THE EULER SCHEME FOR SDE'S DRIVEN BY SEMIMARTINGALES

by

Arturo Kohatsu and Philip Protter
University of Puerto Rico — Purdue University

Technical Report #94-7

Department of Statistics
Purdue University

April 1994

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Arturo Kohatsu[1] and Philip Protter[2]

1. Introduction

We consider here stochastic differential equations driven by semimartingales and the rate of convergence of the Euler scheme approximations to the solutions. It is already known [2] that Euler schemes converge, but a complete characterization of the rate of convergence is still open. We describe here the convergence rate for convergence in probability, in terms of the Lipschitz constant of the coefficients and more importantly the L^2 norm of the time changed increments of the driving semimartingales. In the important special case when the driving semimartingales are diffusions or Lévy processes, we obtain — under mild additional technical assumptions — the same rate of convergence as in the classical case of Brownian motion and Lebesgue measure as differentials.

For all unexplained notation and undefined terms we refer the reader to Protter [4]. A standard reference on rates of convergence for the "classical" case of stochastic differential equations driven by Brownian motion and Lebesgue measure is the recent book by Kloeden and Platen [1]. For a survey paper of recent results see, e.g., Talay [6].

We work only on the time interval $[0,1]$, and in our interpretation of the Euler scheme we follow [2]: let $0 \equiv \tau_0^n < \tau_1^n < \cdots < \tau_k^n \equiv 1$ be stopping times and define

$$\eta_n(t) = \tau_k^n \text{ for } \tau_k^n \leq t < \tau_{k+1}^n.$$

We of course assume $\lim_{n\to\infty} \sup_k \tau_{k+1}^n - \tau_k^n = 0$, a.s. We then have that the solution

$$X_t^n = X_0 + \int_0^t f(X_{\eta_n(s-)}^n) dY_s$$

represents the Euler scheme for the equation

$$X_t = X_0 + \int_0^t f(X_{s-}) dY_s.$$

[1]Supported in part by an ROA grant from NSF, awarded to Grant #9103454
[2]Supported in part by NSF Grant #9103454

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Acknowledgements:

The first author wishes to thank the Mathematics Department of Purdue University for its hospitality during his stay under an ROA grant from NSF; the second author wishes to thank Tom Kurtz and Denis Talay for discussions during his work on this paper, as well as Professor H. H. Kuo for his invitation to the U.S.–Japan Bilateral Seminar.

2. The Quasi-left Continuous Case

In this section, we assume the driving term of the stochastic differential equation is quasi–left continuous, which leads to a much cleaner theory. Quasi–left continuity (QLC) is often verified in practice: for example, Lévy processes are QLC and so are most strong Markov processes. (Any continuous process is *a fortiori* QLC.)

Definition.

A stochastic process Y is *quasi-left continuous* (QLC) if all of its jump times are totally inaccessible stopping times.

See [4, p. 99] for the definition of totally inaccessible stopping times. A canonical example is the jump times of a Poisson process, or more generally of a Lévy process.

Let Y be a QLC special semimartingale with canonical decomposition $Y = M+V$, where M is assumed to be locally square integrable and the finite variation term V is predictable. Let

$$A_t = \langle M, M\rangle_t + \int_0^t |dV_s| + t, \tag{2.1}$$

where $\langle M, M\rangle$ denotes the compensator of the quadratic variation process $[M, M]$ of M. Then A is adapted, continuous, and strictly increasing. Define

$$\gamma_u = \inf\{t > 0 : A_t > u\}. \tag{2.2}$$

Then γ_u is a stopping time for each u, and $u \to \gamma_u$ is also continuous and strictly increasing, and $A_{\gamma_u} = \gamma_{A_u} = u$.

Let F be a process Lipschitz coefficient with (non–random) Lipschitz constant k, and suppose further F is bounded by a constant c. Let X^i denote the unique solution of

$$X_t^i = U_t^i + \int_0^t F(X^i)_{s-} dY_s \tag{2.3}$$

where U^i are adapted, square integrable, càdlàg processes ($i = 1, 2$).

Lemma 2.4

With X^i as in (2.3) we have

$$E\left\{\sup_{s\leq\gamma(u)}|X_s^1-X_s^2|^2\right\}\leq 3e^{\beta(\alpha,k)u}E\left\{\sup_{s\leq\gamma(u)}|U_1^2-U_s^2|^2\right\}$$

where $\beta(a,k)=(12\vee 3\alpha)k^2$, (where "$\vee$" denotes "max") if $\int_0^t|dV_s|\leq\alpha<\infty$ a.s.

Proof:

Recall $Y=M+V$ is the canonical decomposition of Y. Then

$$|X_t^1-X_t^2|^2\leq 3|U_t^1-U_t^2|^2+3\left|\int_0^t F(X^1)_{s-}-F(X^2)_{s-}dM_{s-}\right|^2 + 3\left|\int_0^t F(X^1)_{s-}-F(X^2)_{s-}dV_{s-}\right|^2.$$

Let $Z_t=\sup_{s\leq t}|X_s^1-X_s^2|^2$, and recalling that $\gamma(u)$ is a stopping time, use Doob's martingale quadratic inequality on the second term and Jensen's inequality on the third on the right side above to get ($k=$ Lipschitz constant for F):

$$\begin{aligned}E\{Z_{\gamma(u)}\}&\leq 3E\{\sup_{s\leq\gamma(u)}|U_s^1-U_s^2|^2\}\\&\quad+12k^2E\left\{\int_0^{\gamma(u)}|X_{s-}^1-X_{s-}^2|^2d\langle M,M\rangle_s\right\}\\&\quad+3k^2\alpha E\left\{\int_0^{\gamma(u)}|X_{s-}^1-X_{s-}^2|\,|dV_s|\right\}\\&\leq 3E\{\sup_{s\leq\gamma_u}|U_s^1-U_s^2|^2\}+\beta(\alpha,k)E\left\{\int_0^{\gamma(u)}Z_{s-}dA_s\right\}\end{aligned}$$

where $A_t=\langle M,M\rangle_t+\int_0^t|dV_s|+t$, and $\beta(\alpha,k)=(12\vee 3\alpha)k^2$. By Lebesgue's change of time lemma (cf., e.g., [5, p. 379]) and Fubini's theorem we obtain

$$E\{Z_{\gamma(u)}\}\leq 3E\{\sup_{s\leq\gamma_u}|U_s^1-U_s^2|^2\}+\beta(\alpha,k)\int_0^u E\{Z_{\gamma_{(s-)}}\}ds.$$

Since $u\to E[\sup_{s\leq\gamma_u}|U_s^1-U_s^2|^2\}$ is an increasing function of u, by a version of Gronwall's inequalities (cf., e.g., [3, p. 359]) we have the result. □

Theorem 2.5

Let

$$X_t = X_0 + \int_0^t F(X)_{s-} dY_s,$$
$$X_t^n = X_0 + \int_0^t F(X^n)_{\eta^n(s-)} dY_s.$$

Assume Y is a quasi–left continuous, special semimartingale, and that if $Y = M+V$ is its canonical decomposition, then M is locally square integrable. Also assume F is process Lipschitz and bounded. Let

$$\rho_n(s) = E\{(Y_{\gamma(s-)} - Y_{\eta_n(\gamma(s-))})^2\},$$

$\overline{\rho}_n(t) = \int_0^t \rho_n(s)ds$, *and* $\overline{\rho}_n = \overline{\rho_n}(A_1)$. *Then $X^n - X$ converges to O in ucp at rate* $\sqrt{\overline{\rho}_n}$.

Proof:

Assume first that M is a square integrable martingale and $\int_0^t |dV_s| \leq \alpha$ a.s. we can rewrite

$$\begin{aligned} X_t^n &= X_0 + \int_0^t (F(X^n)_{\eta_n(s-)} - F(X^n)_{s-}) dY_s + \int_0^t F(X^n)_{s-} dY_s \\ &= X_0 + U_t^n + \int_0^t F(X^n)_{s-} dY_s. \end{aligned}$$

Using $(a+b)^2 \leq 2a^2 + 2b^2$, Doob's inequality, and the Lipschitz property of F:

$$\begin{aligned} E\{\sup_{s \leq \gamma_u} |U_s^n|^2\} &\leq (8 \vee 2\alpha) k^2 E\Big\{ \int_0^{\gamma_u} |X^n_{\eta_n(s-)} - X^n_{s-}|^2 dA_s \Big\} \\ &= (8 \vee 2\alpha) k^2 E\Big\{ \int_0^{1v} F(X^n_{\eta_n(s-)})^2 (Y_{s-} - Y_{\eta_n(s-)})^2 dA_s \Big\}, \end{aligned}$$

and since F is assumed bounded by a constant c:

$$\begin{aligned} &\leq (8 \vee 2\alpha) k^2 c E\Big\{ \int_0^{\gamma_u} (Y_s - Y_{\eta_n(s-)})^2 dA_s \Big\} \\ &= \gamma(\alpha, k, c) \int_0^u E\{(Y_{\gamma_s} - Y_{\eta_n(\gamma_{s-})})^2\} ds, \end{aligned}$$

where the last equality is by Lebesgue's change of time lemma and Fubini's theorem. Since the equation for X has $U^0 \equiv 0$, Lemma 2.4 implies

$$E\{\sup_{s \leq \gamma_u} |X_s - X^n_{\eta_n(s)}|^2\} \leq 3 e^{\beta(\alpha,k)u} \delta(\alpha, k, c) \int_0^u \rho_n(s) ds.$$

Since $\overline{\rho_n}(t) = \int_0^t \rho_n(s)ds$ and $A_1 \geq 1$ a.s., we have the result.

To remove the simplifying assumptions, choose $\varepsilon > 0$ and T so large that $P(T < A_1) < \varepsilon$, but M^T is a square integrable martingale and $\int_0^T |dV_s| \leq \alpha$ a.s., for some α, however large. (Recall that $t \to \int_0^t |dV_s|$ is continuous, since Y is QLC and V is predictable.) Let m_n be a sequence increasing to ∞. Then,

$$\begin{aligned}
&P\left(\sup_{s\leq 1} \frac{1}{m_n\sqrt{\overline{\rho_n}}}|X_s - X_s^n| > \delta\right) \\
&\leq P\left(\frac{1}{m_n\sqrt{\overline{\rho_n}}}(X^T - (X^n)^T)_1^* > \delta\right) + P(T < A_1) \\
&\leq \varepsilon + \frac{E\{\sup_{s\leq 1}|X_s^T - (X_s^n)^T|^2\}}{(m_n)^2\overline{\rho_n}\delta^2} \\
&\leq \varepsilon + \frac{3e^{\beta(\alpha,k)}\delta(\alpha,k,c)\overline{\rho_n}}{(m_n)^2\overline{\rho_n}\delta^2},
\end{aligned}$$

where we have used the Chebyshev–Markov inequality. Since the $\overline{\rho_n}$'s cancel, and since ε was arbitrary, we have

$$\frac{1}{m_n\sqrt{\overline{\rho_n}}}(X - X^n)_1^* \text{ tends to } O$$

in probability, hence $X - X^n$ tends to O in ucp at rate $\sqrt{\overline{\rho_n}}$. □

Example (2.6)

Let Y be a Lévy process such that $E(Y_t^2) < \infty$ each t, $Y_0 = 0$. Then Y has a canonical decomposition $Y_t = (Y_t - \xi t) + \xi t$, where ξ is a constant. (For example, Y having bounded jumps would more than suffice!) Then $A_t = \tau t$ for a constant $\tau > 1$, and $\gamma_u = \frac{1}{\tau}u$. One then has

$$\begin{aligned}
\rho_n(s) &= E\left\{\left(Y_{\gamma(s)} - Y_{\eta_n(\gamma(s-))}\right)^2\right\} \\
&= E\left\{\left(Y_{(\frac{s}{\tau})} - Y_{\eta_n(\frac{s}{\tau})}\right)^2\right\} \sim \frac{1}{n}
\end{aligned}$$

and therefore $\overline{\rho_n} \sim \frac{1}{n}$, which means that $X^n - X$ converges to O in ucp at rate $\frac{1}{\sqrt{n}}$, which is the same rate for solutions of stochastic differential equations driven by Brownian motion and Lebesgue measure.

Example (2.7)

Let Y be a diffusion that is the solution of a stochastic differential equation driven by standard Brownian motion and Lebesgue measure. Then since B_t and t have continuous paths, they are *a fortiori* QLC, and one easily checks that with mild assumptions on the coefficients that determine Y, that $X^n - X$ converges to O in ucp again at rate $\frac{1}{\sqrt{n}}$.

We note that the same proofs can be used for systems of SDEs, which yield the same rate of convergence.

3. The General Case

When the jumps of a semimartingale are not necessarily totally inaccessible, then the process A given in (2.1) need not be continuous, and the situation becomes more complicated. For simplicity we again assume we have one driving semimartingale Y and one equation; our results extend trivially to several driving semimartingales and systems.

(3.1)**Assumptions on the driving semimartingale Y:**

We assume Y is a special semimartingale with canonical decomposition $Y = M + V$, where M is assumed to be locally square integrable; the finite variation term V is predictable and also there exists a constant α such that $\int_0^1 |dV_s| \leq \alpha$ a.s. Finally assume Y has square-integrable jumps.

We consider the equations

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-})dY_s, \\ X_t^n &= X_0 + \int_0^t f(X^n_{\eta_n(s-)})dY_s \end{aligned} \tag{3.2}$$

where f is Lipschitz continuous with constant k and *bounded.* (η_n is defined in Section 1 and determines an Euler approximate scheme.)

Let ξ be a constant such that $(12 \vee 3\alpha)k^2 < \frac{1}{\xi}$. Let

$$A_t = \langle M, M\rangle_t + \int_0^t |dV_s| + t. \tag{3.3}$$

Then A is predictable, right continuous, and strictly increasing. Let $T_0 = 0$, and

$$T_{i+1} = \inf\{t > T_i : A_t - A_{T_i} \geq \xi\}. \tag{3.4}$$

Define

$$A_t^i = \langle M, M\rangle_t^{T^{i+1}-} - \langle M, M\rangle_t^{T^i} + \int_{t\wedge T^i}^{t\wedge T^{i+1}-} |dV_s| + (t\wedge T^{i+1} - t\wedge T^i)$$

where $Z^{T-} = Z_t 1_{(t<T)} + Z_{T-}1_{(t\geq T)}$, for a càdlàg process Z and random time $T>0$. (Of course, $s\wedge t = \min(s,t)$.) Next define

$$\gamma^i(u) = \inf\{t>0 : A_t^i > u\}, \tag{3.5}$$

so that γ^i is the right continuous inverse of A^i. Note that $\gamma^i(u)$ is a stopping time for each $u>0$. Let $Y^i = Y^{T^{i+1}-} - Y^{T_i}$ and define also:

$$\rho_n^i(s) = E\{(Y^i_{\gamma_i(s)-} - Y^i_{\eta_n(\gamma_i(s)-)})^2\}, \tag{3.6}$$

and note that $\rho_n^i(s) < \infty$. Also,

$$\overline{\rho}_n^i(s) = \rho_n^i(s) + \int_0^s \rho_n^i(r)dr$$

and

$$\overline{\rho}_n = \sum_{i=1}^{[\alpha/\xi]+1} \overline{\rho}_n^i(\xi), \tag{3.7}$$

where $[\alpha/\xi]$ denotes the integer part of α/ξ.

Theorem 3.8

Let f be Lipschitz continuous and bounded and assume assumptions (3.1) hold. Let X^n, X be solutions of (3.2). Then X^n converges to X uniformly in probability on $[0,1]$ at rate $\sqrt{\rho_n}$. That is, for any sequence m_n increasing to ∞,

$$\lim_{n\to\infty} \frac{1}{m_n\sqrt{\rho_n}} \sup_{t\leq 1} |X_t^n - X_t| = 0,$$

with convergence in probability.

Before proving Theorem (3.8) we establish three lemmas. With the same f and Y, it is convenient to consider more general equations:

$$\begin{cases} X_t^1 = U_t^1 + \int_0^t f(X_{s-}^1)dY_s, \\ X_t^2 = U_t^2 + \int_0^t f(X_{s-}^2)dY_s \end{cases} \tag{3.9}$$

where U^1, U^2 are adapted, càdlàg processes.

Lemma 3.10

Assume $\sup_t \Delta A_t \leq \xi$, *where* $\beta(\alpha,k)\xi < 1$, *and* $\beta(\alpha,k) = (12 \vee 3\alpha)k^2$. *Let* $Z_t = \sup_{s\leq t}|X_s^1 - X_s^2|^2$. *Then*

$$E\{Z_{\gamma(u)}\} \leq \frac{3}{1-\beta(\alpha,k)\xi}\exp\left(\frac{\beta(\alpha,k)}{1-\beta(\alpha,k)\xi}u\right)E\{\sup_{s\leq\gamma(u)}|U_s^1 - U_s^2|^2\}.$$

Proof:

Since

$$|X_t^1 - X_t^2| \leq |3U_t^1 - U_t^2|^2 + 3\left|\int_0^t f(X_{s-}^1) - f(X_{s-}^2)dM_s\right|^2 + 3\left|\int_0^t f(X_{s-}^1) - f(X_{s-}^2)dV_s\right|^2,$$

using Doob's quadratic martingale inequality and Jensen's inequality implies

$$E\{Z_{\gamma(u)}\} \leq 3E\{\sup_{s\leq\gamma(u)}|U_s^1 - U_s^2|^2\} + \beta(\alpha,k)E\left\{\int_0^{\gamma(u)} Z_{s-}dA_s\right\},$$

and by Lebesgue's change of time lemma and Fubini's theorem,

$$\leq 3E\{\sup_{s\leq\gamma(u)}|U_s^1 - U_s^2|^2\} + \beta(\alpha,k)E\{(A_{\gamma(u)} - u)Z_{\gamma(u)-}\} + \beta(\alpha,k)\int_0^u E\{Z_{\gamma(s-)}\}ds.$$

However, $0 \leq A\circ\gamma(u) - u \leq \sup_t \Delta A_t \leq \xi$, whence

$$E\{Z_{\gamma(u)}\} \leq 3E\{\sup_{s\leq\gamma(u)}|U_s^1 - U_s^2|^2\} + \beta(\alpha,k)\xi E\{Z_{\gamma(u)-}\} + \beta(\alpha,k)\int_0^u E\{Z_{\gamma(s)-}\}ds,$$

and since $\beta(\alpha,k)\xi < 1$ and $u \to Z_{\gamma(u)}$ is increasing in u, we obtain

$$\big(1-\beta(\alpha,k)\xi\big)E\{Z_{\gamma(u)}\} \leq 3E\{\sup_{s\leq\gamma(u)}|U_s^1 - U_s^2|^2\} + \beta(\alpha,k)\int_0^u E\{Z_{\gamma(s)-}\}ds.$$

The result now follows by multiplying by $\frac{1}{1-\beta(\alpha,k)\xi}$ and applying Gronwall's inequality. □

Lemma 3.11

Assume $Y = Y^{T_1-}$, $\rho_n(s) := \rho_n^1(s)$, $\gamma(u) := \gamma^1(u)$, *and* $c = \|f\|_{L^\infty}$, *where* T_1, ρ_n^1, *and* γ^1 *are given by (3.4), (3.6), and (3.5). Then,*

$$E\{\sup_{s\leq\gamma(u)} |X_s - X_s^n|^2\} \leq (8 \vee 2\alpha)k^2 c\Big\{\xi\rho_n(u) - \int_0^u \rho_n(s)ds\Big\}.$$

Proof:

We write the equation for X^n as:

$$X_t^n = X_0 + U_t^n + \int_0^t f(X_{s-}^n)dY_s,$$

where $U_t^n = \int_0^t f(X_{\eta_n(s-)}^n) - f(X_{s-}^n)dY_s$. In the corresponding equation for X take $U_s^0 \equiv 0$. By Lemma (3.10) it then suffices to estimate

$$E\{\sup_{s\leq\gamma(u)} |U_s^n|^2\} = E\{\sup_{s\leq\gamma(u)} |U_s^n - U_s^0|^2\}.$$

Using that f is Lipschitz with constant k and also bounded by c, and by Doob and Jensen's inequalities, we have:

$$\begin{aligned}
E\{\sup_{s\leq\gamma(u)} |U_s^n|^2\} &\leq (8k^2 \vee 2\alpha k^2)E\Big\{\int_0^{\gamma(u)} |X_{\eta_n(s-)}^n - X_{s-}^n|^2 dA_s\Big\} \\
&= (8 \vee 2\alpha)k^2 E\Big\{\int_0^{\gamma(u)} f(X_{\eta_n(s-)}^n)^2(Y_{s-} - Y_{\eta_n(s-)})^2 dA_s\Big\} \\
&\leq (8 \vee 2\alpha)k^2 cE\Big\{\int_0^{\gamma(u)} (Y_{s-} - Y_{\eta_n(s-)})^2 dA_s\Big\} \\
&\leq h(\alpha, k, c)\Big[E\{(A_{\gamma(u)} - u)(Y_{\gamma(u-)} - Y_{\gamma(\eta_n(u-))})^2\} \\
&\qquad + E\{\int_0^u (Y_{\gamma(s-)} - Y_{\eta_n(\gamma(s-))})^2 ds\}\Big],
\end{aligned}$$

where $h(\alpha, k, c) = (8 \vee 2\alpha)k^2 c$. Since $0 \leq A_{\gamma(u)} - u \leq \sup_t \Delta A_t \leq \sup_t A_t \leq \xi$, the preceding is:

$$\begin{aligned}
&\leq h(\alpha, k, c)\Big[\xi E\{(Y_{\gamma(u-)} - Y_{\gamma(\eta_n(u-))})^2\} \\
&\qquad + \int_0^u E\{(Y_{\gamma(s-)} - Y_{\eta_n(\gamma(s-))})^2\}ds\Big] \\
&= h(\alpha, k, c)\Big[\xi\rho_n(u) + \int_0^u \rho_n(s)ds\Big]. \quad \square
\end{aligned}$$

Lemma 3.12

Assume $Y = Y^{T_1}$. Then,

$$\lim_{n\to\infty} P(\sup_{s\leq\gamma(u)} |X_s - X_s^n| > \delta) = 0,$$

with rate of convergence $\sqrt{\overline{\rho}_n^1(u)}$.

Proof:

We have

$$\begin{aligned} &P(\sup_{s\leq\gamma(u)} |X_s - X_s^n| > \delta) \\ \leq\; &P(\sup_{s\leq\gamma(u)} |X_s^{T-} - (X_s^n)^{T-}| > \delta) + P(|X_R - X_R^n|^2 > \delta). \end{aligned} \tag{3.13}$$

where $R = \gamma(u) \wedge T$. The first term on the right side of (3.13) tends to O at rate $\sqrt{\overline{\rho_n}^1(u)}$ by Lemma (3.11) and Chebyshev's inequality. For the second term on the right side of (3.13), note that

$$X_T - X_T^n = X_{T-} - X_{T-}^n + \big(f(X_{T-}) - f(X_{\eta_n(T-)}^n)\big)\Delta Y_T,$$

and using the Lipschitz property of f:

$$\begin{aligned} |X_T - X_T^n| &\leq |X_{T-} - X_{T-}^n| + k|X_{T-} - X_{\eta_n(T-)}^n|\,|\Delta Y_T| \\ &\leq \sup_{s\leq\gamma(u)} |X_s^{T-} - (X_s^n)^{T-}|(1 + k|\Delta Y_T|), \end{aligned}$$

and since $1 + k|\Delta Y_T| < \infty$ a.s., $X_T - X_T^n$ tends to O in probability at least as fast as $\sup_{s\leq\gamma(u)} |X_s^{T-} - (X_s^n)^{T-}|$. □

Proof of Theorem 3.8:

Lemmas (3.10), (3.11), and (3.12) establish that for any sequence m_n increasing to ∞, if we assume $Y = Y^{T_1}$, then

$$\lim_{n\to\infty} \frac{1}{m_n\sqrt{\overline{\rho}_n^1(u)}} \sup_{t\leq\gamma^1(u)} |X_t^n - X_t| = 0, \tag{3.14}$$

with convergence in probability, for each $u > 0$. By definition of T_1 and γ^1, if we take $u \geq \xi$, then $\gamma^1(u) \geq T_1$ a.s. Thus we can rewrite (3.14) as:

$$\lim_{n\to\infty} \frac{1}{m_n\sqrt{\overline{\rho}_n^1(u)}} \sup_{t\leq 1} |X_{t\wedge T_1}^n - X_{t\wedge T_1}| = 0 \tag{3.15}$$

with convergence in probability. We now let $Y^2 = Y^{T_2-} - Y^{T_1}$, and consider the equations

$$X_t = X_{t\wedge T_1} + \int_0^t f(X_{s-})dY_s^2,$$
$$X_t^n = X_{t\wedge T_1}^n + \int_0^t f(X_{\eta_n(s-)}^n)dY_s^2, \tag{3.16}$$

which are the same as equations (3.2) but stopped at T_2-. We rewrite these equations as

$$X_t = U_t^0 + \int_0^t f(X_{s-})dY_s^2,$$
$$X_t^n = U_t^n + \int_0^t f(X_{s-}^n)dY_s^2.$$

Since Y is assumed to have square integrable jumps, we can immediately apply Lemma (3.10), and a slight modification of Lemmas (3.11) and (3.12) give the inductive step. This is a finite induction, only needing to be performed at most $[\alpha/\xi]+1$ times. Theorem (3.8) follows. □

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Arturo Kohatsu, CINVESTAV-IPN, Departamento de Matematicas, Avenida Instituto Politecnico Nacional 2508, Colonia san Pedro Zacatenco, Mexico D. F., MEXICO

Philip Protter, Mathematics and Statistics Departments, Purdue University, West Lafayette, IN 47907-1395, USA