}} Y, \quad as \ \ p \to \infty,$$

where, for $U \sim Uniform(0,1)$, Y is defined *in law* as

$$Y = \begin{cases} \pi_0 & \text{if } U \leq 1/2 \\ \{1 + \frac{1-\pi_0}{\pi_0} \exp(\Phi^{-1}(U))^2/2)\}^{-1} & \text{if } U > 1/2. \end{cases}$$

Very interestingly, the limiting distribution has a point mass at π_0. Similarly, the limiting distribution of $\pi_0(\underset{\sim}{Z})$, required for the equality of P-value and $\underset{\sim}{P}(H_0|\underset{\sim}{X}, \Gamma_{\mathcal{MVN}})$, is given as follows.

Proposition 4.

$$\pi_0(\underset{\sim}{Z}) \rightarrow_{\mathcal{D}} V, \quad as \ p \rightarrow \infty,$$

where, for $U \sim Uniform(0,1)$, V is defined (*in law*) as

$$V = \begin{cases} 1 - U & \text{if } U \leq 1/2 \\ \{1 + \frac{U}{1-U}\exp(-(\Phi^{-1}(U))^2/2)\}^{-1} & \text{if } U > 1/2. \end{cases}$$

For the distribution of V under the null hypothesis, the median and the third quartile are 1/2 and 3/4, respectively and the first quartile is $\{1 + 3\exp((\Phi^{-1}(3/4))^2/2)\}^{-1}$ which is approximately 0.295. So the conflict between the P-value and $\underset{\sim}{P}(H_0|\underset{\sim}{X}, \Gamma_{\mathcal{MVN}})$ still exists in the limit, but somewhat marginal (the case of no conflict corresponding to a uniform distribution for $\pi_0(\underset{\sim}{Z})$ in the limit).

4.2 Miscellany

Finally, some interesting properties in one dimension will now be discussed very briefly. Consider the family of Normal priors on H_1, Γ_{NOR}. Then the prior probability on H_0, required for the equality of P-value and the infimum of the posterior probability of H_0, is given by

$$\pi_0(Z) = \begin{cases} 2(1 - \Phi(|Z|)) & \text{if } |Z| \leq 1 \\ (1 - \Phi(|Z|))\{1 - \Phi(|Z|) + \sqrt{e}|Z|e^{-Z^2/2}(2\Phi(|Z|) - 1)\}^{-1} & \text{if } |Z| > 1. \end{cases}$$

Also, the ratio of $\underset{\sim}{P}(H_0|\mathrm{x}, \Gamma_{NOR})$ to the P-value is

$$R(z) = \begin{cases} \frac{\pi_0}{2(1-\Phi(|z|))} & \text{if } |z| \leq 1 \\ \{2(1 - \Phi(|z|))(1 + \frac{1-\pi_0}{\pi_0}\frac{e^{z^2/2}}{\sqrt{e}|z|})\}^{-1} & \text{if } |z| > 1. \end{cases}$$

1. From Theorem 1, the median and the third quartile of $\pi_0(Z)$ are exactly 0.5 and 0.75 respectively and the first quartile is approximately 0.254. Furthermore, the mean is approximately 0.508. Note that these values are almost or exactly the same as the corresponding values of the Uniform distribution on [0,1].

The next result gives an interesting mathematical property of the null distribution of $\pi_0(Z)$.

Proposition 5. The null distribution of $\pi_0(Z)$ is unimodal and the mode is approximately equal to 0.1356.

The proof is complex and is deferred till the appendix. The plot of the density functions of $\pi_0(Z)$ is given in Figure 3.

2. Let us also consider the null distribution of $R(Z)$ in more detail.

Proposition 6. The function $R(z) \geq \pi_0$ for all z. Furthermore, the density function of $Y = R(Z)$, is decreasing in y for $y \geq \pi_0$.

Again, the decreasing density property is interesting. We give the proof in the appendix. The plot of the density function of $R(Z)$ is given in Figure 4.

3. There is also some natural interest in the distribution of $\pi_0(Z)$ under the alternative hypothesis H_1. Note that under the alternative, as $n \to \infty$, $\sqrt{n}|\overline{X}| \to \infty$ almost surely and as $x \to \infty$, $1 - \Phi(x) \sim \frac{1}{x}\phi(x)$ (see Feller(1973)). So

$$\frac{|\sqrt{n}\overline{X}|(1 - \Phi(|\sqrt{n}\ \overline{X}|))}{\phi(|\sqrt{n}\overline{X}|)} \to 1$$

almost surely, as $n \to \infty$. Thus,

$$n\pi_0(\sqrt{n}|\overline{X}|) \to \frac{1}{\sqrt{e}\sqrt{2\pi\theta^2}}$$

almost surely as $n \to \infty$ for the true θ $(\neq 0)$. In a similar fashion, as $n \to \infty$,

$$\frac{R(Z)}{n} \to \frac{\pi_0}{1 - \pi_0}\sqrt{2e\pi\theta^2},$$

almost surely under the true θ. These two results are saying that the more seriously H_0 is false, the more pronounced is the conflict (a smaller and smaller prior probability is required for $\underline{P}(H_0)$ to equal the P-value, etc.). One can regard these as remanifestations of the Lindley phenomenon (see Lindley(1957)).

5 Appendix : Proofs

We will verify Theorems 1,2,3 and 4 together with Lemmas 4 and 5 and Propositions 5 and 6.

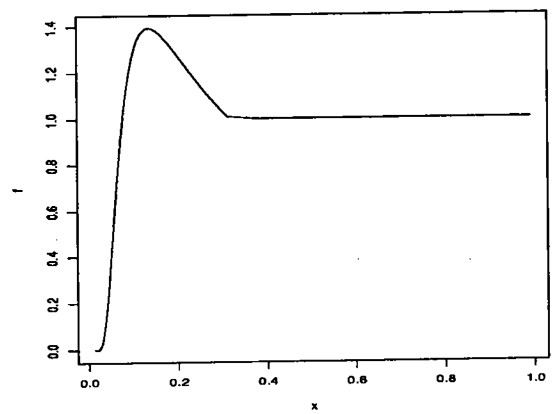

Figure 3: The density function of $\pi_0(Z)$

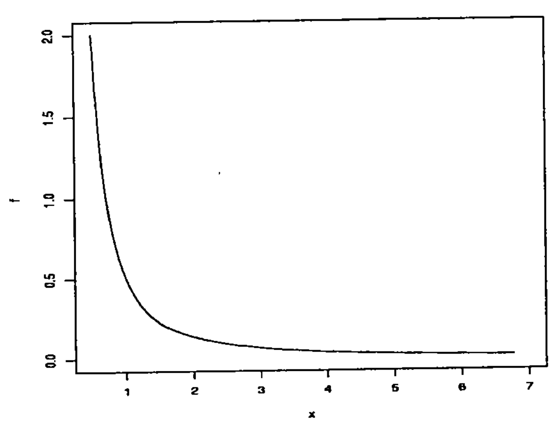

Figure 4: The density function of $R(Z)$ when $\pi_0 = 0.5$

Proof of Theorem 1 :

Since $\pi_0(\underset{\sim}{Z})$ is decreasing in $\|Z\|^2$ by Lemma 2 and for $\underset{\sim}{z}$ such that $\|z\|^2 = p$, $\pi_0(\underset{\sim}{z}) = P(\chi_p^2 \geq p) < \frac{1}{2}$,

$$P(\pi_0(\underset{\sim}{z}) > \frac{1}{2}) = P(P(\chi_p^2 \geq \|z\|^2) > \frac{1}{2}). \tag{2}$$

(Geometric thinking helps understand 2 ; see figure 3). Since the P-value $= P(\chi_p^2 \geq \|Z\|^2)$ is distributed as *Uniform*(0,1), (2) $= \frac{1}{2}$. Similarly,the third quartile is $\frac{3}{4}$. Now, let $q_{1,p}$ denote the first quartile of the null distribution of $\pi_0(\underset{\sim}{Z})$ for any given p. First we show $q_{1,p} < P(\chi_p^2 \geq p)$. If not, then one must have

$$\begin{aligned} \frac{1}{4} &= P(\pi_0(\underset{\sim}{z}) < q_{1,p}) \\ &\geq P(\|Z\|^2 \geq p) \\ &= P(\chi_p^2 \geq p). \end{aligned}$$

But $P(\chi_p^2 \geq p) > \frac{1}{4}$ for all $p \geq 1$ because $P(\chi_p^2 \geq p)$ is increasing in p. Thus $q_{1,p} < P(\chi_p^2 \geq p)$. Hence

$$\begin{aligned} P(\pi_0(\underset{\sim}{Z}) \leq q_{1,p}) &= P(\eta(\|Z\|^2) \leq q_{1,p}) \\ &= P(\|Z\|^2 \geq \eta^{-1}(q_{1,p})), \end{aligned}$$

which implies that $\eta^{-1}(q_{1,p}) = \chi^2_{p,\frac{3}{4}} \Leftrightarrow q_{1,p} = \eta(\chi^2_{p,\frac{3}{4}})$, where $\chi^2_{p,\frac{3}{4}}$ is third quartile of the Chisquare distribution with p degrees of freedom. This completes the proof. □

Proof of Lemma 4 :

Since

$$\lim_{r\to 0} \frac{1}{V(r)} \int_{\|\theta\|^2 \leq r} \exp(-\|\theta - z\|^2/2) d\theta = \exp(-\|z\|^2/2),$$

$\lim_{r\to 0} \inf_{g\in\Gamma_{USS}} B_g(\underset{\sim}{z}) = 1$. So it is sufficient to show that

$$\frac{1}{V(r)} \int_{\|\theta\|^2 \leq r} \exp(-\|\theta - z\|^2/2) d\theta \tag{3}$$

is decreasing in $r > 0$, for $\|z\|^2 \leq p$. Let

$$f(r) = \frac{1}{r^{p/2}} \int_{\|\theta\|^2 \leq r} (2\pi)^{-p/2} \exp(-\|\theta - z\|^2/2) d\theta.$$

Then since the expression in (3) is $f(r^2)$, we will show that $f(r)$ is decreasing in r if $\|z\|^2 \leq p$. Note that

$$\int_{\|\theta\|^2 \leq r} (2\pi)^{-p/2} \exp(-\|\theta - z\|^2/2) d\theta = P(\theta^t\theta \leq r),$$

where $\underset{\sim}{\theta} \sim \mathcal{MVN}(\underset{\sim}{z},\ \mathrm{I}_p)$. Thus $\theta^t\theta$ has a noncentral chisquare distribution with p degrees of freedom and noncentrality parameter $t = \|z\|^2$. Hence

$$\begin{aligned} P(\theta^t\theta \leq r) &= e^{-t/2} \sum_{m=0}^{\infty} \frac{(\frac{t}{2})^m}{m!} \int_0^r \frac{1}{2^{p/2+m}\Gamma(p/2+m)} x^{p/2+m-1} e^{-x/2} dx \\ &= \frac{1}{2} \int_0^r e^{-(x+t)} (\frac{x}{t})^{p/4-1/2} I_{p/2-1}(\sqrt{tx}) dx, \end{aligned}$$

where $I_k(x)$ is the *modified Bessel function* of the first kind and order k (see Tranter(1969)). Then

$$\begin{aligned} \frac{\partial}{\partial r} f(r) &= \frac{1}{2} r^{-p/2-1} [-\frac{p}{2} \int_0^r e^{-(x+t)/2} (\frac{x}{t})^{p/4-1/2} I_{p/2-1}(\sqrt{tx}) dx \\ &\quad + r e^{-(r+t)/2} (\frac{r}{t})^{p/4-1/2} I_{p/2-1}(\sqrt{tr})]. \end{aligned}$$

So

$$\frac{\partial}{\partial r} f(r) < 0 \Leftrightarrow -u(r) + w(r) < 0,$$

where

$$u(r) = \frac{p}{2} \int_0^r e^{-x/2} (\sqrt{x})^{p/2-1} I_{p/2-1}(\sqrt{tx}) dx$$

and

$$w(r) = r e^{-r/2} (\sqrt{r})^{p/2-1} I_{p/2-1}(\sqrt{tr}).$$

Let us set $\nu = p/2 - 1$. The rest of the proof consists of showing that for any $p \geq 1$, $-u(r) + w(r)$ is decreasing in $r > 0$ if $t = \|z\|^2 \leq p$ and that $u(0) - w(0) = 0$. Of these, the second step is trivial. For the first step, note that

$$\frac{\partial}{\partial r} u(r) = \frac{p}{2} e^{-r/2} (\sqrt{r})^{\nu} I_{\nu}(\sqrt{tr})$$

and

$$
\begin{aligned}
\frac{\partial}{\partial r}w(r) &= e^{-r/2}(\sqrt{r})^{\nu}I_{\nu}(\sqrt{tr}) \\
&\quad +re^{-r/2}(\sqrt{r})^{\nu}[-\frac{1}{2}I_{\nu}(\sqrt{tr})+(\frac{p}{4}-\frac{1}{2})\frac{1}{r}I_{\nu}(\sqrt{tr})+\frac{\sqrt{t}}{2\sqrt{r}}I'_{\nu}(\sqrt{tr})],
\end{aligned}
$$

where $I'_{\nu}(\sqrt{tr})=\frac{\partial}{\partial x}I_{\nu}(x)$ at $x=\sqrt{tr}$. Then the sign of $\frac{\partial}{\partial r}(-u(r)+w(r))$ is

$$
sgn\{-\frac{\nu}{2}I_{\nu}(\sqrt{tr})-\frac{r}{2}I_{\nu}(\sqrt{tr})+\frac{\sqrt{tr}}{2}I'_{\nu}(\sqrt{tr})\}. \tag{4}
$$

Now, since (see Tranter(1969)) $I'_{\nu}(x)=\frac{\nu}{x}I_{\nu}(x)+I_{\nu+1}(x)$, the expression in (4) equals

$$
\begin{aligned}
& -\frac{r}{2}I_{\nu}(\sqrt{tr})+\frac{\sqrt{tr}}{2}I_{\nu+1}(\sqrt{tr}) \\
= & \{\frac{r}{2}\sum_{m=0}^{\infty}\frac{(\sqrt{tr})^{2m+\nu}}{2^{2m+\nu}m!\Gamma(m+\nu+1)}(-1+\frac{t}{2(m+\nu+1)})\}.
\end{aligned}
$$

But $t/(2m+2(\nu+1))\leq 1$ if $t\leq 2(\nu+1)(=p)$ for all $m\geq 0$. Hence $-u(r)+w(r)$ is decreasing in $r>0$ if $\|z\|^2\leq p$. This proves the first step. □

Proof of Theorem 2 :

Since the P-value $=P(\chi_p^2\geq\|z\|^2)$ is *Uniformly* distributed on [0,1], if we can show that $\pi_0(\underset{\sim}{z})\leq\frac{1}{2}$ for all $\|z\|^2>p$, it will follow as before that $P(\pi_0(\underset{\sim}{Z})\geq\frac{1}{2})=P(P(\chi_p^2\geq\|Z\|^2)\geq\frac{1}{2})=\frac{1}{2}$ and also, $P(\pi_0(\underset{\sim}{Z})\geq\frac{3}{4})=\frac{1}{4}$. So we will show that $\pi_0(\underset{\sim}{z})\leq\frac{1}{2}$ for all $\|z\|^2>p$ which is equivalent to $\delta(t)>1$, where $\delta(t)$ is defined as (1) in section (2.2). Note that $\delta(t)=\delta_1(t)P(\chi_p^2\leq t)/(t^{p/2-1}Q(t))$, where

$$
\delta_1(t)=\frac{t^{p/2-1}e^{-t/2}}{P(\chi_p^2\geq t)}.
$$

Now, since $\delta_1(t)$ is increasing in $t>0$ by Lemma 1, $\delta_1(t)>\delta_1(p)$ for $t>p$. Let us consider now an upper bound of $t^{p/2-1}Q(t)$. Towards this end, note that

$$\begin{aligned} & t^{p/2-1}\frac{1}{V(r)}\int_{\|\theta\|\le r}\exp(-\|\theta - z\|^2/2)d\theta \\ = & \frac{p\Gamma(p/2)2^{p/2}}{2r^p}\sum_{j=0}^{\infty} f_{p+2j}(t)\frac{1}{j!2^j}\int_0^{r^2} x^{p/2+j-1}e^{-x/2}dx \\ = & I, \ \ say, \end{aligned}$$

where

$$f_{p+2j}(t) = \frac{e^{-t/2}t^{p/2+j-1}}{\Gamma(p/2+j)2^{p/2+j}},$$

the density function of the Chisquare distribution with $p+2j$ degrees of freedom. But note that $f_{p+2j}(t) \le f_p(p)$ for all $j \ge 0$ and $t > p$. Thus

$$\begin{aligned} I \ \le & \ \frac{p\Gamma(p/2)2^{p/2}}{2r^p} f_p(p)\sum_{j=0}^{\infty}\frac{1}{j!2^j}\int_0^{r^2} x^{p/2+j-1}e^{-x/2}dx \\ = & \ \frac{p\Gamma(p/2)2^{p/2}}{2r^p} f_p(p)\int_0^{r^2} x^{p/2-1}e^{-x/2}\sum_{j=0}^{\infty}\frac{(x/2)^j}{j!}dx \\ = & \ p^{p/2-1}e^{-p/2}. \end{aligned}$$

Hence

$$\delta(t) \ge \delta_1(p)\frac{P(\chi_p^2 \le p)}{p^{p/2-1}e^{-p/2}},$$

where $\delta_1(p) = \frac{p^{p/2-1}e^{-p/2}}{P(\chi_p^2 \ge p)}$. So

$$\delta(t) \ge P(\chi_p^2 \le p)/P(\chi_p^2 \ge p),$$

which is greater than 1 because $P(\chi_p^2 \le p) \ge \frac{1}{2}$. This completes the proof as explained earlier. □

Proof of Lemma 5 :
Note that

$$u'(x) = -(x-z)e^{-(x-z)^2/2} - (x+z)e^{-(x+z)^2/2}.$$

Let $v_1(x) = -(x-z)e^{-(x-z)^2/2}$ and let $v_2(x) = (x+z)e^{-(x+z)^2/2}$. Without loss of generality assume $z > 0$. Then

$$\lim_{x\to-\infty} v_1(x) = \lim_{x\to\infty} v_1(x) = 0.$$

Also,

$$\frac{\partial}{\partial x}v_1(x) = 0 \Leftrightarrow x = z-1 \quad or \quad x = z+1$$

and

$$v_1(z-1) > 0 \quad and \quad v_1(z+1) < 0.$$

Since $v_1(x) = v_2(-x)$, it is easy to see that v_1 and v_2 meet only once at 0 if $z \le 1$ and they cross three times at $-\zeta$, 0, ζ for some $0 < \zeta < z$ if $z > 1$. Furthermore, for $z \le 1$, $u'(x) < 0$ for $x > 0$. Thus $\frac{\partial}{\partial\sigma}H(\sigma) \le 0$ for $z \le 1$. For $z > 1$, $u'(x)$ changes sign once and in the direction from the positive to the negative if at all for $x > 0$. We will prove that $\frac{\partial}{\partial\sigma}H(\sigma)$ changes sign exactly once by showing $\frac{\partial}{\partial\sigma}H(\sigma) < 0$ for all $\sigma > 0$ or $\frac{\partial}{\partial\sigma}H(\sigma) \ge 0$ for all $\sigma > 0$ are each impossible. Furthermore, each statement will be proved by contradiction.

(i)Suppose that $\frac{\partial}{\partial\sigma}H(\sigma) < 0$ for all $\sigma > 0$. Then $\sup_{\sigma>0} H(\sigma) = \frac{1}{2}u(0)$. But since $u'(x)$ changes sign in the direction of the positive to the negative, $u(x)$ is maximized at $\zeta > 0$. Then

$$\begin{aligned}\sup_{\sigma>0} H(\sigma) &= \sup_{\sigma>0}\int_0^\infty f(y)u(\sigma y)dy \\ &= \int_0^\infty f(y)\max_{x>0} u(x)dy \\ &= \frac{1}{2}u(\zeta) \\ &\ne \frac{1}{2}u(0).\end{aligned}$$

(ii)Suppose that $\frac{\partial}{\partial\sigma}H(\sigma) \ge 0$ for all $\sigma > 0$. Then

$$\begin{aligned}\sup_{\sigma>0} H(\sigma) &= \lim_{\sigma\to\infty} H(\sigma) \\ &= \lim_{\sigma\to\infty}\int_0^\infty f(y)u(\sigma y)dy \\ &= 0.\end{aligned}$$

This implies that $H(\sigma) = 0$ for all σ, which is impossible.
Hence $\frac{\partial}{\partial \sigma} H(\sigma)$ changes sign exactly once. □

Proof of Theorem 3 :

Since the P-value $= P(\chi_p^2 \geq \|z\|^2)$ is *Uniformly* distributed on [0,1], if we show that $\pi_0(z)$ is decreasing in $\|z\|^2$, our result will follow as before. Note that for $|z| > 1$,

$$\pi_0(z) = \{1 + \frac{e^{-z^2/2}(2\Phi(|z|) - 1)}{(1 - \Phi(|z|))(2H(\hat{\sigma}))}\}^{-1}.$$

It is easy to verify by taking derivatives that

$$\frac{e^{-z^2/2}(2\Phi(|z|) - 1)}{1 - \Phi(|z|))}$$

is increasing in $z > 0$. Next, we will show that $H(\hat{\sigma}) = \max_{\sigma>0} H(\sigma)$ is decreasing in z for $z > 1$. But

$$\begin{aligned} \max_{\sigma>0} H(\sigma) &= \max_{\sigma>0} \int_0^\infty f(y) u(\sigma y) dy \\ &= \int_0^\infty f(y) \max_{x>0} u(x) dy \end{aligned}$$

Then since $\max_{x>0} u(x)$ is decreasing in z for $z > 1$ by virtue of Lemma 7, the proof is complete. □

Proof of Theorem 4:

Since $P(\chi_p^2 \geq p) < 1/2$ and $R(\underset{\sim}{z})$ is increasing in $\|z\|^2$, $R(\underset{\sim}{z}) > 2\pi_0$ if $\|z\|^2 > p$ for any given π_0. Hence

$$\begin{aligned} P(R(\underset{\sim}{Z}) \leq 2\pi_0) &= P(\frac{\pi_0}{P(\chi_p^2 \geq \|Z\|^2)} \leq 2\pi_0) \\ &= \frac{1}{2}. \end{aligned}$$

Hence the median of the distribution of $R(\underset{\sim}{Z})$ is $2\pi_0$ for any given π_0. Similarly, it can be shown that the first quartile is $\frac{4}{3}\pi_0$. Finally, let $q_{3,p}$ denote the third quartile of the distribution. Since $P(\chi_p^2 \leq p)$ is decreasing in p,

$$\begin{aligned} P(R(\underset{\sim}{Z}) \leq \frac{\pi_0}{P(\chi_p^2 \geq p)}) &= P(\|Z\|^2 \leq p) \\ &= P(\chi_p^2 \leq p) \\ &< \frac{3}{4} \end{aligned}$$

for any given p. So $q_{3,p} > \pi_0/P(\chi_p^2 \geq p)$. Thus

$$\begin{aligned} & P(R(\underset{\sim}{Z}) < q_{3,p}) = \frac{3}{4} \\ \iff & P(R(\underset{\sim}{Z}) > q_{3,p}) = \frac{1}{4} \\ \iff & P(\psi(\|Z\|^2) \geq q_{3,p}) = \frac{1}{4}, \end{aligned} \tag{5}$$

where

$$\psi(\|z\|^2) = \{\{1 + \frac{1-\pi_0}{\pi_0}(e\|z\|^2/p)^{-p/2}\exp(\|z\|^2/2)\}P(\chi_p^2 \geq \|z\|^2)\}^{-1}.$$

Since $\psi(\|z\|^2)$ is increasing in $\|z\|^2$, (5) implies $q_{3,p} = \psi(\sqrt{\chi^2_{p,\frac{3}{4}}})$, where $\chi^2_{p,\frac{3}{4}}$ is the third quartile of Chisquare distribution with p degrees of freedom. □

Proof of Proposition 5:

A detailed proof is available in Oh(1993); it is indeed involved. The main steps are the following :

Let $f(y)$ denote the density function of the distribution of $Y = \pi_0(Z)$. It is obvious that f is constant on $[\xi, 1]$ by the fact that the P-value$= 2(1-\Phi(|Z|))$ has a uniform distribution on [0,1], where $\xi = 2(1-\Phi(1))$. That is, $f(y) = 1$ for $y \geq \xi$. Furthermore,

$$f(y) = -2\phi(\eta^{-1}(y))\frac{1}{\eta'(\eta^{-1}(y)} \quad \text{for } 0 \leq y \leq \xi,$$

where

$$\eta(w) = \{1 + \sqrt{e}we^{-w^2/2}(2\Phi(w)-1)/(1-\Phi(w))\}^{-1}.$$

Since $\eta(w)$ is a decreasing function of w for $w > 1$ by Lemma 2, $\eta^{-1}(y)$ is also a decreasing function of $y > 0$. Then the shape of $f(y)$ for $0 < y < \xi$ is the same as the reversed shape of $\gamma(y)$ for $1 < y < \infty$, where

$$\gamma(y) = -2\phi(y)\frac{1}{\eta'(y)}.$$

Thus, it is sufficient to show that $\gamma(y)$ is a unimodal function for $y > 1$. Setting $k = \frac{\sqrt{e}}{2}\sqrt{2\pi}$,

$$\gamma(y) = \frac{2}{k}\frac{(1-\Phi(y)+ky\phi(y)(2\Phi(y)-1))^2}{(1-y^2)(1-\Phi(y))(2\Phi(y)-1)+y\phi(y)}.$$

<u>step 1</u> Let $a(y) = 1-\Phi(y)$ and let $b(y) = 2\Phi(y)-1$. Then the sign of $\gamma'(y)$,

$$sgn(\gamma'(y)) = sgn\{A(y) + k\phi(y)(1-y^2)B(y)\},$$

where

$$A(y) = -y\phi(y)^2 + 2ky^2\phi(y)^3 + ya(y)^2b(y)k\phi(y)a(y)b(y)^2$$

and

$$B(y) = y\phi(y)b(y) - y^2a(y)b(y)^2 - \frac{a(y)}{k}.$$

<u>step 2</u> There is a M such that $0 < M < \infty$ and if $y \geq M$, then $A < 0$ and $B > 0$. So $sgn(\gamma'(y)) < 0$ if $y \geq M$. In this step, it is used that (see Feller(1973))

$$\begin{aligned} a(y) &= 1-\Phi(y) \\ &< \frac{\phi(y)}{y} \end{aligned}$$

and

$$a(y) < \phi(y)(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5}) .$$

<u>step 3</u> For $0 \leq y < M$, it is shown that $\gamma(y)$ has a unique maxima at $\hat{y}$, where $\hat{y}$ is approximately equal to 1.65279. This implies that f is unimodal and the mode is $h(\hat{y})$, which is about 0.135634. Moreover,

$$\gamma(y) < \frac{(1-\Phi(y))(1+ky\phi(y)\frac{(2\Phi(y)-1)}{1-\Phi(y)})}{2\Phi(y)-1)+2y^2(1-\Phi(y))}$$

which goes to 0 as y goes to ∞. So, $\lim_{x\to 0} f(x) = 0$. □

Proof of Proposition 6 :

Again, a detailed proof is available in Oh(1993). The main steps are the following :

Let $f_R(y)$ denote the density of the null distribution of $R(Z)$. Then

$$f_R(y) = 2\phi(R^{-1}(y))\frac{1}{R'(R^{-1}(y))}.$$

It is easy to see that $f_R(y) = \pi_0/y^2$ for $\pi_0 \le y \le \pi_0/(2(1-\Phi(1)))$. Now, let $g(y) = 2\phi(y)\frac{1}{R'(y)}$, $y > 1$. Since R is increasing, the shape of $f_R(y)$ for $y > \pi_0/(2(1-\Phi(1)))$ is the same as that of $g(y)$ for $y > 1$. It will be shown that $g(y)$ is a decreasing function. Substituting the formula for $R'(y)$ to $g(y)$, we get

$$g(y) = \frac{4\phi(y)(1-\Phi(y))^2(1+ce^{y^2/2}/y)^2}{\phi(y)(1+ce^{y^2}/y) - c(1-\Phi(y))e^{y^2/2}(y^2-1)/y^2},$$

where $c = \frac{1-\pi_0}{\pi_0\sqrt{e}}$. Then

$$\begin{aligned} sgn(g'(y)) &= sgn\{c(1-\Phi(y))^2\frac{e^{y^2/2}(y^2-1)}{y} - \phi(y)^2(1+2c\frac{e^{y^2/2}}{y}+c^2\frac{e^{y^2}}{y^2}) \\ &\quad +c\phi(y)(1-\Phi(y))\frac{e^{y^2/2}(y^2-1)}{y^2} \\ &\quad +c^2\phi(y)(1-\Phi(y))\frac{e^{y^2}(y^2-1)}{y^3} + c(1-\Phi(y))^2\frac{e^{y^2/2}}{y^3}\} \\ &= sgn\{I\}, \quad say. \end{aligned}$$

Since $(1-\Phi(y)) < \phi(y)/y$ for $y > 0$, we have

$$\begin{aligned} I &< -c\phi(y)^2\frac{e^{y^2/2}}{y^3} - \phi(y)^2 - c\phi(y)^2\frac{e^{y^2/2}}{y^3} - c^2\phi(y)^2\frac{e^{y^2}}{y^4} \\ &\quad -c^2(1-\Phi(y))^2\frac{e^{y^2}}{y^2} + c\phi(y)^2\frac{e^{y^2}}{y^5} \\ &< 0, \end{aligned}$$

Q.E.D. □

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Table 1: Summary of the distribution of $R(\underset{\sim}{Z})$ for given p

p	π_0	$q_1 = \frac{4}{3}\pi_0$	$median = 2\pi_0$	q_3	mean	standard dev.
1	0.1	0.1333	0.2	0.392296	0.331084	0.331402
	0.2	0.266667	0.4	0.786274	0.680976	0.721988
	0.3	0.4	0.6	1.18195	1.0543	1.19204
	0.4	0.53333	0.8	1.57932	1.45775	1.77316
	0.5	0.666667	1.0	1.97841	1.90153	2.51812
	0.6	0.8	1.2	2.37923	2.40247	3.52347
	0.7	0.93333	1.4	2.78179	2.99156	4.99142
	0.8	1.06667	1.6	3.1861	3.73694	7.44869
	0.9	1.2	1.8	3.59216	4.85038	13.0156
2	0.1	0.1333	0.2	0.379029	0.305551	0.273561
	0.2	0.266667	0.4	0.762499	0.629941	0.597835
	0.3	0.4	0.6	1.150491	0.977718	0.990307
	0.4	0.53333	0.8	1.543083	1.35545	1.478234
	0.5	0.666667	1.0	1.940357	1.77319	2.10704
	0.6	0.8	1.2	2.342399	2.24747	2.959827
	0.7	0.93333	1.4	2.749295	2.80882	4.210419
	0.8	1.06667	1.6	3.161133	3.52438	6.310970
	0.9	1.2	1.8	3.578004	4.60313	11.077554

[continued]

p	π_0	$q_1 = \frac{4}{3}\pi_0$	$median = 2\pi_0$	q_3	mean	standard dev.
3	0.1	0.1333	0.2	0.371244	0.288444	0.253819
	0.2	0.266667	0.4	0.748467	0.596536	0.553712
	0.3	0.4	0.6	1.13181	0.92889	0.915955
	0.4	0.53333	0.8	1.52144	1.29214	1.365933
	0.5	0.666667	1.0	1.91749	1.69639	1.945945
	0.6	0.8	1.2	2.32013	2.15822	2.733347
	0.7	0.93333	1.4	2.72953	2.70807	3.889904
	0.8	1.06667	1.6	3.14585	3.41278	5.835943
	0.9	1.2	1.8	3.56928	4.48007	10.257338
5	0.1	0.1333	0.2	0.362418	0.268387	0.238783
	0.2	0.266667	0.4	0.732482	0.556884	0.518792
	0.3	0.4	0.6	1.110438	0.870238	0.854971
	0.4	0.53333	0.8	1.496541	1.21527	1.270573
	0.5	0.666667	1.0	1.891056	1.60232	1.804941
	0.6	0.8	1.2	2.294261	2.04834	2.529210
	0.7	0.93333	1.4	2.706448	2.5843	3.592978
	0.8	1.06667	1.6	3.127920	3.27781	5.384883
	0.9	1.2	1.8	3.558993	4.33736	9.462099

[continued]

p	π_0	$q_1 = \frac{4}{3}\pi_0$	$median = 2\pi_0$	q_3	mean	standard dev.
7	0.1	0.1333	0.2	0.35734	0.260892	0.228896
	0.2	0.266667	0.4	0.723251	0.541694	0.497002
	0.3	0.4	0.6	1.098044	0.847128	0.818605
	0.4	0.53333	0.8	1.482047	1.18398	1.216052
	0.5	0.666667	1.0	1.875604	1.56254	1.726690
	0.6	0.8	1.2	2.279075	1.99973	2.418872
	0.7	0.93333	1.4	2.692841	2.52645	3.435688
	0.8	1.06667	1.6	3.117300	3.2103	5.148891
	0.9	1.2	1.8	3.552872	4.25996	9.047361
10	0.1	0.1333	0.2	0.3526	0.256339	0.218418
	0.2	0.266667	0.4	0.714609	0.532352	0.474445
	0.3	0.4	0.6	1.08641	0.832712	0.781832
	0.4	0.53333	0.8	1.4684	1.16414	1.16207
	0.5	0.666667	1.0	1.86101	1.53684	1.6511
	0.6	0.8	1.2	2.26469	1.96756	2.31464
	0.7	0.93333	1.4	2.67991	2.48699	3.29021
	0.8	1.06667	1.6	3.10718	3.1622	4.93494
	0.9	1.2	1.8	3.54702	4.20057	8.67767

Table 2: $P(R(\underset{\sim}{Z}) > 1)$ for given π_0 and p

π_0	$p=1$	$p=3$	$p=5$	$p=7$	$p=10$
0.1	0.0499319	0.0247353	0.017106	0.0133107	0.0101689
0.2	0.184968	0.160925	0.150345	0.14399	0.137851
0.3	0.299715	0.291994	0.287059	0.283858	0.280638
0.4	0.4	0.4	0.399789	0.399297	0.398606
0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.6	0.6	0.6	0.6	0.6
0.7	0.7	0.7	0.7	0.7	0.7
0.8	0.8	0.8	0.8	0.8	0.8
0.9	0.9	0.9	0.9	0.9	0.9

Table 3: $P(1/1.1 < R(\underset{\sim}{Z}) < 1.1)$ for given π_0 and p

π_0	$p=1$	$p=3$	$p=5$	$p=7$	$p=10$
0.1	0.023203	0.0174094	0.0144012	0.0125348	0.0107241
0.2	0.0476125	0.0519383	0.0530215	0.0535289	0.05399338
0.3	0.0591603	0.0669701	0.0695881	0.0710193	0.0723246
0.4	0.0763636	0.0.0770048	0.0786186	0.0799068	0.0813215
0.5	0.0954545	0.0954545	0.0954545	0.0954545	0.0954545
0.6	0.114545	0.114545	0.114545	0.114545	0.114545
0.7	0.133636	0.133636	0.133636	0.133636	0.133636
0.8	0.152727	0.152727	0.152727	0.152727	0.152727
0.9	0.171818	0.171818	0.171818	0.171818	0.171818

Table 4: $P(1/1.2 < R(\underset{\sim}{Z}) < 1.2)$ for given π_0 and p

π_0	$p=1$	$p=3$	$p=5$	$p=7$	$p=10$
0.1	0.0445157	0.0337059	0.0281191	0.0246629	0.0213152
0.2	0.0909508	0.0992301	0.0101305	0.102276	0.103049
0.3	0.114288	0.127732	0.132721	0.135448	0.137935
0.4	0.146667	0.14942	0.152226	0.154163	0.156166
0.5	0.18333	0.18333	0.18333	0.183438	0.183735
0.6	0.22	0.22	0.22	0.22	0.22
0.7	0.256667	0.256667	0.256667	0.256667	0.256667
0.8	0.29333	0.29333	0.29333	0.29333	0.29333
0.9	0.25	0.25	0.25	0.25	0.25